6 Appendix

6.1 The PATIPPET filter

We let $\boldsymbol{\mu} = \begin{pmatrix} \bar{\phi} \\ \bar{\theta} \end{pmatrix}$ denote the posterior mean and $\mathbf{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}$ denote the posterior covariance. The expressions for the evolution of the PATIPPET filter, which we derive in the following section, are:

$$\begin{cases} d\boldsymbol{\mu} = & \begin{pmatrix} \bar{\theta} \\ 0 \end{pmatrix} dt + (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_t) \cdot (dN_t - \Lambda dt) \\ d\mathbf{V} = & \begin{pmatrix} 2V^{12} + \sigma^2 & V^{22} \\ V^{22} & \sigma_\theta^2 \end{pmatrix} dt + \left(\hat{\mathbf{V}} - \mathbf{V}_t\right) \cdot (dN_t - \Lambda dt) \end{cases} \tag{9}$$

where we define

$$\begin{cases} \Lambda := & \sum_{i=0,1,\dots} \Lambda_i \hat{\theta}_i \\ \hat{\boldsymbol{\mu}} = & \frac{1}{\Lambda} \sum_{i=0,1,\dots} \Lambda_i \begin{pmatrix} K_i^{12} + \hat{\phi}_i \hat{\theta}_i \\ K_i^{22} + \hat{\theta}_i^2 \end{pmatrix} \\ \hat{\mathbf{V}} := & \frac{1}{\Lambda} \sum_{i=0,1,\dots} \Lambda_i \Big(\hat{\theta}_i \mathbf{K}_i + \hat{\theta}_i (\hat{\boldsymbol{\mu}}_i - \boldsymbol{\mu}_{t+})(\hat{\boldsymbol{\mu}}_i - \boldsymbol{\mu}_{t+})^T \\ & + (\hat{\boldsymbol{\mu}}_i - \boldsymbol{\mu}_{t+}) \begin{pmatrix} K_i^{21} & K_i^{22} \end{pmatrix} + \begin{pmatrix} K_i^{12} \\ K_i^{22} \end{pmatrix} (\hat{\boldsymbol{\mu}}_i - \boldsymbol{\mu}_{t+})^T \Big) \end{cases} \tag{10}$$

and where

$\mathbf{K}_0 := \mathbf{V}$, $\mathbf{K}_i := \left(\mathbf{P}_i + \mathbf{V}^{-1}\right)^{-1}$ for $i > 0$.

K_i^{kl} denotes the entries in $\mathbf{K}_i$.

$\Lambda_0 := \lambda_0$, $\Lambda_i := \lambda_i \varphi(\phi_i | \bar{\phi}, v_i^{-1} + (V^{11})^{-1})$ for $i > 0$.

$\hat{\boldsymbol{\mu}}_i = \begin{pmatrix} \hat{\phi}_i \\ \hat{\theta}_i \end{pmatrix} := \mathbf{K}_i \left(\begin{pmatrix} v_i^{-1} \phi_i \\ 0 \end{pmatrix} + \mathbf{V}^{-1} \boldsymbol{\mu} \right)$ for $i > 0$, and $\hat{\boldsymbol{\mu}}_0 := \boldsymbol{\mu}$.

$\mathbf{P}_i := \begin{pmatrix} v_i^{-1} & 0 \\ 0 & 0 \end{pmatrix}$

6.2 Derivation of differential equations and update equations.

Here we derive the PATIPPET filter; the PIPPET filter can be derived similarly or as a special case of PATIPPET.

Snyder [23] provides a partial differential equation describing the evolution of a probability distribution on a continuously stochastically evolving state that drives the emission of point process events. If the evolution of the underlying state is described by a Gauss-Markov diffusion process:

$$d\mathbf{x} = \mathbf{A}\mathbf{x}dt + \mathbf{B}d\mathbf{W}_t \tag{11}$$

and events are generated at rate $\lambda(\mathbf{x})$, then the evolution of the probability distribution $p_t(\mathbf{x})$ is described by

$$dp_t(\mathbf{x}) = \mathcal{L}[p_t(\mathbf{x})]dt + p_t(\mathbf{x})\left(\frac{\lambda(\mathbf{x})}{\Lambda} - 1\right) \cdot (dN_t - \Lambda dt) \tag{12}$$

