MATH521
Numerical Analysis of Partial Differential Equations
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Stability

General Idea Imagine we want to solve the problem $T(u) = 0$. This function T would be

$$T(u) = \begin{pmatrix} -\Delta u - f \\ u\big|_{\partial\Omega} - g \end{pmatrix} \tag{1}$$

for the Poisson-Dirichlet problem, but T could also represent any other PDE, ODE or maybe an integral, ...

Numerically, we solve a discrete problem $T^h(u^h) = 0$ instead which gives us a numerical solution u^h. For instance, the above problem discretised with finite differences yields the left hand side

$$T^h(u^h) = L^h u^h - f^h \tag{2}$$

of the 'big linear system', where $h > 0$ is a mesh parameter. For our Poisson-Dirichlet solver, this would usually be $h = \max\{\, h_1, h_2 \,\}$.

Clearly, the discrete problem must meet some minimum requirements to be useful at all. In particular, we require the problem $T^h(u^h) = 0$ to be well-posed. Recall that this means that a discrete solution u^h (i) exists, (ii) is unique and (iii) depends continuously on the discrete data (independent of the grid spacing h, provided that this is sufficiently small). This is called *stability* of the numerical scheme.

Formal Definition The numerical scheme T^h is said to be *stable* (with respect to h) if there are constants $h_0 > 0$ and $C > 0$ such that $T^h(u^h) = 0$ has a unique solution for all $h \in \,]0, h_0]$ and if additionally the stability inequality

$$\|u^h - v^h\| \leq C\|T^h(u^h) - T^h(v^h)\|$$

holds for all discrete functions v^h and all $h \in \,]0, h_0]$.

Note that since $T^h(u^h) = 0$, we could have equally written $\|T^h(u^h) - T^h(v^h)\| = \|T^h(v^h)\|$, but the above notation is probably more illustrative.

Your Task Is our finite-difference approximation of the 1D Poisson-Dirichlet problem stable and if so, what is the stability constant C?

It's sufficient if you consider the 1D problem

$$\begin{aligned} -u'' &= f \qquad \text{in }]0, 1[\\ u(0) = u(1) &= 0 \end{aligned}$$

because the analysis of the 2D case is essentially the same.

Let's break this down into three steps:

1. Does the discrete problem, i.e. the 'big linear system' have a unique solution?
2. What does the stability inequality mean explicitly for this (linear) problem?
3. What's the link between the constant C and the eigenvalues of the 'big matrix'? Compute C!

Hint: The $(N-1)$ eigenvalues of the 'big' $(N-1) \times (N-1)$ matrix L^h which approximates the negative second derivative using N subintervals are

$$\lambda_i = \frac{4}{h^2} \sin^2\left(\frac{N-i}{N}\frac{\pi}{2}\right) \qquad i = 1, \ldots, N-1.$$

You don't have to prove this, but can you make any hypothesis on what the eigenvectors of this matrix are?