THE UNIVERSITY OF BRITISH COLUMBIA

MATH 521: Numerical Analysis of Partial Differential Equations

Predicting the Walkability of a Slackline

Project Report

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Abstract

Slacklining is a new sport gaining popularity. Its physics are not yet well-studied. This paper proposes a model for the slackline dynamics, which leads to a nonlinear wave equation. This equation is then solved numerically using a Finite Element approach.

1 Introduction

1.1 What is slacklining?

Slacklining is a recent sport which mainly consists in balancing on a one-inch wide webbing made from polyamide or polyester fibers. For more information about the basics of the sport please see source [1]. For more advanced knowledge about the gear used and rigging techniques please refer to [2]. Many beginner-oriented tutorials are also available on the Internet.

1.2 Objectives of this project

The main objective of this report is to quantify the response of the line to a 'mistake' of the slackliner, i.e. the amplitude of the oscillations following a back-and-forth side movement of the slackliner's feet.

To my knowledge, this work is mainly new. In fact, the only published paper treating the dynamics of a slackline that I could find considers the movements of the slackliner, not the whole line [3].

1.3 The walkability of a slackline

Analyzing the oscillations of the line should allow to explore ways of limiting their propagation and growth, thus making the slackline easier to walk. Of course the easiness of crossing a slackline is a very subjective notion, called the walkability. To quantify it anyway, I define a test problem: the slackliner is placed at the middle of the line, the mistake is a sinusoid with a frequency of 10 Hz and amplitude corresponding to $1/10^{\text{th}}$ of the weight of the slackliner, during 0.1 s (one period). The output of this test is the maximal horizontal displacement of the slackline. The total simulation time must allow to capture this maximal displacement, which may occur a few seconds after the mistake. I will use 4 seconds as the videos obtained from the simulations indicate that it is enough for the kind of lengths I am considering.

2 Derivation of the governing equation

2.1 Modeling of the physical system

A slackline can be approximated by a one dimensional string, moving in a three-dimensional space. For simplicity, the material can be assumed to be perfectly elastic, with a linear strain-stress relationship. The mass of the webbing is considered, as well as the mass of the slackliner, treated as a distributed mass. I will vary the length of the gap and slackline as well as the position of the slackliner.
Usually, manufacturers do not give the stiffness of their product, they prefer to give the percentage of elongation (strain) at 10 kN. Assuming a linear stress-strain relationship, we have $k = \frac{10^6}{elasticity\ [\%\ at\ 10\ kN]} = \frac{10^6}{12}\ N$, the stiffness of the webbing. The other paramaters are the mass of the slackliner $m_{body} = 55\ kg$, the width over which it is distributed $w_{body} = 0.6\ m$ and the linear mass density of the webbing $\mu_{webbing} = 0.052\ kg/m$.
Supposing the problem is solved for $t \in [0, T]$, the domains associated with this model are $\Omega =]0, L_{gap}[$, $\Sigma = \{0, L_{gap}\} \times]0, T[$ and $Q = \Omega \times]0, T[$.

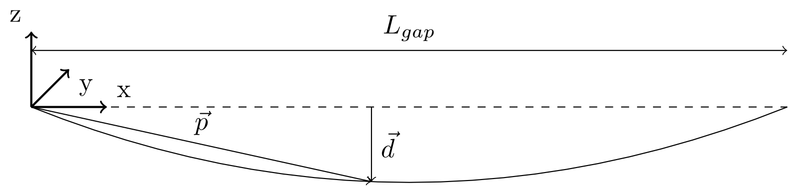

Figure 1: A slackline model

2.2 Strong Formulation

Let us consider the equilibrium of an infinitesimal element of the string, as shown in Figure 2. s is a curvilinear co-ordinate that runs along the string, thus $\vec{p}(s)$ (refer to Figure 1) can be seen as absolute the position, while $\vec{d}(x)$ is the displacement from the configuration shown as a dashed line. I will need the following relations to switch between those two descriptions: $\vec{p}(x,t) = x\hat{e}_x + \vec{d}(x,t)$, $ds = \left\| \frac{\partial \vec{p}}{\partial x} \right\| dx = J dx$, where $J = \left\| \frac{\partial \vec{p}}{\partial x} \right\| = \left\| \hat{e}_x + \frac{\partial \vec{d}}{\partial x} \right\|$.

