Consistency

General Idea Imagine we want to solve the problem $T(u) = 0$. This function T would be

$$T(u) = \begin{pmatrix} -\Delta u - f \\ u\big|_{\partial\Omega} - g \end{pmatrix} \tag{1}$$

for the Poisson-Dirichlet problem, but T could also represent any other PDE, ODE or maybe an integral, ...

Numerically, we solve a discrete problem $T^h(u^h) = 0$ instead which gives us a numerical solution u^h. For instance, the above problem discretised with finite differences yields the left hand side

$$T^h(u^h) = L^h u^h - f^h \tag{2}$$

of the 'big linear system', where $h > 0$ is a mesh parameter. For our Poisson-Dirichlet solver, this would usually be $h = \max\{ h_1, h_2 \}$.

Clearly, there must be some relationship between the continuous problem $T(u) = 0$ and the discrete problem $T^h(u^h) = 0$ for T^h to be a sensible approximation of T. If we plug the (usually unknown) exact solution $\bar{u}$ of the exact problem $T(u) = 0$ into the discrete formulation, then this *residual* $T^h(\bar{u})$ should ideally be zero, but at least small. 'Small' means that as we refine the mesh ($h \to 0$), the numerical scheme T^h should become an increasingly accurate approximation of T with perfect agreement in the hypothetical limit $h = 0$. This is called *consistency* of the numerical scheme.

Formal Definition The numerical scheme T^h is said to be

1. *consistent*, if for every solution $\bar{u}$ of $T(u) = 0$
$$\|T^h(\bar{u}) - T^h(u^h)\| \to 0 \qquad \text{as } h \to 0.$$
2. *consistent of order $O(h^p)$*, if additionally
$$\|T^h(\bar{u}) - T^h(u^h)\| = O(h^p) \qquad \text{as } h \to 0.$$

Note that since $T^h(u^h) = 0$, we could have equally written $\|T^h(\bar{u}) - T^h(u^h)\| = \|T^h(\bar{u})\|$, but the above notation is probably more illustrative.

Your Task What is the order of consistency of our finite-difference scheme for the 1D Poisson-Dirichlet problem? Under what regularity assumption on the exact solution $\bar{u}$, i.e. how many continuous derivatives do we need?

It's sufficient if you consider the 1D problem

$$\begin{aligned} -u'' &= f \qquad \text{in }]0,1[\\ u(0) = u(1) &= 0 \end{aligned}$$

because the analysis of the 2D case is essentially the same.

Let's break this down into three steps:

1. What does $\|T^h(\bar{u}) - T^h(u^h)\|$ mean explicitly for this problem?
2. Compute the difference between $-u''(x)$ and its finite-difference approximation at a fixed point $x \in]0,1[$!
3. Now take the maximum difference on the domain.