Finally, let us add some a few remarks regarding nonconforming approximations. In this case we are solving the discrete problem

$$B^h(u^h, v^h) = \langle f^h, v^h \rangle_{V^{h*},V^h}, \qquad \forall v^h \in V^h$$

where $V^h \not\subset V$. We need the extra assumptions that B^h can be defined for arguments from V and conversely that B can be defined for arguments from V^h. Additionally we impose continuity and coercivity of B^h

$$\begin{aligned} |B^h(u,v)| &\le C^h \|u\|_h \|v\|_h, && \forall u, v \in V + V^h \\ B^h(u^h, u^h) &\ge c^h \|u^h\|_h^2, && \forall u^h \in V^h \end{aligned}$$

where $\|\cdot\|_h$ is some norm on $V + V^h$, the space of all linear combinations $\lambda v + \mu v^h$ with $\lambda, \mu \in \mathbb{R}$, $v \in V$, $v^h \in V^h$. The inhomogeneity f^h is assumed to be continuous in this norm on the space V^h.

We will also need the corresponding operator norm, defined by

$$\|f^h\|_{h*} = \sup_{v^h \in V^h} \frac{|\langle f^h, v^h \rangle_{V^{h*},V^h}|}{\|v^h\|_h}.$$

2.3.20 Lemma (STRANG's Second Lemma) *The error $e^h = \bar{u} - u^h$ of the possibly non-conforming finite element approximation satisfies the estimate*

$$\|e^h\|_h \le c \Big(\underbrace{\inf_{v^h \in V^h} \|\bar{u} - v^h\|_h}_{\text{approximation error}} + \underbrace{\|B^h(\bar{u}, \cdot) - f^h\|_{h*}}_{\text{consistency error/ non-conformity error}} \Big)$$

with a constant $c > 0$ that is independent of $\bar{u}, u^h$ and h.

Proof. Let $v^h \in V^h$ be arbitrary.

- $B^h(u^h - v^h, u^h - v^h) = B(u^h, u^h - v^h) - B(v^h, u^h - v^h) = \langle f^h, u^h - v^h \rangle_{V^{h*},V^h} - B(v^h, u^h - v^h)$

- $c^h \|u^h - v^h\|_h^2 \le B^h(u^h - v^h, u^h - v^h) + 0$

$$= B^h(\bar{u}, u^h - v^h) - B^h(v^h, u^h - v^h) + \langle f^h, u^h - v^h \rangle_{V^{h*},V^h} - B^h(\bar{u}, u^h - v^h)$$

$$= \underbrace{B^h(\bar{u} - v^h, u^h - v^h)}_{\le C^h \|\bar{u} - v^h\|_h \|u^h - v^h\|_h} + \underbrace{\langle f^h - B^h(\bar{u}, \cdot), u^h - v^h \rangle_{V^{h*},V^h}}_{\le \|f^h - B^h(\bar{u}, \cdot)\|_{h*} \|u^h - v^h\|_h}$$

- $\|e^h\|_h = \|u^h - \bar{u}\|_h$

$$\begin{aligned} &\le \|u^h - v^h\|_h + \|v^h - \bar{u}\|_h \\ &\le \frac{C^h}{c^h} \|\bar{u} - v^h\|_h + \frac{1}{c^h} \|f^h - B^h(\bar{u}, \cdot)\|_{h*} + \|v^h - \bar{u}\|_h \\ &\le c \left(\|\bar{u} - v^h\|_h + \|B^h(\bar{u}, \cdot) - f^h\|_{h*} \right) \end{aligned}$$

$\square$