MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
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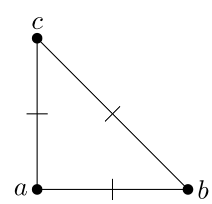
Homework Assignment 6

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **2 marks** | Given three points $a, b, c \in \mathbb{R}^2$ that are not collinear (not all on one line) and that are sorted in anticlockwise order, we define

$$T = \Delta(a, b, c) \qquad \text{(the triangle with these vertices)}$$
$$P = P_2(T)$$
$$L = \left\{ p \mapsto p(a), \quad p \mapsto p(b), \quad p \mapsto p(c), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{a+b}{2}\right), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{b+c}{2}\right), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{c+a}{2}\right) \right\} \subset P^*$$



(a) Show that prescribed data for

$$p(a), \quad p(b), \quad p(c), \quad \frac{\partial p}{\partial n}\left(\frac{a+b}{2}\right), \quad \frac{\partial p}{\partial n}\left(\frac{b+c}{2}\right) \quad \text{and} \quad \frac{\partial p}{\partial n}\left(\frac{c+a}{2}\right)$$

uniquely determines any $p \in P$. You don't have to show that such p always exists.

NB: This is the so-called Morley element and of great importance for nonconforming approximations of the 4^{th}-order plate equation. H^2-conforming approximations are quite demanding:

- If P should be a space $P_k(T)$ of polynomials, then this must be at least quintic ($k \geq 5$)! The classical quintic element for this purpose is the Argyris element.
- Alternatively, the triangle T could be subdivided into three smaller triangles along the angle bisectors. The functions in P are then defined as cubic C^1-splines: three cubics, each defined on one of the subtriangles, stitched together in a continuously differentiable fashion. This is the so-called Hsieh-Clough-Tocher (macro-)element.

The Morley element is clearly a lot simpler than those H^2-conforming approximations. It works surprisingly well for the plate equation (with modified Galerkin equations), even though it's not even H^1-conforming — see part (b)!

After these preliminary remarks, let's answer part (a). We set all of the above six degrees of freedom to zero and show that this already implies $p \equiv 0$ on T.

1. On each triangle edge, p is a parabola with identical function values (zero) at the two endpoints. Hence, its vertex is located at the edge midpoint. Consequently, the tangential derivative of p vanishes in the three edge midpoints.
2. Since the normal derivatives vanish in the edge midpoints, too, the entire gradient is zero in these three points. But the gradient of a quadratic polynomial is a linear polynomial, which is fully determined by its values in the three edge midpoints. Hence, $\nabla p \equiv 0$.
3. Now it follows that p is constant, and due to the prescribed values in the corner points, this constant must be zero.

(b) Now let Ω^h be a domain with a regular triangulation $\mathcal{T}^h$ such that

$$\bar{\Omega}^h = \bigcup_{T\in\mathcal{T}^h} T.$$

Is the space

$$V^h = \left\{ v^h : \bar{\Omega}^h \to \mathbb{R} \;\middle|\; v^h\big|_T \in P_2(T), v^h \text{ is continuous in all vertices}, \frac{\partial v^h}{\partial n} \text{ is continuous in all edge midpoints} \right\}$$

H^1-conforming, i.e. is $V^h \subset H^1(\Omega^h)$?

Hint: Check if there may be any jumps of v^h across triangle edges.

Consider a quadrilateral domain Ω^h composed of only two triangles T_+ and T_- with the common edge $[0,1]\times\{0\}$, where T_+ is located above the x_1-axis ($x_2 \geq 0$ on T_+) and T_- is located below the x_1-axis ($x_2 \leq 0$ on T_-).

We define

$$v^h = \begin{cases} x_1(1-x_1) & \text{for } x_2 \geq 0 \\ 0 & \text{for } x_2 < 0. \end{cases}$$

Clearly, $v^h\big|_{T_\pm} \in P_2(T_\pm)$. Also,

$$\begin{aligned} v^h\big|_{T_+}(0,0) &= v^h\big|_{T_-}(0,0) \\ v^h\big|_{T_+}(1,0) &= v^h\big|_{T_-}(1,0) \\ \frac{\partial v^h\big|_{T_+}}{\partial x_2}(1/2,0) &= \frac{\partial v^h\big|_{T_-}}{\partial x_2}(1/2,0) \end{aligned}$$

and thus $v^h \in V^h$.

