MATH521 ~~[scribbled out]~~ Homework Assignment #3 R Jan 25

① We consider the weak & strong formulations of a 1D BVP:

$$\begin{cases} -u'' = f & \text{in } (-1,1) \\ u(-1) = 0 = u(1) \end{cases} \quad (S) \quad \text{where } u \in C^2(]-1,1[) \cap C([-1,1])$$

$$\int_{-1}^{1} u'v'\,dx = \langle f, v \rangle_{H^{-1}, H_0^1} \quad (W) \quad \text{where } u \in H_0^1(]-1,1[) \text{ and } (W) \text{ holds } \forall\, v \in H_0^1(]-1,1[)$$

Problem: Let $u = \frac{1}{2}(1-|x|)$. Show that u satisfies (W) but not (S), and find the forcing term f.

Solution: → u does not have two continuous derivatives, thus it cannot solve (S).

→ u does, however, have piecewise continuous 1^{st} derivatives:

$$u = \begin{cases} \frac{1}{2}(1+x), & x \in [-1,0] \\ \frac{1}{2}(1-x), & x \in]0,1] \end{cases} \longrightarrow u' = \begin{cases} \frac{1}{2}, & x \in [-1,0[\\ -\frac{1}{2}, & x \in]0,1] \end{cases}$$

Thence we have: $\int_{-1}^{1} u'v'\,dx = \int_{-1}^{0} \frac{1}{2}v'\,dx + \int_{0}^{1} -\frac{1}{2}v'\,dx = \frac{1}{2}[v]_{\cancel{-1}}^{0} - \frac{1}{2}[v]_{0}^{\cancel{1}}$ ← as $v \in H_0^1(]-1,1[)$

$= v(0) \quad \forall\, v \in H_0^1(]-1,1[)$

~~And~~ And so it follows that: $v(0) = \langle f, v \rangle_{H^{-1}, H_0^1}$, and so $f = \delta(x)$ is the Dirac δ (generalized) function.

$$\boxed{\text{Thus, we have that } u \text{ satisfies } (W) \text{ w/ } f = \delta(x).}$$

Learning Progress

<u>Most important thing learned:</u> Definitely dealing w/ sparse matrices in MATLAB, and in particular learning about assembling multi-dimensional operators through Kronecker products.

<u>'Aha' moment:</u> Question one was illuminating, as we actually got to work through a weak solution example (which ended up having a non-trivial RHS $f = \delta(x)$.)