Homework Assignment 10

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **2 marks** | On the assignment page you can find videos of four animated solutions of the parabolic problem

$$\begin{aligned} \partial_t u(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

with the data from Assignment 9. However, the initial condition has been replaced with the function

$$u_0(x) = \begin{cases} 50 & \text{if } |x-(1,1)^\top| < 0.5 \\ 20 & \text{elsewhere} \end{cases}$$

Explain your observations.

Since this initial condition is discontinuous, its decomposition in terms of the eigenfunctions of the Laplacian contains significant contributions from high-frequency modes.

Crank-Nicolson scheme, coarse mesh The numerical solution shows spurious high-frequency modes (corresponding to eigenvalues $\lambda \ll -1$) for positive times. These artificial local maxima and minima are unphysical artefacts since they violate the parabolic maximum principle (applied to a subdomain in the top left corner where $f \equiv 0$). Only after approximately three seconds (video time) they are no longer visible.

The Crank-Nicolson method is only simply A-stable, and its stability function $|R(\lambda\Delta t)$ is only marginally below than 1 for such eigenvalues. This explains the slow decay rate of the high-frequency modes. A discrete maximum principle could be enforced by mass lumping and sufficiently small time steps.

Crank-Nicolson scheme, fine mesh The spurious noise decays far more slowly and is still clearly visible even after sixteen seconds (= five seconds in physical time).

Since the largest eigenvalue of the stiffness matrix is of the order $1/h^2$, the highest frequencies for this mesh with $h \approx 1/200$ are approximately 16 times larger than the highest frequencies for the coarser mesh with $h \approx 1/50$. For these values, the growth factor is even closer to the limit

$$\lim_{\lambda\to-\infty} |R(\lambda\Delta t)| = 1.$$

Therefore, Crank-Nicolson scheme hardly dampens these high-frequency modes. An even smaller time step size would be required to enforce the maximum principle.

Theta scheme, fine mesh The spurious oscillations now decay very rapidly, but they are still present.

For $\theta = 0.55$, this time-stepping method is strongly A-stable. This means that even in the limit $\lambda \to -\infty$ the growth factor is bounded by a constant $c < 1$—all frequencies decay at least at this rate.

Backward Euler scheme, fine mesh No more spurious oscillations are present.

The backward Euler method is even L-stable. This means that

$$\lim_{\lambda\to-\infty} |R(\lambda\Delta t)| = 0$$

and modes of the highest frequencies experience strongest damping. With mass lumping, the maximum principle would be preserved at the discrete level, independent of the time step size Δt. Even with the consistent mass matrix used here, there are at least no visible violations of the maximum principle.

Question 2 | **1 mark** | We have seen that the homogeneous wave equation

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \partial_t u(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \end{aligned} \tag{W}$$

with propagation speed $c > 0$ can equivalently be re-written as

$$\begin{aligned} \partial_t u - v &= 0 && \text{in } Q =]0,T[\times \Omega \\ \partial_t v - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ v(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \\ v &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega. \end{aligned} \tag{W'}$$

Discretising with the θ-method in time and linear finite elements in space leads to the coupled system for the vectors of nodal values $\vec{u}_+^h$ and $\vec{v}_+^h$

$$\begin{aligned} M^h \vec{u}_+^h - \theta \Delta t M^h \vec{v}_+^h &= M^h \vec{u}_\circ^h + (1-\theta) \Delta t M^h \vec{v}_\circ^h \\ \theta \Delta t c^2 K^h \vec{u}_+^h + M^h \vec{v}_+^h &= -(1-\theta) \Delta t c^2 K^h \vec{u}_\circ^h + M^h \vec{v}_\circ^h \end{aligned}$$

which has to be solved at every time step. Show that this is equivalent to the two smaller, successively solvable problems

$$\begin{aligned} \left(M^h + (\theta \Delta t c)^2 K^h \right) \vec{u}_+^h &= M^h \left(\vec{u}_\circ^h + \Delta t \vec{v}_\circ^h \right) - \left(\theta (1-\theta) (\Delta t c)^2 \right) K^h \vec{u}_\circ^h \\ M^h \vec{v}_+^h &= M^h \vec{v}_\circ^h - \Delta t c^2 K^h \left(\theta \vec{u}_+^h + (1-\theta) \vec{u}_\circ^h \right). \end{aligned}$$

The coupled system corresponds to the augmented matrix

$$\left(\begin{array}{cc|c} M^h & -\theta \Delta t M^h & M^h \vec{u}_\circ^h + (1-\theta) \Delta t M^h \vec{v}_\circ^h \\ \theta \Delta t c^2 K^h & M^h & -(1-\theta) \Delta t c^2 K^h \vec{u}_\circ^h + M^h \vec{v}_\circ^h \end{array} \right) \begin{array}{c} (1) \\ (2) \end{array}$$

Add $\theta \Delta t \times (2)$ to (1):

$$\left(\begin{array}{cc|c} M^h + (\theta \Delta t c)^2 K^h & 0 & M^h \left(\vec{u}_\circ^h + \Delta t \vec{v}_\circ^h \right) - \left(\theta (1-\theta) (\Delta t c)^2 \right) K^h \vec{u}_\circ^h \\ \theta \Delta t c^2 K^h & M^h & -(1-\theta) \Delta t c^2 K^h \vec{u}_\circ^h + M^h \vec{v}_\circ^h \end{array} \right)$$

Question 3 | **2 marks**

(a) Download and complete the FEniCS script `hw10.py` to solve Problem (W) with the data provided.

(b) Solve the wave equation

- with the (symplectic) implicit midpoint rule
- with the backward EULER method
- with the forward EULER method

and look at the solutions in ParaView.

Hint: Use the 'Warp by Scalar' filter, re-scale the colour map to the range $[-1, 1]$ and tick the box 'enable opacity mapping for surfaces' in the colour map editor.

For each of the three time stepping schemes, create a graph with curves of the total energy $E(u(t), v(t)) = T(v(t)) + V(u(t))$, the kinetic energy $T(v(t)) = \frac{1}{2}\|v(t)\|_{L^2}^2$ and the potential energy $V(u(t)) = \frac{c^2}{2}\|\nabla u(t)\|_{L^2}^2$ as functions of time. Please submit these plots and interpret the results:

In this case, the implicit midpoint rule is equivalent to the CRANK-NICOLSON scheme. It is non-dissipative and as a symplectic method it preserves quadratic invariants of the differential equation like the total energy (Hamiltonian). As expected, the kinetic and potential energy of this wave change over time.

The backward EULER method is dissipative and hence the total energy decays exponentially.

The forward EULER method is anti-dissipative and hence the total energy grows exponentially.

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
