(k=1)*

2.3.22 Theorem (Convergence in the L^2-Norm) *Let $V^h \subset V = H_0^1(\Omega)$ be a conforming finite element space of piecewise polynomial functions of degree k. Then the error $e^h = u^h - \bar{u}$ for the approximation of the* POISSON-DIRICHLET *problem satisfies the a priori estimate*

$$\|e^h\|_{L^2} \leq ch^2 \|\nabla^2 \bar{u}\|_{L^2(\Omega)}$$

and we even have

$$\|e^h\|_{L^2} \leq ch^2 \|f\|_{L^2(\Omega)}.$$

Proof. This proof relies on the AUBIN-NITSCHE trick: consider the so-called dual problem

for a symmetric B (like $\int_\Omega \nabla u \cdot \nabla v \, dx$) $B(v,z) = B(z,v)$

$$\underbrace{B(v, z)}_{\text{test fcn} \quad \text{solution}} = \left\langle \frac{e^h}{\|e^h\|_{L^2}}, v \right\rangle_{L^2} \xleftarrow{\;\int_\Omega e^h / \|e^h\| \, v \, dx\;}, \qquad \forall v \in H_0^1(\Omega).$$

In our setting, the strong formulation of this problem reads

$$\begin{aligned} -\Delta z &= \frac{e^h}{\|e^h\|_{L^2}} \qquad &&\text{in } \Omega \\ z &= 0 &&\text{on } \partial\Omega \end{aligned}$$

and from (2.8) we conclude that the solution z belongs to $H_0^1(\Omega) \cap H^2(\Omega)$ with

$$\|z\|_{H^2} \leq c \left\| \frac{e^h}{\|e^h\|_{L^2}} \right\|_{L^2} = c.$$

Now, using the test function $v = e^h$ in the dual problem, we obtain

$$\|e^h\|_{L^2} = \int_\Omega \frac{e^h}{\|e^h\|_{L^2}} e^h \, dx = B(e^h, z) = B(e^h, z) - \underbrace{B(e^h, I^h z)}_{= 0 \text{ (Galerkin orthogonality)}}$$

$$= B(e^h, z - I^h z) \leq \underbrace{\|e^h\|_B}_{\leq ch \|\nabla^2 \bar{u}\|_{L^2}} \underbrace{\|z - I^h z\|_B}_{\leq ch \|\nabla^2 z\|_{L^2}},$$

(Thm. 2.3.21) (Thm 2.3.15) □

In many practical applications, estimates in the L^2-norm or in the energy norm would not be desirable, as they inherently include some averaging of the error over the entire domain. Even local singularities where the solution blows up may still give a finite error.

In structural engineering, for instance, a local spike in the stress acting on a building or a bridge may result in the failure of the structure. In such settings, pointwise estimates, just like for the finite difference method, are more desirable:

* The theorem is also true for higher-order finite elements, but without extra smoothness of the true solution $\bar{u}$, higher-order elements don't usually attain a higher rate of convergence.