MATH521
Numerical Analysis of Partial Differential Equations

Winter 2017/18, Term 2
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Forward Euler Method

General Framework

If we use the method of lines to discretise a homogeneous, time-dependent PDE, then we're confronted with a system of ODEs of the form

$$\dot{u} = Au \qquad u(0) = u_0 \tag{1}$$

(e.g. $A = (M^h)^{-1}K^h$ for the homogeneous heat equation with a finite-element discretisation in space). If we use ROTHE's method to discretise a homogeneous, time-dependent PDE, then we're confronted with an ODE in a function space of the form

$$\dot{u} = Au \qquad u(0) = u_0 \tag{2}$$

(e.g. $A = \Delta$ in $H^1_0(\Omega)$ for the homogeneous heat equation with homogeneous DIRICHLET conditions at all times).

However, we'll just look at the simple scalar ODE

$$\dot{U} = \lambda U \qquad U(0) = U_0 \tag{3}$$

today. Recall from your linear algebra or dynamical systems courses that this scalar ODE already captures all the important features of the ODEs (1) and (2), since for a diagonalisable matrix / operator A, ... we can expand u in terms of the eigenfunctions v_i of A: $u = \sum_i a_i v_i \Rightarrow$

$\dot{u} = Au$

$\sum_i a_i \dot{v}_i = A(\sum_i a_i v_i) = \sum_i a_i A v_i = \sum_i \lambda_i a_i v_i \Rightarrow$

The system is fully determined by the eigenfunction-ODEs

$\dot{v}_i = \lambda_i v_i$

In the following we will always assume that our PDEs are equipped with homogeneous DIRICHLET boundary conditions in space.

Exercise 1 What values of λ are you interested in when you study the homogeneous heat equation

$$\partial_t u - \Delta u = 0?$$

How does $|U(t)|$ evolve in time for such values of λ?

Hint: If $-\Delta f = \mu f$ for $f \in H^1_0(\Omega)$, then μ ... ? Coercivity: $\int_\Omega (-\Delta f)(f)\,dx = \int_\Omega |\nabla f|^2 dx \geq c \int_\Omega |f|^2 dx \Rightarrow$

all eigenvalues of the negative Laplacian have $\mu \geq c > 0$.

The heat equation is of the form $\dot{u} = Au$ if we choose $A = \Delta$, which has all -ve eigenvalues $\lambda = -\mu \leq -c < 0$.

$\Rightarrow |U(t)| = |e^{\lambda t} U_0| = e^{\lambda t} |U_0| \longrightarrow 0$ for -ve $\lambda = -\mu$

Exercise 2 What values of λ are you interested in when you study the homogeneous wave equation

$$\partial_{tt}u - \Delta u = 0?$$

How does $|U(t)|$ evolve in time for such values of λ?

Hint: Like for second-order ODEs, introduce an auxiliary variable $v = \partial_t u$ and re-write the wave equation as a system that's first order in time. Also use the hint from Exercise 1.

$$\left.\begin{array}{l}\partial_t v - \Delta u = 0 \\ \partial_t u \quad\;\; = v\end{array}\right\} \iff \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & id \\ \Delta & 0 \end{pmatrix}}_{A}\begin{pmatrix} u \\ v \end{pmatrix}$$

Eigenvalue problem for A: $A\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} v \\ \Delta u \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow$$

$$\Delta u = \lambda v = \lambda^2 u \overset{\text{Ex 1}}{=} -\mu u \Rightarrow$$

$$\lambda = \pm\sqrt{\mu}\, i$$

$$|U(t)| = |e^{\lambda t} U_0| = |e^{\pm\sqrt{\mu}\, i t}||U_0| = |U_0|$$

In Exercises 1 and 2 you have looked at characteristic features of the heat equation and the wave equation. Now that you know how the numerical solutions of these equations *should* behave, we'll check if they actually *do* behave that way, or under what conditions.

Numerical Solution of the Test ODE

The test problem

$$\dot{U} = \lambda U \qquad U(0) = U_0$$

has the analytical solution

$$U(t) = e^{\lambda t} U_0$$

where $\lambda \in \mathbb{C}$.

