MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
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Homework Assignment 8

Please submit the following files as indicated below: source code PDF file image file video file

If you haven't done so already, install ParaView on your computer. This is already included in many Linux distributions. For other operating systems, visit `https://www.paraview.org/download/`.

Question 1 | **1 mark** | This assignment is dedicated to a posteriori error estimates for the problem

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{P}$$

where Ω is the unit square $]0,1[^2$.

We want to solve this problem because we are interested in the average of u over the set $R = \left]\frac{1}{2}, 1\right[\times \left]0, \frac{1}{2}\right[$.

Following the dual weighted residual method, what is the dual problem that you have to solve for z? Write down its weak formulation, then its strong formulation. Don't forget to specify what space the solution and the test functions belong to for the weak formulation.

Hint: Indicator function.

The quantity of interest is

$$J(u) = \frac{1}{|R|} \int_R u \,\mathrm{d}x = \int_\Omega \frac{\mathbb{1}_R}{|R|} u \,\mathrm{d}x$$

Weak formulation: find $z \in H_0^1(\Omega)$ such that for all test functions $v \in H_0^1(\Omega)$

$$\begin{aligned} B(v, z) &= J(v) \\ \int_\Omega \nabla z \cdot \nabla v \,\mathrm{d}x &= \int_\Omega \frac{\mathbb{1}_R}{|R|} v \,\mathrm{d}x \end{aligned}$$

Strong formulation:

$$\begin{aligned} -\Delta z = \frac{\mathbb{1}_R}{|R|} &= \begin{cases} 4 & x \in \left]\frac{1}{2}, 1\right[\times \left]0, \frac{1}{2}\right[\\ 0 & \text{otherwise} \end{cases} && \text{in } \Omega \\ z &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Question 2 | **4 marks** | If the source term in Problem (P) is given as

