1 Classification of PDEs

Partial differential equations (PDEs) model a vast range of natural phenomena and industrial problems. Even though analytical solutions are not normally available, the equation itself may already reveal crucial information on some characteristic features of the (unknown) analytical solution. Our main objective is to design numerical approximations that reflect these properties accurately.

The theoretical part of this course is devoted to classifying PDEs based on such characteristic features.

1.1 Basic Properties

Often, we will focus on the most important special case of a (quasi-)linear, scalar PDE of second order in two variables:

$$\underbrace{a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2}}_{\text{principle part}} + a_1\frac{\partial u}{\partial x_1} + a_2\frac{\partial u}{\partial x_2} + au = f \qquad \text{in } \underset{\text{open \& connected}}{\Omega} \subseteq \mathbb{R}^2, \tag{$\star$}$$

where not all of a_{11}, a_{12}, a_{22} are simultaneously equal to zero.

We also use the short operator notation for PDEs

unknown sol^n

$$\underset{\text{diff. operator}}{L}u = \underset{\text{inhomogeneity}}{f} \qquad \text{in } \Omega.$$

1.1.1 Definition (Linear and Nonlinear PDEs) The PDE ($\star$) is said to be

- linear: L is indep. of u (coeffs. are fcns. of $x_{1,2}$ only)
- semi-linear: coeffs. of principle part are fcns of x_1, x_2 only
- quasi-linear: coeffs. depend on $\vec{x}, u$, but still in the form of ($\star$)
- fully non-linear: $F(x, u, \nabla u, \nabla^2 u) = 0$ cannot be written in the form ($\star$)
  N.B. ∇^2 means HESSIAN, not Laplacian (which we write Δ)

Otherwise, if a second-order PDE

$$F(x, u, \nabla u, \nabla^2 u) = 0$$

cannot be written in the form ($\star$), then it is said to be *fully nonlinear.*

In practice, the domain Ω is usually bounded, or unbounded in one direction only (namely, the 'time-direction'). While the PDE describes the behaviour of a solution u in the interior of Ω, additional conditions prescribe certain values on the boundary $\partial\Omega$. The question what kind of boundary conditions are 'admissible' for a given PDE is not straightforward to answer. The answer will depend on the type and the exact form of the PDE. First, we clarify what 'admissible' means in this context:

1.1.2 Definition (Well-Posed Problem) A PDE equipped with boundary conditions is said to be *well-posed in the sense of* HADAMARD, if we have:

- Existence: $\exists$ a solution
- Uniqueness: any existent sol^n must be unique
- Continuous dependence on IC's/BC's ("data"): small perturbations in data (f in $(\star)$) lead to small perturbations in the solution u.

In addition to these three properties, we are also interested in the regularity or smoothness of solutions, i.e. how many derivatives they possess.

1.2 Second-Order PDEs

There are three types of PDEs: *elliptic*, *parabolic* and *hyperbolic* equations. Strictly speaking, this classification only works for certain classes of PDEs, and therefore we continue to focus on the class of (quasi-)linear, scalar PDEs of second order in two variables $(\star)$.

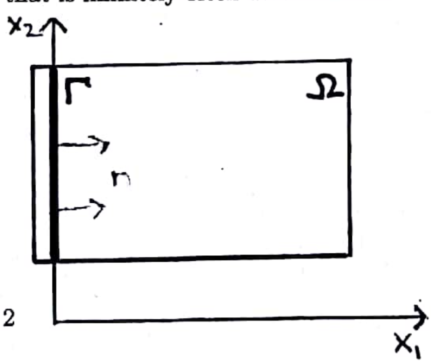
The classification of these PDEs into elliptic, parabolic and hyperbolic equations is based on a so-called CAUCHY *initial value problem*

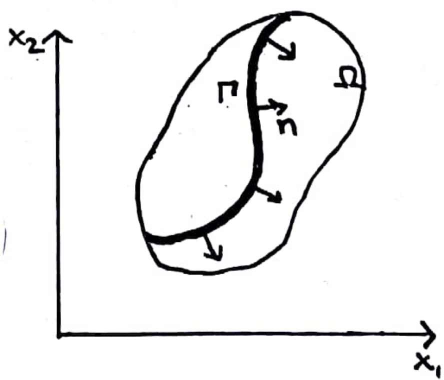
$$
\begin{aligned}
a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + a_1\frac{\partial u}{\partial x_1} + a_2\frac{\partial u}{\partial x_2} + au &= f && \text{in } \Omega && (\star)\\
u &= g && \text{on } \Gamma && (\mathrm{C1})\\
\frac{\partial u}{\partial n} &= h && \text{on } \Gamma, && (\mathrm{C2})
\end{aligned}
$$

where we assume that the 'initial curve' $\Gamma \subset \Omega$ is of the class C^∞, i.e. Γ has a parameterisation

$$x(\tau) = (x_1(\tau), x_2(\tau))$$

that is infinitely often differentiable.





