Topology
Homework 5

1. Let X be a separable space and let $\mathcal{U}$ be a family of pairwise disjoint open sets in X. Show that $\mathcal{U}$ cannot be uncountable.

2. Let X be a topological space and let $\mathcal{B}$ be a base in X. Show that X is Lindelöf iff for any $\mathcal{U} \subseteq \mathcal{B}$ with $\bigcup \mathcal{U} = X$, there is a countable subfamily $\mathcal{U}'$ with $\bigcup \mathcal{U}' = X$.

3. Prove that any subspace X of the Sorgenfrey line is Lindelöf. *Hint:* Take an open family $\mathcal{U}$ with $X \subseteq \mathcal{U}$, i.e. $\mathcal{U}$ is an open cover of X. By the previous problem, you can assume that any element of $\mathcal{U}$ is of the form $[a, b)$. Consider the family $\mathcal{U}' = \{(a, b) : [a, b) \in \mathcal{U}\}$ and show that $X \setminus \bigcup \mathcal{U}'$ is at most countable. Then notice that $\mathcal{U}'$ consists of open sets in the Euclidean topology on $\mathbb{R}$ and hence there is a countable subfamily $\mathcal{V} \subseteq \mathcal{U}'$ with $\bigcup \mathcal{V} = \bigcup \mathcal{U}'$.

4. Let (X, ρ) be a separable metric space, where $\rho(x, y) \leq 1$ (there is no loss of generality making this assumption). Let $\{d_0, d_1, d_2 \ldots\}$ be a countable dense subset of X. We define a function $h : X \to [0, 1]^{\mathbb{N}}$ by $h(x) = (\rho(x, d_0), \rho(x, d_1), \rho(x, d_2), \ldots)$. Show that h is a homeomorphic embedding. *Hint:* To prove the continuity of h^{-1} it may be convenient to use convergent sequences (this is allowed because we work with metric spaces).

5. Prove that if X is a Tychonoff space and there is a base $\mathcal{B}$ of cardinality κ in X, then X embeds into the cube $[0, 1]^\kappa$ (i.e. the product of κ-many intervals). *Hint:* Consider the family of pairs $\mathcal{U} = \{(U, V) \in \mathcal{B} \times \mathcal{B} : \ \exists\, f_{U,V} : X \to [0, 1]$ continuous such that $f(U) \subseteq \{0\}$ and $f(X \setminus V) \subseteq \{1\}\}$. Then use a similar reasoning as in the proof of Urysohn's metrization theorem.

6. Recall that a family $\mathcal{F}$ of subsets of X is *centered* if for any finite subcollection $F_1, \ldots, F_n \in \mathcal{F}$ we have $\bigcap_{i=1}^{n} F_i \neq \emptyset$. Show that the following are equivalent:
   (i) X is compact
   (ii) If $\mathcal{F}$ is a centered family of closed subsets of X then $\bigcap \{F : F \in \mathcal{F}\} \neq \emptyset$
   (iii) If $\mathcal{C}$ is a centered family of subsets of X then $\bigcap \{\overline{C} : C \in \mathcal{C}\} \neq \emptyset$

7. Let τ be topology on X such that the space (X, τ) is compact T_2. Prove that if $\tau' \supsetneq \tau$ (i.e. τ' is essentialy finer topology than τ) then (X, τ') is not compact and if $\tau'' \subsetneq \tau$ (i.e. τ' is essentialy weaker topology than τ) then τ'' is not T_2. *Hint:* Consider the identity map $X \to X$.

8. Which space Z_0, Z_1, Z_2, Z_3 from exercise 1, problem set no. 4, is compact. Why/Why not?

9. Let K be a compact Hausdorff space. Show that the projection $p_1 : X \times K \to X$ is closed for any topological space X. *Hint:* Let $F \subseteq X \times K$ be compact and $x_0 \notin p_1(F)$. Observe that the set $\{x_0\} \times K$ is compact and does not intersect F.