$\vec{T}$ represents the tension in the string, which may vary both in magnitude and direction along the string. μ_{tot} is the linear mass density of both the webbing and the slackliner. It varies along the line. $\vec{M}$ models the mistakes the slackliner makes, i.e. the forces he applies on the line (excluding his own weight).

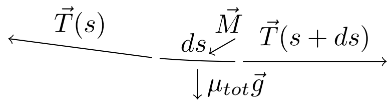

Figure 2: Equilibrium of an infinitesimal element

Newton's second law can be applied:

$$\mu_{tot} \frac{\partial^2 \vec{p}}{\partial t^2} ds = \mu_{tot} \vec{g} ds - \vec{T}(s) + \vec{T}(s+ds) + \vec{M} ds$$

Dividing both sides by ds and recalling the definition of a derivative yields:

$$\mu_{tot}(s) \frac{\partial^2 \vec{p}}{\partial t^2}(s,t) = \mu_{tot}(s) \vec{g} + \frac{\partial \vec{T}}{\partial s}(s,t) + \vec{M}(s,t) \tag{1}$$

The material law is $\vec{T} = k\epsilon\hat{t}$, where ϵ is the strain of the string and $\hat{t}$ the unit tangent vector along the line. To find ϵ consider three string configurations: 0 is the undeformed configuration (0 strain), 1 the configuration shown in Figure 1 as a dashed line assuming a uniform strain, and 2 is the final state. The strain in configuration 1 is $\epsilon_1 = \frac{L_{gap} - L_{slackline}}{L_{slackline}}$. The strain in configuration 2 when referring to configuration 1 as the undeformed configuration is $\epsilon_{12} = \frac{ds - dx}{dx} = J - 1$. Finally, the strain in the final configuration (2) when referring to the undeformed configuration (0) can be expressed as:

$$\epsilon = \epsilon_2 = \epsilon_1 + \epsilon_{12} + \epsilon_1 \epsilon_{12} = \frac{L_{gap}}{L_{slackline}} J - 1$$

However, a string cannot produce a force when compressed, so we have:

$$\epsilon = max\left(0, \frac{L_{gap}}{L_{slackline}} J - 1\right)$$

The unit tangent vector is simply:

$$\hat{t} = \frac{\frac{\partial \vec{p}}{\partial x}}{\left\| \frac{\partial \vec{p}}{\partial x} \right\|} = \frac{\hat{e}_x + \frac{\partial \vec{d}}{\partial x}}{J}$$

An equation with variable x is more useful than an equation in s, so by noting that $\frac{\partial^2 \vec{p}}{\partial t^2} = \frac{\partial^2 \vec{d}}{\partial t^2}$, $\frac{\partial}{\partial s} = \frac{1}{J}\frac{\partial}{\partial x}$, $\mu_{tot}(s) = \frac{\mu_{tot}(x)}{J}$, $\vec{M}(s,t) = \frac{\vec{M}(x,t)}{J}$ equation 1 can be rewritten as:

$$\mu_{tot}(x)\frac{\partial^2 \vec{d}}{\partial t^2}(x,t) = \mu_{tot}(x)\vec{g} + \frac{\partial \vec{T}}{\partial x}(x,t) + \vec{M}(x,t)$$

$$\frac{\partial \vec{T}}{\partial x} = k\left(\frac{\partial \epsilon}{\partial x}\hat{t} + \epsilon\frac{\partial \hat{t}}{\partial x}\right) = k\left[\left(\frac{\partial \epsilon}{\partial x}\frac{1}{J} + \epsilon\frac{\partial}{\partial x}\left(\frac{1}{J}\right)\right)\left(\hat{e}_x + \frac{\partial \vec{d}}{\partial x}\right) + \frac{\epsilon}{J}\frac{\partial^2 \vec{d}}{\partial x^2}\right]$$

$$\mu_{tot}(x) = \frac{\mu_{webbing}L_{slackline}}{L_{gap}} + \frac{m_{body}(x)}{w_{body}}$$