$$f(x) = a(a+1)x_1^{a-1}x_2(1-x_2) + 2x_1(1-x_1^a)$$

for $a \geq 1$, then the analytical solution is

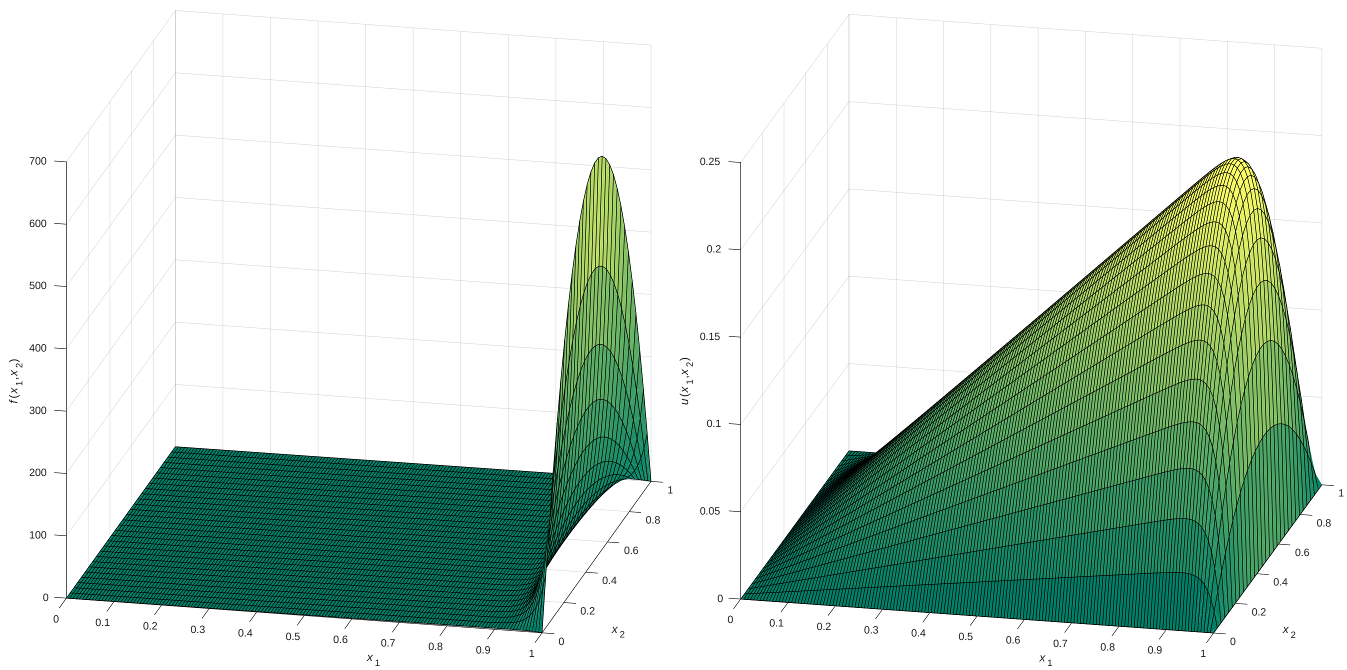
$$\bar{u}(x) = x_1(1-x_1^a)x_2(1-x_2)$$

which has an average of

$$\frac{3a - 2 + 2^{1-a}}{24a + 48}$$

over the set R.

For large a, this problem is numerically challenging: observe that f becomes very large near the right boundary, while it remains comparatively small elsewhere in the domain. As a result, the solution $\bar{u}$ exhibits a sharp boundary layer near $x_1 = 1$.



Source term f (left) and analytical solution $\bar{u}$ (right) for $a = 50$.

(a) Download the FEniCS script `hw8.py` and complete the missing commands. This script should evaluate the a posteriori estimator $\eta_{L^2} \approx \|u^h - \bar{u}\|_{L^2} = \|e^h\|_{L^2}$ as derived in class (or on Canvas, under modules). Solve Problem (P) on the given grids to complete the following table:

h	$\|e^h\|_{L^2}$	η_{L^2}	$\eta_{L^2}/\|e^h\|_{L^2}$
$\frac{1}{64}\sqrt{2}$ (64 × 64 grid)	0.002420	0.066093	27.3
$\frac{1}{128}\sqrt{2}$ (128 × 128 grid)	0.000648	0.017056	26.3
$\frac{1}{256}\sqrt{2}$ (256 × 256 grid)	0.000162	0.004375	27.0

Is the error overestimated or underestimated by η_{L^2}? By what factor, approximately?

These errors are overestimated by a factor of approximately 27.

NB: Since we derived this error estimator by applying a global upper bound to estimate the interpolation error in the dual problem, this is a rather coarse estimate. Anyways, in practice we're more interested in the distribution of the error over the triangles (cf part (c)) and not so much in its absolute magnitude.

(b) Compute a posteriori estimators $\eta_J \approx |J(u^h) - J(\bar{u})| = |J(e^h)|$ for the error in the average solution value on R, using both the expensive Strategy 1 and the cheap Strategy 2 to approximate the dual weights (cf p 55 in the notes). Complete the following table:

h	$\lvert J(e^h)\rvert$	$\lvert\eta_{J,1}\rvert$	$\lvert\eta_{J,2}\rvert$	$\lvert\eta_{J,1}/J(e^h)\rvert$	$\lvert\eta_{J,2}/J(e^h)\rvert$
$\frac{1}{64}\sqrt{2}$ (64 × 64 grid)	0.000567	0.0007784	0.0007133	1.37	1.26
$\frac{1}{128}\sqrt{2}$ (128 × 128 grid)	0.000157	0.0001876	0.0001709	1.19	1.09
$\frac{1}{256}\sqrt{2}$ (256 × 256 grid)	0.000040	0.000046	0.000041	1.14	1.04

Is the error overestimated or underestimated? By what factor, approximately?

The expensive estimator overestimates the true error by up to 37%, but this is improving on finer meshes.

The cheap estimator overestimates the true error by only 26% max, also improving significantly on the finer meshes.

NB: By introducing an absolute value into all local error indicators, we first overestimate the true error. However, the cheap approximation for the dual weights usually underestimates the true interpolation error of the exact dual solution. Overestimation and underestimation appear to cancel partially, and this could explain why the cheap estimator is actually better here than the expensive one.

