Homework Assignment 7

Please submit the following files as indicated below: source code PDF file image file video file

Install FEniCS and ParaView on your computer. Please do this as soon as possible so that you have sufficient time for troubleshooting, if needed. For this assignment, only FEniCS is required, but if you want to visualise your numerical solutions, then you will need ParaView, too. Both FEniCS and ParaView are free/libre and open source software.

1. Visit `https://fenicsproject.org/download/`.
2. The Docker option is usually the most convenient choice (unless you're running Ubuntu). Follow the instructions to install Docker, then FEniCS.
3. (Optional) Install ParaView. This is already included in many Linux distributions. For other operating systems, visit `https://www.paraview.org/download/`.
4. If you run into any issues or if you don't have administrator privileges on your computer, please contact your department's IT support. I might be able to help if you're running Linux.

You won't have to use any complicated Docker commands. To run a FEniCS script called `ft01_python.py`

- open a terminal window and navigate to the folder where this script is located
- type `fenicsproject run` and wait for a few moments
- type `python ft01_python.py`
- to run the script again, call `python ft01_python.py` again
- once you're done, type `exit`

Note that any plotting commands in `ft01_python.py` will not work if you use the Docker option described here. Instead, you will have to write the data of your numerical solution to a file and open this with ParaView (but you don't have to plot anything in this assignment).

Question 1 | **2 marks** | Let $D > 0$, $a \in \mathbb{R}^2$, $r \geq 0$, $f \in L^2(\Omega)$ and $g \in H^{3/2}(\partial\Omega)$ (this means that g can be obtained as the restriction to $\partial\Omega$, aka trace, of a function $g \in H^2(\Omega)$), where $\Omega \subset \mathbb{R}^2$ is a convex, polygonal domain.

Derive a priori error estimates in the H^1-norm and the L^2-norm for the steady reaction-advection-diffusion problem

$$\begin{aligned} -D\Delta u + \operatorname{div}(au) + ru &= f \qquad \text{in } \Omega \\ u &= g \qquad \text{on } \partial\Omega \end{aligned}$$

discretised with conforming linear finite elements and exact integration.

Note that the bilinear form corresponding to this elliptic operator is not symmetric.

I'm going to derive these estimates in four steps:

1. Derive an equivalent PDE with homogeneous boundary conditions.
2. Check that the assumptions of Céa's lemma are satisfied if we use the H^1-norm.
3. Use Céa's lemma to derive the H^1-estimate.
4. Use the Aubin-Nitsche trick to derive the L^2-estimate.

The last two steps will also use the interpolation estimates from 2.3.15 and the bound on the H^2-norm of the solution of an elliptic PDE from 2.1.16.

Step 1: We define $w = u - g$ and after substituting $u = w + g$ in the PDE we obtain the problem

$$\begin{aligned} -D\Delta w + a\cdot\nabla w + rw &= f + D\Delta g - a\cdot\nabla g - rg && \text{in } \Omega \\ w &= 0 && \text{on } \partial\Omega \end{aligned}$$

(note that $\operatorname{div} a = 0$). Let's call this right hand side s. According to the given assumptions on f and g, s is an L^2-function.

The weak formulation reads: find $w \in H_0^1(\Omega)$ such that for all test functions $v \in H_0^1(\Omega)$

$$\begin{aligned} \int_\Omega D\nabla w\cdot\nabla v \,\mathrm{d}x + \int_\Omega (a\cdot\nabla w)v \,\mathrm{d}x + \int_\Omega rwv \,\mathrm{d}x &= \int_\Omega sv \,\mathrm{d}x \\ B(w,v) &= \int_\Omega sv \,\mathrm{d}x. \end{aligned}$$

Step 2:

- B is continuous with respect to the H^1-norm:

$$\begin{aligned} |B(w,v)| &\le \left|\int_\Omega D\nabla w\cdot\nabla v \,\mathrm{d}x\right| + \left|\int_\Omega (a\cdot\nabla w)v \,\mathrm{d}x\right| + \left|\int_\Omega rwv \,\mathrm{d}x\right| \\ &\le D\|\nabla w\|_{L^2}\|\nabla v\|_{L^2} + |a|_\infty\|\nabla w\|_{L^2}\|v\|_{L^2} + r\|w\|_{L^2}\|v\|_{L^2} \\ &\le C\|w\|_{H^1}\|v\|_{H^1} \qquad \text{for all } w, v \in H_0^1(\Omega) \end{aligned}$$

- B is coercive with respect to the H^1-norm: First observe that the advection term is antisymmetric, since integration by part gives

$$\int_\Omega (a\cdot\nabla w)v \,\mathrm{d}x = \int_\Omega (va)\cdot\nabla w \,\mathrm{d}x = -\int_\Omega \operatorname{div}(va)w \,\mathrm{d}x = -\int_\Omega (a\cdot\nabla v)w \,\mathrm{d}x.$$

  In particular,

$$\int_\Omega (a\cdot\nabla w)w \,\mathrm{d}x = 0.$$

  In Corollary 2.1.13 we have already shown that with no advection term

$$B(w,w) \ge c\|w\|_{H^1}^2 \qquad \text{for all } w \in H_0^1(\Omega).$$

Step 3:

