Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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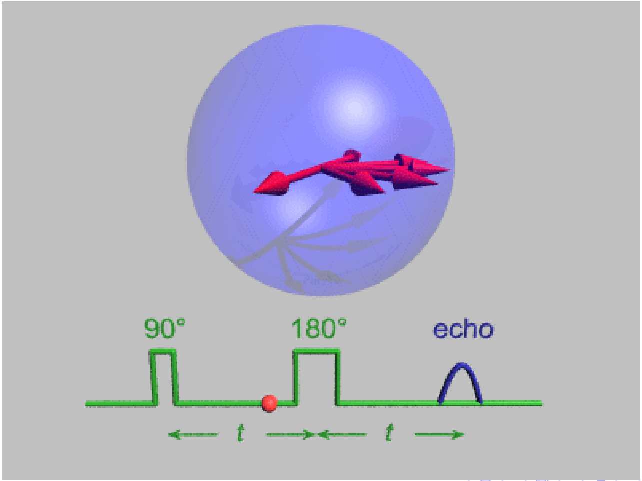
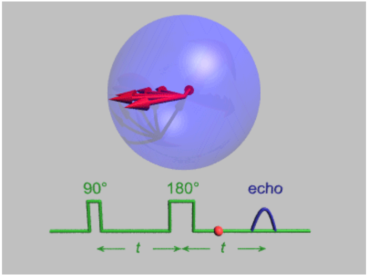
TODO: uncomment all “pause” statements

1. Animation of a typical “spin echo” MRI sequence
2. **TODO: re-compile with animated gif**



Magnetic Resonance Visualised

1. In actuality, spins (water molecules) do not truly fully refocus
2. Relative angular frequency depends on local magnetic field, and therefore spins dephase at different rates at different locations
3. In particular, the **diffusion** of spins during the scan ($\approx 40\,\text{ms}$) leads to a net lost in signal: <u>the "echo" is weaker</u>



1. We model this system through the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

   where:

$$\mathcal{M} = M_x + iM_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. The initial transverse magnetization $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. Boundary conditions are typically zero Neumann or periodic

Magnetic Resonance Visualised

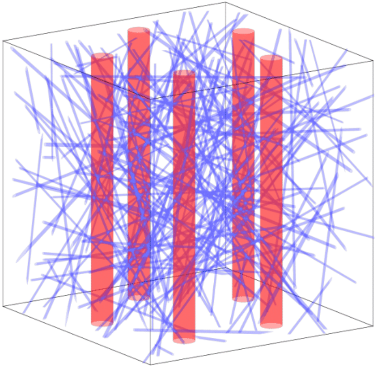


Figure: Cubic imaging voxel filled with randomly oriented microvasculature

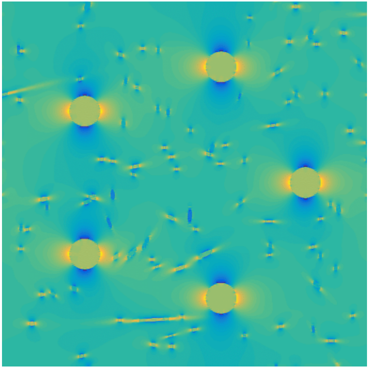


Figure: Cross section of ω corresponding to the microvasculature filled voxel

Operator Splitting Methods

1. One effective method of solving the BT equation is via *operator splitting methods*
2. First, we re-write the BT PDE in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

   where

$$H = -D\Delta + \Gamma.$$

3. Then, the general solution $\mathcal{M}$ may then be written as

$$\mathcal{M} = e^{-Ht}\mathcal{M}_0$$

   where e^{-Ht} is the *evolution operator*

Operator Splitting Methods

1. Now, the evolution operator may be *split* using the approximation

$$
\begin{aligned}
e^{-Ht} &= e^{D\Delta t - \Gamma t} \\
&\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3)
\end{aligned}
$$

2. Although e^{-Ht} has no closed form, the split operators do:

$$
\begin{aligned}
e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\
e^{D\Delta t}\mathcal{M} &= \Phi * \mathcal{M}
\end{aligned}
$$

where $\odot$ is the Hadamard (pointwise) product, $*$ is the spatial convolution, and Φ is a Gaussian smoothing kernel with $\sigma = \sqrt{2Dt}$

1. The BT equation can also be solved using FEM
2. First, we let $u = M_x$ and $v = M_y$ and re-write the complex Bloch-Torrey PDE as a pair of coupled real PDE's:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v, & u(\mathbf{x}, 0) = M_x(\mathbf{x}, 0) \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u, & v(\mathbf{x}, 0) = M_y(\mathbf{x}, 0) \end{cases}$$

3. Writing the pair of PDE's as a linear system, we have

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D\Delta - R & \omega \\ -\omega & D\Delta - R \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D\Delta - R & \omega \\ -\omega & D\Delta - R \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

1. If we take the inner product of both sides with (u, v), we have that

$$\int_\Omega uu_t + vv_t \, \mathrm{d}x = \int_\Omega Du\Delta u - Ru^2 + \omega uv - \omega vu + Dv\Delta v - Rv^2 \, \mathrm{d}x$$

   Integrating by parts, we have that

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_\Omega u^2 + v^2 \, \mathrm{d}x\right) = \int_\Omega -D\left(||\nabla u||^2 + ||\nabla v||^2\right) - R\left(u^2 + v^2\right) \mathrm{d}x$$
$$\leq 0$$

2. Now, $\int_\Omega u^2 + v^2 \, \mathrm{d}x = \|\mathcal{M}\|_{L^2}^2$, and so $\|\mathcal{M}\|_{L^2}$ decreases with time

1. For this reason, the second order and strongly A-stable time stepping $\mathrm{TR\text{-}BDF2}$ was used

Operator Splitting vs. FEM

1. Comparing the solution of the BT equation with splitting methods vs. FEM...
2. **TODO: do this**