where $\Lambda := \mathbb{E}[\lambda(\mathbf{x})]$ (with $\mathbb{E}$ denoting expectation under distribution $p_t(\mathbf{x})$), dN_t is the increment in the event count over each dt time step (assumed to be either 1 or 0 with probability 1), and $\mathcal{L}$ is the Kolmogorov forward operator associated with (11):

$$\mathcal{L}[p(\mathbf{x})] = -\sum_i \frac{\partial}{\partial x_i}[\mathbf{A}\mathbf{x}]_i p(\mathbf{x}) + \frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j}[\mathbf{B}\mathbf{B}^T]_{ij} p(\mathbf{x}) \tag{13}$$

Here we project p onto a Gaussian distribution at each time step by matching mean $\boldsymbol{\mu}$ and covariance $\mathbf{V}$, which is also the projection with minimal KL divergence. We do this by finding the differentials of these moments of p_t and using them to drive the evolution of these two variables:

$$\begin{aligned}
d\boldsymbol{\mu}_t =& \boldsymbol{\mu}_{t+} - \boldsymbol{\mu}_t = \int_{\mathbf{x}} \mathbf{x} p_{t+}(\mathbf{x}) d\mathbf{x} - \int_{\mathbf{x}} \mathbf{x} p_t(\mathbf{x}) d\mathbf{x} \\
=& \int_{\mathbf{x}} \mathbf{x} \left(p_{t+}(\mathbf{x}) - p_t(\mathbf{x})\right)(\mathbf{x}) d\mathbf{x} = \int_{\mathbf{x}} \mathbf{x} dp_t(\mathbf{x}) d\mathbf{x} \\
=& \int_{\mathbf{x}} \mathbf{x} \mathcal{L}[p_t(\mathbf{x})] dt d\mathbf{x} + (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_t) \cdot (dN_t - \Lambda dt)
\end{aligned} \tag{14}$$

where we define $\hat{\boldsymbol{\mu}} := \mathbb{E}\left[\mathbf{x}\lambda(\mathbf{x})\right]$, and

$$d\mathbf{V}_t = \mathbf{V}_{t+} - \mathbf{V}_t = \int_{\mathbf{x}} [[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 p_{t+}(\mathbf{x}) d\mathbf{x} - \int_{\mathbf{x}} [[\mathbf{x} - \boldsymbol{\mu}_t]]^2 p_t(\mathbf{x}) d\mathbf{x}$$

where $[[\mathbf{x}]]^2$ denotes $\mathbf{x}\mathbf{x}^T$.

$$
\begin{aligned}
d\mathbf{V}_t &= \int_{\mathbf{x}} [[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 \left(p_{t+}(\mathbf{x}) - p_t(\mathbf{x})\right) d\mathbf{x} \\
&\quad + \int_{\mathbf{x}} \left([[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 - [[\mathbf{x} - \boldsymbol{\mu}_t]]^2\right) p_t(\mathbf{x}) d\mathbf{x} \\
&= \int_{\mathbf{x}} [[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 dp_t(\mathbf{x}) - [[\boldsymbol{\mu}_{t+} - \boldsymbol{\mu}_t]]^2 \\
&= \int_{\mathbf{x}} [[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 \mathcal{L}[p_t(\mathbf{x}|N_t)] dt d\mathbf{x} + \left(\hat{\mathbf{V}} - \mathbf{V}_t\right) \cdot (dN_t - \Lambda dt)
\end{aligned} \tag{15}
$$

where we define $\hat{\mathbf{V}} := \mathbb{E}\left[[[\mathbf{x} - \boldsymbol{\mu}_{t+}]]^2 \lambda(\mathbf{x})\right]$.

