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Representation of Joints in Multibody Systems

Zur Darstellung der Verbindungen zwischen den Körpern eines Mehrkörpersystems werden Mode-Vektoren definiert. Mit ihrer Hilfe lassen sich gebundene Relativbewegungen der Körper des Systems sowie generalisierte eingeprägte Kräfte einfach angeben. Die Darstellung kann zum Aufbau von Gelenk-Bibliotheken für Mehrkörperprogramme verwendet werden. Damit ist der Benutzer des Programms von der Formulierung kinematischer Gleichungen zur Beschreibung der im System eingeschränkten Bewegungen befreit. Die eingeführten Begriffe werden an einem Beispiel erläutert.

A description of the interconnections between the bodies of a multibody system is given in terms of mode vectors representing constrained relative motion and generalized applied and constraint interaction across joints. The representation can be applied to develop joint libraries for general purpose multibody programs thus releasing the user from formulating constraint equations and force laws describing the joints. An example provides the physical interpretation of the concepts introduced here.

Для представления связей между телами системы многих тел определяются векторы мод. С их помощью можно легко представлять связанные относительные движения тел системы, а также обобщённые действующие силы. Такое представление можно использовать для создания библиотек-связей для программ систем многих тел с тем, чтобы при использовании программ избежать составление кинематических уравнений для описания связанных движений в системе. Введённые понятия объясняются на примере.

1. Introduction

A multibody system as considered here is a system of N rigid or gyrostatic bodies enumerated $i = 1, 2, \dots, N$. A reference frame representing any overall motion of the system in a nominal configuration is labeled $i = 0$. The N bodies interact with each other and with points in the reference frame across $N_J \geqq N$ joints labeled $s = 1, 2, \dots, N_J$. The term "joint" is considered to represent the entity of interactions between two bodies or one body and any points in the reference frame. With this labeling convention it is simple to distinguish two basic topological configurations of multibody systems. If $N = N_J$ the system has a tree configuration; cutting any of the N joints causes the system to fall into two parts. Systems with closed circuits are characterized by $N_J > N$. Their treatment is considerably more complicated, for the mobility of the system (the number of positional and motional degrees of freedom cannot be determined as easily as in the tree case. For tree systems, one just adds the numbers of degrees of freedom) in relative motion across the joints to get the number of degrees of freedom of the whole system. In case of closed circuit systems one must also consider the kinematic consistency conditions for the motions of bodies which form the circuits. Combining the consistency conditions with the constraints on relative motion across the joints, one gets a set of possibly redundant constraint equations. To determine the number of degrees of freedom the rank of a matrix must be calculated.

The mid sixties saw an increasing interest in computer oriented formulation of multibody system equations and the development of the corresponding general purpose multibody computer programs [1]. Such programs are capable of generating and integrating multibody system equations of motion based on a simple description of the system elements, i.e. of bodies, reference frame and joints. Whereas the description of bodies and reference frame is straightforward, the representation of joints is less clear and unique. Sometimes it is intermingled with the definition of generalized coordinates and speeds to be used for describing the system motion [2] and often the class of joints, which can be handled by a multibody formalism is restricted to simple cases as lower kinematic pairs or revolute and prismatic joints, e.g. [3]. A description of the constraints on the system motion by selection of generalized coordinates can be a drawback in case of systems with closed kinematic chains. In such cases this choice is influenced not only by the joint constraints but also by constraints resulting from the kinematic consistency conditions. Just defining the number of independent generalized coordinates and speeds can be a nontrivial problem. Both tasks, computation of the number of degrees of freedom and definition of independent coordinates can be handled in a formal way by the computer [4]. Thus the user should not be asked to consider such problems when formulating his input, if simplicity of the data describing the system is considered to be an essential feature of a general purpose multibody program.

Here we propose a representation of the joints in a multibody system which
— covers a general class of joints between the system bodies;
— is based on a description of the system motion in terms of relative variables between interacting bodies;
— can be applied to develop libraries of joints to be used in general purpose multibody programs based on formalisms using relative variables as the ones described in [4] and [5].

Our notation is as follows. Vectors considered as invariants are represented by any letter printed in bold type. (In figures the letters are underlined). Normal font denotes scalars, parameters and variables. Arrays of vectors and numbers (i.e. matrices) are not distinguished notationally from single quantities. To signal that a symbol is to be interpreted as a vector array or matrix rather than an individual vector or number, we define those quantities by equations of the type $A = [A_{ij}]$ or $\boldsymbol{e} = [\boldsymbol{e}_\alpha]$. Singly subscripted quantities are formed into a column matrix (or array) always. Row matrices (or arrays) are written as transposes of column matrices (or arrays) denoted by superscript T. Greek subscripts as used here always have the range 1, 2, 3. Unit matrices are denoted by symbol E.

2. Unconstrained relative motion and interaction

Consider two bodies of a multibody system interacting across a joint as represented in Fig. 1. Either of the two bodies may be the reference frame $i = 0$. To define relative motion and interaction across the joint unambiguously a direction is assigned to the joint, visualized in Fig. 1 by the curved arrow. The numerical labels of the bodies connected by the joint are associated with the label s of the joint by denoting the body from which the joint arrow originates by $f = f(s)$ while label $t = t(s)$ refers to the body to which the joint arrow points.

