Homework Assignment 10

Please submit the following files as indicated below:

Question 1 | 2 marks On the assignment page you can find videos of four animated solutions of the parabolic problem

$$\begin{aligned} \partial_t u(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

with the data from Assignment 9. However, the initial condition has been replaced with the function

$$u_0(x) = \begin{cases} 50 & \text{if } |x - (1,1)^\top| < 0.5 \\ 20 & \text{elsewhere} \end{cases}$$

Explain your observations.

Solution. First, we note that since our initial condition in Assignment 9 was $u_0 = 20$ in Ω, the only change in the initial condition is that now $u_0 = 20$ throughout Ω, except for those points contained in the region $|x-(1,1)^\top| < 0.5$ wherein $u_0 = 50$.

Now, some initial observations:

- Our initial data has a discontinuous "hot spot" in the north-eastern corner of the domain.
- The forcing term applies gaussianly distributed heat centred about the south-western corner of the domain for the first second of the five second simulation.
- The heat equation will smooth out the discontinuity immediately in the continuous case due to the sharp discontinuity containing high frequency eigenmodes which are damped exponentially fast. Ideal numerical schemes should mimic this effect.
- Failure of the numerical scheme in handling the discontinuity effectively will likely lead to ringing of the solution, possible with extreme "wiggles".

TODO: Interpret wiggles vs. methods

□

Question 2 | 1 mark We have seen that the homogeneous wave equation

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \partial_t u(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \end{aligned} \tag{W}$$

with propagation speed $c > 0$ can equivalently be re-written as

$$\begin{aligned} \partial_t u - v &= 0 && \text{in } Q =]0,T[\times \Omega \\ \partial_t v - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ v(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \\ v &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega. \end{aligned} \tag{W'}$$

Discretising with the θ-method in time and linear finite elements in space leads to the coupled system for the vectors of nodal values $\vec{u}_+^h$ and $\vec{v}_+^h$

$$\begin{aligned} M^h \vec{u}_+^h - \theta \Delta t M^h \vec{v}_+^h &= M^h \vec{u}_\circ^h + (1-\theta)\Delta t M^h \vec{v}_\circ^h \\ \theta \Delta t c^2 K^h \vec{u}_+^h + M^h \vec{v}_+^h &= -(1-\theta)\Delta t c^2 K^h \vec{u}_\circ^h + M^h \vec{v}_\circ^h \end{aligned}$$

which has to be solved at every time step. Show that this is equivalent to the two smaller, successively solvable problems

$$\begin{aligned} \left(M^h + (\theta \Delta t c)^2 K^h\right) \vec{u}_+^h &= M^h \left(\vec{u}_\circ^h + \Delta t \vec{v}_\circ^h\right) - \left(\theta\,(1-\theta)\,(\Delta t c)^2\right) K^h \vec{u}_\circ^h \\ M^h \vec{v}_+^h &= M^h \vec{v}_\circ^h - \Delta t c^2 K^h \left(\theta \vec{u}_+^h + (1-\theta)\,\vec{u}_\circ^h\right). \end{aligned}$$

Solution. If we multiply the second equation of the coupled system by $\theta \Delta t$ and add it to the first equation of the coupled system and simply, we get

$$\begin{aligned} \left(M^h + (\theta \Delta t c)^2 K^h\right) \vec{u}_+^h &= M^h \left(\vec{u}_\circ^h + \Delta t \vec{v}_\circ^h\right) + \left(\theta^2\,(\Delta t c)^2 - \theta\,(\Delta t c)^2\right) K^h \vec{u}_\circ^h \\ &= M^h \left(\vec{u}_\circ^h + \Delta t \vec{v}_\circ^h\right) - \left(\theta\,(1-\theta)\,(\Delta t c)^2\right) K^h \vec{u}_\circ^h, \end{aligned} \tag{1}$$

which is the first desired equation of the new uncoupled system.

Now, if we simply subtract $\theta \Delta t c^2 K^h \vec{u}_+^h$ from both sides of the second equation of the coupled system and simply, we get

$$\begin{aligned} M^h \vec{v}_+^h &= M^h \vec{v}_\circ^h + \left(\Delta t c^2 K^h \theta - \Delta t c^2 K^h\right) \vec{u}_\circ^h - \left(\theta \Delta t c^2 K^h\right) \vec{u}_+^h \\ &= M^h \vec{v}_\circ^h - \Delta t c^2 K^h \left(\theta \vec{u}_+^h + (1-\theta) \vec{u}_\circ^h\right), \end{aligned} \tag{2}$$

which is the second desired equation of the new uncoupled system.

$\square$

Question 3 | 2 marks

(a) Download and complete the FEniCS script `hw10.py` to solve Problem (W) with the data provided.

(b) Solve the wave equation

- with the (symplectic) implicit midpoint rule
- with the backward EULER method
- with the forward EULER method

and look at the solutions in ParaView.

Hint: Use the 'Warp by Scalar' filter, re-scale the colour map to the range $[-1, 1]$ and tick the box 'enable opacity mapping for surfaces' in the colour map editor.

For each of the three time stepping schemes, create a graph with curves of the total energy $E(u(t), v(t)) = T(v(t)) + V(u(t))$, the kinetic energy $T(v(t)) = \frac{1}{2}\|v(t)\|_{L^2}^2$ and the potential energy $V(u(t)) = \frac{c^2}{2}\|\nabla u(t)\|_{L^2}^2$ as functions of time. Please submit these plots and interpret the results:

Solution. **TODO: SUBMIT plots/data, and interpret plots** $\square$

Your Learning Progress What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