Integrating by parts (or following [26]), we can calculate the appropriate integrals of $\mathcal{L}[p_t(\mathbf{x}|N_t)]$, arriving at a general expression for the variational Bayesian filter for point process data:

$$
\begin{cases}
d\boldsymbol{\mu}_t = & \mathbf{A}\boldsymbol{\mu}_t dt + (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_t) \cdot (dN_t - \Lambda dt) \\
d\mathbf{V}_t = & (\mathbf{A}\mathbf{V}_t + \mathbf{V}_t\mathbf{A}^T + \mathbf{B}\mathbf{B}^T) dt + \left(\hat{\mathbf{V}} - \mathbf{V}_t\right) \cdot (dN_t - \Lambda dt)
\end{cases} \tag{16}
$$

From (4), the PATIPPET generative model is described by the Gauss-Markov diffusion process (11) with

$$
\mathbf{x} = \begin{pmatrix} \phi \\ \theta \end{pmatrix} \text{ and } \boldsymbol{\mu} = \begin{pmatrix} \bar{\phi} \\ \bar{\theta} \end{pmatrix}
$$

$$
\mathbf{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{pmatrix}
$$

$$
\mathbf{A} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{B} := \begin{pmatrix} \sigma & 0 \\ 0 & \sigma_\theta \end{pmatrix}.
$$

Plugging into (16), we have

$$
\begin{cases}
d\boldsymbol{\mu}_t = & \begin{pmatrix} \bar{\theta} \\ 0 \end{pmatrix} dt + (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}_t) \cdot (dN_t - \Lambda dt) \\
d\mathbf{V} = & \begin{pmatrix} 2V^{12} + \sigma^2 & V^{22} \\ V^{22} & \sigma_\theta^2 \end{pmatrix} dt + \left(\hat{\mathbf{V}} - \mathbf{V}_t\right) \cdot (dN_t - \Lambda dt)
\end{cases} \tag{17}
$$

We complete the derivation by calculating Λ, $\hat{\boldsymbol{\mu}}$, and $\hat{\mathbf{V}}$. This proceeds by first deriving a simple expression for $p(\mathbf{x})\lambda(\mathbf{x})$ as a sum of scaled normal distributions.

Let $\|x\|_A^2$ denote $x^T Ax$. We will make use of the following result, a generalized form of a well-known result about quadratic forms that allows us to write products of multivariate normal distributions as normal distributions (see [96] for proof and similar application):

$$\|x-a\|_A^2 + \|x-b\|_B^2 = \|a-b\|_{A(A+B)^{-1}B}^2 + \|x-(A+B)^{-1}(Aa+Bb)\|_{A+B}^2 \tag{18}$$

In the PATIPPET generative model, events are generated at rate

$$\lambda(\mathbf{x}) = \theta\left(\lambda_0 + \sum_{i=1,2,\cdots} \frac{\lambda_i}{\sqrt{2\pi v_i}} e^{-\frac{1}{2}\|\mathbf{x}-\mathbf{x}_i\|_{\mathbf{P}_i}^2}\right)$$

$$\mathbf{P}_i = \begin{pmatrix} v_i^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \ \mathbf{x}_i = \begin{pmatrix} \phi_i \\ 0 \end{pmatrix}.$$

$p(\mathbf{x})$ is assumed (forced) to be Gaussian, so we can write:

$$p(\mathbf{x}) = \frac{1}{\sqrt{2\pi|\mathbf{V}|}} e^{-\frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_{\mathbf{V}^{-1}}^2}.$$

We calculate:

$$\begin{aligned} p(\mathbf{x})\lambda(\mathbf{x}) =& \frac{\theta}{\sqrt{2\pi|\mathbf{V}|}} e^{-\frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_{\mathbf{V}^{-1}}^2} \left(\lambda_0 + \sum_{i=1,2,\cdots} \frac{\lambda_i}{\sqrt{2\pi v_i}} e^{-\frac{1}{2}\|\mathbf{x}-\mathbf{x}_i\|_{\mathbf{P}_i}^2}\right) \\ =& \frac{\lambda_0\theta}{\sqrt{2\pi|\mathbf{V}|}} e^{-\frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_{\mathbf{V}^{-1}}^2} + \theta \sum_{i=1,2,\cdots} \frac{\lambda_i}{2\pi\sqrt{v_i|\mathbf{V}|}} e^{-\frac{1}{2}\|\mathbf{x}-\mathbf{x}_i\|_{\mathbf{P}_i}^2 - \frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_{\mathbf{V}^{-1}}^2} \end{aligned}$$