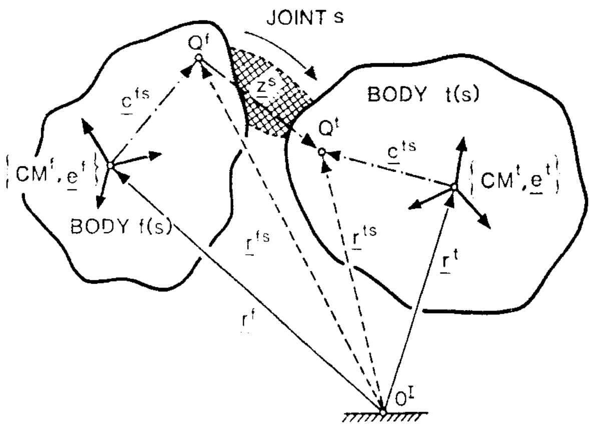

Fig. 1. Geometry of two interacting bodies

Relative displacement across the joint is measured by the vector

$$\boldsymbol{z}^s = \boldsymbol{r}^{ts} - \boldsymbol{r}^{fs} . \tag{2.1}$$

It is the displacement between the two attachment points of the joint located on the bodies by body fixed vectors $\boldsymbol{c}^{fs}$ and $\boldsymbol{c}^{ts}$. Relative orientation across the joint is described by direction cosine matrix B^s as

$$\boldsymbol{e}^t = B^s \boldsymbol{e}^f , \qquad B^s = [B^s_{\alpha\beta}] . \tag{2.2}$$

Symbols $\boldsymbol{e}^i = [\boldsymbol{e}^i_\alpha]$, $i = t$ or $i = f$ denote vector arrays whose elements are the base-vectors of the body fixed frames $\{CM^i, \boldsymbol{e}^i\}$ located at the center of mass CM^i.

The relative velocity of the attachment points of the joint is

$$\dot{\boldsymbol{z}}^s = \dot{\boldsymbol{r}}^{ts} - \dot{\boldsymbol{r}}^{fs} . \tag{2.3}$$

Let vector $\boldsymbol{z}^s$ be resolved in basis $\boldsymbol{e}^f$ as

$$\boldsymbol{z}^s = \boldsymbol{e}^{f\mathrm{T}} z^s , \qquad z^s = [z^s_\alpha] , \tag{2.4}$$

and denote the relative derivative of $\boldsymbol{z}^s$ with respect to $\boldsymbol{e}^f$ by $\boldsymbol{V}^s$,

$$\frac{{}^f\mathrm{d}\boldsymbol{z}^s}{\mathrm{d}t} = \boldsymbol{V}^s = \boldsymbol{e}^{f\mathrm{T}} V^s , \qquad V^s = [V^s_\alpha] = [\dot z^s_\alpha] . \tag{2.5}$$

Denoting the absolute angular velocity of body i by $\boldsymbol{\omega}^i$ one gets

$$\dot{\boldsymbol{z}}^s = \boldsymbol{V}^s + \boldsymbol{\omega}^f \times \boldsymbol{z}^s . \tag{2.6}$$

Vector $\boldsymbol{V}^s$ or its coordinates V^s are a measure of the relative velocity across joint s. The relative angular velocity of the two interacting bodies is defined to be

$$\boldsymbol{\Omega}^s = \boldsymbol{\omega}^t - \boldsymbol{\omega}^f = \boldsymbol{e}^{f\mathrm{T}} \Omega^s , \qquad \Omega^s = [\Omega^s_\alpha] . \tag{2.7}$$

In view of Eqs. 2.2, 2.7 and 2.5 the quantities describing relative position and velocity across joint s satisfy the kinematical equations of motion

$$\dot B^s = -B^s \tilde\Omega^s , \qquad \dot z^s = V^s \tag{2.8a}$$

with the tilde operator defined in [4]. Matrices B^s and z^s may be represented by a suitable set of six position variables

$$P^s = [P^s_j] , \qquad j = 1, 2, \ldots, 6 , \tag{2.9}$$

e.g. by three angles ϑ^s_α parametrizing matrix B^s plus three coordinates z^s_α. The velocity across the joint is given by the six velocity variables

$$G^s = [G^s_j] = \begin{bmatrix} \Omega^s \\ V^s \end{bmatrix} , \qquad j = 1, 2, \ldots, 6 . \tag{2.10}$$

In terms of variables P^s and G^s the kinematical equations of motion, Eq. 2.8a have the general form

$$\dot P^s = \hat X^s_I G^s . \tag{2.8b}$$

The specific form of 6 × 6-matrix $\hat{X}_I^s = \hat{X}_I^s(P^s)$ depends on the variables used to parametrize matrices B^s and z^s.

The complete interaction between the two bodies generally involves a complex of surface and body forces and torques. Because the bodies are rigid the net effect of the whole complex can be represented by a single force $\boldsymbol{F}^s$ and a single torque $\boldsymbol{L}^s$:

$$\boldsymbol{L}^s = \boldsymbol{e}^{f\mathrm{T}} L^s\,, \qquad L^s = [L_\alpha^s]\,, \qquad \boldsymbol{F}^s = \boldsymbol{e}^{f\mathrm{T}} F^s\,, \qquad F^s = [F_\alpha^s]\,. \tag{2.11}$$

The reference point for reducing the system of interactions is point Q^t and the positive senses of $\boldsymbol{L}^s$ and $\boldsymbol{F}^s$ are defined by saying that they are applied on body t by body f. Thus interactions $+\boldsymbol{L}^s$ and $+\boldsymbol{F}^s$ act on body t while $-\boldsymbol{L}^s$ and $-\boldsymbol{F}^s$ act on body f — see Fig. 2. All of the six interactions L_α^s and F_α^s now are collected in 6 × 1-matrix

$$\Lambda^s = [\Lambda_j^s] = \begin{bmatrix} L^s \\ F^s \end{bmatrix}, \qquad j = 1, 2, \dots, 6\,. \tag{2.12}$$

In case of unconstrained motion across the joint all the elements of matrices P^s and G^s are required to describe the motion and all of the six elements of Λ^s are known applied forces and torques.