Finally, the strong formulation the problem is:

$$\begin{aligned}
\mu_{tot}\frac{\partial^2 \vec{d}}{\partial t^2} &= \mu_{tot}\vec{g} + \vec{M} + k\left[\left(\epsilon\frac{\partial}{\partial x}\left(\frac{1}{J}\right) + \frac{1}{J}\frac{\partial \epsilon}{\partial x}\right)\left(\hat{e}_x + \frac{\partial \vec{d}}{\partial x}\right) + \frac{\epsilon}{J}\frac{\partial^2 \vec{d}}{\partial x^2}\right] && \text{in } Q\\
\vec{d} &= \vec{0} && \text{on } \Sigma\\
\vec{d}(0) &= \vec{d}_0 && \text{in } \Omega\\
\frac{\partial \vec{d}}{\partial t}(0) &= \vec{v}_0 && \text{in } \Omega
\end{aligned} \tag{S}$$

I will choose $\vec{d}_0 = \vec{d}_{eq}$, the static equilibrium position and $\vec{v}_0 = \vec{0}$. This way the system is initially undisturbed.
Please note that this equation assumes the body to be at the same x-position at all times, hence the slackliner is not allowed to walk. To simulate walking one would need to include the linear mass density inside the time derivative.
This a non-linear hyperbolic equation, second-order in both time and space. With the appropriate boundary and initial conditions proposed above this is a well-posed problem.

To prepare for future developments, I introduce the velocity field $\vec{v} = \frac{\partial \vec{d}}{\partial t}$. (S) can now be rewritten as a system of two hyperbolic equations, each of them first order in time:

$$\begin{aligned}
\frac{\partial \vec{d}}{\partial t} &= \vec{v} && \text{in } Q\\
\mu_{tot}\frac{\partial \vec{v}}{\partial t} &= \mu_{tot}\vec{g} + \vec{M} + k\left[\left(\epsilon\frac{\partial}{\partial x}\left(\frac{1}{J}\right) + \frac{1}{J}\frac{\partial \epsilon}{\partial x}\right)\left(\hat{e}_x + \frac{\partial \vec{d}}{\partial x}\right) + \frac{\epsilon}{J}\frac{\partial^2 \vec{d}}{\partial x^2}\right] && \text{in } Q\\
\vec{d} &= \vec{0} && \text{on } \Sigma\\
\vec{d}(0) &= \vec{d}_{eq} && \text{in } \Omega\\
\vec{v}(0) &= \vec{0} && \text{in } \Omega
\end{aligned} \tag{S2}$$

2.3 Weak Formulation

Let $< .,. >$ denote the L^2 scalar product in Ω, $V = \{\vec{u} = (u_1, u_2, u_3)^T | u_i \in H_0^1(\Omega)\}$ and $W = \{\vec{u} \in L^2(0,T,V) | \frac{\partial \vec{u}}{\partial t} \in L^2(0,T,V^*)\}$. Integrating (S2) against test functions after multiplying the second equation by $\frac{J}{k}$ and using $-\int_0^{L_{gap}} \frac{\partial^2 \vec{d}}{\partial x^2}\epsilon\vec{w}dx = \int_0^{L_{gap}} \frac{\partial \vec{d}}{\partial x}\frac{\partial}{\partial x}(\epsilon\vec{w})dx$ (integration by parts with zero boundary conditions) yields the following weak formulation:

Find $\vec{d}, \vec{v} \in W$ such that

$$\begin{aligned}
\left\langle \frac{\partial \vec{d}}{\partial t}, \vec{z} \right\rangle - \langle \vec{v}, \vec{z} \rangle &= 0 & \forall \vec{z} \in V, \forall t \in]0, T[\\
\left\langle \mu_{tot} \frac{J}{k} \frac{\partial \vec{v}}{\partial t}, \vec{w} \right\rangle + \left\langle \frac{\partial \vec{d}}{\partial x}, \frac{\partial}{\partial x}(\epsilon \vec{w}) \right\rangle & & \\
- \left\langle \mu_{tot} \frac{J}{k} \vec{g} + \frac{J}{k} \vec{M} + \left[J\epsilon \frac{\partial}{\partial x}\left(\frac{1}{J}\right) + \frac{\partial \epsilon}{\partial x} \right] \left(\hat{e}_x + \frac{\partial \vec{d}}{\partial x} \right), \vec{w} \right\rangle &= 0 & \forall \vec{w} \in V, \forall t \in]0, T[\\
\vec{d}(0) &= \vec{d}_{eq} & \text{in } \Omega \\
\vec{v}(0) &= \vec{0} & \text{in } \Omega
\end{aligned} \tag{W}$$

3 Finite Element Analysis

The Finite Element Analysis of this problem is performed using FEniCS. I use Rothe's method to discretise the weak formulation (W). This allows both the mesh and the time step size to change from one time step to another. I was planning to take advantage of this but did not have time to implement an adaptive method.