Applying (18),

$$\begin{aligned} p(\mathbf{x})\lambda(\mathbf{x}) =& \frac{\lambda_0\theta}{\sqrt{2\pi|\mathbf{V}|}} e^{-\frac{1}{2}\|\mathbf{x}-\boldsymbol{\mu}\|_{\mathbf{V}^{-1}}^2} \\ &+ \theta \sum_{i=1,2,\cdots} \lambda_i \left(\frac{1}{\sqrt{2\pi(v_i^{-1}+V^{-1})}} e^{-\frac{1}{2}\|\mathbf{x}_i-\boldsymbol{\mu}\|_{\mathbf{P}_i\mathbf{K}_i(\mathbf{V}^1 1)^{-1}}^2}\right)\left(\frac{1}{\sqrt{2\pi\frac{v_i|\mathbf{V}|}{v_i^{-1}+(\mathbf{V}^1 1)^{-1}}}} e^{-\frac{1}{2}\|\mathbf{x}-\mathbf{K}_i(\mathbf{P}_i\mathbf{x}_i+\mathbf{V}^{-1}\boldsymbol{\mu})\|_{\mathbf{K}_i^{-1}}^2}\right) \end{aligned} \tag{19}$$

where we define $\mathbf{K}_i := (\mathbf{P}_i + \mathbf{V}^{-1})^{-1}$. These two final terms are both expressions for normal distributions, so we can rewrite (19) as

$$p(\mathbf{x})\lambda(\mathbf{x}) = \lambda_0\theta\varphi(\mathbf{x}|\boldsymbol{\mu}, \mathbf{V}) + \theta \sum_{i=1,2,\cdots} \lambda_i\varphi(\phi_i|\bar{\phi}, v_i^{-1} + (V^{11})^{-1})\varphi(\mathbf{x}|\mathbf{K}_i(\mathbf{P}_i\mathbf{x}_i + \mathbf{V}^{-1}\boldsymbol{\mu}), \mathbf{K}_i) \tag{20}$$

We simplify this expression by defining $\Lambda_i := \lambda_i\varphi(\phi_i|\bar{\phi}, v_i^{-1} + (V^{11})^{-1})$ for $i > 0$, and setting $\Lambda_0 := \lambda_0$ and $\mathbf{K}_0 = \mathbf{V}$. We define $\hat{\boldsymbol{\mu}}_i := \begin{pmatrix} \hat{\phi}_i \\ \hat{\theta}_i \end{pmatrix} := \mathbf{K}_i(\mathbf{P}_i\mathbf{x}_i + \mathbf{V}^{-1}\boldsymbol{\mu})$ for $i > 0$ and set $\hat{\boldsymbol{\mu}}_0 := \boldsymbol{\mu}$. This lets us write

$$p(\mathbf{x})\lambda(\mathbf{x}) = \sum_{i=0,1,\cdots} \Lambda_i\theta\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i, \mathbf{K}_i) \tag{21}$$

We use this expression and the moments of normal distributions to calculate Λ, $\hat{\boldsymbol{\mu}}$, and $\hat{\mathbf{V}}$:

$$\Lambda := \mathbb{E}_p\left[\lambda(\mathbf{x})\right] = \sum_{i=0,1,\cdots} \Lambda_i \int \theta\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i, \mathbf{K}_i)d\mathbf{x} = \sum_{i=0,1,\cdots} \Lambda_i\hat{\theta}_i \tag{22}$$

$$\hat{\boldsymbol{\mu}} := \frac{1}{\Lambda}\mathbb{E}\left[\mathbf{x}\lambda(\mathbf{x})\right] = \frac{1}{\Lambda}\sum_{i=0,1,\cdots} \Lambda_i \int \begin{pmatrix} \phi\theta \\ \theta^2 \end{pmatrix} \varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i, \mathbf{K}_i)d\mathbf{x}$$

This expression picks out non-central second moment terms of each normal distributions in (21), each of which can be written in terms of the covariance matrix and mean of the distribution. Using K_i^{kl} to denote the entries in $\mathbf{K}_i$, we can write

$$\hat{\boldsymbol{\mu}} = \frac{1}{\Lambda}\sum_{i=0,1,\cdots} \Lambda_i \begin{pmatrix} K_i^{12} + \hat{\phi}_i\hat{\theta}_i \\ K_i^{22} + \hat{\theta}_i^2 \end{pmatrix} \tag{23}$$

