following a posteriori error estimates hold:

$$\|e^h\|_B \leq \eta_B(u^h) = c\sqrt{\sum_{T\in\mathcal{T}^h} h_T^2\left(\|r_T\|^2_{L^2(T)} + \|r_{\partial T}\|^2_{L^2(\partial T\setminus\partial\Omega)}\right)}$$

$$\|e^h\|_{L^2} \leq \eta_{L^2}(u^h) = c\sqrt{\sum_{T\in\mathcal{T}^h} h_T^4\left(\|r_T\|^2_{L^2(T)} + \|r_{\partial T}\|^2_{L^2(\partial T\setminus\partial\Omega)}\right)}$$

These estimators are reliable and, for linear finite elements, at least asymptotically efficient in the sense

$$\eta_B(u^h) \leq c\|e^h\|_B + c\sqrt{\sum_{T\in\mathcal{T}^h} h_T^2\|f\|^2_{L^2(T)}}$$

$$\eta_{L^2}(u^h) \leq c\|e^h\|_{L^2} + c\sqrt{\sum_{T\in\mathcal{T}^h} h_T^4\|f\|^2_{L^2(T)}}.$$

Proof. $\|e^h\|_{L^2} = \int_\Omega \frac{e^h \cdot e^h}{\|e^h\|_{L^2}}\,dx = J(e^h) = J(u^h) - J(\bar{u})$

(2.25)

$$= \sum_{T\in\mathcal{T}^h} \int_T (-\Delta u^h - f)(z - I^h z)\,dx + \int_{\partial T\setminus\partial\Omega} \frac{1}{2}[\partial_n u^h](z - I^h z)\,ds$$

$z = I^h z = 0$ on $\partial\Omega$

Cauchy-Schwarz

$$\leq \sum_{T\in\mathcal{T}^h} \|-\Delta u^h - f\|_{L^2(T)} \|z - I^h z\|_{L^2(T)} + \frac{1}{2}\|[\partial_n u^h]\|_{L^2(\partial T\setminus\partial\Omega)} \|z - I^h z\|_{L^2(\partial T)}$$

interpolation estimates

$$\leq \sum_{T\in\mathcal{T}^h} \|-\Delta u^h - f\|_{L^2(T)} \underbrace{c h_T^2 \|\nabla^2 z\|_{L^2(T)}}_{2.3.14} + \frac{1}{2}\|[\partial_n u^h]\|_{L^2(\partial T\setminus\partial\Omega)} \underbrace{c h_T^{3/2} \|\nabla^2 z\|_{L^2(T)}}_{\text{works similarly}}$$

$$= c\sum_{T\in\mathcal{T}^h} h_T^2 \left(\|r_T\|_{L^2(T)} + \|r_{\partial T}\|_{L^2(\partial T\setminus\partial\Omega)}\right) \|\nabla^2 z\|_{L^2(T)}$$

Cauchy-Schwarz: $\sum_i a_i b_i \leq \sqrt{\sum_i a_i^2}\sqrt{\sum_i b_i^2}$ & $(a+b)^2 \leq 2(a^2+b^2)$

$$\leq c\sqrt{\sum_{T\in\mathcal{T}^h} h_T^4\left(\|r_T\|^2_{L^2(T)} + \|r_{\partial T}\|^2_{L^2(\partial T\setminus\partial\Omega)}\right)}\ \underbrace{\|\nabla^2 z\|_{L^2(\Omega)}}_{\leq c\ (2.1.16)},$$

□

2.3.28 Remark (Adaptive Finite Element Methods) Here we have used a posteriori error estimates to compute an approximation of the unknown error. Another application is goal-oriented mesh refinement: the goal is to compute $J(u)$ as accurately as possible, while making optimal use of the limited computational resources that are available. E.g. if the memory allows for a computation with N nodes in the mesh, then one would start with computing u^h on a coarse mesh. The error identity (2.25)

$$J(u^h) - J(\bar{u}) = \eta(u^h) = \sum_{T\in\mathcal{T}^h} \eta_T(u^h)$$

gives local error indicators $\eta_T(u^h)$ on every triangle T. One can then refine only those triangles with the largest error indicators, solve the problem again on the locally refined mesh and continue with this procedure until the maximum number N of nodes has been reached.