Math 521 HW 2 R Jan 18

① Let $\Omega \subset \mathbb{R}^2$ be a 2D domain, $\vec{a}: \Omega \to \mathbb{R}^2$ a 2D continuously differentiable vector field, $D > 0$ a constant, and $g: \partial\Omega \to \mathbb{R}$ a continuous function.

Steady advection-diffusion: $\begin{cases} \operatorname{div}(u\vec{a}) - \operatorname{div}(D\nabla u) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ (*)

ⓐ Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be a strong soln to (*).
Show that u is bounded by it's bdry. values,
$$\min_{\vec{s} \in \partial\Omega} g(\vec{s}) \le u(\vec{x}) \le \max_{\vec{s} \in \partial\Omega} g(\vec{s}) \quad \forall \vec{x} \in \bar{\Omega}$$

Provided that $\operatorname{div} \vec{a} = 0$ ("incompressibility" condition).

Solution: • First, we note that: $\operatorname{div}(u\vec{a}) = u \operatorname{div}\vec{a} + \vec{a} \cdot \nabla u$ for incompressible $\vec{a}$.
$= \vec{a} \cdot \nabla u$

• Then, we rewrite (*) as: $\begin{cases} \vec{a} \cdot \nabla u - D\nabla \cdot \nabla u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$ (+)

(where $\nabla \cdot \vec{f}$ is short for $\operatorname{div} \vec{f}$).

• Next ~~First~~, we note that $L = (\vec{a} \cdot \nabla u - D\nabla \cdot \nabla u)$ is an elliptic operator of the form
$$L = b_{11} \frac{\partial^2}{\partial x_1^2} + 2b_{12} \frac{\partial^2}{\partial x_1 \partial x_2} + b_{22} \frac{\partial^2}{\partial x_2^2} + b_1 \frac{\partial}{\partial x_1} + b_2 \frac{\partial}{\partial x_2}$$
where $b_{11} = b_{22} = -D$, $b_{12} = 0$, $b_1 = a_1$ and $b_2 = a_2$ where $\vec{a} = (a_1, a_2)$.
We see it is elliptic as $b_{12}^2 - b_{11}b_{22} = -D^2 < 0$.

• Now, we may use the Elliptic Maximum Principle (Theorem 2.1.2 in the notes):
Given an elliptic operator L of the above form and $u \in C^2(\Omega) \cap C(\bar{\Omega})$, then
$$Lu \le 0 \text{ in } \Omega \Rightarrow \max_{\vec{x} \in \bar{\Omega}} u(\vec{x}) \le \max_{\vec{x} \in \partial\Omega} u(\vec{x})$$

• Finally, since we have $Lu = 0$ in our case, we have:
$\to Lu = 0 \Rightarrow \max_{\vec{x} \in \bar{\Omega}} u(\vec{x}) = \max_{\vec{x} \in \partial\Omega} u(\vec{x}) = \max_{\vec{s} \in \partial\Omega} g(\vec{s})$,
$\to L(-u) = 0 \Rightarrow \max_{\bar{\Omega}} -u = \min_{\bar{\Omega}} u = \max_{\partial\Omega} -u = \min_{\partial\Omega} g$

$\boxed{\Leftrightarrow \min_{\vec{s} \in \partial\Omega} g(\vec{s}) \le u(\vec{x}) \le \max_{\vec{s} \in \partial\Omega} g(\vec{s})}$ ∎

(b) Why do we have to assume $\operatorname{div}\vec{a} = 0$ to derive these bounds?

→ For this derivation, we rely on the Elliptic Maximum Principle (Theorem 2.1.2 in our notes), which strictly requires that there is no term with any u, only with (mixed) 1st & 2nd derivatives.

→ Since $\operatorname{div}(u\vec{a}) = u \operatorname{div}\vec{a} + \vec{a}\cdot\nabla u$ contains a term w/ u in it, it must drop out in order to apply the theorem.

→ Intuitively, $\vec{a}$ must be divergenceless or else we could construct a counter-example, such as if the flow field $\vec{a}$ had a sink in it then u (a concentration) could easily fall to zero, and if g is positive on the boundary then we clearly do not fit the conclusion of the theorem.

(2) (See attached)

Learning Progress

Most important thing learned?

→ Question 1a is definitely an important result, and I think that's the biggest takeaway from this.

Substantial new insight?

→ Not sure I'd call it an "insight", but the ordering of MATLAB 2D arrays for finite differencing makes much more sense to me now, and is intuitive.