Consistency, Stability and Convergence

Here's a quick summary of the jigsaw teamwork on consistency, stability and convergence of a numerical scheme.

General Framework

These three notions do not only apply to numerical methods for PDEs, but pretty much any discretisation scheme, e.g. an ODE solver, a quadrature formula,...

Therefore, I'm using the very generic notation

$$T(u) = 0$$

for the exact problem which we want to solve and

$$T^h(u^h) = 0$$

for its discretised version with some discretisation parameter (e.g. grid spacing, time step) *h*.

The 3 Fundamental Notions of Numerical Analysis

1. Is the discretisation scheme T^h a good approximation of the exact problem T? **(Consistency)**
2. Is the discrete problem well-posed and does a small residual $T^h(v^h) = \mathbf{small}$ guarantee a small error $\|v^h - \bar{u}\| = \mathbf{small}$? **(Stability)**
3. Is the discrete solution u^h a good approximation of the exact solution $\bar{u}$? **(Convergence)**

These three notions describe *properties of a discretisation scheme*.

Condition of a Problem

When we talk of well-posedness ('stability') of the original problem $T(u) = 0$ we use the term **condition**. You can think of the condition of a problem as a more qualitative statement on 'how well-posed' it is.

- Well-posedness: either yes or no
- Condition: well conditioned - poorly conditioned - extremely ill-conditioned
  (all of these describe well-posed problems, but an extremely ill-conditioned problem would be very close to the limit of not being well-posed anymore)

In contrast to consistency, stability and convergence, the condition is a *property of the exact problem*, independent of any numerical method. E.g. we will sometimes look at the condition of our 'big linear

systems', which tells us how difficult they are to solve.

consistency.pdf (https://canvas.ubc.ca/courses/2337/files/588729/download?wrap=1)  **(https://canvas.ubc.ca/courses/2337/files/588729/download?wrap=1)**

stability.pdf (NB: there was a factor of 4 missing in the eigenvalues on the printed handout) (https://canvas.ubc.ca/courses/2337/files/588731/download?wrap=1)  **(https://canvas.ubc.ca/courses/2337/files/588731/download?wrap=1)**

convergence.pdf (https://canvas.ubc.ca/courses/2337/files/588732/download?wrap=1)  **(https://canvas.ubc.ca/courses/2337/files/588732/download?wrap=1)**

The questions on these worksheets are addressed in the notes.