A Posteriori Error Estimation

We can derive a posteriori error estimates

- for the error of a quantity of interest $J(u)$, or
- for the error measured in a global (energy, L^2, H^1, ...) norm.

A posteriori estimates are important because they allow us to conclude

- how big the error is
- where the error comes from, i.e. where we should refine the mesh to get a solution of higher accuracy.

Here we'll look at Poisson's equation with Dirichlet boundary conditions. If we had a different equation, then we would get different cell residuals (= the PDE in strong form) and different jump residuals (= boundary terms from integration by parts).

Error of a Quantity of Interest

In this case, J is a given functional, here assumed to be linear.

1. Use the numerical solution u^h to compute, on each triangle T, the residuals
$r_T = -\Delta u^h - f$
$r_{\partial T} = \frac{1}{2} h_T^{-1/2} \left[\partial_n u^h\right]$
2. Solve the dual problem with J on the right hand side to compute a numerical solution z^h. Use the expensive Strategy 1 or the cheap Strategy 2 (p 55 of the notes) to obtain a good approximation of the exact dual solution z and its interpolant $I^h z$. Then use this approximate z to obtain approximations of the dual weights
$w_T = z - I^h z$
$w_{\partial T} = h_T^{1/2} (z - I^h z)$
3. Plug all of the above into the general a posteriori error identity
$J(u^h) - J(\bar{u}) = \sum_{T \in \mathcal{T}^h} \int_T r_T w_T \,\mathrm{d}x + \int_{\partial T \setminus \partial\Omega} r_{\partial T} w_{\partial T} \,\mathrm{d}s$

NB: Since we only compute approximate weights, we also just get an approximation of the error. The expensive Strategy 1 usually gives really good approximations, while the cheap Strategy 2 usually underestimates the true error by up to one order of magnitude.

In fact, we don't actually care whether $J(u^h)$ is too big or too small. In practice, we compute an a posteriori estimate for the absolute value of the error:

$$|J(u^h) - J(\bar{u})| \leq \sum_{T \in \mathcal{T}^h} \left| \int_T r_T w_T \, \mathrm{d}x + \int_{\partial T \setminus \partial\Omega} r_{\partial T} w_{\partial T} \, \mathrm{d}s \right|$$

If you want to know what triangles contribute most to the error, look at the error indicators

$$\eta_T = \left| \int_T r_T w_T \, \mathrm{d}x + \int_{\partial T \setminus \partial\Omega} r_{\partial T} w_{\partial T} \, \mathrm{d}s \right|$$

If η_T is large on a triangle T, then this triangle contributes strongly to the overall error and hence it should be refined. If η_T is close to zero on a triangle T, then this triangle hardly contributes to the overall error and refining this triangle would not lead to a significantly smaller error.

Error in a Global Norm

We consider the error measured in the L^2-norm. We can still use the framework of the dual weighted residual method like for errors of quantities of interest, by defining our quantity of interest as

$$J(u) = \int_T \frac{e^h}{\|e^h\|_{L^2}} u \, \mathrm{d}x.$$

We use this particular J because $\|e^h\|_{L^2} = J(u^h) - J(\bar{u})$ and the above error identity now gives us an expression for the L^2-norm of the error in terms of the residuals and dual weights.

Note that this is just an analytical trick. Since the error is unknown in practice, we cannot actually solve a dual problem with this right hand side. This is where a posteriori error estimates of global norms differ from a posteriori estimates for quantities of interest. Since we cannot compute the dual solution z, we have to use interpolation estimates to get an upper bound for the dual weights.

1. Use the numerical solution u^h to compute, on each triangle T, the residuals
   $r_T = -\Delta u^h - f$
   $r_{\partial T} = \frac{1}{2} h_T^{-1/2} \left[\partial_n u^h\right]$
2. Use the following interpolation estimates for the dual weights:
   $\|z - I^h z\|_{L^2(T)} \leq c h_T^2 \|\nabla^2 z\|_{L^2(T)}$
   $\|z - I^h z\|_{L^2(\partial T)} \leq c h_T^{3/2} \|\nabla^2 z\|_{L^2(T)}$
3. Plug all of the above into the general a posteriori error identity
   $$J(u^h) - J(\bar{u}) = \sum_{T \in \mathcal{T}^h} \int_T (-\Delta u^h - f)(z - I^h z) \, \mathrm{d}x + \int_{\partial T \setminus \partial\Omega} \frac{1}{2} \left[\partial_n u^h\right] (z - I^h z) \, \mathrm{d}s$$
   to obtain the a posteriori estimate
   $$\|e^h\|_{L^2}^2 \leq c \sum_{T \in \mathcal{T}^h} h_T^4 \left(\|r_T\|_{L^2(T)}^2 + \|r_{\partial T}\|_{L^2(\partial T \setminus \partial\Omega)}^2 \right).$$
   aPosterioriL2.pdf (https://canvas.ubc.ca/courses/2337/files/928602/download?wrap=1)

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If you want to know what triangles contribute most to the L^2-error, look at the error indicators

$$\eta_T^2 = h_T^4 \left(\|r_T\|^2_{L^2(T)} + \|r_{\partial T}\|^2_{L^2(\partial T \setminus \partial \Omega)} \right)$$