By differentiating once, the CAUCHY condition (C1) gives the derivative $\frac{\partial u}{\partial t}$ in tangential direction along Γ. Therefore, (C1) and (C2) combined already prescribe both the function values u and the complete gradient $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right)^{\mathsf{T}}$ on the curve Γ.

Our objective is to find all second, third and higher partial derivatives of u on Γ, so that the TAYLOR series

$$u(x) = \sum_{k,l=0}^{\infty} \frac{(x_1 - x_1^*)^k (x_2 - x_2^*)^l}{(k+l)!} \frac{\partial^k}{\partial x_1^k} \frac{\partial^l}{\partial x_2^l} u(x^*) \tag{1.1}$$

around a point $x^* \in \Gamma$ would then give a solution to the CAUCHY problem for the PDE ($\star$) in a neighbourhood of Γ.

Since both first derivatives $\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ are given on Γ, we differentiate in tangential direction:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial u}{\partial x_1} &= \frac{\partial^2 u}{\partial x_1^2}\frac{\mathrm{d}x_1}{\mathrm{d}\tau} + \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}\tau} \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial u}{\partial x_2} &= \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\mathrm{d}x_1}{\mathrm{d}\tau} + \frac{\partial^2 u}{\partial x_2^2}\frac{\mathrm{d}x_2}{\mathrm{d}\tau}.
\end{aligned}$$

Let us introduce some short-hand notations for the second partial derivatives of the solution u:

$$p := \frac{\partial^2 u}{\partial x_1^2} \qquad q := \frac{\partial^2 u}{\partial x_1 \partial x_2} \qquad r := \frac{\partial^2 u}{\partial x_2^2}.$$

Two equations for these three unknowns p, q, r are given by the differentiated CAUCHY conditions, and the PDE ($\star$) adds a third equation. This yields the following linear system:

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} & \frac{\mathrm{d}x_2}{\mathrm{d}\tau} & 0 \\ 0 & \frac{\mathrm{d}x_1}{\mathrm{d}\tau} & \frac{\mathrm{d}x_2}{\mathrm{d}\tau} \\ a_{11} & 2a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial u}{\partial x_1} \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial u}{\partial x_2} \\ f - a_1\frac{\partial u}{\partial x_1} - a_2\frac{\partial u}{\partial x_2} - au \end{pmatrix},$$

where the determinant of the coefficient matrix—let's call it B—is found to be

$$\det B = a_{11}\left(\frac{\mathrm{d}x_2}{\mathrm{d}\tau}\right)^2 - 2a_{12}\left(\frac{\mathrm{d}x_2}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\right) + a_{22}\left(\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\right)^2. \tag{1.2}$$

1st Case: $\det B \neq 0$ on all of Γ The linear system admits a unique solution. Hence, it determines the second derivatives p, q, r on Γ.

The third partial derivatives $\frac{\partial p}{\partial x_1}, \frac{\partial q}{\partial x_1}, \frac{\partial r}{\partial x_1}$ on Γ are found by differentiating the linear system with respect to x_1, which yields another linear system with exactly the same coefficient matrix B. The same holds true for the remaining third partial derivatives $\frac{\partial p}{\partial x_2}, \frac{\partial q}{\partial x_2}, \frac{\partial r}{\partial x_2}$ and then all higher derivatives. We have reached our objective to describe u in a neighbourhood of Γ through the TAYLOR series (1.1).

2nd Case: $\det B = 0$ in a point $x^* \in \Gamma$ The linear system does not admit a unique solution. Hence, the second partial derivatives of u in x^* cannot be determined from the CAUCHY conditions.

More specifically, the equation

$$a_{11}\left(\frac{\mathrm{d}x_2}{\mathrm{d}\tau}\right)^2 - 2a_{12}\left(\frac{\mathrm{d}x_2}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\right) + a_{22}\left(\frac{\mathrm{d}x_1}{\mathrm{d}\tau}\right)^2 = 0 \tag{1.3}$$

determines the slope of curves $(x_1(\tau), x_2(\tau))$ through the point $x^* \in \Gamma$, along which the TAYLOR series approach breaks down. In other words, on these critical curves a.k.a. *characteristics* of the operator L (or, actually, its principal part L_0) the solution u cannot be derived from the given data. Instead, u or its derivatives may possess discontinuities along the characteristics.