The third-order expression for $\hat{\mathbf{V}}$ can also be written in terms of covariance matrices and means since the central third moments of normal distributions are zero.

$$
\begin{aligned}
\hat{\mathbf{V}} :=& \frac{1}{\Lambda}\mathbb{E}_p\left[[[\mathbf{x}-\boldsymbol{\mu}_{t+}]]^2\lambda(\mathbf{x})\right] \\
=& \frac{1}{\Lambda}\sum_{i=0,1,\cdots}\Lambda_i\int[[\mathbf{x}-\boldsymbol{\mu}_{t+}]]^2\theta\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i,\mathbf{K}_i)d\mathbf{x} \\
=& \sum_{i=0,1,\cdots}\Lambda_i\Big[\hat{\theta}_i\int[[\mathbf{x}-\hat{\boldsymbol{\mu}}_i]]^2\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i,\mathbf{K}_i)d\mathbf{x}\cdots \\
& + \hat{\theta}_i[[\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+}]]^2\cdots \\
& + (\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+})\int(\mathbf{x}-\hat{\boldsymbol{\mu}}_i)^T(\theta-\hat{\theta}_i)\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i,\mathbf{K}_i)d\mathbf{x}\cdots \\
& + \left(\int(\mathbf{x}-\hat{\boldsymbol{\mu}}_i)(\theta-\hat{\theta}_i)\varphi(\mathbf{x}|\hat{\boldsymbol{\mu}}_i,\mathbf{K}_i)d\mathbf{x}\right)(\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+})^T\Big] \\
=& \frac{1}{\Lambda}\sum_{i=0,1,\cdots}\Lambda_i[\hat{\theta}_i\mathbf{K}_i+\hat{\theta}_i[[\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+}]]^2\cdots \\
& + (\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+})\begin{pmatrix} K_i^{21} & K_i^{22}\end{pmatrix} + \begin{pmatrix} K_i^{12} \\ K_i^{22}\end{pmatrix}(\hat{\boldsymbol{\mu}}_i-\boldsymbol{\mu}_{t+})^T]
\end{aligned}
\tag{24}
$$

Expressions (22), (23), and (24) coupled with (17) constitute the PATIPPET filter.

The PIPPET filter can be derived as a special case of the PATIPPET filter by setting $\sigma_\theta = 0$, $\theta_0 = 1$, and all terms in $\mathbf{V}$ to zero except V. However, this requires finessing various degeneracies, e.g. wherever $\mathbf{V}$ is inverted. More straightforward is to follow the same process as above, starting from the PIPPET generative model (1) and (2). Either way ultimately yields the PIPPET filter (3).

For multiple event streams j,:

$$
dp_t(\mathbf{x}) = \mathcal{L}[p_t(\mathbf{x})]dt + p_t(\mathbf{x})\sum_j(\lambda_j(\mathbf{x})-\mathbb{E}_p[\lambda_j(\mathbf{x})])\cdot(\mathbb{E}_p[\lambda_j(\mathbf{x})]^{-1}dN_j - dt)
\tag{25}
$$

This follows directly from application of the derivation above to equation (5) in [97] with a discrete spatial dimension. By the methods above, it yields the mPIPPET filter (8) and the mPATIPPET filter:

$$
\begin{cases}
d\boldsymbol{\mu}_t = & \begin{pmatrix}\bar{\theta} \\ 0\end{pmatrix} dt + \sum_j\left(\hat{\boldsymbol{\mu}}^j-\boldsymbol{\mu}_t\right)\cdot(dN_t^j-\Lambda^j dt) \\
d\mathbf{V} = & \begin{pmatrix} 2V^{12}+\sigma^2 & V^{22} \\ V^{22} & \sigma_\theta^2\end{pmatrix} dt + \sum_j\left(\hat{\mathbf{V}}^j-\mathbf{V}_t\right)\cdot(dN_t^j-\Lambda^j dt)
\end{cases}
\tag{26}
$$

6.3 Simulation parameters.

All code used to create figures in this manuscript is available at `https://github.com/joncannon/PIPPET`.

