Céa's Lemma and the Two Strang Lemmas

So far, we have analysed the **error of interpolation problems** (using the Bramble-Hilbert lemma).

However, we are actually interested in the **error of the Galerkin approximation** in an elliptic PDE problem.

Let's consider the case of linear finite elements.

If the exact solution $\bar{u}$ of a PDE were known, we could interpolate it with the piecewise linear function $I^h\bar{u}$ and the interpolation error would converge as follows:

$$\|I^h\bar{u}-\bar{u}\|_{L^2(\Omega)} \leq ch^2\|\nabla^2\bar{u}\|_{L^2(\Omega)}$$

$$\|\nabla(I^h\bar{u}-\bar{u})\|_{L^2(\Omega)} \leq ch\|\nabla^2\bar{u}\|_{L^2(\Omega)}$$

In reality, however, the interpolant of the exact solution, $I^h\bar{u}$ is just as unknown as the exact solution itself. Following the Galerkin approach, we actually compute a numerical solution u^h by solving the Galerkin equations

$$B(u^h,v^h)=\langle f,v^h\rangle \qquad \forall v^h\in V^h$$

and the numerical solution u^h is probably not the same as $I^h\bar{u}$.

Best Approximation

Recall the equivalence between Galerkin orthogonality

$$B(u^h-\bar{u},v^h)=0 \qquad \forall v^h\in V^h$$

and the best approximation property

$$\|u^h-\bar{u}\|_B=\inf_{v^h\in V^h}\|v^h-\bar{u}\|_B.$$

(For those of you who haven't come across infimums yet: in this context you may simply replace $\inf$ with $\min$.)

In plain English: if we use the natural energy norm that the PDE comes with to measure the error, then the finite-element solution u^h is the best possible approximation to the exact solution from the subspace V^h. Other functions from V^h, e.g. the interpolant $v^h=I^h\bar{u}$, are less accurate than u^h:

$$\|u^h-\bar{u}\|_B\leq\|I^h\bar{u}-\bar{u}\|_B.$$

Above we have seen that even the interpolant $I^h\bar{u}$ converges to $\bar{u}$, so in particular the even better finite-element solution u^h must converge to $\bar{u}$, too.

Quasi-Best Approximation

We may not always want to use the energy norm of the problem to measure the error, but maybe the H^1-norm. In scalar products other than the "energy product" (which may not even be a scalar product, since it could e.g. be non-symmetric), the error is usually not orthogonal to all the test functions v^h and u^h is usually not the best approximation to $\bar{u}$ as soon as we use a different norm.

Céa's lemma states that as long as B remains coercive and continuous with respect to that other norm, u^h is still almost as good as the best approximation, it might just be off by a multiplicative constant C that comes from the coercivity and continuity inequalities:

$\|u^h - \bar{u}\| \leq C \inf_{v^h \in V^h} \|v^h - \bar{u}\|$.

Strang's First Lemma

While Céa's lemma is already a generalisation of the best approximation theorem (for norms other than the energy norm), Strang's first lemma is another generalisation of Céa's lemma (where the finite-element solution comes from perturbed Galerkin equations, e.g. due to inexact integration).

The proof is still very similar and uses the same assumptions, it just contains (quite a few) more terms.

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The additional terms on the right hand side are the quadrature errors, e.g.

inexact stiffness matrix - exact stiffness matrix

$$\|\tilde{K}^h - K^h\|$$

or

inexact load vector - exact load vector

$$\|\tilde{f}^h - f^h\|.$$

Strang's Second Lemma

If you ever use a non-conforming approximation, e.g.

- a non-convex, non-polygonal domain where the discrete domain Ω^h is not fully contained in Ω
- shape and test functions that are not as smooth as required by the weak formulation, such as
  - piecewise constant functions (finite volume method) for a second-order PDE, which actually needs H^1

- the quadratic Morley element (not even continuous) for the plate equation, which actually needs H^2

then you need Strang's second lemma.

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