

Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation



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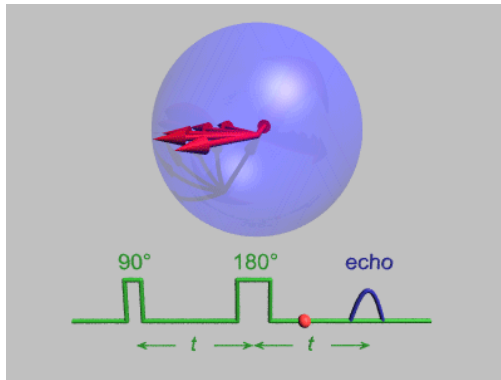
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- 1 Animation of a typical “spin echo” MRI sequence

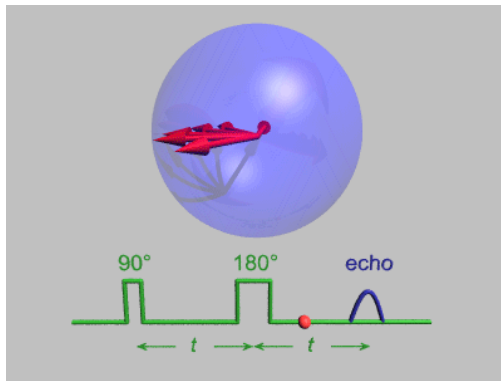
① Spins (water molecules) do not truly refocus:

- ① Precession rate depends on position
- ② Diffusion plays an important role

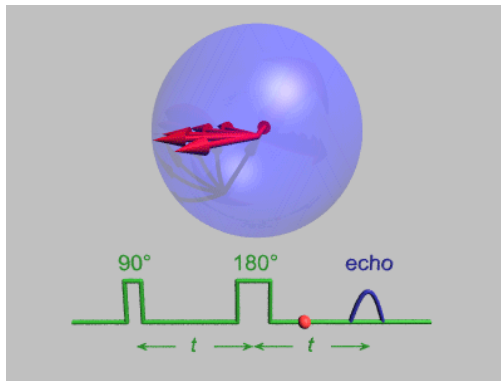


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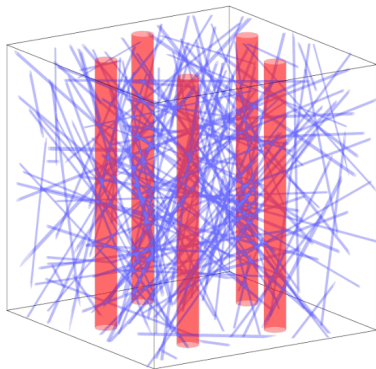


Figure: Simulated imaging voxel

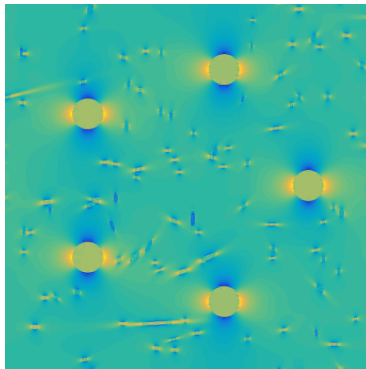


Figure: Cross section of precession rate

- 1 Evolution of the magnetization is modelled by the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D \Delta \mathcal{M} - \Gamma \mathcal{M}$$

where:

$$\begin{aligned}\mathcal{M} &= M_x + i M_y \\ \Gamma(\mathbf{x}) &= R(\mathbf{x}) + i \omega(\mathbf{x})\end{aligned}$$

- 2 IC: $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
- 3 BC: zero Neumann or periodic
- 4 Note:

$$D = 0 \Rightarrow \mathcal{M}(\mathbf{x}, t) = \mathcal{M}_0(\mathbf{x}) e^{-\Gamma(\mathbf{x})t}$$

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- 1 First, the BT PDE is rewritten in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

where

$$H = -D\Delta + \Gamma.$$

- 2 Then, the general solution \mathcal{M} may then be written as

$$\mathcal{M} = e^{-Ht} \mathcal{M}_0$$

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- ① Now, the evolution operator may be *split* using the approximation

$$\begin{aligned} e^{-Ht} &= e^{D\Delta t - \Gamma t} \\ &\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3) \end{aligned}$$

- ② Although e^{-Ht} has no closed form, the split operators do:

$$\begin{aligned} e^{-\Gamma t/2} \mathcal{M} &= e^{-\Gamma(x)t/2} \odot \mathcal{M} \\ e^{D\Delta t} \mathcal{M} &= \Phi * \mathcal{M} \end{aligned}$$

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- 1 The BT equation can also be solved using FEM
- 2 First, let $u = M_x$ and $v = M_y$ and write:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u \end{cases}$$

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- 1 Applying the method of lines:

$$M^h \mathbf{u}_t = -(DK^h + R^h) \mathbf{u} + W^h \mathbf{v}$$

$$M^h \mathbf{v}_t = -(DK^h + R^h) \mathbf{v} - W^h \mathbf{u}$$

where:

$$R_{ij}^h := \int R \phi_i \phi_j dx$$

$$W_{ij}^h := \int \omega \phi_i \phi_j dx$$

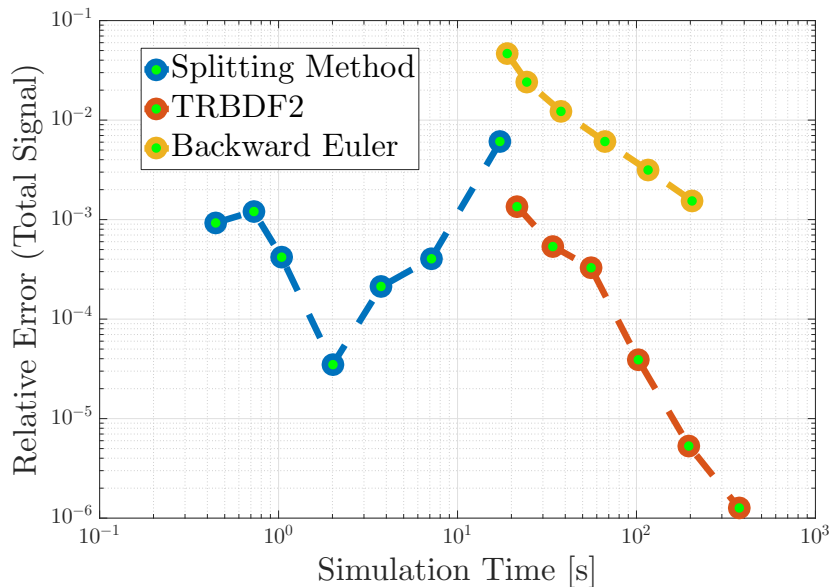
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- 1 Solutions to the Bloch-Torrey equation decay exponentially in time and smooth high frequency modes
- 2 Strongly A-stable time stepping methods are required

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Operator Splitting versus FEM



- ① Splitting methods:
 - + Fast
 - Resolution limitations
 - ± Periodic boundary conditions

- ② Finite element methods:
 - Slow
 - + Flexible boundary conditions
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