MATH521
Numerical Analysis of Partial Differential Equations
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Convergence

General Idea Imagine we want to solve the problem $T(u) = 0$. This function T would be

$$T(u) = \begin{pmatrix} -\Delta u - f \\ u\big|_{\partial\Omega} - g \end{pmatrix} \tag{1}$$

for the Poisson-Dirichlet problem, but T could also represent any other PDE, ODE or maybe an integral, ...

Numerically, we solve a discrete problem $T^h(u^h) = 0$ instead which gives us a numerical solution u^h. For instance, the above problem discretised with finite differences yields the left hand side

$$T^h(u^h) = L^h u^h - f^h \tag{2}$$

of the 'big linear system', where $h > 0$ is a mesh parameter. For our Poisson-Dirichlet solver, this would usually be $h = \max\{\, h_1, h_2 \,\}$.

Clearly, our final objective is to find a numerical solution u^h which is a good approximation of the exact solution $\bar{u}$ of $T(u) = 0$. In other words, the *error* $e^h = u^h - \bar{u}$ should be small. 'Small' means that as we refine the mesh ($h \to 0$), the numerical solution u^h should become an increasingly accurate approximation of $\bar{u}$ with perfect agreement in the hypothetical limit $h = 0$. This is called *convergence* of the numerical scheme.

Formal Definition The numerical scheme T^h is said to be

1. *convergent*, if there exists a constant $h_0 > 0$ such that $T^h(u^h) = 0$ has a unique solution for all $h \in]0, h_0]$ and if with the solution $\bar{u}$ of $T(u) = 0$
$$\|u^h - \bar{u}\| \to 0 \qquad \text{as } h \to 0.$$
2. *convergent of order* $O(h^p)$, if additionally
$$\|u^h - \bar{u}\| = O(h^p) \qquad \text{as } h \to 0.$$

Your Task Ultimately, we want to determine the order of convergence of our finite-difference scheme for the Poisson-Dirichlet problem, however, you will need the results of other groups to answer that question. Conversely, they will have to learn about convergence, so be prepared to explain what convergence means!

Meanwhile, I'd like you to look at something else, namely the *conditioning* of the 'big linear system' $L^h u^h = f^h$. In reality, there will always be some perturbation on the right hand side since we only have about 16 decimal places available in our computer and everything below is chopped off, or maybe the source term f^h is an extremely complicated function which we can only evaluate up to a pretty coarse accuracy. Hence, even though we want to solve

$$L^h u^h = f^h$$

we're actually solving this 'big linear system' with a different right hand side $f^h + \delta f^h$, which also gives rise to an error δu^h in the solution:

$$L^h(u^h + \delta u^h) = f^h + \delta f^h.$$

Assuming that $\|(L^h)^{-1}\| \leq C$ for some hopefully not too large constant $C > 0$

1. how does the absolute error $\|\delta u^h\|$ depend on the absolute perturbation $\|\delta f^h\|$?
2. how does the relative error $\|\delta u^h\| / \|u^h\|$ depend on the relative perturbation $\|\delta f^h\| / \|f^h\|$?
3. which of the two errors could be quite large under (what?) bad circumstances?