Numerical methods in magnetic resonance imaging: the Bloch-Torrey equation

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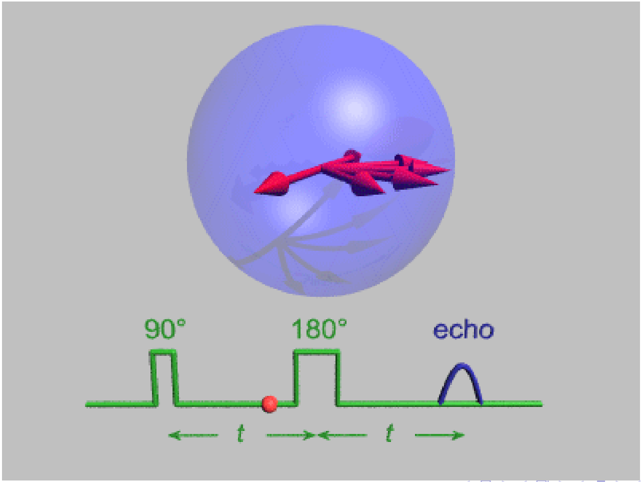
TODO: uncomment all “pause” statements

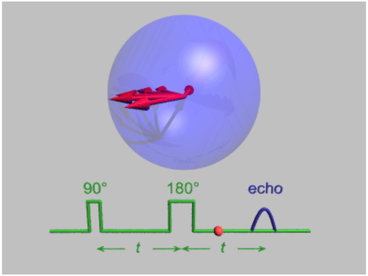
Magnetic Resonance Visualised

1. Animation of a typical “spin echo” MRI sequence
2. **TODO: re-compile with animated gif**



Magnetic Resonance Visualised

1. In actuality, spins (water molecules) do not truly fully refocus
2. Relative angular frequency depends on local magnetic field, and therefore spins dephase at different rates at different locations
3. In particular, the **diffusion** of spins during the scan ($\approx 40\,\text{ms}$) leads to a net lost in signal: the "echo" is weaker



The Bloch-Torrey Equation

1. We model this system through the **Bloch-Torrey equation**

$$\frac{\partial \mathcal{M}}{\partial t} = D\Delta\mathcal{M} - \Gamma\mathcal{M}$$

   where:

$$\mathcal{M} = M_x + iM_y$$
$$\Gamma(\mathbf{x}) = R(\mathbf{x}) + i\omega(\mathbf{x})$$

2. The initial magnetization $\mathcal{M}(\mathbf{x}, 0) = \mathcal{M}_0(\mathbf{x})$ is given
3. Boundary conditions are typically zero Neumann or periodic

Operator Splitting Methods

1. One effective method of solving the BT equation is via *operator splitting methods*
2. First, we re-write the BT PDE in the more suggestive form

$$\frac{\partial \mathcal{M}}{\partial t} = H\mathcal{M}$$

   where

$$H = -D\Delta + \Gamma.$$

3. Then, the general solution $\mathcal{M}$ may then be written as

$$\mathcal{M} = e^{-Ht}\mathcal{M}_0$$

   where e^{-Ht} is the *evolution operator*

1. Now, the evolution operator may be *split* using the approximation

$$
\begin{aligned}
e^{-Ht} &= e^{D\Delta t - \Gamma t} \\
&\approx e^{-\Gamma t/2} e^{D\Delta t} e^{-\Gamma t/2} + \mathcal{O}(t^3)
\end{aligned}
$$

2. Although e^{-Ht} has no closed form, the split operators do:

$$
\begin{aligned}
e^{-\Gamma t/2}\mathcal{M} &= e^{-\Gamma(\mathbf{x})t/2} \odot \mathcal{M} \\
e^{D\Delta t}\mathcal{M} &= \Phi * \mathcal{M}
\end{aligned}
$$

where $\odot$ is the Hadamard (pointwise) product, $*$ is the spatial convolution, and Φ is a Gaussian smoothing kernel with $\sigma = \sqrt{2Dt}$

Finite Element Methods

1. The BT equation can also be solved using FEM
2. First, we let $u = M_x$ and $v = M_y$ and re-write the complex Bloch-Torrey PDE as a pair of coupled real PDE's:

$$\frac{\partial u}{\partial t} = D\Delta u - Ru + \omega v$$
$$\frac{\partial v}{\partial t} = D\Delta v - Rv - \omega u$$

with

$$u(\mathbf{x}, 0) = M_x(\mathbf{x}, 0)$$
$$v(\mathbf{x}, 0) = M_y(\mathbf{x}, 0)$$

Finite Element Methods

1. Applying the method of lines, the pair of PDE's becomes

$$M^h \mathbf{u}_t = -(DK^h + R^h)\mathbf{u} + W^h \mathbf{v}$$
$$M^h \mathbf{v}_t = -(DK^h + R^h)\mathbf{v} - W^h \mathbf{u}$$

   where $R^h_{ij} := \int R\,\phi_i\,\phi_j\,\mathrm{d}x$, $W^h_{ij} := \int \omega\,\phi_i\,\phi_j\,\mathrm{d}x$, and M^h and K^h are the usual mass and stiffness matrices

2. M^h, K^h, and R^h are symmetric positive definite; W^h is symmetric
3. In choosing a time discretisation, first consider the block system:

$$\begin{pmatrix} M^h & 0 \\ 0 & M^h \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = -\begin{pmatrix} A^h & -W^h \\ W^h & A^h \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

   where $A^h := DK^h + R^h$

Finite Element Methods

$$\begin{pmatrix} M^h & 0 \\ 0 & M^h \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = - \begin{pmatrix} A^h & -W^h \\ W^h & A^h \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

1. If we take the inner product of both sides with $\left(\mathbf{u}^T, \mathbf{v}^T\right)$, we have that

$$\begin{aligned} \frac{\partial}{\partial t}\left(\mathbf{u}^T M^h \mathbf{u} + \mathbf{v}^T M^h \mathbf{v}\right) &= -\left(\mathbf{u}^T A^h \mathbf{u} + \mathbf{v}^T A^h \mathbf{v}\right) \quad \text{by symmetry of } W^h \\ &\leq 0 \quad \text{by positive definiteness of } A^h \end{aligned}$$

2. Now, $\|\mathcal{M}\|_{L^2}^2 = \mathbf{u}^T M^h \mathbf{u} + \mathbf{v}^T M^h \mathbf{v}$, and so $\|\mathcal{M}\|_{L^2}$ decreases with time
3. For this reason, the second order and strongly A-stable time stepping TR-BDF2 was used