(c) For the convergence studies above we have refined the entire mesh from 64×64 to 128×128 to 256×256. The second mesh is four times larger, the third mesh even 16 times larger than the coarsest one. This makes uniform mesh refinement very expensive. We can probably compute a solution that is just as accurate as the solution on the 256×256 mesh, by refining only those triangles on the 64×64 mesh with a noteworthy contribution to the overall error.

Solve Problem (P) on the 64×64 mesh and plot the numerical solutions u^h and z^h, the cell residuals $\|r_T\|_{L^2}$, the dual weights $\|w_T\|_{L^2}$ (approximated with either the expensive or the cheap strategy) and the local error indicators η_T. What triangles of the 64×64 mesh would you refine to compute the average of u over R more accurately? A rough description like 'near the left boundary' will do. Also give a brief reason for your answer:

I'd refine all triangles near the right boundary except for maybe the top few. As can be seen from the plot of the local error indicators η_T, only these contribute significantly to this error. This is where both

- the residuals are large, i.e. the boundary layer is not well resolved by u^h, and
- the quantity $J(u)$ would be perturbed sensitively by large residuals, as indicated by the dual weights.

NB: We're not using the solution u^h from outside R to compute J. However, even outside of R we can't allow for huge errors in u^h: we still have to solve the equation somewhat accurately there, since large errors would also pollute the solution inside R due to diffusion (or e.g. advection, not present here). The dual problem automatically determines where good accuracy is needed to compute the given J and where not. From the plot of the dual weights we see that our average functional used with Poisson's equation requires highest accuracy inside R, and to a lesser extent in its immediate surroundings.

(Optional) Bonus Question | 1 bonus mark | Derive an a priori estimate for the error in the above quantity of interest. Does it agree with the numerical results from Q2(b)?

For this particular problem we have already derived the a priori L^2 error estimate in class. We could re-use this to obtain with Cauchy-Schwarz

$$|J(e^h)| = \left| \int_\Omega \frac{\mathbb{1}_R}{|R|} e^h \,\mathrm{d}x \right| \leq \left\| \frac{\mathbb{1}_R}{|R|} \right\|_{L^2} \|e^h\|_{L^2} \leq Ch^2 \frac{\|f\|_{L^2}}{|R|^{3/2}}.$$

NB: This error bound is deteriorating as $|R| \to 0$ and the limiting case of a set of measure zero (e.g. average over a line R, function value in a point R) it is no longer valud. This suggests that such quantities of interest probably converge slower than h^2! Indeed, we've seen in Theorem 2.3.23 that — assuming sufficient regularity — point values only converge at a rate of $h^2|\ln h|$.

In general you may not have this L^2-estimate available, and maybe you're not even interested in it. All you have is the error estimate derived using CÉA's lemma or one of STRANG's lemmas, which would measure the error e.g. in the H^1 or energy norm. Then we apply the AUBIN-NITSCHE trick with the dual problem

$$B(v, z) = J(v) \qquad \forall v \in H_0^1(\Omega)$$

for $v = e^h$ (note that $\|z\|_{H^2} \leq C \left\| \frac{\mathbb{1}_R}{|R|} \right\|_{L^2} = C|R|^{-3/2}$):

$$\begin{aligned}
|J(e^h)| &= |B(e^h, z)| && \text{(AUBIN-NITSCHE trick / dual problem)} \\
&= |B(e^h, z - I^h z)| && \text{(GALERKIN orthogonality)} \\
&\leq \|e^h\|_B \|z - I^h z\|_B && \text{(CAUCHY-SCHWARZ)} \\
&\leq Ch\|f\|_{L^2} Ch\|\nabla^2 z\|_{L^2} && \text{(energy error estimate \& interpolation error estimate)} \\
&\leq Ch^2\|f\|_{L^2} \left\| \frac{\mathbb{1}_R}{|R|} \right\|_{L^2} && \text{(elliptic } H^2\text{-regularity, see above)} \\
&= Ch^2 \frac{\|f\|_{L^2}}{|R|^{3/2}}.
\end{aligned}$$

Yes, indeed, the table in Q2(b) shows that $|J(e^{h/2})| \approx \frac{1}{4}|J(e^h)|$

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
