with

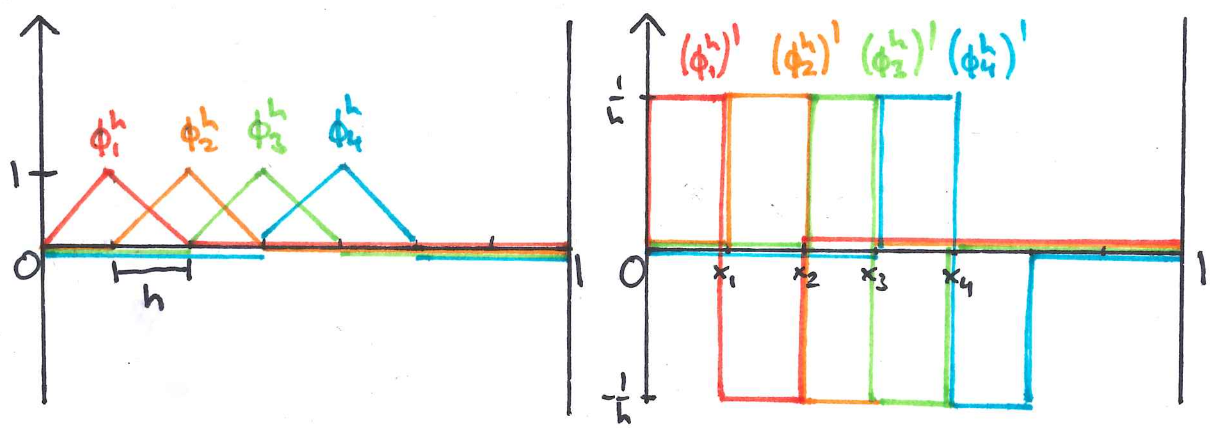
2.3.1 Example (Linear Finite Elements in 1D) In one dimension, the model problem reads: find $u \in H_0^1(]0,1[)$ such that for all $v \in H_0^1(]0,1[)$:

$$\int_0^1 u'v'\,\mathrm{d}x = \int_0^1 fv\,\mathrm{d}x.$$

We discretise this problem on the equidistant grid

$$0, h, 2h, 3h, \dots, (N-1)h, 1$$

with N subintervals and grid spacing $h = 1/N$.



Galerkin equations:

$$\sum_{j=1}^{N-1}\left(\int_0^1 (\phi_i^h)'(\phi_j^h)'\,dx\right)u_j^h = \int_0^1 f\phi_i\,dx \qquad \text{for all } i=1,\dots,N-1$$

Left hand side:

hat functions don't overlap unless they're immediate neighbours

$$\int_0^1 (\phi_i^h)'(\phi_j^h)'\,dx = \begin{cases} 0 & \text{if } |i-j|>1 \\ \int_0^1 [(\phi_i^h)']^2\,dx = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2}\,dx = \frac{2}{h} & \text{if } i=j \\ \int_0^1 (\phi_i^h)'(\phi_{i+1})'\,dx = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right)dx = -\frac{1}{h} & \text{if } j=i+1 \\ \int_0^1 (\phi_i^h)'(\phi_{i-1})'\,dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)dx = -\frac{1}{h} & \text{if } j=i-1 \end{cases}$$

Right hand side:

$$\int_0^1 f\phi_i\,dx = \int_{x_{i-1}}^{x_i} f\phi_i\,dx + \int_{x_i}^{x_{i+1}} f\phi_i\,dx$$

$$\approx \begin{cases} (x_i - x_{i-1})\, f\left(\frac{x_{i-1}+x_i}{2}\right)\phi_i\left(\frac{x_{i-1}+x_i}{2}\right) \\ \quad + (x_{i+1}-x_i)\, f\left(\frac{x_i+x_{i+1}}{2}\right)\phi_i\left(\frac{x_i+x_{i+1}}{2}\right) & \text{[midpoint rule]} \\ \frac{x_i - x_{i-1}}{2}\left(f(x_{i-1})\phi_i(x_{i-1}) + f(x_i)\phi_i(x_i)\right) \\ \quad + \frac{x_{i+1}-x_i}{2}\left(f(x_i)\phi_i(x_i) + f(x_{i+1})\phi_i(x_{i+1})\right) & \text{[trapezoidal rule]} \end{cases}$$

$$= \begin{cases} \frac{h}{2}\left(f\left(\frac{x_{i-1}+x_i}{2}\right) + f\left(\frac{x_i+x_{i+1}}{2}\right)\right) & \text{[midpoint rule]} \\ h\, f(x_i) & \text{[trapezoidal rule]} \end{cases}$$

NB: If the trapezoidal rule is used to evaluate the source term, then the discrete problem is equivalent to a finite difference approximation. Other quadrature formulae give a different right hand side in the discrete linear system.

Big linear system (using the trapezoidal rule):

$$\frac{1}{h}\begin{pmatrix} 2 \\ -1 & 2 & -1 \\ & -1 & 2 & -1 \\ & & & \ddots \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1^h \\ u_2^h \\ u_3^h \\ \vdots \\ u_{N-1}^h \end{pmatrix} = h\begin{pmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_{N-1}) \end{pmatrix}$$

where $x_i = ih \quad (i = 1, \dots, N-1)$.