3.1 Semi-discretisation in time

In this work, the midpoint rule is used as the time-stepping scheme. As it is a symplectic method, the total energy of the system will be conserved (no numerical dissipation). Its other main advantages are its second-order accuracy, A-stability and rather low computational expense. However, it is not strictly A-stable, which means that numerical errors may not decay, and it is quite strongly dispersive. This last issue should not be too visible as the solution is smooth, it has no discontinuity.
Applying the midpoint rule to (W) (omitting the initial conditions, which will simply be used to initialise the iteration) gives:

$$\left\langle \frac{\vec{d}_+ - \vec{d}_\circ}{\Delta t}, \vec{w} \right\rangle - \left\langle \frac{\vec{v}_+ + \vec{v}_\circ}{2}, \vec{w} \right\rangle = 0 \tag{3}$$

$$\left\langle \mu_{tot} \frac{J_m}{k} \frac{\vec{v}_+ - \vec{v}_\circ}{\Delta t}, \vec{w} \right\rangle + \frac{1}{2} \left\langle \frac{\partial}{\partial x}(\vec{d}_+ + \vec{d}_\circ), \frac{\partial}{\partial x}(\epsilon_m \vec{w}) \right\rangle$$
$$- \left\langle \mu_{tot} \frac{J_m}{k} \vec{g} + \frac{J_m}{k} \vec{M} + \left[J_m \epsilon_m \frac{\partial}{\partial x}\left(\frac{1}{J_m}\right) + \frac{\partial \epsilon_m}{\partial x} \right] \left(\hat{e}_x + \frac{1}{2} \frac{\partial}{\partial x}(\vec{d}_+ + \vec{d}_\circ) \right), \vec{w} \right\rangle = 0 \quad \forall \vec{w} \in V \tag{4}$$

where J_m and ϵ_m are evaluated at the midpoint. Similarly $\vec{M}$ is evaluated at $t_m = t_\circ + \frac{\Delta t}{2}$.
The same test functions can be used in both equations because they belong to the same function space. (3) can be replaced by $2\frac{\mu_{tot} J_m}{\Delta t k}$(3)+(4):

$$\left\langle 2\frac{\mu_{tot} J_m}{\Delta t k} \left(\frac{\vec{d}_+ - \vec{d}_\circ}{\Delta t} - \vec{v}_\circ \right), \vec{w} \right\rangle + \frac{1}{2} \left\langle \frac{\partial}{\partial x}(\vec{d}_+ + \vec{d}_\circ), \frac{\partial}{\partial x}(\epsilon_m \vec{w}) \right\rangle$$
$$- \left\langle \mu_{tot} \frac{J_m}{k} \vec{g} + \frac{J_m}{k} \vec{M} + \left[J_m \epsilon_m \frac{\partial}{\partial x}\left(\frac{1}{J_m}\right) + \frac{\partial \epsilon_m}{\partial x} \right] \left(\hat{e}_x + \frac{1}{2} \frac{\partial}{\partial x}(\vec{d}_+ + \vec{d}_\circ) \right), \vec{w} \right\rangle = 0 \quad \forall \vec{w} \in V \tag{5}$$

These equations can now be solved separately: first solve the non-linear equation (5) for $\vec{d}_+$, then the linear equation (4) for $\vec{v}_+$.

An appropriate time step size has to be chosen. As I am using an implicit time stepping scheme, there is no restriction for stability, I just have to make sure that I am capturing the 10 Hz mistake correctly. To do so I will pick a time step no bigger than 0.01 s.

