Homework Assignment 10

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **2 marks** | On the assignment page you can find videos of four animated solutions of the parabolic problem

$$\begin{aligned} \partial_t u(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

with the data from Assignment 9. However, the initial condition has been replaced with the function

$$u_0(x) = \begin{cases} 50 & \text{if } |x - (1,1)^\top| < 0.5 \\ 20 & \text{elsewhere} \end{cases}$$

Explain your observations.

Question 2 | **1 mark** | We have seen that the homogeneous wave equation

$$\begin{aligned} \partial_t^2 u - c^2 \Delta u &= 0 && \text{in } Q = \,]0,T[\, \times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \partial_t u(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \end{aligned} \tag{W}$$

with propagation speed $c > 0$ can equivalently be re-written as

$$\begin{aligned} \partial_t u - v &= 0 && \text{in } Q = \,]0,T[\, \times \Omega \\ \partial_t v - c^2 \Delta u &= 0 && \text{in } Q = \,]0,T[\, \times \Omega \\ u(0) &= u_0 && \text{in } \Omega \\ v(0) &= v_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \\ v &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega. \end{aligned} \tag{W'}$$

Discretising with the θ-method in time and linear finite elements in space leads to the coupled system for the vectors of nodal values $\vec{u}^h_+$ and $\vec{v}^h_+$

$$\begin{aligned} M^h \vec{u}^h_+ - \theta \Delta t M^h \vec{v}^h_+ &= M^h \vec{u}^h_\circ + (1-\theta)\Delta t M^h \vec{v}^h_\circ \\ \theta \Delta t c^2 K^h \vec{u}^h_+ + M^h \vec{v}^h_+ &= -(1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + M^h \vec{v}^h_\circ \end{aligned}$$

which has to be solved at every time step. Show that this is equivalent to the two smaller, successively solvable problems

$$\begin{aligned} \left(M^h + (\theta \Delta t c)^2 K^h\right) \vec{u}^h_+ &= M^h \left(\vec{u}^h_\circ + \Delta t \vec{v}^h_\circ\right) - \left(\theta\,(1-\theta)\,(\Delta t c)^2\right) K^h \vec{u}^h_\circ \\ M^h \vec{v}^h_+ &= M^h \vec{v}^h_\circ - \Delta t c^2 K^h \left(\theta \vec{u}^h_+ + (1-\theta)\,\vec{u}^h_\circ\right). \end{aligned}$$

Question 3 | **2 marks**

(a) Download and complete the FEniCS script `hw10.py` to solve Problem (W) with the data provided.

(b) Solve the wave equation

- with the (symplectic) implicit midpoint rule
- with the backward EULER method
- with the forward EULER method

and look at the solutions in ParaView.

Hint: Use the 'Warp by Scalar' filter, re-scale the colour map to the range $[-1, 1]$ and tick the box 'enable opacity mapping for surfaces' in the colour map editor.

For each of the three time stepping schemes, create a graph with curves of the total energy $E(u(t), v(t)) = T(v(t)) + V(u(t))$, the kinetic energy $T(v(t)) = \frac{1}{2}\|v(t)\|_{L^2}^2$ and the potential energy $V(u(t)) = \frac{c^2}{2}\|\nabla u(t)\|_{L^2}^2$ as functions of time. Please submit these plots and interpret the results:

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

Any new discoveries or achievements towards the objectives of your course project?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?