MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
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Homework Assignment 8

Please submit the following files as indicated below: source code PDF file image file video file

If you haven't done so already, install ParaView on your computer. This is already included in many Linux distributions. For other operating systems, visit `https://www.paraview.org/download/`.

Question 1 | **1 mark** | This assignment is dedicated to a posteriori error estimates for the problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{P}$$

where Ω is the unit square $]0,1[^2$.

We want to solve this problem because we are interested in the average of u over the set $R = \left]\frac{1}{2},1\right[\times \left]0,\frac{1}{2}\right[$.

Following the dual weighted residual method, what is the dual problem that you have to solve for z? Write down its weak formulation, then its strong formulation. Don't forget to specify what space the solution and the test functions belong to for the weak formulation.

Hint: Indicator function.

Question 2 | **4 marks** | If the source term in Problem (P) is given as

$$f(x) = a(a+1)x_1^{a-1}x_2(1-x_2) + 2x_1(1-x_1^a)$$

for $a \geq 1$, then the analytical solution is

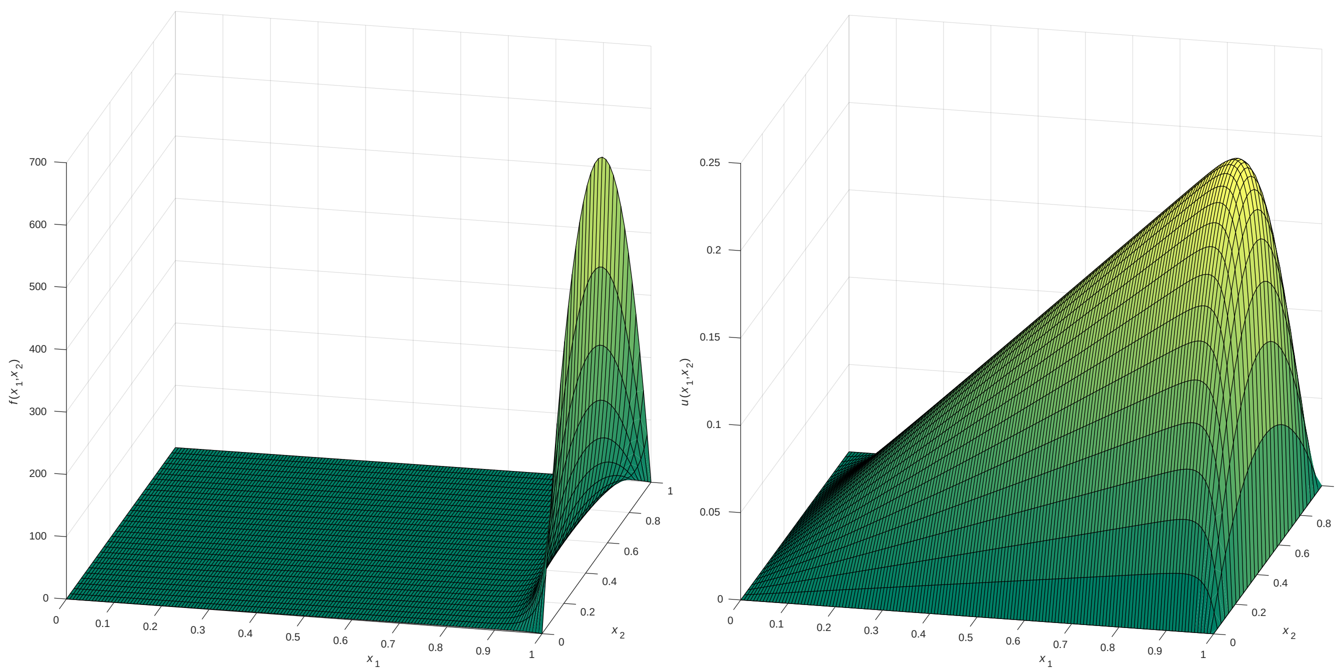
$$\bar{u}(x) = x_1(1-x_1^a)x_2(1-x_2)$$

which has an average of

$$\frac{3a-2+2^{1-a}}{24a+48}$$

over the set R.

For large a, this problem is numerically challenging: observe that f becomes very large near the right boundary, while it remains comparatively small elsewhere in the domain. As a result, the solution $\bar{u}$ exhibits a sharp boundary layer near $x_1 = 1$.



Source term f (left) and analytical solution $\bar{u}$ (right) for $a = 50$.

(a) Download the FEniCS script `hw8.py` and complete the missing commands. This script should evaluate the a posteriori estimator $\eta_{L^2} \approx \|u^h - \bar{u}\|_{L^2} = \|e^h\|_{L^2}$ as derived in class (or on Canvas, under modules). Solve Problem (P) on the given grids to complete the following table:

h	$\|e^h\|_{L^2}$	η_{L^2}	$\eta_{L^2}/\|e^h\|_{L^2}$
$\frac{1}{64}\sqrt{2}$ (64 × 64 grid)			
$\frac{1}{128}\sqrt{2}$ (128 × 128 grid)			
$\frac{1}{256}\sqrt{2}$ (256 × 256 grid)			

Is the error overestimated or underestimated by η_{L^2}? By what factor, approximately?

(b) Compute a posteriori estimators $\eta_J \approx |J(u^h) - J(\bar{u})| = |J(e^h)|$ for the error in the average solution value on R, using both the expensive Strategy 1 and the cheap Strategy 2 to approximate the dual weights (cf p 55 in the notes). Complete the following table:

h	$\lvert J(e^h)\rvert$	$\lvert\eta_{J,1}\rvert$	$\lvert\eta_{J,2}\rvert$	$\lvert\eta_{J,1}/J(e^h)\rvert$	$\lvert\eta_{J,2}/J(e^h)\rvert$
$\frac{1}{64}\sqrt{2}$ (64×64 grid)					
$\frac{1}{128}\sqrt{2}$ (128×128 grid)					
$\frac{1}{256}\sqrt{2}$ (256×256 grid)					

Is the error overestimated or underestimated? By what factor, approximately?

(c) For the convergence studies above we have refined the entire mesh from 64×64 to 128×128 to 256×256. The second mesh is four times larger, the third mesh even 16 times larger than the coarsest one. This makes uniform mesh refinement very expensive. We can probably compute a solution that is just as accurate as the solution on the 256×256 mesh, by refining only those triangles on the 64×64 mesh with a noteworthy contribution to the overall error.

Solve Problem (P) on the 64×64 mesh and plot the numerical solutions u^h and z^h, the cell residuals $\|r_T\|_{L^2}$, the dual weights $\|w_T\|_{L^2}$ (approximated with either the expensive or the cheap strategy) and the local error indicators η_T. What triangles of the 64×64 mesh would you refine to compute the average of u over R more accurately? A rough description like 'near the left boundary' will do. Also give a brief reason for your answer:

(Optional) Bonus Question | **1 bonus mark** | Derive an a priori estimate for the error in the above quantity of interest. Does it agree with the numerical results from Q2(b)?

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