3. Constraints across joints

Constraints on the motion across joints result either from direct physical contact of the bodies or by means of kinematically constraining mechanisms, whose bodies have been idealized as "inertialess" by lumping their mass and inertia properties with the bodies connected. Examples of the former type resulting in any number of degrees of freedom in relative motion are given in Fig. 3. In particular rotation and translation may be coupled as in case of the screwmotion shown in Fig. 3c. An example of an interconnection with an essentially "inertialess" intermediate body (the spider) is the Cardan or universal joint shown in Fig. 4. Of course the bodies can interact without constraining relative motion, e.g. by a spring as shown in Fig. 5.

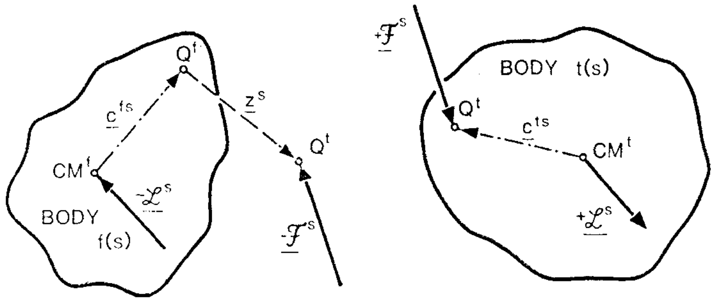

Fig. 2. Free body diagram showing interactions across joint s

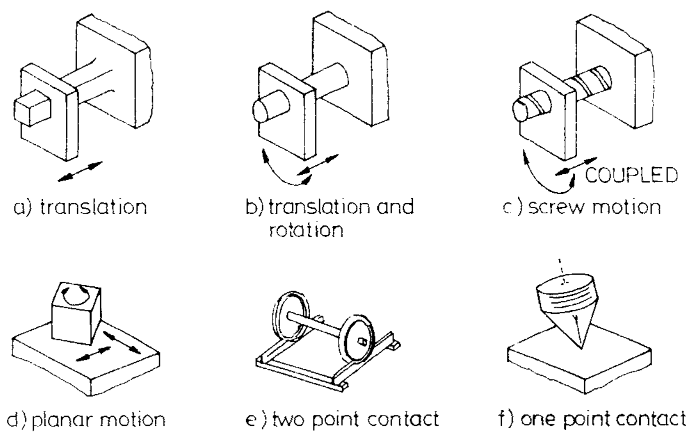

Fig. 3. Examples of constrained motion

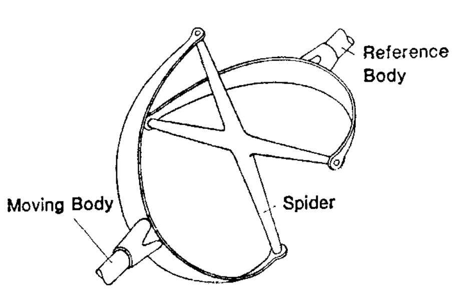

Fig. 4. Universal joint

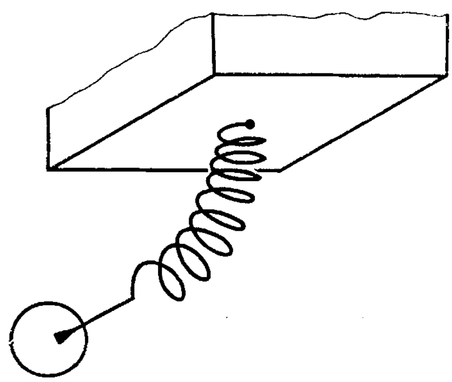

Fig. 5. Unconstrained relative motion

Positional constraints

Because of n_{cp}^s holonomic constraints on the position variables P^s

$$f_p^s(P^s, t) = 0\,, \qquad f_p^s = [f_{pj}^s]\,, \qquad j = 1, 2, \dots, n_{cp}^s\,, \tag{3.1}$$

the relative position of the two bodies can be described by

$$n_{fp}^s = 6 - n_{cp}^s \tag{3.2}$$

joint position state variables

$$p^s = [p_j^s]\,, \qquad j = 1, 2, \dots, n_{fp}^s\,, \tag{3.3}$$

corresponding to the n_{fp}^s positional degrees of freedom in the relative motion across joint s. The n_{cp}^s relations, Eq. 3.1 are the implicit forms of the position constraint equations. Their explicit forms may be written as

$$P^s = P^s(p^s, t)\,. \tag{3.4a}$$

Equations 3.4a are the solutions of the in general nonlinear implicit constraint equations, Eq. 3.1 and represent the six position variables P_j^s in terms of time t and n_{fp}^s parameters p^s which cannot be determined from Eq. 3.1, which is to say that they are the joint position state variables. Instead of Eq. 3.4a we prefer to write

$$B^s = B^s(p^s, t)\,, \qquad z^s = z^s(p^s, t) \tag{3.4b}$$

thus avoiding the intermediate step of representing the twelve elements of B^s and z^s in terms of six variables P_j^s. In multibody system simulation one needs matrices B^s and z^s rather than variables P_j^s to generate the equations of motion.

