Interpolation Error on 'Special Triangles'

Discretisation Errors of Finite Element Methods

Today we learnt about three sources of errors in a finite-element discretisation:

1. Galerkin approximation error: we "truncate" the infinite-dimensional function space V and only use $V^h \subset V$ for the shape and test functions
   - functions in V are approximated with piecewise polynomial functions from V^h
   - the (convex) domain Ω is approximated with a polygon $\Omega^h \subset \Omega$
2. quadrature error: we generally don't solve the Galerkin equations exactly, but some perturbed Galerkin equations where integral expressions have been replaced with quadrature formulae
3. conformity error: if we use a non-conforming approximation $V^h \not\subset V$, then the error analysis becomes fairly complicated

To analyse the first of these three errors, we studied how well functions in V can be approximated by functions in V^h.

The Bramble-Hilbert Lemma

The Bramble-Hilbert lemma is an abstract result from approximation theory giving an estimate for functionals with certain properties. We apply this lemma to functionals involving the interpolation error $I^h v - v$.

v = some function from V

$I^h v$ = a piecewise polynomial function from V^h that interpolates v

Interpolation Error for Linear Finite Elements on a 'Special Triangle'

Since finite element methods are very flexible and versatile, their consistency proofs have to apply to a very general setting, too. Therefore, they often require a number of technicalities. Unfortunately this means that the basic strategy, which is not actually that complicated, is uneasy to understand if we looked at the most general setting first.

So let's find the interpolation error when we use

- linear finite elements
- on a 'special triangle' that is just a downsized version of the reference triangle, but without any shear component in the transformation $F_T : \hat{T} \to T : \hat{x} \mapsto A_T\hat{x} + b_T$. We just consider the special case

$$A_T = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}$$

You used a change of variables to derive that the norms of any sufficiently regular function $w(x) = w(F_T(\hat{x})) = \hat{w}(\hat{x})$ scale as follows:

$$\|w\|_{L^2(T)} = h\|\hat{w}\|_{L^2(\hat{T})}$$

$$\|\nabla w\|_{L^2(T)} = \|\hat{\nabla}\hat{w}\|_{L^2(\hat{T})}$$

$$\|\nabla^2 w\|_{L^2(T)} = \frac{1}{h}\|\hat{\nabla}^2\hat{w}\|_{L^2(\hat{T})}$$

$$\|\nabla^i w\|_{L^2(T)} = h^{1-i}\|\hat{\nabla}^i\hat{w}\|_{L^2(\hat{T})}$$

Now we apply the Bramble-Hilbert lemma to the functional

$$F(\hat{v}) = \|I^h\hat{v} - \hat{v}\|_{L^2(\hat{T})}$$

the interpolation error on the reference triangle measured in the L^2-norm. Recall that we interpolate the function $\hat{v}$ with linear finite elements, so the interpolation error is zero if the input function is already a linear polynomial, $\hat{v} \in P_1(\hat{T})$. You can check that all three assumptions of the Bramble-Hilbert lemma are satisfied. Hence, provided that $\hat{v} \in H^2(\hat{T})$,

$$\|I^h\hat{v} - \hat{v}\|_{L^2(\hat{T})} \leq c\|\hat{\nabla}^2\hat{v}\|_{L^2(\hat{T})}.$$

Now we apply the scaling that you derived on both sides to obtain the following inequality on the actual triangle T:

$$\|I^h v - v\|_{L^2(T)} \leq ch^2\|\nabla^2 v\|_{L^2(T)}.$$

In plain English: if we interpolate a H^2-function v on a triangle T with linear finite elements, then the interpolation error in the L^2-norm is of order $O(h^2)$.

In the same fashion we obtain

$$\|\nabla(I^h v - v)\|_{L^2(T)} \leq ch\|\nabla^2 v\|_{L^2(T)}$$

$$\|\nabla^2(I^h v - v)\|_{L^2(T)} \leq c\|\nabla^2 v\|_{L^2(T)}$$

i.e. the gradient of the interpolant converges to the exact gradient only at a linear rate and the Hessian may not converge at all.

General Finite Elements and General Triangles

The general interpolation result can be found in Theorem 2.3.14. Applied to linear finite elements ($k=1$), this theorem yields (using $i=0,1,2$)

$$\|I^h v - v\|_{L^2(T)} \leq ch^2 \|\nabla^2 v\|_{L^2(T)}$$

$$\|\nabla(I^h v - v)\|_{L^2(T)} \leq c\frac{h^2}{r} \|\nabla^2 v\|_{L^2(T)}$$

$$\|\nabla^2(I^h v - v)\|_{L^2(T)} \leq c\frac{h^2}{r^2} \|\nabla^2 v\|_{L^2(T)}$$

where r is the incircle radius of the triangle T. If you prescribe a minimum angle for the triangle as $h \to 0$ or, equivalently, an upper bound for the ratio $h/r \leq c$, these general inequalities give the same convergence rates as the ones we derived for 'special triangles'.

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