We were allowed to assume that we can solve the Galerkin equations in a fully conforming fashion with no quadrature errors, i.e. both $\bar{u}$ and u^h come from the same affine space $g + H_0^1(\Omega)$:

$$\begin{aligned} \bar{w} &= \bar{u} - g \\ w^h &= u^h - g \end{aligned}$$

such that $w^h - \bar{w} = u^h - \bar{u}$. However, I'm going to look at the more general and more realistic case here, where we have to interpolate the boundary data g with a piecewise linear function $I^h g$. Then

$$\begin{aligned} \bar{w} &= \bar{u} - g \\ w^h &= u^h - I^h g \end{aligned}$$

and $w^h - \bar{w} = u^h - \bar{u} - (I^h g - g)$. Just drop the second terms below (except for the last two lines) to recover the special case that was already sufficient for this assignment question.

$$
\begin{aligned}
\|u^h - \bar u\|_{H^1} &\le \|w^h - \bar w\|_{H^1} + \|I^h g - g\|_{H^1} && (\Delta \text{ inequality})\\
&\le C\|I^h \bar w - \bar w\|_{H^1} + \|I^h g - g\|_{H^1} && (\text{C\'{E}A's lemma})\\
&\le C\sqrt{\|I^h \bar w - \bar w\|_{L^2}^2 + \|\nabla(I^h \bar w - \bar w)\|_{L^2}^2} + \sqrt{\|I^h g - g\|_{L^2}^2 + \|\nabla(I^h g - g)\|_{L^2}^2} && (\text{definition of } \|\cdot\|_{H^1})\\
&\le C\sqrt{(ch^2\,\|\nabla^2 \bar w\|_{L^2})^2 + (ch\,\|\nabla^2 \bar w\|_{L^2})^2} + \sqrt{(ch^2\,\|\nabla^2 g\|_{L^2})^2 + (ch\,\|\nabla^2 g\|_{L^2})^2} && (2.3.15)\\
&\le Ch\left(\left\|\nabla^2 \bar w\right\|_{L^2} + \left\|\nabla^2 g\right\|_{L^2}\right) && (\text{dominant terms})\\
&= Ch\left(\left\|\nabla^2 (\bar u - g)\right\|_{L^2} + \left\|\nabla^2 g\right\|_{L^2}\right) && (\text{backsubstitution})\\
&\le Ch\left(\left\|\nabla^2 \bar u\right\|_{L^2} + \left\|\nabla^2 g\right\|_{L^2}\right) && (\Delta \text{ inequality})\\
&\le Ch\left(\|f\|_{L^2} + \|g\|_{H^2}\right) && (\|\nabla^2\cdot\|_{L^2} \le \|\cdot\|_{H^2} \;\&\; 2.1.16)
\end{aligned}
$$

Step 4:

We consider the following dual problem: find $z \in H_0^1(\Omega)$ such that for all test functions $v \in H_0^1(\Omega)$

$$
B(v, z) = \int_\Omega \frac{e^h}{\|e^h\|_{L^2}} v \,\mathrm{d}x
$$

where $e^h = u^h - \bar u$.

FYI, since the diffusion and reaction terms are symmetric, while the advection term is antisymmetric, this dual problem is essentially the same steady reaction-advection-diffusion problem

$$
\begin{aligned}
-D\Delta z - \operatorname{div}(az) + rz &= \frac{e^h}{\|e^h\|_{L^2}} && \text{in } \Omega\\
z &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

except that the advective flux now points in the opposite direction.

$$
\begin{aligned}
\|u^h - \bar u\|_{L^2} &= \|e^h\|_{L^2}\\
&= B(e^h, z) && (\text{dual problem})\\
&= B(e^h, z - I^h z) && (\text{GALERKIN orthogonality})\\
&\le C\|e^h\|_{H^1}\|z - I^h z\|_{H^1} && (\text{continuity})\\
&\le C\Big(ch\left(\|f\|_{L^2} + \|g\|_{H^2}\right)\Big)\Big(ch\|\nabla^2 z\|_{L^2}\Big) && (H^1\text{-estimate \& 2.3.15})\\
&\le Ch^2\left(\|f\|_{L^2} + \|g\|_{H^2}\right) && (2.1.16)
\end{aligned}
$$

Question 2 | 3 marks | Work through the introductory FEniCS tutorial on Poisson's equation, available at `https://fenicsproject.org/pub/tutorial/html/._ftut1004.html`. Modify the code to test your a priori estimates for the H^1 and L^2-error from Question 1 (for nonzero a and r, please!). Your data:

$$\bar{u}(x_1, x_2) = 1 + x_1^2 + 2x_2^2 \qquad D = 10$$

$$f(x_1, x_2) = x_1^2 + 2x_2^2 + 2x_1 + 4x_2 - 59 \qquad a = (1, 1)^\top$$

$$g(x_1, x_2) = \bar{u}(x_1, x_2) \qquad r = 1$$

Solve the reaction-advection-diffusion problem for different grid spacings to complete the following table with data from your numerical experiments:

h	$\lVert u^h - \bar{u} \rVert_{H^1}$	$\lVert u^h - \bar{u} \rVert_{L^2}$
$\frac{1}{8}\sqrt{2}$ (8 × 8 grid)	0.1616	0.008210
$\frac{1}{16}\sqrt{2}$ (16 × 16 grid)	0.0807	0.002052
$\frac{1}{32}\sqrt{2}$ (32 × 32 grid)	0.0403	0.000513
$\frac{1}{64}\sqrt{2}$ (64 × 64 grid)	0.0202	0.000128

Do these data support your results from Question 1?

Absolutely: when the grid spacing h is halved, the H^1-error decays by a factor of approximately 2, while the L^2-error decays by a factor of approximately 4.

Hint: For advection-dominated problems, this discretisation scheme with linear finite elements behaves like a central-differencing scheme. Therefore, don't make the advection velocity too large.

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
