Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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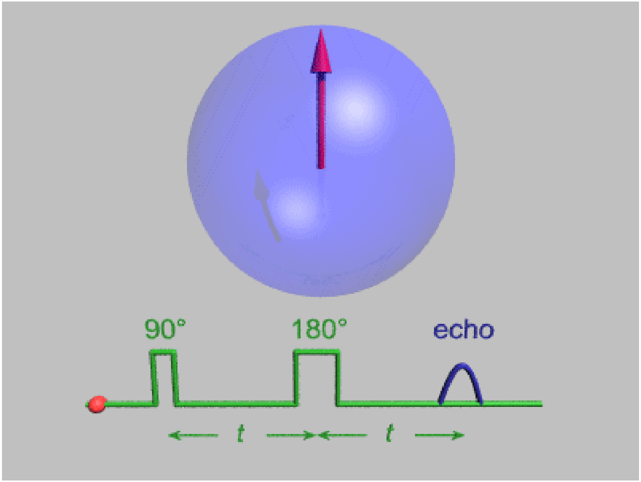
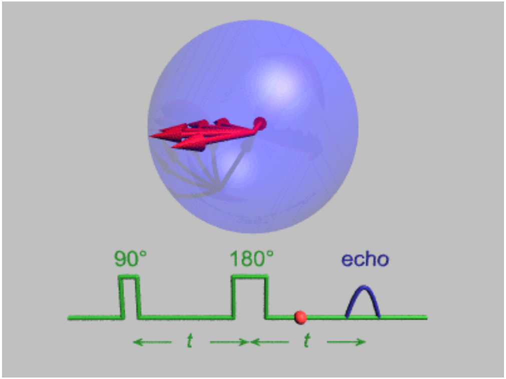
1. Animation of a typical "spin echo" MRI sequence



1. Spins (water molecules) do not truly refocus:
   1. Precession rate depends on position
   2. Diffusion plays an important role



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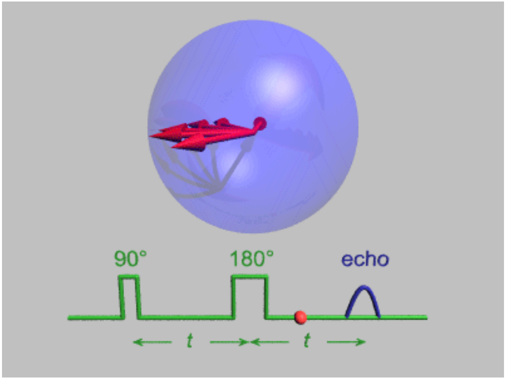

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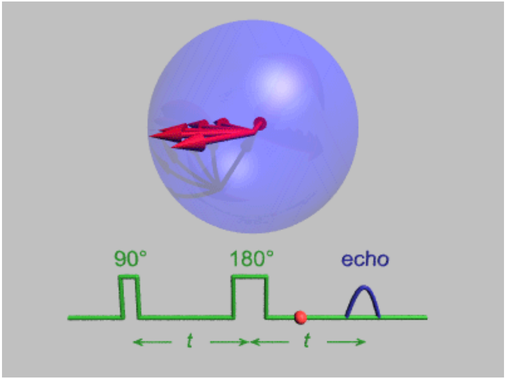

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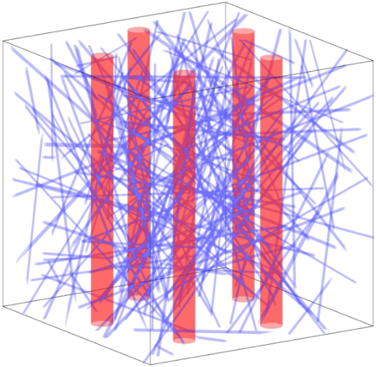

Figure: Simulated imaging voxel

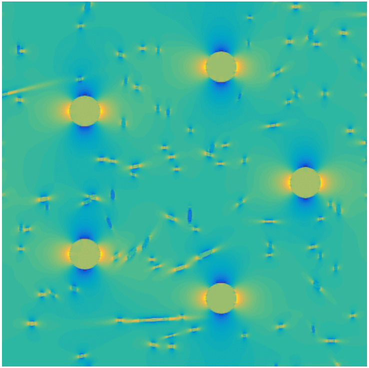

Figure: Cross section of precession rate

1. Evolution of the magnetization is modelled by the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

where:

$$\mathcal{M} = M_x + i\,M_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. IC: $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. BC: zero Neumann or periodic
4. Note:

$$D = 0 \;\Rightarrow\; \mathcal{M}(\mathbf{x}, t) = \mathcal{M}_0(\mathbf{x})e^{-\Gamma(\mathbf{x})t}$$

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The Bloch-Torrey Equation

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1. First, the BT PDE is rewritten in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

where

$$H = -D\Delta + \Gamma.$$

2. Then, the general solution $\mathcal{M}$ may then be written as

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Operator Splitting Methods

1. Now, the evolution operator may be *split* using the approximation

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\begin{aligned}
e^{-Ht} &= e^{D\Delta t - \Gamma t} \\
&\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3)
\end{aligned}
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2. Although e^{-Ht} has no closed form, the split operators do:

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e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\
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$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u \end{cases}$$

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1. Applying the method of lines:

$$M^h \mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v}$$
$$M^h \mathbf{v}_t = -(DK^h + R^h)\mathbf{v} - W^h \mathbf{u}$$

where:

$$R^h_{ij} := \int R\,\phi_i\,\phi_j\,\mathrm{d}x$$
$$W^h_{ij} := \int \omega\,\phi_i\,\phi_j\,\mathrm{d}x$$

Time Stepping Methods

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2. Strongly A-stable time stepping methods are required

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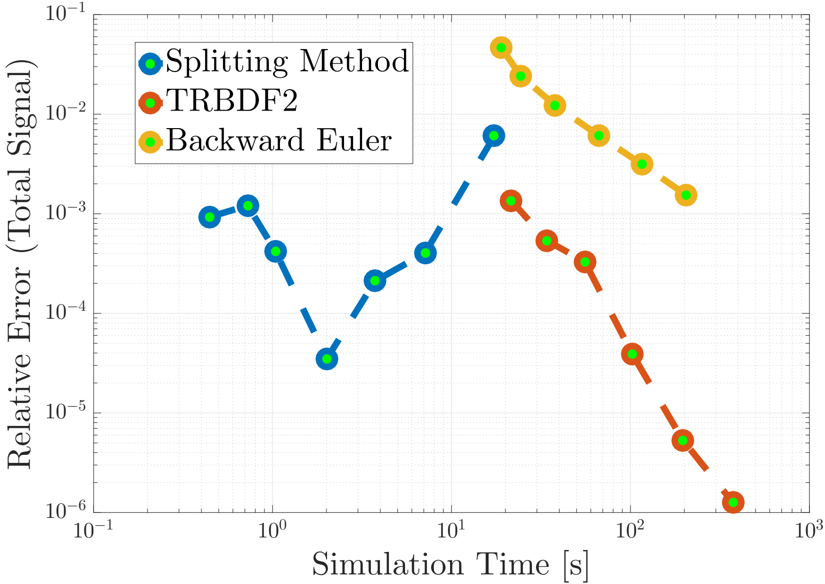
Operator Splitting versus FEM



Conclusion

1. Splitting methods:
   - $+$ Fast
   - $-$ Resolution limitations
   - $\pm$ Periodic boundary conditions
2. Finite element methods:
   - $-$ Slow
   - $+$ Flexible boundary conditions
   - $+$ Adaptivity?

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