All of the three Runge-Kutta methods presented on page 64 in the notes yield iterations of the form

$$Q(\lambda\Delta t)U_+ = P(\lambda\Delta t)U_\circ.$$

Here, P and Q are polynomials. Consequently, over one time step, each method approximates the exponential function from the analytical solution with a rational function $e^z \approx R(z) = \frac{P(z)}{Q(z)}$:

$$U_+ = R(\lambda\Delta t)U_\circ$$

and thus, for the k-th time step,

$$U_k = R(\lambda\Delta t)^k U_0.$$

Exercise 3 What are P, Q and R for the forward Euler scheme?

FE: $P(z) = 1 + z$ $\quad Q(z) = 1$ $\quad R(z) = 1 + z$

BE: $P(z) = 1$ $\quad Q(z) = 1 - z$ $\quad R(z) = \frac{1}{1-z}$

CN: $P(z) = 1 + \frac{1}{2}z$ $\quad Q(z) = 1 - \frac{1}{2}z$ $\quad R(z) = \frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$

Stability

Exercise 4 Independent of the chosen time-stepping scheme, why is the condition $|R(\lambda\Delta t)| \le 1$ equivalent to saying that the iterates $|U_k|$ remain bounded?

From the previous page: $U_k = R(\lambda\Delta t)^k U_0 \Rightarrow$

$|U_k| = |R(\lambda\Delta t)|^k |U_0|$ → diverges if $|R(\lambda\Delta t)| > 1$

→ $= |U_0|$ if $|R(\lambda\Delta t)| = 1$

→ $\to 0$ if $|R(\lambda\Delta t)| < 1$

Definition A RUNGE-KUTTA method is said to be *A-stable* if for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z \le 0$

$$|R(z)| \le 1$$

(so that the iterates of the homogeneous problem remain bounded, $\sup_k |U_k| < \infty$).

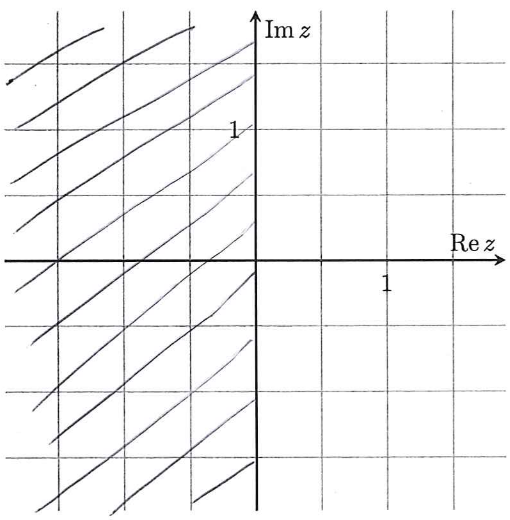
Exercise 5

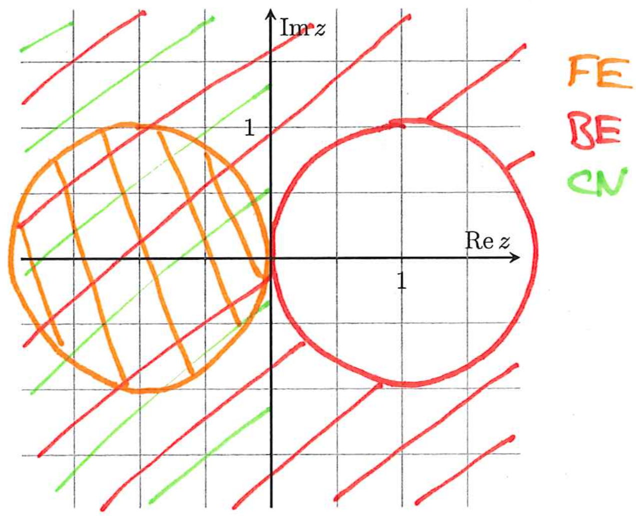
if $z = a + bi$: $|e^z| = |e^a| \underbrace{|e^{bi}|}_{=1} = |e^a| \le 1 \Leftrightarrow a \le 0$

(a) In the left coordinate system, sketch the region of all $z \in \mathbb{C}$ where $|e^z| \le 1$.