Velocity constraints

Differentiating the constraint equations, Eq. 3.1 one finds using Eq. 2.8b

$$w_p^{s\mathsf{T}}\, G^s = \zeta_p^s \tag{3.5a}$$

with matrices

$$w_p^{s\mathsf{T}} = \left[\frac{\partial f_{pj}^s}{\partial P_k}\right] \hat{X}_I^s\,, \qquad j = 1, 2, \dots, n_{cp}^s\,, \qquad k = 1, 2, \dots, 6\,, \tag{3.5b}$$

$$\zeta_p^s = -\left[\frac{\partial f_{pj}^s}{\partial t}\right], \qquad j = 1, 2, \dots, n_{cp}^s\,, \tag{3.5c}$$

of orders $n_{cp}^s \times 6$ and $n_{cp}^s \times 1$. In general the elements of w_p^s and ζ_p^s are nonlinear functions of p_j^s and t but Eq. 3.5a is linear in the velocities G^s. It represents the n_{cp}^s holonomic constraints on the velocity variables. These are accompanied by n_{cg}^s nonholonomic constraints. As all known nonholonomic constraints are linear in the velocity variables [6, page 499] they can be written in the form

$$w_g^{s\mathsf{T}}\, G^s = \zeta_g^s \tag{3.6}$$

where matrices $w_g^{s\mathsf{T}}$ and ζ_g^s are of orders $n_{fg}^s \times 6$ and $n_{fg}^s \times 1$, respectively. Introducing the partitioned matrices

$$w^{s\mathsf{T}} = \begin{bmatrix} w_p^{s\mathsf{T}} \\ w_g^{s\mathsf{T}} \end{bmatrix}, \qquad \zeta_s = \begin{bmatrix} \zeta_p^s \\ \zeta_g^s \end{bmatrix} \tag{3.7}$$

all of the constraint equations on the velocity variables can be combined as

$$w^{s\mathsf{T}} G^s = \zeta^s\,. \tag{3.8}$$

Matrices $w^{s\mathsf{T}}$ and ζ^s are of orders $n_c^s \times 6$ and $n_c^s \times 1$ where

$$n_c^s = n_{cp}^s + n_{cg}^s = 6 - n_f^s\,. \tag{3.9}$$

Obviously n_f^s is the number of motional degrees of freedom across joint s.

Equation 3.8, the most general implicit form of the velocity constraint equations, is a set of n_c^s linear equations for the 6 unknowns G^s. Its solution can be written as

$$G^s = \varphi^s g^s + \bar{\varphi}^s \bar{g}^s\,. \tag{3.10a}$$

Here $\varphi^s g^s$ is the solution of the homogeneous equation $w^{s\mathsf{T}} G^s = 0$ and matrix

$$g^s = [g_j^s]\,, \qquad j = 1, 2, \dots, n_f^s\,, \tag{3.11}$$

contains n_f^s arbitrary parameters which cannot be determined from the constraint equations, Eq. 3.8, i.e. the joint velocity state variables. Because the g^s may be given any values one concludes from $w^{s\mathsf{T}} \varphi^s g^s = 0$

$$w^{s\mathsf{T}} \varphi^s = 0 \quad \text{or} \quad \varphi^{s\mathsf{T}} w^s = 0\,. \tag{3.12}$$

The other term $\bar{\varphi}^s \bar{g}^s$ in Eq. 3.10 is a particular solution of the nonhomogeneous equation and thus satisfies

$$w^{s\mathsf{T}} \bar{\varphi}^s \bar{g}^s = \zeta^s\,. \tag{3.13}$$

Selecting $6 \times n_c^s$-matrix $\bar{\varphi}^s$ such that

$$w^{s\mathsf{T}}\bar{\varphi}^s = E \tag{3.14}$$

one concludes $\bar{g}^s = \zeta^s$. Now partition the n_c^s elements of $\bar{g}^s$ into n_{ck}^s known kinematic excitation time functions

$$\varkappa^s = [\varkappa_j^s], \qquad j = 1, 2, \dots, n_{ck}^s, \tag{3.15}$$

and into n_{cl}^s zeros [0] corresponding to locked motions, where

$$n_{ck}^s + n_{cl}^s = n_c^s. \tag{3.16}$$

Thus one can write

$$\bar{g}^s = \zeta^s = \begin{bmatrix} \varkappa^s(t) \\ [0] \end{bmatrix}. \tag{3.17}$$

Partitioning $\bar{\varphi}^s$ correspondingly into $\bar{\varphi}_k^s$ and $\bar{\varphi}_l^s$ one gets instead of Eq. 3.10a

$$G^s = \varphi^s g^s + \bar{\varphi}_k^s \varkappa^s(t) + \bar{\varphi}_l^s[0]. \tag{3.10b}$$

Equations 3.10 a, b provide a nice interpretation of the constraints across joints. The columns of 6×6-matrix

$$\hat{\varphi}^s = [\varphi^s, \bar{\varphi}^s] = [\varphi^s, \bar{\varphi}_k^s, \bar{\varphi}_l^s] \tag{3.18}$$

define a basis of the sixdimensional linear vector space. When velocity G^s, an element of this space, is represented with respect to that basis, its components are devided into the n_f^s unknown velocities g^s and the n_c^s known velocities $\bar{g}^s$. The latter may be subdivided into n_{ck}^s known kinematic excitation time functions $\varkappa^s$ and into n_{cl}^s zeros for the locked motions. Therefore the columns of $\hat{\varphi}^s$ are called the mode vectors representing the free and the constrained modes of motion across the joint, the latter subdivided into kinematically excited and locked modes. For most technically important joints the free and constrained mode vectors are identified easily by inspection.

