The Aubin-Nitsche Trick

Convergence Rates of Function Values and Gradients

From Corollary 2.3.15 (Interpolation Error with Finite Elements in 2D) we know that the function values of the interpolation error converge at a higher rate than the derivatives of the error, and the derivatives converge at a higher rate than the second derivatives etc.

When we derive an a priori error estimate for the finite-element solution of a second-order elliptic equation, we have to use a norm which also includes gradients, such as the energy norm of the problem or the H^1-norm. We wouldn't be allowed to use the L^2-norm, since the bilinear form B is not bounded with respect to the L^2-norm - a necessary requirement for Céa's lemma or the Strang lemmas.

For linear finite elements, we have shown

$$\|u^h - \bar{u}\|_B \leq O(h)$$

provided that the exact solution $\bar{u} \in H^2(\Omega)$. Hence, the gradients of u^h converge to the gradients of $\bar{u}$ at a linear rate. According to Poincaré's inequality, the function values of u^h must converge to the function values of $\bar{u}$ at least at the same rate.

At least? If the function values of *interpolants* converge faster than the gradients of interpolants, do the function values of the *finite-element solution* also converge faster than its gradients?

Yes, indeed, they do. To go from an a priori error estimate in the energy or H^1-norm to an a priori error estimate in the L^2-norm, we use the Aubin-Nitsche trick.

The Dual (or Adjoint) Problem

If our (linear) PDE has the weak formulation

$$\textbf{Find } u \in V \textbf{ such that } \forall v \in V: \qquad B(u, v) = \langle f, v \rangle$$

then we can define new elliptic operator by swapping the positions of the solution and the test functions in B. If the bilinear form is symmetric, e.g. for the Laplacian with Dirichlet boundary conditions, then this doesn't change anything at all. First (or other odd) derivatives in the differential operator, however, lead to nonsymmetric terms in B and then the new elliptic operator is no longer the same as the one with u and v in their regular positions.

With certain data g on the right hand side, the weak form of a PDE with the dual (aka adjoint) differential operator reads

$$\textbf{Find } u \in V \textbf{ such that } \forall v \in V: \qquad B(v,u) = \langle g, v\rangle$$

To avoid any confusion with the solution of the original PDE problem, we call the solution of this dual problem z instead of u:

$$\textbf{Find } z \in V \textbf{ such that } \forall v \in V: \qquad B(v,z) = \langle g, v\rangle$$

The Aubin-Nitsche Trick

The Aubin-Nitsche Trick is a technique for proving faster convergence in weaker norms. It involves the following steps:

- Write down the dual problem with a cleverly chosen source term g
- Use Galerkin orthogonality to add a term with an interpolant of z
- Obtain an extra h from the interpolation error $z - I^h z$

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