Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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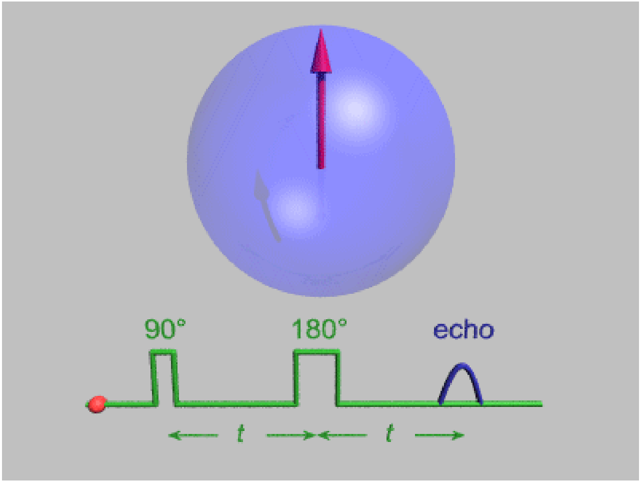
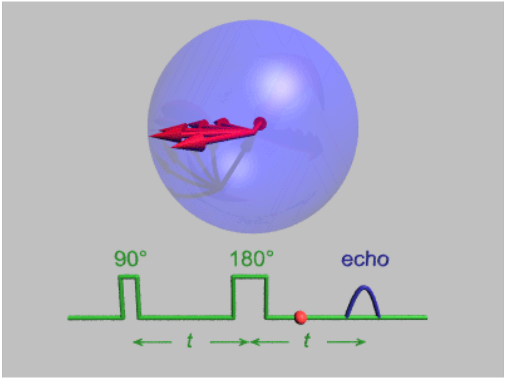
1. Animation of a typical “spin echo” MRI sequence



1. Spins (water molecules) do not truly refocus:
   1. Precession rate depends on position
   2. Diffusion plays an important role



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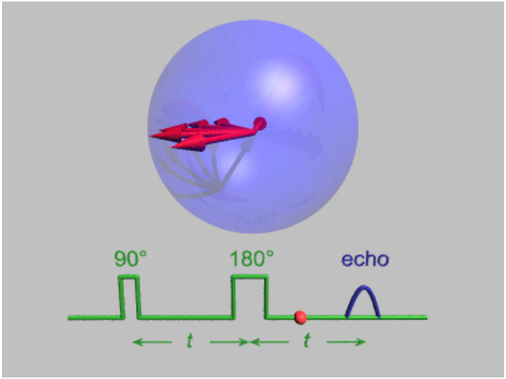

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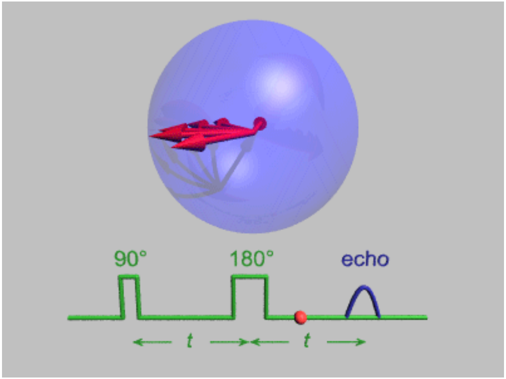

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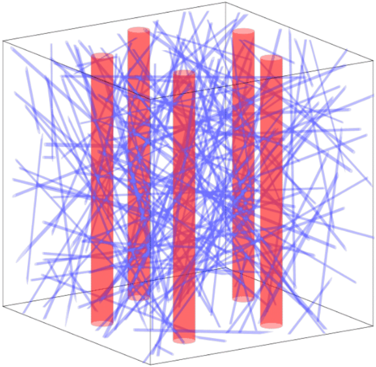