From $\hat{\varphi}^s$ a second basis for the sixdimensional vector space now is constructed, called the dual basis and denoted $\hat{\psi}^s$. It is defined by

$$(\hat{\psi}^s)^\mathsf{T} = (\hat{\varphi}^s)^{-1}. \tag{3.19}$$

In particular, if $\hat{\varphi}^s$ is orthogonal which implies $\hat{\varphi}^{s\mathsf{T}} = \hat{\varphi}^{s^{-1}}$ one concludes $\hat{\psi}^s = \hat{\varphi}^s$. Now $\hat{\psi}^s$ is partitioned in the same way as $\hat{\varphi}^s$ into n_f^s columns ψ^s and n_c^s columns $\bar{\psi}^s$ to get

$$\hat{\psi}^s = [\psi^s, \bar{\psi}^s]. \tag{3.20a}$$

Equation 3.20a implies for the submatrices

$$\left.\begin{aligned} &\varphi^{s\mathsf{T}}\psi^s = E, \quad &&\varphi^{s\mathsf{T}}\bar{\psi}^s = 0, \quad &&\psi^{s\mathsf{T}}\varphi^s = E, \quad &&\bar{\psi}^{s\mathsf{T}}\varphi^s = 0, \\ &\bar{\varphi}^{s\mathsf{T}}\psi^s = 0, \quad &&\bar{\varphi}^{s\mathsf{T}}\bar{\psi}^s = E, \quad &&\psi^{s\mathsf{T}}\bar{\varphi}^s = 0, \quad &&\bar{\psi}^{s\mathsf{T}}\bar{\varphi}^s = E. \end{aligned}\right\} \tag{3.20b}$$

In view of Eqs. 3.20b and 3.14 one concludes

$$\bar{\psi}^s = w^s. \tag{3.21}$$

Now premultiply velocity G^s as given by Eq. 3.10a to get in view of Eqs. 3.20b

$$\psi^{s\mathsf{T}}G^s = g^s, \qquad \bar{\psi}^{s\mathsf{T}}G^s = \bar{g}^s = \zeta^s. \tag{3.22}$$

Thus the free and the constrained mode velocities, being the coordinates of velocity G^s in basis $[\varphi^s, \bar{\varphi}^s]$, appear, when projecting G^s onto the dual base vectors ψ^s, $\bar{\psi}^s$. This is an important property of nonorthogonal vector bases: The coordinates of the vectors with respect to one basis are the projections of the vector onto the dual basis. In case of orthogonal vector bases the difference between the original and the dual basis disappears and the components of a vector are identical with its projections onto the base vectors.

Kinematical equations of motion

From Eq. 3.5a one concludes by analogy with Eq. 3.10a

$$G^s = \varphi_p^s g_p^s + \bar{\varphi}_p^s \bar{g}_p^s. \tag{3.23}$$

Subscript p has been added to signal that holonomic constraints have been taken into account only. A set of n_{fp}^s arbitrary variables g_p^s of course is

$$g_p^s = C\dot{p}^s \tag{3.24}$$

where $C = C(p^s, t)$ is an arbitrary nonsingular $n_{fp}^s \times n_{fp}^s$-matrix. Introducing Eq. 3.24 into Eq. 3.23 and equating the G^s given by the resulting relation and Eq. 3.10a yields

$$\varphi_p^s C\dot{p}^s + \bar{\varphi}_p^s \bar{g}_p^s = \varphi^s g^s + \bar{\varphi}^s \bar{g}^s. \tag{3.25}$$

Defining base vectors ψ_p^s and $\bar{\psi}_p^s$ for holonomic constraints by analogy to the general case described by Eqs. 3.19 and 3.20, it follows when Eq. 3.25 is premultiplied by $\psi_p^{s\mathsf{T}}$ that $C\dot{p}^s = \psi_p^{s\mathsf{T}}(\varphi^s g^s + \bar{\varphi}^s \bar{g}^s)$. Because C has been defined to be any nonsingular matrix it finally is found that

$$\dot{p}^s = C^{-1}\psi_p^{s\mathsf{T}}(\varphi^s g^s + \bar{\varphi}^s \bar{g}^s). \tag{3.26}$$

This is the general form of the kinematical equations in terms of joint state variables p^s and g^s namely

$$\dot{p}^s = Y^s_{I1}(p^s, t)\, g^s + Y^s_{I2}(p^s, t)\,. \tag{3.27}$$

Matrix Y^s_{I1} is of order $n^s_{fp} \times n^s_f$ while Y^s_{I2} is of order $n^s_{fp} \times 1$. This implies that Y^s_{I1} is square if there are only holonomic constraints. In particular for the latter case $\dot{p}^s = C^{-1}g^s$. The freedom in the choice of C reflects the fact that one can select any set of variables g^s once P^s, G^s and p^s have been defined. Specifically one might define $C = E$ for holonomic systems to get $\dot{p}^s = g^s$. When dealing with the kinematics of specific joints one should not apply the general formal procedure. The kinematical equations, Eq. 3.27 are found easily once the position and velocity variables have been defined.

