Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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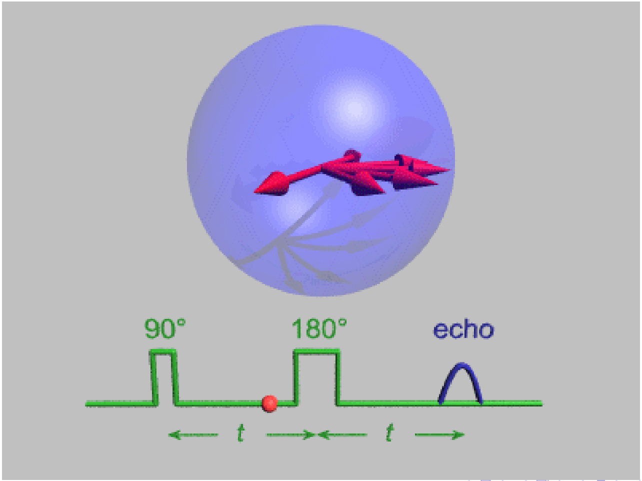
1. Background
   - Magnetic Resonance Visualised
   - The Bloch-Torrey Equation
   - Problem Geometry
2. Solving the Bloch-Torrey Equation
   - Operator Splitting Methods
   - Finite Element Methods
   - Time Stepping
3. Results
   - Operator Splitting vs. FEM

TODO: uncomment all "pause" statements

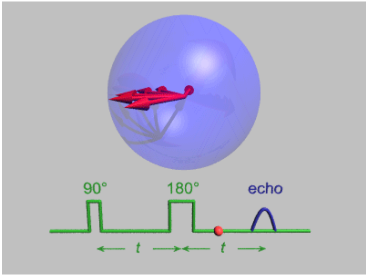
Magnetic Resonance Visualised

1. Animation of a typical “spin echo” MRI sequence
2. **TODO: re-compile with animated gif**



Magnetic Resonance Visualised

1. In actuality, spins (water molecules) do not truly fully refocus
2. Relative angular frequency depends on local magnetic field, and therefore spins dephase at different rates at different locations
3. In particular, the **diffusion** of spins during the scan ($\approx 40\,\mathrm{ms}$) leads to a net lost in signal: the "echo" is weaker



The Bloch-Torrey Equation

1. This system is modelled through the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

   where:

$$\mathcal{M} = M_x + iM_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. The initial transverse magnetization $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. Boundary conditions are typically zero Neumann or periodic

Magnetic Resonance Visualised

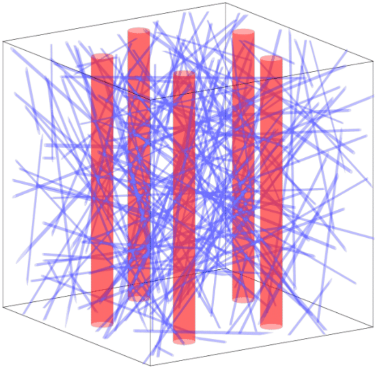


Figure: Cubic imaging voxel filled with randomly oriented microvasculature

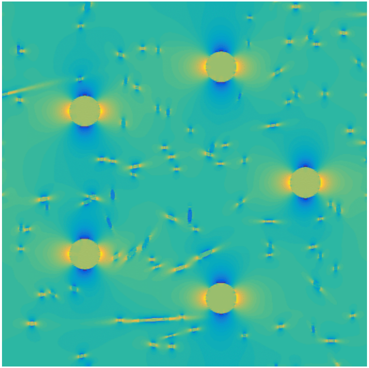


Figure: Cross section of ω corresponding to the microvasculature filled voxel

Operator Splitting Methods

1. One effective method of solving the BT equation is via *operator splitting methods*
2. First, the BT PDE is rewritten in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

where

$$H = -D\Delta + \Gamma.$$

3. Then, the general solution $\mathcal{M}$ may then be written as

$$\mathcal{M} = e^{-Ht}\mathcal{M}_0$$

where e^{-Ht} is the *evolution operator*

Operator Splitting Methods

1. Now, the evolution operator may be *split* using the approximation

$$
\begin{aligned}
e^{-Ht} &= e^{D\Delta t - \Gamma t} \\
&\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3)
\end{aligned}
$$

2. Although e^{-Ht} has no closed form, the split operators do:

$$
\begin{aligned}
e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\
e^{D\Delta t}\mathcal{M} &= \Phi * \mathcal{M}
\end{aligned}
$$

where $\odot$ is the Hadamard (pointwise) product, $*$ is the spatial convolution, and Φ is a Gaussian smoothing kernel with $\sigma = \sqrt{2Dt}$

1. The BT equation can also be solved using FEM
2. First, let $u = M_x$ and $v = M_y$ and rewrite the complex Bloch-Torrey PDE as a pair of coupled real PDE's:

$$\begin{cases} \frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v, & u(\mathbf{x}, 0) = M_x(\mathbf{x}, 0) \\ \frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u, & v(\mathbf{x}, 0) = M_y(\mathbf{x}, 0) \end{cases}$$

3. Writing the pair of PDE's as a linear system results in

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} D\Delta - R & \omega \\ -\omega & D\Delta - R \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

1. Applying the method of lines, the pair of PDE's becomes

$$M^h \mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v}$$
$$M^h \mathbf{v}_t = -(DK^h + R^h)\mathbf{v} - W^h \mathbf{u}$$

where $R^h_{ij} := \int R\,\phi_i\,\phi_j\,\mathrm{d}x$, $W^h_{ij} := \int \omega\,\phi_i\,\phi_j\,\mathrm{d}x$, and M^h and K^h are the usual mass and stiffness matrices

2. M^h, K^h, and R^h are symmetric positive definite; W^h is symmetric
3. In choosing a time discretisation, first consider the block system:

$$\begin{pmatrix} M^h & 0 \\ 0 & M^h \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = -\begin{pmatrix} A^h & -W^h \\ W^h & A^h \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

where $A^h := DK^h + R^h$ is symmetric positive definite

Time Stepping Methods

$$\begin{pmatrix} M^h & 0 \\ 0 & M^h \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = - \begin{pmatrix} A^h & -W^h \\ W^h & A^h \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

1. Solutions to the Bloch-Torrey equation decay exponentially in time at a rate roughly equal to $R(\mathbf{x})$
2. The time discretisation scheme should therefore be at least strongly A-stable to reflect this
3. For this reason, the strongly A-stable and second order accurate time stepping scheme $\mathrm{TR\text{-}BDF2}$ was used

Operator Splitting vs. FEM

1. Comparing the solution of the BT equation with splitting methods vs. FEM...
2. **TODO: do this**