Figure: Simulated imaging voxel

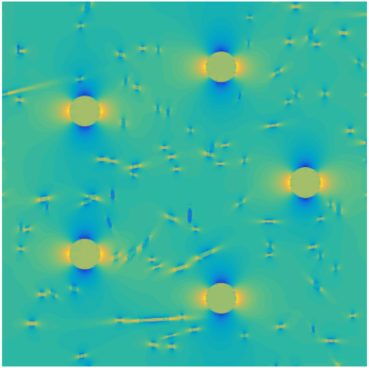

Figure: Cross section of precession rate

The Bloch-Torrey Equation

1. Evolution of the magnetization is modelled by the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

where:

$$\mathcal{M} = M_x + i\,M_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. IC: $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. BC: zero Neumann or periodic
4. Note:

$$D = 0 \;\Rightarrow\; \mathcal{M}(\mathbf{x}, t) = \mathcal{M}_0(\mathbf{x})e^{-\Gamma(\mathbf{x})t}$$

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$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

where

$$H = -D\Delta + \Gamma.$$

2. Then, the general solution $\mathcal{M}$ may then be written as

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1. Now, the evolution operator may be *split* using the approximation

$$\begin{aligned} e^{-Ht} &= e^{D\Delta t - \Gamma t} \\ &\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3) \end{aligned}$$

2. Although e^{-Ht} has no closed form, the split operators do:

$$\begin{aligned} e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\ e^{D\Delta t}\mathcal{M} &= \Phi * \mathcal{M} \end{aligned}$$

Operator Splitting Methods

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Finite Element Methods

1. The BT equation can also be solved using FEM
2. First, let $u = M_x$ and $v = M_y$ and write:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u \end{cases}$$

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1. Applying the method of lines:

$$
\begin{aligned}
M^h \mathbf{u}_t &= -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v} \\
M^h \mathbf{v}_t &= -(DK^h + R^h)\mathbf{v} - W^h \mathbf{u}
\end{aligned}
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where:

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\begin{aligned}
R^h_{ij} &:= \int R\,\phi_i\,\phi_j\,\mathrm{d}x \\
W^h_{ij} &:= \int \omega\,\phi_i\,\phi_j\,\mathrm{d}x
\end{aligned}
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Time Stepping Methods

$$M^h \mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v}$$
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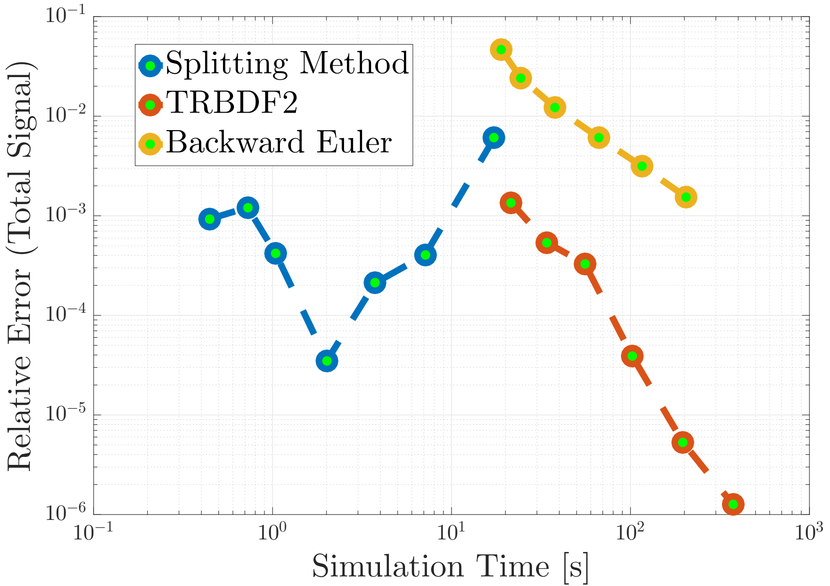
Operator Splitting versus FEM



Conclusion

1. Splitting methods:
   - $+$ Fast
   - $-$ Resolution limitations
   - $\pm$ Periodic boundary conditions
2. Finite element methods:
   - $-$ Slow
   - $+$ Flexible boundary conditions
   - $+$ Adaptivity?

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