Interactions

We now use basis $[\psi^s, \bar{\psi}^s]$ to represent another element of the sixdimensional vector space, the interaction Λ^s,

$$\Lambda^s = \psi^s\lambda^s + \bar{\psi}^s\bar{\lambda}^s\,. \tag{3.28}$$

Matrices

$$\lambda^s = [\lambda^s_j]\,, \qquad j = 1, 2, \dots, n^s_f; \qquad \bar{\lambda}^s = [\bar{\lambda}^s_j]\,, \qquad j = 1, 2, \dots, n^s_c\,, \tag{3.29}$$

contain the components of interaction Λ^s with respect to basis $\hat{\psi}^s$. To get an interpretation of the elements of matrices λ^s and $\bar{\lambda}^s$, premultiply Eq. 3.28 by $\varphi^{s\mathsf{T}}$ and $\bar{\varphi}^{s\mathsf{T}}$, respectively, to get in view of Eq. 3.20b

$$\lambda^s = \varphi^{s\mathsf{T}}\Lambda^s\,, \qquad \bar{\lambda}^s = \bar{\varphi}^{s\mathsf{T}}\Lambda^s\,. \tag{3.30}$$

That is, the elements of λ^s are the projections of the interaction Λ^s onto the free modes. These are *exactly the vectors* on which the free-mode velocity variables act, so the λ^s elements are known from the nature of the devices producing torques or forces in these modes. In analytical mechanics they are refered to as generalized applied forces. On the other hand, the elements of $\bar{\lambda}^s$ are projections of the interaction onto the constrained modes, which is to say the unknown generalized constraint forces. In the case of orthogonal modes, vector bases $\hat{\varphi}^s$ and $\hat{\psi}^s$ are identical and the generalized forces can be interpreted in the familiar way appropriate for vectors in Cartesian frames. However, in general the modes $\hat{\varphi}^s$ form a nonorthogonal basis. In such cases it is important to recognize that $\lambda^s, \bar{\lambda}^s$ are projection of Λ^s onto basis $[\varphi^s, \bar{\varphi}^s]$.

It is helpful to relate the generalized forces λ^s and $\bar{\lambda}^s$ introduced here to the Cartesian coordinates of the applied interaction forces and torques. Denoting these by Λ^s_e (eingeprägte Kräfte und Momente) and Λ^s_z (Zwangskräfte und -momente) respectively, one gets

$$\Lambda^s = \Lambda^s_e + \Lambda^s_z\,. \tag{3.31}$$

From the principles of d'Alembert or Jourdain one concludes $\varphi^{s\mathsf{T}}\Lambda_z = 0$ yielding

$$\lambda^s = \varphi^{s\mathsf{T}}\Lambda^s_e \tag{3.32}$$

whereas from Eqs. 3.30 and 3.31

$$\bar{\lambda}^s = \bar{\varphi}^{s\mathsf{T}}(\Lambda^s_e + \Lambda^s_z)\,. \tag{3.33}$$

To develop an expression for Λ^s_z in terms of the generalized constraint forces $\bar{\lambda}^s$ one concludes from Eqs. 3.28, 3.31 and 3.32

$$\Lambda^s_z = \psi^s\lambda^s + \bar{\psi}^s\bar{\lambda}^s - \Lambda^s_e = (\psi^s\varphi^{s\mathsf{T}} - E)\,\Lambda^s_e + \bar{\psi}^s\bar{\lambda}^s\,. \tag{3.34}$$

Thus the Cartesian coordinates of the constraint forces and torques Λ^s_z can be gotten easily once the generalized constraint forces $\bar{\lambda}^s$ are known.

It may be worth mentioning that the generalized constraint forces as defined here differ from the Lagrange multipliers μ appearing in Lagrange's equations of type one. The latter are defined in terms of Λ_z exclusively, i.e. $\mu = \bar{\varphi}^{\mathsf{T}}\Lambda_z$, whereas the $\bar{\lambda}$ defined by Eq. 3.30 may contain contributions from applied forces as well — see Eq. 3.33.

4. Description of joints

In terms of the quantities introduced heretofore any joints may be described to a multibody formalism as follows:

1. Definition of joint position and velocity state variables p^s and g^s used to describe relative motion across joint and kinematical equations they satisfy. These have the general from given by Eq. 3.27 relating n^s_f velocity variables g^s to n^s_{fp} derivatives $\dot{p}^s$ of position variables. Functions Y^s_{I2} may appear only in case of nonholonomic constraints and kinematic excitations.
2. Implicit and explicit forms of positional constraint equations, Eqs. 3.1 and 3.4.
3. Mode vectors $\hat{\varphi}^s$ and dual vectors $\hat{\psi}^s$.
4. Kinematic excitation time functions $\varkappa^s(t)$, if any.
5. Laws for the applied interactions Λ^s_e from which the generalized applied forces λ^s appearing in the state space representation of the system motion can be gotten easily.

