Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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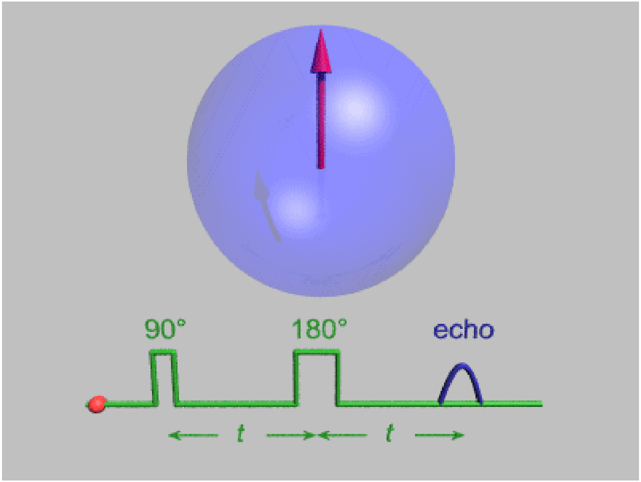
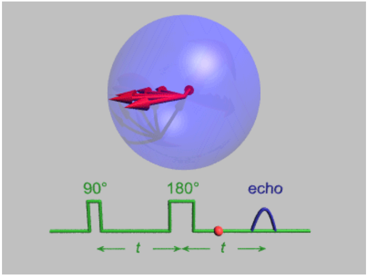
TODO: uncomment all "pause" statements

1. Animation of a typical “spin echo” MRI sequence



1. In actuality, spins (water molecules) do not truly fully refocus
2. Relative angular frequency depends on local magnetic field, and therefore spins dephase at different rates at different locations
3. In particular, the **diffusion** of spins during the scan ($\approx 40\,\text{ms}$) leads to a net lost in signal: the "echo" is weaker



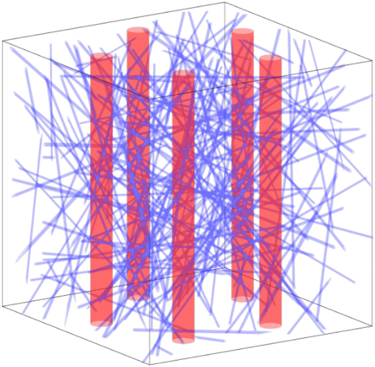


Figure: Cubic imaging voxel filled with randomly oriented microvasculature

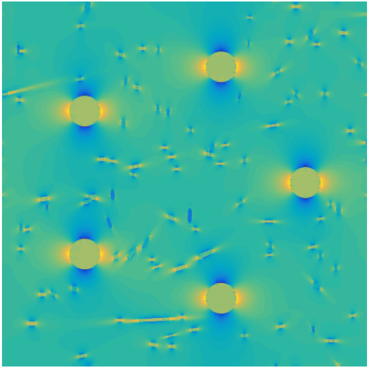


Figure: Cross section of ω corresponding to the microvasculature filled voxel

The Bloch-Torrey Equation

1. Evolution of the transverse magnetization through time is modelled with the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

where:

$$\mathcal{M} = M_x + i\,M_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. The initial transverse magnetization $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. Boundary conditions are typically zero Neumann or periodic
4. Note:

$$D = 0 \;\Rightarrow\; \mathcal{M}(\mathbf{x}, t) = \mathcal{M}_0(\mathbf{x})e^{-\Gamma(\mathbf{x})t}$$

1. One effective method of solving the BT equation is via *operator splitting methods*
2. First, the BT PDE is rewritten in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

where

$$H = -D\Delta + \Gamma.$$

3. Then, the general solution $\mathcal{M}$ may then be written as

$$\mathcal{M} = e^{-Ht}\mathcal{M}_0$$

where e^{-Ht} is the *evolution operator*

1. Now, the evolution operator may be *split* using the approximation

$$\begin{aligned} e^{-Ht} &= e^{D\Delta t - \Gamma t} \\ &\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3) \end{aligned}$$

2. Although e^{-Ht} has no closed form, the split operators do:

$$\begin{aligned} e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\ e^{D\Delta t}\mathcal{M} &= \Phi * \mathcal{M} \end{aligned}$$

where $\odot$ is the Hadamard (pointwise) product, $*$ is the spatial convolution, and Φ is a Gaussian smoothing kernel with $\sigma = \sqrt{2Dt}$

Finite Element Methods

1. The BT equation can also be solved using FEM
2. First, let $u = M_x$ and $v = M_y$ and rewrite the complex Bloch-Torrey PDE as a pair of coupled real PDE's:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v, & u(\mathbf{x}, 0) = M_x(\mathbf{x}, 0) \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u, & v(\mathbf{x}, 0) = M_y(\mathbf{x}, 0) \end{cases}$$

1. Applying the method of lines, the pair of PDE's becomes

$$M^h\mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h\mathbf{v}$$
$$M^h\mathbf{v}_t = -(DK^h + R^h)\mathbf{v} - W^h\mathbf{u}$$

where $R^h_{ij} := \int R\,\phi_i\,\phi_j\,\mathrm{d}x$, $W^h_{ij} := \int \omega\,\phi_i\,\phi_j\,\mathrm{d}x$, and M^h and K^h are the usual mass and stiffness matrices

Time Stepping Methods

$$M^h \mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v}$$
$$M^h \mathbf{v}_t = -(DK^h + R^h)\mathbf{v} - W^h \mathbf{u}$$

1. Solutions to the Bloch-Torrey equation decay exponentially in time at a rate roughly equal to $R(\mathbf{x})$
2. The time discretization scheme should therefore be at least strongly A-stable to reflect this
3. For this reason, the strongly A-stable and second order accurate time stepping scheme $\mathrm{TR\text{-}BDF2}$ was used

Operator Splitting versus FEM

1. Comparing the solution of the BT equation with splitting methods versus FEM...
2. **TODO: do this**