MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
Timm Treskatis

Homework Assignment 3

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | 2 marks | We consider the following two formulations of a one-dimensional boundary-value problem with a given right hand side f:

Find a function $u \in C^2(]-1,1[) \cap C([-1,1])$ such that

$$\begin{aligned} -u'' &= f \quad \text{in }]-1,1[\\ u(-1) &= u(1) = 0 \end{aligned} \tag{S}$$

Find a function $u \in H_0^1(]-1,1[)$ such that for all $v \in H_0^1(]-1,1[)$

$$\int_{-1}^{1} u'v' \, \mathrm{d}x = \langle f, v \rangle_{H^{-1},H_0^1} \tag{W}$$

We have learnt that if u is a solution to Problem (S), then u is also a solution to Problem (W) — simply multiply (S) with a test function and integrate by parts. In this exercise you will see an example of a weak solution of the above boundary-value problem that is not a strong solution.

Select *one* of these two functions

$$\text{either } u(x) = \frac{1}{2}(1-|x|) \qquad \text{or } u(x) = \frac{1}{2}(1-|x|)x.$$

Note that neither of these two functions is twice continuously differentiable, so they cannot solve Problem (S).

Show that your selected function solves Problem (W), though. What is the source term f?

(i) $u(x) = \frac{1}{2}(1-|x|)$

- This function u is in $L^2(]-1,1[)$, since

$$\int_{-1}^{1} u(x)^2 \, \mathrm{d}x < \infty.$$

- The (weak) derivative of u is

$$u'(x) = -\frac{1}{2}\operatorname{sgn} x$$

  and u' is an L^2-function as well.
- $u(-1) = u(1) = 0$, hence $u \in H_0^1(]-1,1[)$.
- Now let v be any test function from $H_0^1(]-1,1[)$.

$$\begin{aligned} \int_{-1}^{1} u'v' \, \mathrm{d}x &= \int_{-1}^{1} -\frac{1}{2}(\operatorname{sgn} x)v' \, \mathrm{d}x \\ &= \int_{-1}^{0} \frac{1}{2}v' \, \mathrm{d}x - \int_{0}^{1} \frac{1}{2}v' \, \mathrm{d}x \\ &= \frac{1}{2}\left(v(0) - v(-1)\right) - \frac{1}{2}\left(v(1) - v(0)\right) \\ &= v(0) \\ &= \langle f, v \rangle_{H^{-1},H_0^1} \end{aligned}$$

Therefore, f is not a proper (L^2-)function, but the delta-'function' centred at the origin, $f = \delta_0$.

(NB: It follows from SOBOLEV embeddings that all H_0^1-functions over a 1D domain are continuous, therefore the expression $v(0)$ is well-defined and this f is in $H^{-1}(]-1,1[)$. If we had a 2D domain, then functions must have more than one derivative for guaranteed continuity, e.g. functions in H^2. In 2D, this delta-'function' would not be in H^{-1}, but it would be in H^{-2}.)

(ii) $u(x) = \frac{1}{2}(1-|x|)x$

- This function u is in $L^2(]-1,1[)$, since

$$\int_{-1}^{1} u(x)^2 \, \mathrm{d}x < \infty.$$

- The (weak) derivative of u is

$$u'(x) = -\frac{1}{2}|x| + \frac{1}{2}(1-|x|) = \frac{1}{2} - |x|$$

  and u' is an L^2-function as well.