To find the initial condition $\vec{d}_{eq}$ (this curve is often refered to as the elastic catenary), simply note that it is such that $\vec{v}_+ = \vec{v}_\circ = \vec{0}$, and $\vec{d}_+ = \vec{d}_\circ = \vec{d}_{eq}$. In this case (4) and (5) actually give the same following equation:

$$\left\langle \frac{\partial \vec{d}_{eq}}{\partial x}, \frac{\partial}{\partial x}(\epsilon \vec{w}) \right\rangle - \left\langle \mu_{tot} \frac{J}{k} \vec{g} + \left[J\epsilon \frac{\partial}{\partial x}\left(\frac{1}{J}\right) + \frac{\partial \epsilon}{\partial x} \right] \left(\hat{e}_x + \frac{1}{2} \frac{\partial \vec{d}_{eq}}{\partial x} \right), \vec{w} \right\rangle = 0 \qquad \forall \vec{w} \in V$$

There is no ambiguity on J and ϵ here. $\vec{M}$ no longer appears because I will always choose $\vec{M}(0) = \vec{0}$ so that the slackline is truly still at the initial time.

3.2 Semi-discretisation in space

The spatial discretisation is done by FEniCS, the only thing to specify is the mesh and discrete vector space. The mesh is composed of intervals initially of the same size. I then refine the mesh around the slackliner because this is where the steepest gradients will occur. I use a Continuous Galerkin method, with piecewise second order polynomials as the discrete function space. This is an H^1-conforming approximation.
To solve this non-linear problem, FEnicS implements Newton's method. Unfortunately, this method will not converge (at least from a zero initial guess) if the tension force is exactly 0 in compression. So the expression of the strain has to be corrected to $\epsilon = max\left(10^{-12}, \frac{L_{gap}}{L_{slackline}} J - 1\right)$.
To find the appropriate number of cells, I did a convergence study for $L_{gap} = 100\ m$ and $L_{slackline} = 102\ m$. I picked the sag as a simple indicator of convergence. The error is calculated against the value found for a very fine mesh. The results are shown on Figure 3.

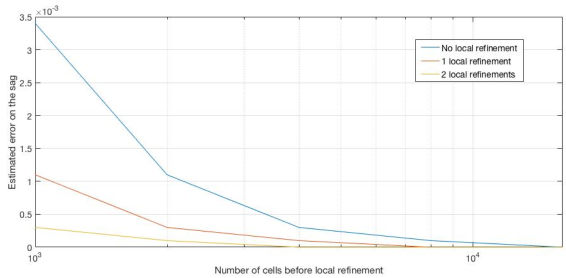

Figure 3: Convergence study

I conclude that using a mesh of initially 4000 cells and refining it twice around the slackliner is a reasonable choice. Actually, it seems like refining the mesh around the slackliner only is as efficient as refining the mesh everywhere! However, I do not want the cells to have very different sizes, so I do not refine the mesh locally more than twice. This is also because this study was done with the static calculations and I want to use this mesh for the wave equation too.
Another set of calculations for a 20 m gap confirmed that 40 cells per meter is a good choice.
My results also agree with [4], an online application which uses Python to integrate the equation representing an elastic catenary.

3.3 Results

Let $f = \frac{L_{slackline}}{L_{gap}}$. The standing tension is the maximal tension in the slackline at equilibrium. The walkability indicator is defined in 1.3. A lower indicator should indicate a more walkable slackline. The results are condensed in Table 1.

4 Discussion

The results are rather intuitive: increasing the gap length increases the amplitude of the oscillations, increasing the tension decreases these amplitudes.

$L_{gap}[m]$	f [-]	Standing tension [kN]	Walkability indicator [mm]
20	1.02	1.07	2.13
	1	1.46	1.96
	0.98	2.29	1.78
50	1.02	1.09	2.92
	1	1.48	2.79
	0.98	2.31	2.36
100	1.02	1.12	4.19
	1	1.51	3.68
	0.98	2.33	3.16

Table 1: Results

However, this choice of walkability indicator is too simplistic: it is known by experience that a looser slackline (i.e. when the tension is reduced) is not necessarily less walkable, even though it can easily move further to the sides, because the movements are also slower. Indeed, the velocity of a wave on a string is given by $\sqrt{\frac{T}{\mu}}$, with T the tension [5]. So, the higher the tension, the faster the mistakes, shakes, come back to the slackliner. Less time is left to react.

5 Conclusion

Predicting the walkability of a slackline is a somewhat difficult task, this report is a first step in this direction. Putting a number on how hard it is to cross a slackline involves studying a nonlinear vectorial partial differential equation to explore the dynamic response of the system to various excitations. This work did not thoroughly explore the various choices of excitations and observable quantities to properly define the walkability.

6 References

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