Homework Assignment 9

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **1 mark** | We consider the initial boundary value problem for the heat equation

$$\begin{aligned} \partial_t u(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

where u is a temperature field, u_0 an initial temperature distribution, the diffusion-like parameter $a > 0$ the heat conductivity of the material, f a source term e.g. due to thermal radiation and $T > 0$ a final time. The homogeneous Neumann boundary conditions mean that the domain Ω is perfectly insulated so that no thermal energy is radiated into the environment.

The θ-method is a class of Runge-Kutta schemes for integrating ODEs of the form

$$\dot{U} = F(t, U)$$

by using the iteration

$$U_+ = U_\circ + \Delta t \left(\theta F(t_+, U_+) + (1-\theta) F(t_\circ, U_\circ)\right).$$

The parameter $\theta \in [0,1]$ can be interpreted as the 'degree of implicitness', since $\theta = 0$ gives the forward Euler method, $\theta = \frac{1}{2}$ the Crank-Nicolson method (aka implicit trapezium rule in the ODE context) and $\theta = 1$ the backward Euler method.

For the discretisation in space, we apply linear finite elements. Both the time step size Δt and the spatial triangulation $\mathcal{T}^h$ are fixed.

Show that in this setting both the method of lines and Rothe's method lead to the same discrete problems

$$\left(M^h + \theta \Delta t a K^h\right) \vec{u}^h_+ = \left(M^h - (1-\theta)\Delta t a K^h\right) \vec{u}^h_\circ + \Delta t \left(\theta \vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ\right).$$

You don't have to include any details about the components of the discrete vectors and matrices. We all know what they are!

Method of Lines 1. Semi-discretise in space

$$\begin{aligned} M^h \partial_t \vec{u}^h + a K^h \vec{u}^h &= \vec{f}^h(t) \\ \vec{u}^h(0) &= \vec{u}^h_0 \end{aligned}$$

2. Fully discretise in time

$$M^h \frac{\vec{u}^h_+ - \vec{u}^h_\circ}{\Delta t} + a K^h \left(\theta \vec{u}^h_+ + (1-\theta)\vec{u}^h_\circ\right) = \theta \vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ$$

Rothe's Method 1. Semi-discretise in time

$$\begin{aligned} \frac{u_+ - u_\circ}{\Delta t} - a\Delta\left(\theta u_+ + (1-\theta)u_\circ\right) &= \theta f_+ + (1-\theta) f_\circ \\ \frac{\partial\left(\theta u_+ + (1-\theta)u_\circ\right)}{\partial n} &= 0 \end{aligned}$$

2. Fully discretise in space

$$M^h \frac{\vec{u}_+^h - \vec{u}_\circ^h}{\Delta t} + aK^h \left(\theta \vec{u}_+^h + (1-\theta)\vec{u}_\circ^h\right) = \theta \vec{f}_+^h + (1-\theta)\vec{f}_\circ^h$$

Question 2 | 4 marks

(a) The FEniCS script `hw9.py` implements the backward Euler method for Problem (H). Starting from room temperature ($u_0 \equiv 20$), the bottom left corner of a metal piece Ω with conductivity parameter $a = 0.1$ is held over a flame for one second, then the flame is extinguished. This is modelled by

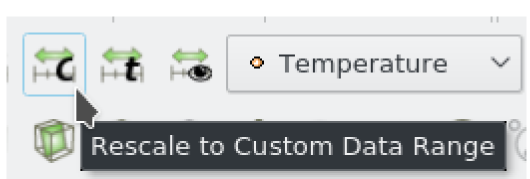
$$f(t,x) = \begin{cases} 200e^{-5x_1^2-2x_2^2} & t \leq 1 \\ 0 & t > 1 \end{cases}$$

Complete the missing commands to compute the evolution of the temperature field over the first five seconds using a time step size of $\Delta t = 10^{-2}$.

Save your results as a video, using a frame rate such that the video time is equal to the physical time. You don't have to submit any other files for this part of Question 2.

Hint: Open the PVD-file in ParaView, click the 'Apply'-button, select a reasonable colour map

and then re-scale the colour values



to the range $[20, 160]$. Use the same range for the following questions, too.

(b) Generalise this script to implement the θ-method from Question 1. Check whether setting $\theta = 1$ still gives you the same results. Using the same parameters as in Question 2(a), solve the problem with the CRANK-NICOLSON method and the forward EULER method. What do you observe?

The CRANK-NICOLSON and backward EULER solutions are hardly distinguishable by visual inspection.

The forward EULER solution contains exponentially increasing modes that also change sign with every time step. After 139 time steps, the diverging nodal values of the numerical solution have outgrown the range of floating-point numbers. FEniCS returns `nan`'s which result in error messages in ParaView.

(c) Solve the problem with the forward EULER method again up to time $T = 0.1$, once with $\Delta t = 1.25 \times 10^{-4}$ and once with $\Delta t = 10^{-4}$. Explain your observations, using the relevant terminology.

The forward EULER method is not A-stable, since its stability region is smaller than the left half of the complex plane. Iterates remain bounded if time steps are smaller than ch^2, where c is a positive constant and h the mesh size. Here we have $h \approx O(10^{-2})$ so the largest possible time step that results in a bounded solution is likely of the order $\Delta t \approx O(10^{-4})$. (Its exact value can be found by computing the largest eigenvalue of the matrix aK^h, cf Example 3.2.5.)

For $\Delta t = 1.25 \times 10^{-4}$ the numerical solution still contains an exponentially growing high-frequency mode. For the even smaller time step $\Delta t = 10^{-4}$ no more unbounded modes are detected, indicating that the largest possible time step lies in between these values.

Relevant terminology that should be used: A-stability, stability region

Irrelevant terminology that should not be used: numerical (anti-)dissipation (NB: the eigenvalues of this equation have no imaginary part since there is no advection term, so there are no wave-like solutions for which numerical dissipation would play a role.)

(d) If you could choose among the time-stepping schemes considered above and other time-discretisation schemes you know of, which one would you choose for this problem? First formulate some reasonable objectives for your simulation, then explain which method attains these to the greatest extent.

Sample Answer 1 The objective of this simulation is to compute the temperature distribution at steady-state, i.e. as $t \to \infty$.

There are no particular requirements for high-accuracy in time, since the evolution of the solution in time is not of interest. Also, since f is piecewise constant in time (and not, e.g. a sine-function, exponential function, quadratic function etc), any consistent time-stepping scheme will capture the total influx of thermal energy accurately.

However, to allow for large time steps towards the steady-state solution, strong A-stability or even L-stability is desirable.

Therefore, the backward EULER method is ideally suited for this problem. I would also choose my time steps such that one iterate is computed at $t = 1$ where $f(t)$ is discontinuous. Time steps can be enlarged the closer the solution gets to attaining a steady state.

Sample Answer 2 The objective of this simulation is to compute the evolution of the temperature distribution, also for other source terms and initial conditions.

The time-stepping scheme should be at least second-order accurate.

Also, since other source terms may not be turned off after a certain time of heating/cooling, the method should be strictly A-stable to guarantee a bounded solution.

Other initial conditions may also be composed of high-frequency eigenfunctions of the Laplacian. Then strong A-stability is required to capture their exponential decay correctly.

Under these restrictions, the TR-BDF2 or fractional-step θ-schemes are both good candidates. If computational resources impose serious limitations, then the TR-BDF2 method with its lower cost per time step would be preferred.

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?