MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
Timm Treskatis

Homework Assignment 4

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **4 marks** | Today we will solve the steady advection-diffusion equation in 1D

$$
\begin{aligned}
au' - Du'' &= 0 \qquad \text{in }]0,1[\\
u(0) &= 0 \\
u(1) &= 1
\end{aligned}
$$

where, for simplicity, we assume the advection velocity $a \geq 0$ and the diffusivity $D > 0$ to be constant throughout the domain.

(a) Discretise the advection-diffusion problem using the $N+1$ grid points

$$x = 0, h, 2h, 3h, \dots, (N-1)h, 1,$$

(where $h = 1/N$) with the second-order consistent central difference approximation

$$u'(x) \approx \frac{u(x+h) - u(x-h)}{2h}.$$

Write down the 'big linear system' $L^h_{\mathrm{CD}} u^h = f^h$ (the first two rows and the last row suffice).

For what range of h-values does the matrix L^h_{CD} satisfy the M-matrix criterion 'weakly chained diagonally dominant & L-matrix'?

Hint: The identity

$$|\alpha + \beta| + |\alpha - \beta| = 2 \max \{ |\alpha|, |\beta| \}$$

may be useful.

(b) Leaving everything else unchanged, discretise the transport term with the first-order consistent upwind[1] differencing scheme

$$u'(x) \approx \frac{u(x) - u(x-h)}{h}$$

instead. Write down the 'big linear system' $L^h_{\mathrm{UD}} u^h = f^h$.

For what range of h-values does the matrix L^h_{UD} satisfy the M-matrix criterion?

[1] Upwind differencing uses a one-sided difference quotient. The two-point stencil covers the point x itself and the nearest point in 'upwind' direction, where the flow is coming from.

(c) Download the file `advection_diffusion.m` and read the output of `help advection_diffusion`. Everything else is intentionally obfuscated so that you still have to do (a) and (b) yourself! You can however use this function to check your answers for the admissible ranges of h in (a) and (b).

What do you observe if the matrix L^h is not an M-matrix? Use your knowledge from lectures and the technical language you have learnt so far to *describe* and *explain* your observations.

(d) Even though the upwind differencing scheme is only first-order consistent as an approximation of the equation

$$au' - Du'' = 0$$

it is second-order consistent as an approximation of a slightly different equation,

$$au' - (D + \tilde{D})u'' = 0,$$

provided that $u \in C^4([0,1])$. Calculate this number $\tilde{D}$.

Hint: Use TAYLOR expansions as done in class or video #7 to determine the truncation error

$$au'(x) \quad - \quad (\text{upwind difference approximation of this term})\,.$$

$\tilde{D}$ will depend on h.

Your Primer Talk | **1 mark** | Open the primer talk assignment on Canvas, where you can find a worksheet that will help you prepare an effective primer talk. You will give your 2-3 minute talk on Thursday, 8 February during regular lecture times. You don't have to use any visual aids for this very short talk, but you may if you wish.

Please give me the following information on your talk when you submit this assignment and I will give you 1 mark for that:

- Are you going to use the document camera (which I use to project my paper notes onto the screen)?

  ◯ Yes / Probably ◯ No

  If you're planning to connect your own device to the projector, please tick 'No'.

- Purpose of your talk (from the talk preparation worksheet): By the end of this talk, the audience should be able to...

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?