- $u(-1) = u(1) = 0$, hence $u \in H_0^1(]-1,1[)$.
- Now let v be any test function from $H_0^1(]-1,1[)$.

$$\begin{aligned}
\int_{-1}^{1} u'v' \, \mathrm{d}x &= \int_{-1}^{1} (\frac{1}{2} - |x|)v' \, \mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2}v' \, \mathrm{d}x + \int_{-1}^{0} xv' \, \mathrm{d}x - \int_{0}^{1} xv' \, \mathrm{d}x \\
&= \frac{1}{2}\left(v(1) - v(-1)\right) + \left(xv\Big|_{-1}^{0} - \int_{-1}^{0} v \, \mathrm{d}x \right) - \left(xv\Big|_{0}^{1} - \int_{0}^{1} v \, \mathrm{d}x \right) \\
&= -\int_{-1}^{0} v \, \mathrm{d}x + \int_{0}^{1} v \, \mathrm{d}x \\
&= \int_{-1}^{1} (\operatorname{sgn} x) v \, \mathrm{d}x \\
&= \langle f, v \rangle_{H^{-1}, H_0^1}
\end{aligned}$$

Therefore, this f is a proper function (in L^2), namely the sign-function.

Question 2 | **3 marks** | In this assignment, we will solve the POISSON-DIRICHLET problem

$$-\Delta u = f \quad \text{in } \Omega \qquad\qquad u = g \quad \text{on } \partial\Omega$$

on a rectangular domain with the finite difference method.

(a) Implement a function `discretisePoisson` with inputs

`f` and `g`: function handles for f and g

`msh`: the output of `meshRectangle`

and outputs

`A`: a sparse $(\texttt{msh.N(1)}-1)(\texttt{msh.N(2)}-1)\times(\texttt{msh.N(1)}-1)(\texttt{msh.N(2)}-1)$ array

`b`: a $(\texttt{msh.N(1)}-1)(\texttt{msh.N(2)}-1)\times 1$ array

that assembles the linear system for the POISSON-DIRICHLET problem, as derived in class.

Note that, using the notation from the cheat sheet,

$$L_1^h = \frac{1}{h_1^2} I_{N_2-1} \otimes L_{N_1-1} \qquad L_2^h = \frac{1}{h_2^2} L_{N_2-1} \otimes I_{N_1-1}$$

and the boundary conditions are included by adding the vector

$$\frac{1}{h_1^2} g_W^h \otimes e_{N_1-1}^{(1)} + \frac{1}{h_1^2} g_E^h \otimes e_{N_1-1}^{(N_1-1)} + \frac{1}{h_2^2} e_{N_2-1}^{(1)} \otimes g_S^h + \frac{1}{h_2^2} e_{N_2-1}^{(N_2-1)} \otimes g_N^h$$

to the discrete source term. I_n denotes the $n\times n$ identity matrix, L_n the $n\times n$ second-difference matrix (with the $-1\ 2\ -1$ pattern), $e_n^{(i)}$ the vector of length n with entry 1 in its i-th component and 0 elsewhere. $\otimes$ represents the KRONECKER product. This allows for a very efficient implementation with few lines of code in GNU Octave or MATLAB. No `for` loops are required, only built-in functions which are already optimised for fast execution.

(b) Write a script `hw3.m` to solve the above boundary value problem on the rectangle $\Omega =]0,1[\times]2,3[$ with $f(x_1,x_2) = 40\pi^2 \sin(2\pi x_1)\cos(6\pi x_2)$ and $g(x_1,x_2) = \sin(2\pi x_1)\cos(6\pi x_2)$. Use the same meshing parameters as in `hw1.m` and solve the linear system with the built-in \ command.

Provide a surface plot of the numerical solution u^h including its boundary values and a plot of the sparsity pattern of `A`. Check if both graphs agree with your expectations (zoom into the sparsity plot to see the detail).

Hint: In GNU Octave / MATLAB, the commands `speye`, `spdiags`, `kron` and `spy` may be helpful. Avoid commands like `eye` or `diag` for large matrices, since they also store all zeros. Else your program would take minutes to run (instead of a fraction of a second) and use up all memory.

Your Project Proposal Please upload your proposal as a one-page PDF document. We do not need large line spacing or wide margins, but please do not make the font size smaller than 10pt. Before submission, please check carefully that you have covered all requirements as defined in the rubric for this assessment component.

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?