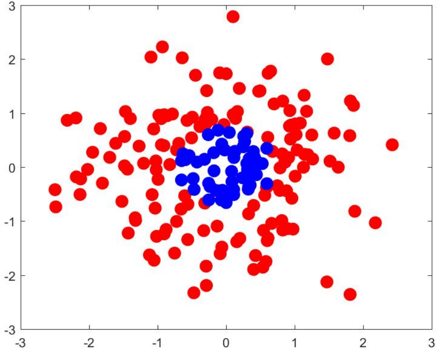
09 - Introduction to Nonlinear Models

Numerical Methods for Deep Learning

February 7, 2018

Why nonlinear problems

Impossible to find a linear separator between points



Goal/Trick

Embed the point in higher dimension or move the points to make them linearly separable

Why nonlinear problems

Consider regression try to find $\mathbf{W}$ such that

$$\mathbf{C} = \mathbf{Y}\mathbf{W}$$

Assume $\mathbf{C} \in R^m$, $\mathbf{Y} \in R^{m \times n}$ and $m \gg n$.
If $\mathbf{Y}$ is full rank impossible to fit the data.

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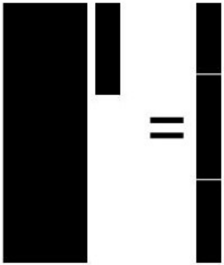

Why nonlinear problems

Use a linear transformation of the points

$$\mathbf{C} = (\mathbf{YK})\hat{\mathbf{W}}$$

Assume $\mathbf{C} \in R^m$, $\mathbf{K} \in n \times s$ is an arbitrary matrix.
$(\mathbf{YK}) \in R^{m \times s}$ and s arbitrary.
If $\mathbf{Y}$ is full rank over-determined (still) impossible to fit the data.

Why nonlinear problems

The size of $\mathbf{YK}$ is $m \times s$ but the problem is still over-determined

$$\mathrm{rank}(\mathbf{YK}) = n \ll m$$

Why nonlinear problems

$$\mathrm{rank}(\mathbf{YK}) = n \ll m$$

But if we use a "smart" nonlinear function f then

$$\mathrm{rank}\left(f(\mathbf{YK} + b)\right) = \min(m, s)$$

The fundamental theorem

Given the data $\mathbf{Y} \in R^{m \times n}$ and $\mathbf{C} \in R^{m}$ with $m > n$ There is a transformation f and a matrix $\mathbf{K} \in R^{n \times s}$ such that $f(\mathbf{YK})$ is of full rank m and therefore we can find a vector $\mathbf{W}$ such that

$$f(\mathbf{YK} + b)\mathbf{W} = \mathbf{C}$$

The fundamental theorem

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- How to find f?
- How to find $\mathbf{K}$ and b?

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- How to find $\mathbf{K}$ and b?

Early days - motivated by neurons

- f is a smooth step function $\tanh(t)$
- Better choices for f is the relu $f = \max(x, 0)$
- $\mathbf{K}, b$ can be
  - Random (random kernels recently branded as extreme learning machines)
  - Optimized (learned) to fit the data

Random kernels

Choose $\mathbf{K}$ randomly and solve the least-squares/classification problem

- Can interpolate any function
- May require very large $\mathbf{K}$ (size of the data)
- May not generalize well
- Large dense linear algebra
- Very easy to program.
- Can serve as a benchmark to more sophisticated methods

Random kernels - programming

Homework assignment I

- Regression: Given the data $\{\mathbf{Y}, \mathbf{C}\}$ write a code that computes the weights such that

$$f(\mathbf{YK})\mathbf{W} \approx \mathbf{C}$$

- Classification: Given the data $\{\mathbf{Y}, \mathbf{C}\}$ write a code that computes the weights such that $f(\mathbf{YK})\mathbf{W}$ classifies $\mathbf{C}$
- Try different activation functions and different size of $\mathbf{K}$.

Test your code on a validation set and report.

Prize to the best winning team