All of these data do not depend on topological aspects of the multibody system and thus can be organized in libraries for joint kinematics and interactions conveniently. For introducing the positional constraint equations into a general purpose multibody computer program one does not need the intermediate step representing twelve variables $B^s_{\alpha\beta}$ and z^s_α in terms of six position variables P^s_j. Thus the explicit constraint equations given by Eq. 3.4b should be available in the joint kinematics library rather than Eq. 3.4a and their implicit form given by Eq. 3.1 should be modified accordingly into $f^s_p(B^s, z^s, t) = 0$.

5. An example

Most of the technologically important joints yield orthogonal modes. Here we give an example in which it is advantageous to use nonorthogonal mode vectors. Consider the interconnection mechanism shown in Fig. 6, [7]. It enforces plane translation of the body $t = 2$ (the sphere) with respect to the 1, 2-plane of the frame fixed in body $f = 1$, a through. The intermediate body sliding in the trough is modelled massless. It contains a second trough, whose axis is inclined to the 2-axis by an angle α. The motion in the direction of axis 1 is dynamic, described by variables s and $\dot{s}$. The motion in the direction of the skewed trough is kinematically excited, the position and velocity functions for this motion being given by variables $\eta(t)$ and $\varkappa(t)$ satisfying

$$\eta(t) = \eta_0 + \int_{t_0}^{t} \varkappa(\tau)\, d\tau \quad \text{where} \quad \eta_0 = \eta(t_0)\,. \tag{5.1}$$

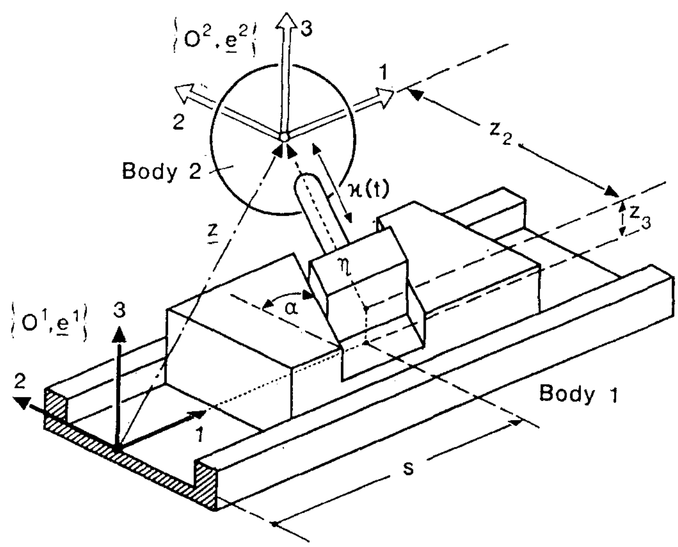

Fig. 6. Planar translation with kinematic excitation

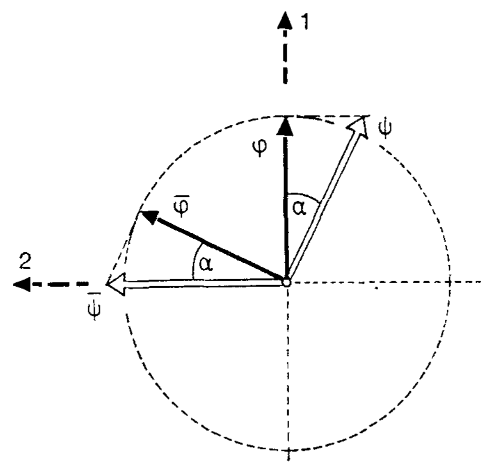

Fig. 7. Translational part of free and kinematically exited modes in 1, 2-plane

The attachment points of the joint are points O^1 and O^2. We assume that they are identical with the centers of mass CM^i, $i = 1, 2$, which implies that the attachment point vectors are $\mathbf{c}^{is} = 0$ and that vector $\mathbf{z}^s$ extends from $O^1 = CM^1$ to $O^2 = CM^2$. Suppressing joint index s for notational simplicity the kinematics across the joint are described by:

State variables

$$p = s\,, \qquad g = \dot{s}\,. \tag{5.2}$$

Explicit positional constraint equations

$$z = \begin{bmatrix} s + \eta \sin\alpha \\ \eta \cos\alpha \\ z_3 \end{bmatrix}, \qquad B = E \qquad \textit{in which } z_3 \textit{ is constant.} \tag{5.3}$$

Mode vectors

$$\varphi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\varphi}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \quad \bar{\varphi}_l = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.4a}$$

$$\varphi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -\tan\alpha \\ 0 \end{bmatrix}, \quad \bar{\varphi}_k = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/\cos\alpha \\ 0 \end{bmatrix}, \quad \bar{\varphi}_l = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5.4b}$$

These are gotten differentiating Eq. 5.3 for z, using Eqs. 2.10, 3.10 and considering the definition of g given as Eq. 5.2. Obviously there are nonorthogonal modes. The translational part of the mode vectors spanning the 1, 2-plane is depicted in Fig. 7.

Kinematical equations of motion

$$\dot{p} = g\,. \tag{5.5}$$

Interaction forces and torques.