(b) In the right coordinate system, sketch the region of all $z \in \mathbb{C}$ where $|R(z)| \le 1$ (for the forward EULER method).

(c) Is the forward EULER method A-stable? If no, under what condition is $|R(\lambda\Delta t)| \le 1$?





↑ z's, for which the exact solution remains bounded

↑ z's, for which the numerical solution remains bounded

(b) FE: $|1+z| \overset{!}{\le} 1 \Leftrightarrow \sqrt{(1+\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2} \le 1$

BE: $\left|\frac{1}{1-z}\right| \overset{!}{\le} 1 \Leftrightarrow \dots \Leftrightarrow \sqrt{(\mathrm{Re}\,z - 1)^2 + (\mathrm{Im}\,z)^2} \ge 1$

CN: $\left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right| \overset{!}{\le} 1 \Leftrightarrow \dots \Leftrightarrow \mathrm{Re}\,z \le 0$

(c) FE: not A-stable. $|R(\lambda\Delta t)| = |1 + \lambda\Delta t| \le 1 \Leftrightarrow \dots \Leftrightarrow \Delta t \le 2\frac{-\mathrm{Re}\,\lambda}{|\lambda|^2}$

BE CN: A-stable

Exercise 6 If we add an inhomogeneity to our test problem

$$\dot{U} = \lambda U + F \qquad U(0) = U_0$$

then the aforementioned RUNGE-KUTTA methods define the iteration

$$U_k = R(\lambda\Delta t)^k U_0 + \frac{\Delta t}{Q(\lambda\Delta t)} \sum_{j=0}^{k-1} R(\lambda\Delta t)^j \bar{F}_{k-1-j}$$

where $\bar{F}_j$ is a convex combination of inhomogeneities evaluated for some times $t \in [j\Delta t, (j+1)\Delta t]$. (For the forward EULER method $\bar{F}_j = F(j\Delta t)$.) We assume that F is bounded.

Why do we now need the stricter condition $|R(\lambda\Delta t)| \leq 1 - c\Delta t$ with some constant $c > 0$ to ensure that the iterates $|U_k|$ still remain bounded even for the inhomogeneous problem?

Hint: Geometric series $\sum_{j=0}^{n} a^k = \frac{1-a^n}{1-a}$.

$$|U_k| \leq |R(\lambda\Delta t)|^k \, |U_0| + \frac{\Delta t}{|\text{const}|}\left(\sup_k |F_k|\right) \sum_{j=0}^{k-1} |R(\lambda\Delta t)|^j$$

$$= \underbrace{|R(\lambda\Delta t)|^k}_{\longrightarrow 0} \, |U_0| + \frac{\cancel{\Delta t}}{|\text{const}|}\left(\sup_k |F_k|\right) \underbrace{\frac{1-|R(\lambda\Delta t)|^{k-1}}{1-|R(\lambda\Delta t)|}}_{\leq \frac{1}{c\cancel{\Delta t}}}$$

NB: Even if we're solving the homogeneous problem, there's practically going to be a non-zero F: numerical errors since we can't solve the problem at time step k exactly. If we just have an explicit method where we only *evaluate* the right hand side, F is likely going to be small. If we have an implicit method for a linear PDE where we have to *solve a linear system* for U^k, F is likely going to be a bit bigger. If we have an implicit method for a nonlinear PDE where we have to *solve a nonlinear system* for U^k using e.g. NEWTON's method, F is likely going to be even bigger, depending on how strict our stopping criterion is.

Definition A RUNGE-KUTTA method is said to be *strictly A-stable* if there exists $c > 0$ such that for all $z \in \mathbb{C}$ with $\operatorname{Re} z \leq 0$

$$|R(z)| \leq 1 - c\Delta t$$

(so that the iterates remain bounded even in the presence of an inhomogeneity or numerical errors F, $\sup_k |U_k| \leq U_0 + C \sup_k |F_k| < \infty$).

Exercise 7 Have a look at your answer to Exercise 5 to check if the forward EULER method is strictly A-stable.