PIPPET simulations were conducted by numerical simulation of (3) with $dt = .001$ and initialized with $\phi_0 = 0$ and $V_0 = .0002$. Parameters for the simulations shown in each figure are listed below, with t_n used to denote simulated event times (in units of seconds).

Figure 2: $\phi_1 = .5$, $v_1 = .0005$, $\lambda_1 = 1$, $\mu_t = .43$, $V_t = .001$, $\lambda_0 = 0$ or $.5$, except as otherwise specified.

Figure 3A:

$$
\begin{aligned}
\{t_n\} =& \{0, .150, .5, .75, .9, 1.25\} \\
\{\phi_i\} =& \{0, .15, .25, .4, .5, .65, .75, .9, 1, 1.15, 1.25, 1.4\} \\
\{v_i\} =& \{.0001, .0005, .0001, .0005, .0001, .0005, .0001, .0005, .0001, .0005\} \\
\{\lambda_i\} =& \{.05, .01, .05, .01, .05, .01, .05, .01, .05, .01\} \\
\lambda_0 =& .01 \\
\sigma =& .05
\end{aligned}
$$

Figure 3B: Same as Figure 3A, but with $t_3 = .45$ (50 ms negative event time shift).

Figure 3C: Same as Figure 3A, but with $\{t_n\} = \{0, .15, .5, .7, .85, 1.2\}$ (50 ms negative phase shift).

Figure 4A: Same as Figure 3, but with $\{t_n\} = \{0, .150, .65, .9, 1.15, 1.25\}$.

Figure 4B: Same as Figure 4A, but with $\sigma = .3$.

Figure 4A: Same as Figure 4A, but with additional tap times and tap feedback expectations:

$$
\begin{aligned}
\{t_n^{tap}\} =& \{\phi_i^{tap}\} = \{0, .5, 1\} \\
v_i^{tap} =& .0005 \\
\lambda_i^{tap} =& .05 \\
\lambda_0^{tap} =& .01
\end{aligned}
$$

PATIPPET simulations were conducted by numerical simulation of (4) with $dt = .001$. Parameters for the simulations shown in each figure are listed below.

Figure 5:

$$
\begin{aligned}
t_n =& \frac{n}{1.2Hz} \\
\phi_i =& i \\
v_i =& .005 \\
\lambda_i =& .02 \\
\lambda_0 =& .0001 \\
\sigma =& .05 \\
\sigma_\theta =& .05 \\
\boldsymbol{\mu}_0 =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{V}_0 =& \begin{pmatrix} .001 & 0 \\ 0 & .04 \end{pmatrix}
\end{aligned}
$$

Figure 6: In four simulations, we set the inter-onset interval Δ to $.4s$, $0,7s$, $1.0s$, and $1.3s$. In each simulation, we set the perturbation δ to $\frac{\Delta}{25}$.

$$
\begin{aligned}
\{t_n\} =& \{\Delta, 2\Delta, 3\Delta, 4\Delta + \delta\} \\
\phi_i =& i \\
v_i =& .0002 \\
\lambda_i =& .02 \\
\lambda_0 =& 10^{-5} \\
\sigma =& .01 \\
\sigma_\theta =& .01 \\
\boldsymbol{\mu}_0 =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{V}_0 =& \begin{pmatrix} 10^{-4} & 0 \\ 0 & 10^{-4} \end{pmatrix}
\end{aligned}
$$

Figure 7A:

$$\begin{aligned}
\phi_i =& .25i \\
v_i =& .0001 \\
\lambda_i =& 1 \\
\lambda_0 =& .0001 \\
\sigma =& .015 \\
\sigma_\theta =& .2 \\
\boldsymbol{\mu}_0 =& \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\mathbf{V}_0 =& \begin{pmatrix} .0001 & 0 \\ 0 & .005 \end{pmatrix}
\end{aligned}$$

Left: $\{t_n\} = \{.25, .5, .75, 1\}$. Right: $\{t_n\} = \{1\}$.

Figure 7B: Same as Figure 7A, but with $\lambda_i = 0$ and $\lambda_0 = 4$.