Homework Assignment 9

Question 1 | 1 mark We consider the initial boundary value problem for the heat equation

$$\begin{aligned} \partial_t u(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

where u is a temperature field, u_0 an initial temperature distribution, the diffusion-like parameter $a > 0$ the heat conductivity of the material, f a source term e.g. due to thermal radiation and $T > 0$ a final time. The homogeneous Neumann boundary conditions mean that the domain Ω is perfectly insulated so that no thermal energy is radiated into the environment.

The θ-method is a class of Runge-Kutta schemes for integrating ODEs of the form

$$\dot{U} = F(t, U)$$

by using the iteration

$$U_+ = U_\circ + \Delta t\,(\theta F(t_+, U_+) + (1-\theta)F(t_\circ, U_\circ)).$$

The parameter $\theta \in [0,1]$ can be interpreted as the 'degree of implicitness', since $\theta = 0$ gives the forward Euler method, $\theta = \frac{1}{2}$ the Crank-Nicolson method (aka implicit trapezium rule in the ODE context) and $\theta = 1$ the backward Euler method.

For the discretisation in space, we apply linear finite elements. Both the time step size Δt and the spatial triangulation $\mathcal{T}^h$ are fixed.

Show that in this setting both the method of lines and Rothe's method lead to the same discrete problems

$$\left(M^h + \theta\Delta t a K^h\right)\vec{u}^h_+ = \left(M^h - (1-\theta)\Delta t a K^h\right)\vec{u}^h_\circ + \Delta t\left(\theta\vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ\right).$$

You don't have to include any details about the components of the discrete vectors and matrices. We all know what they are!

Solution. **Method of lines.** First, we semi-discretise the problem (H) in space using linear finite elements to obtain the weak form of the method of lines

$$\begin{aligned} M^h\partial_t\vec{u}^h + aK^h\vec{u}^h &= \vec{f}^h && \text{in } Q^h =]0,T[\times\Omega^h \\ \vec{u}^h(0) &= \vec{u}^h_0 && \text{in } \Omega^h. \end{aligned} \tag{1}$$

If we now write $U = \vec{u}^h$ and $F(t, \vec{u}^h) = (M^h)^{-1}(\vec{f}^h - aK^h\vec{u}^h)$, we can write the θ-method scheme for the initial value problem (1) as

$$\vec{u}^h_+ = \vec{u}^h_\circ + \Delta t\left(\theta\,(M^h)^{-1}(\vec{f}^h_+ - aK^h\vec{u}^h_+) + (1-\theta)(M^h)^{-1}(\vec{f}^h_\circ - aK^h\vec{u}^h_\circ)\right),$$

or, multiplying through by M^h and rearranging, we have

$$\left(M^h + \theta\Delta t a K^h\right)\vec{u}^h_+ = \left(M^h - (1-\theta)\Delta t a K^h\right)\vec{u}^h_\circ + \Delta t\left(\theta\vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ\right)$$

as desired.

Rothe's method. First, we semi-discretise equation (H) in time using the θ-method by writing $U = u$ and $F(t,u) = f + a\Delta u$. Since the time step Δt is fixed, the resulting scheme is given by

$$u_+ = u_\circ + \Delta t\Big[\theta(f_+ + a\Delta u_+) + (1-\theta)(f_\circ + a\Delta u_\circ)\Big].$$

Rearranging, we have

$$(1 - \theta\Delta t a)\, u_+ = (1 - (1-\theta)\Delta t a)\, u_\circ + \Delta t\,(\theta f_+ + (1-\theta) f_\circ)\,.$$

Now, using the fact that the spatial triangulation $\mathcal{T}^h$ is invariant with time, we can safely multiply both sides through using test functions from the same linear finite element space and integrate in the standard way to obtain

$$\left(M^h + \theta\Delta t a K^h\right)\vec{u}^h_+ = \left(M^h - (1-\theta)\Delta t a K^h\right)\vec{u}^h_\circ + \Delta t\left(\theta \vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ\right)$$

as desired.

$\square$

Question 2 | 4 marks

(a) The FEniCS scridpt `hw9.py` implements the backward EULER method for Problem (H). Starting from room temperature ($u_0 \equiv 20$), the bottom left corner of a metal piece Ω with conductivity parameter $a = 0.1$ is held over a flame for one second, then the flame is extinguished. This is modelled by

$$f(t,x) = \begin{cases} 200e^{-5x_1^2-2x_2^2} & t \leq 1 \\ 0 & t > 1 \end{cases}$$

Complete the missing commands to compute the evolution of the temperature field over the first five seconds using a time step size of $\Delta t = 10^{-2}$.

Save your results as a video, using a frame rate such that the video time is equal to the physical time. You don't have to submit any other files for this part of Question 2.

Hint: Open the PVD-file in ParaView, click the 'Apply'-button, select a reasonable colour map and then re-scale the colour values to the range $[20, 160]$. Use the same range for the following questions, too.

Solution. **TODO: see attached video**

□

(b) Generalise this script to implement the θ-method from Question 1. Check whether setting $\theta = 1$ still gives you the same results. Using the same parameters as in Question 2(a), solve the problem with the Crank-Nicolson method and the forward Euler method. What do you observe?

(c) Solve the problem with the forward Euler method again up to time $T = 0.1$, once with $\Delta t = 1.25 \times 10^{-4}$ and once with $\Delta t = 10^{-4}$. Explain your observations, using the relevant terminology.

(d) If you could choose among the time-stepping schemes considered above and other time-discretisation schemes you know of, which one would you choose for this problem? First formulate some reasonable objectives for your simulation, then explain which method attains these to the greatest extent.

Your Learning Progress What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?