The interaction Λ across the joint is assumed to result from a spring interconnecting the bodies as shown in Fig. 8 and from an actuator yielding the kinematic excitation time function $\varkappa(t)$. The applied interactions are forces $F_{e\alpha}$ resulting from the spring and torques $L_{e\alpha}$. The latter appear because the interactions have been defined to be reduced to point Q^t at the tip of vector z^s — see Fig. 2. Also translation in direction e_1^1 is unconstrained resulting in

$$F_{z1} = 0\,. \tag{5.6}$$

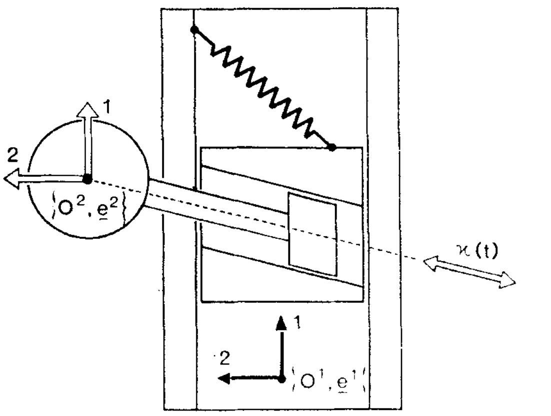

Fig. 8. Planar translation with kinematic excitation under action of a spring

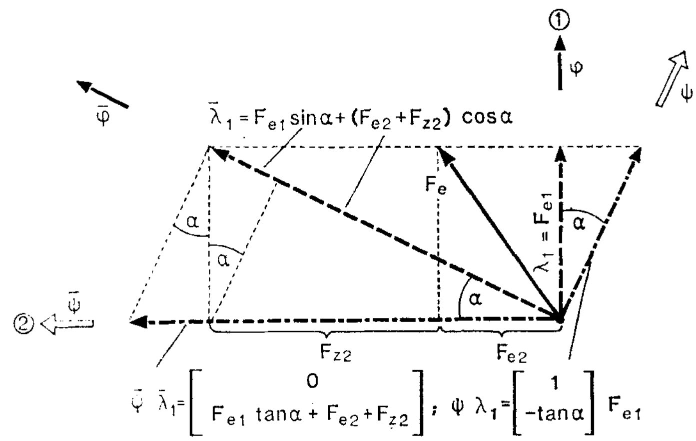

Fig. 9. Applied and constraint forces in 1, 2-plane for interconnection shown in Fig. 8

Thus

$$\Lambda_e = \begin{bmatrix} L_{e1} \\ L_{e2} \\ L_{e3} \\ F_{e1} \\ F_{e2} \\ F_{e3} \end{bmatrix}, \quad \Lambda_z = \begin{bmatrix} L_{z1} \\ L_{z2} \\ L_{z3} \\ 0 \\ F_{z2} \\ F_{z3} \end{bmatrix}. \tag{5.7}$$

With the free mode vector φ given by Eq. 5.4a one concludes that

$$\varphi^T \Lambda = F_{e1} \tag{5.8}$$

and the generalized forces λ and $\bar{\lambda}$ become

$$\lambda = F_{e1}\,, \tag{5.9}$$

$$\bar{\lambda} = \begin{bmatrix} F_{e1} \sin\alpha + (F_{e2} + F_{z2}) \cos\alpha \\ L_{e1} + L_{z1} \\ L_{e2} + L_{z2} \\ L_{e3} + L_{z3} \\ F_{e3} + F_{z3} \end{bmatrix} \tag{5.10}$$

The constraint forces $F_{z\alpha}$ and torques $L_{z\alpha}$ are gotten easily once the elements of matrix $\bar{\lambda}$ are known, [4,7]. In particular, the first element $\bar{\lambda}_1$ represents the force required to maintain a given kinematic excitation $\varkappa(t)$. It is composed of applied forces $F_{e1} \sin\alpha$ and $F_{e2} \cos\alpha$ resulting from the spring and of the actuator force $F_{z2} \cos\alpha$.

One may be inclined to suspect that $\bar{\lambda}_1$ results in a nonzero projection of a constraint force onto the free mode φ, but this is not really true. The projection of the generalized force $\bar{\lambda}_1$ onto φ does not vanish (see Fig. 9), but it does not contain any constraint action. On the other hand, the example demonstrates that the projections of action Λ onto the constrained modes $\bar{\varphi}$ must not contain constraint forces and torques exclusively. The expression for $\bar{\lambda}_1$ contains applied forces F_{e1} and F_{e2} in addition to constraint force F_{z2}. Finally, it is important to recognize that components λ and $\bar{\lambda}$ along axes ψ and $\bar{\psi}$ are not identical with the projections of Λ onto these axes, in case of nonorthogonal vector bases. This is visualized for the present example in Fig. 9 for force components $F_e = [F_{e1}, F_{e2}]^T$ and F_{z2} acting on the body in the 1,2-plane. Generalized forces $\lambda = \lambda_1$ and $\bar{\lambda}_1$ appear as the projections of action Λ onto the mode vectors φ and $\bar{\varphi}$, but it is the sum of vectors $\psi\lambda_1 + \bar{\psi}\bar{\lambda}_1$ acting along the axes of $\hat{\psi}$ which yields action $\Lambda = [F_{e1}, F_{e2} + F_{z2}]^T$.

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The basic idea of using the theory of linear vector spaces to represent the joints in a multibody system has been first proposed by R. E. ROBERSON, University of California, San Diego, more than ten years ago. It has been elaborated in a cooperation of the first author of this paper and R. E. ROBERSON on the development of multibody formalisms. The contributions of R. E. ROBERSON to the concepts presented here are acknowledged gratefully. A detailed discussion of how to use this description of joints in multibody formalisms is given in a book by R. E. ROBERSON and R. SCHWERTASSEK to be published within the next months.

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