However, $v^h \notin H^1(\Omega^h)$, since $\frac{\partial v^h}{\partial x_2}$ "has a delta-ridge over the edge where $x_2 = 0$" (which is not an L^2 function).

(Optional) Slightly more formally:

$$\forall \phi \in C_0^\infty([-1,1]): \quad \int_{-1}^{1} v^h \frac{\partial \phi}{\partial x_2}\, \mathrm{d}x_2 = \int_{-1}^{0} v^h \frac{\partial \phi}{\partial x_2}\, \mathrm{d}x_2 + \int_{0}^{1} v^h \frac{\partial \phi}{\partial x_2}\, \mathrm{d}x_2 = \int_{0}^{1} x_1(1-x_1) \frac{\partial \phi}{\partial x_2}\, \mathrm{d}x_2 = -x_1(1-x_1)\phi(0),$$

but there is no function $u \in L^2(\Omega^h)$ (no weak derivative of v^h in $L^2(\Omega^h)$) that would make

$$\forall \phi \in C_0^\infty([-1,1]): \quad -x_1(1-x_1)\phi(0) = -\int_{-1}^{1} u\phi\, \mathrm{d}x_2.$$

(The required $u(x) = x_1(1-x_1)\delta_0(x_2)$ is not a function.)

Question 2 | **3 marks** | We will now complete our finite-element solver for the linear elasticity problem

$$\begin{aligned} -c\Delta u + au &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

(a) Remove lines 1-10 from `discretiseLinearElasticity.m` and uncomment the sections of code that are currently commented out. Complete the missing commands, including the subfunction `assembleStiffness`. Also inspect the `assembleLoad` subfunction.

(b) Write a script `hw6.m` which

- solves the linear elasticity problem on Ω^h, which you may choose from `kiwi.mat`, `maple.mat`, `pi.mat`, `ubc.mat`. You may also select your own data for $f(x_1, x_2)$, $g(x_1, x_2)$, a and c.

  Hint: You have to set `GammaD = @(x1,x2) true(size(x1))`. For debugging, you might want to use `video10.mat` and check the sparsity patterns of the various matrices.

- calculates the L^2, H^1 and energy norms

$$\|u^h\|_{L^2} = \sqrt{\int_{\Omega^h} |u^h|^2 \, \mathrm{d}x}$$

$$\|u^h\|_{H^1} = \sqrt{\|u^h\|_{L^2}^2 + \|\nabla u^h\|_{L^2}^2} = \sqrt{\int_{\Omega^h} |u^h|^2 \, \mathrm{d}x + \int_{\Omega^h} |\nabla u^h|^2 \, \mathrm{d}x}$$

$$\|u^h\|_B = \sqrt{B(u^h, u^h)} = \sqrt{c\int_{\Omega^h} |\nabla u^h|^2 \, \mathrm{d}x + a\int_{\Omega^h} |u^h|^2 \, \mathrm{d}x}$$

  of the solution, where B is the bilinear form corresponding to the elliptic operator

- creates undistorted plots of the mesh, the force f and the solution u^h

(c) What problem do you solve numerically when you set `GammaD = @(x1,x2) false(size(x1))`? Analyse the code to infer its weak formulation:

The equation itself is completely unaffected. The only difference is that the hat functions centred on boundary nodes are no longer eliminated, neither as test functions nor as shape functions, i.e. we have to remove the subscript 0 from the space $H_0^1(\Omega)$. The weak form reads: find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$

$$c\int_{\Omega} \nabla u \cdot \nabla v \, \mathrm{d}x + a\int_{\Omega} uv \, \mathrm{d}x = \int_{\Omega} fv \, \mathrm{d}x.$$

(Undo integration by parts to see that this is the weak form of the linear elasticity problem with homogeneous Neumann boundary conditions.)

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?