This singular case could be ruled out by choosing an 'initial curve' Γ that is nowhere tangential to a characteristic, to avoid the situation where the tangential vectors $\frac{\mathrm{d}x}{\mathrm{d}\tau}$ of Γ and $\frac{\mathrm{d}x}{\mathrm{d}\tau}$ of a characteristic coincide.

To calculate the characteristics explicitly, we can re-arrange (1.3) (provided that $\frac{\mathrm{d}x_1}{\mathrm{d}\tau} \neq 0$) to obtain the quadratic equation

$$a_{11}\left(\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right)^2 - 2a_{12}\left(\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right) + a_{22} = 0$$

with solutions

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{a_{12}}{a_{11}} \pm \frac{\sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}}.$$

Recall that (1.3) describes conic sections (ellipse, parabola and hyperbola), here in the variables $\frac{\mathrm{d}x_2}{\mathrm{d}\tau}$ and $\frac{\mathrm{d}x_1}{\mathrm{d}\tau}$. This motivates the following nomenclature:

1.2.1 Definition (Elliptic, Parabolic and Hyperbolic Equations) In a point $x^* \in \Omega$ the PDE $(\star)$ is said to be

elliptic if the discriminant $a_{12}^2 - a_{11}a_{22} < 0$, i.e. there are no characteristics through x^*,

parabolic if the discriminant $a_{12}^2 - a_{11}a_{22} = 0$, i.e. there is one characteristic through x^*,

hyperbolic if the discriminant $a_{12}^2 - a_{11}a_{22} > 0$, i.e. there are two characteristics through x^*.

It is important to note that this criterion gives a pointwise classification of a PDE, since the coefficients a_{11}, a_{12}, a_{22} generally depend on x and maybe even on the solution u itself. Hence, a single equation may be elliptic in some parts of the domain Ω, parabolic in other regions and hyperbolic elsewhere. In this course, we normally study equations that are uniformly (i.e. everywhere) elliptic, parabolic or hyperbolic, such as PDEs with constant coefficients.

1.2.2 Example (Prototypes of Elliptic, Parabolic and Hyperbolic Equations) The most fundamental representatives of linear, second-order PDEs are

- POISSON's equation $-\Delta u = f$, which is elliptic,
- the heat equation $\partial_t u - \partial_x^2 u = f$, which is parabolic,
- the wave equation $\partial_t^2 u - \partial_x^2 u = f$, which is hyperbolic.

1.3 Conservation Equations

Very often, PDE problems model physical, chemical or biological systems which are governed by certain *conservation laws*, such as

- conservation of mass
- conservation of energy
- conservation of momentum
- conservation of charge
- conservation of the number of individuals in a population

In mathematical terms, conservation of a (mass, energy, momentum ...) density u implies that if this quantity is transported through the domain with a flux F, the density increases inside any arbitrary control volume V if there is a net influx across the boundary of V and it decreases otherwise:

$$\int_V \frac{\partial u}{\partial t}\,\mathrm{d}x = -\int_{\partial V} F\cdot n\,\mathrm{d}s.$$

Provided that the flux F and the control volume V satisfy the assumptions of the divergence theorem, we obtain

$$\int_V \frac{\partial u}{\partial t}\,\mathrm{d}x + \int_V \operatorname{div} F\,\mathrm{d}x = 0.$$

Since this balance has to hold for arbitrary such volumes, we have derived the continuity equation

$$\frac{\partial u}{\partial t} + \operatorname{div} F = 0. \tag{1.4}$$

There are two particularly important fluxes:

- advective flux $F = ua$, with an advection (velocity) field a,
- diffusive flux $F = -D\nabla u$, with a positive diffusivity D. More generally, D could also be a positive definite matrix, a so-called diffusion tensor.

Sometimes, the quantity u is not actually conserved, but sources or sinks such as chemical reactions or external forces give rise to an inhomogeneity r on the right-hand side of the equation.

Below, we list some prototypical conservation equations:

Unsteady advection-diffusion-reaction equation (parabolic)

$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

Steady advection-diffusion-reaction equation (elliptic)

$$\operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

Unsteady advection equation (hyperbolic)

$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) = 0$$

Steady advection equation (hyperbolic)

$$\operatorname{div}(ua) = 0$$

Note that all first-order PDEs are generally defined to be hyperbolic.

Many advection fields in applications are incompressible, which means that $\operatorname{div} a = 0$. Then the above advective terms can also be written as

$$\operatorname{div}(ua) = u \operatorname{div} a + a \cdot \nabla u = a \cdot \nabla u.$$