Let us now move on to errors due to inexact numerical integration. Whenever we apply a quadrature formula to assemble e.g. the stiffness matrix, mass matrix or load vector, we obtain a perturbed bilinear form $\tilde{B}$ instead of $B$ and a perturbed right hand side $\tilde{f}$ instead of $f$ and we are solving the perturbed GALERKIN equations

$$\tilde{B}(u^h, v^h) = \langle \tilde{f}, v^h \rangle_{V^*,V}, \qquad \forall v^h \in V^h$$

instead. To make sure that this problem still possesses a unique solution, we need the additional assumptions that $\tilde{B}$ is continuous and coercive as well

$$\begin{aligned} |\tilde{B}(u^h, v^h)| &\le \tilde{C}\|u^h\|\|v^h\|, \qquad && \forall u^h, v^h \in V^h \\ \tilde{B}(u^h, u^h) &\ge \tilde{c}\|u^h\|^2, && \forall u^h \in V^h \end{aligned}$$

and that $\tilde{f}$ is continuous, just like their unperturbed counterparts $B$ and $f$.

2.3.19 Lemma (STRANG's First Lemma) *The error $e^h = \bar{u} - u^h$ of the perturbed (but otherwise conforming) GALERKIN approximation satisfies the estimate*

$$\|e^h\|_h \le c\left(\inf_{v^h \in V^h} \Big(\underbrace{\|\bar{u} - v^h\|_h}_{\substack{\text{approximation error} \\ \text{due to} \\ V \to V^h}} + \underbrace{\|B(v^h, \cdot) - \tilde{B}(v^h, \cdot)\|_*\Big) + \|f - \tilde{f}\|_*}_{\substack{\text{quadrature errors} \\ \text{due to} \\ K^h \to \tilde{K}^h,\ M^h \to \tilde{M}^h,\ f^h \to \tilde{f}^h}}\right)$$

*with a constant $c > 0$ that is independent of $u, u^h$ and $h$.*

*Proof.* Let $v^h \in V^h$ be arbitrary.

- $B(\bar{u} - v^h, u^h - v^h) = B(\bar{u}, u^h - v^h) - B(v^h, u^h - v^h) = \langle f, u^h - v^h \rangle - B(v^h, u^h - v^h)$

  $\tilde{B}(u^h - v^h, u^h - v^h) = \tilde{B}(u^h, u^h - v^h) - \tilde{B}(v^h, u^h - v^h) = \langle \tilde{f}, u^h - v^h \rangle - \tilde{B}(v^h, u^h - v^h)$

- $$\begin{aligned} \tilde{c}\|u^h - v^h\|^2 &\le \tilde{B}(u^h - v^h, u^h - v^h) + 0 \\ &= \underbrace{B(\bar{u} - v^h, u^h - v^h)}_{\le C\|\bar{u} - v^h\|\|u^h - v^h\|} + \underbrace{B(v^h, u^h - v^h) - \tilde{B}(v^h, u^h - v^h)}_{\le \|B(v^h,\cdot) - \tilde{B}(v^h,\cdot)\|_*\|u^h - v^h\|} + \underbrace{\langle \tilde{f}, u^h - v^h \rangle - \langle f, u^h - v^h \rangle}_{\le \|\tilde{f} - f\|_*\|u^h - v^h\|} \end{aligned}$$

  with the operator norm

  $$\|f\|_* = \sup_{v^h \in V^h} \frac{|\langle f, v^h \rangle|}{\|v^h\|}$$





- $$\begin{aligned} \|e^h\| &= \|u^h - \bar{u}\| \\ &\le \|u^h - v^h\| + \|v^h - \bar{u}\| \\ &\le \frac{C}{\tilde{c}}\|\bar{u} - v^h\| + \frac{1}{\tilde{c}}\|B(v^h,\cdot) - \tilde{B}(v^h,\cdot)\|_* + \frac{1}{\tilde{c}}\|\tilde{f} - f\|_* + \|v^h - \bar{u}\| \\ &\le C\left(\|\bar{u} - v^h\| + \|B(v^h,\cdot) - \tilde{B}(v^h,\cdot)\|_* + \|f - \tilde{f}\|_*\right) \end{aligned}$$ □