Finite Elements for Poisson's Equation in 1D

Here are my model answers for today's little exercise.

1D linear FE.pdf (https://canvas.ubc.ca/courses/2337/files/698637/download?wrap=1) (https://canvas.ubc.ca/courses/2337/files/698637/download?wrap=1)

To go from a weak formulation of a linear elliptic PDE to a linear system of equations, finite element methods apply the following steps:

- start with the weak form of the exact problem

  $$\textbf{find } u \in V \textbf{ such that for all test functions } v \in V \qquad B(u,v) = \langle f, v\rangle$$

- replace the infinite-dimensional function space V (here: $H_0^1(]0,1[)$) with a finite-dimensional subspace V^h (here: the space of continuous functions which are linear on each subinterval of the grid)

  $$\textbf{find } u^h \in V^h \textbf{ such that for all test functions } v^h \in V^h \qquad B\left(u^h, v^h\right) = \left\langle f, v^h\right\rangle$$

- choose a basis for the subspace V^h (here: the hat functions $\phi_i^h, i = 1, \ldots, N-1$)

  $$\textbf{find coefficients } u_1^h, \ldots, u_{N-1}^h \in \mathbb{R} \textbf{ such that for all hat functions } \phi_i^h, i = 1, \ldots, N -$$

  $$B\left(\sum_{j=1}^{N-1} u_j^h \phi_j^h, \phi_i^h\right) = \left\langle f, \phi_i^h\right\rangle$$

- these are $N-1$ linear equations for the $N-1$ unknown nodal values $u_1^h, \ldots, u_{N-1}^h$

  $$\sum_{j=1}^{N-1} B\left(\phi_j^h, \phi_i^h\right) u_j^h = \left\langle f, \phi_i^h\right\rangle$$

  $$\sum_{j=1}^{N-1} k_{ij}^h u_j^h = f_i^h \qquad (\textbf{where } k_{ij}^h = B\left(\phi_j^h, \phi_i^h\right) \textbf{ and } f_i^h = \left\langle f, \phi_i^h\right\rangle)$$

  $$K^h \vec{u}^h = \vec{f}^h$$

  (and this is the big linear system with a matrix K^h and right hand side $\vec{f}^h$)

To compute the L^2-norm of the numerical solution u^h, we need another big matrix:

1D mass matrix.pdf (https://canvas.ubc.ca/courses/2337/files/709463/download?wrap=1) (https://canvas.ubc.ca/courses/2337/files/709463/download?wrap=1)