Topology
Homework 8

1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by the formula $f(x) = \ln(1 + e^x)$ satisfies $|f(x) - f(y)| < |x - y|$ for $x \neq y$ but it does not have a fixed point.

2. Let (X, d) be a complete metric space and let $f : X \to X$ be an expansive map, i.e. for some $e > 1$ we have $d(f(x), f(y)) \geq e \cdot d(x, y)$. Prove that if $f(X) \subseteq X$ then f has exactly one fixed point.

3. Show that any countable complete metric space has an isolated point.

4. Show that Baire category theorem holds for arbitrary compact Hausdorff space X, i.e. if X is compact Hausdorff then the countable union of closed nowhere dense subsets of X, is co-dense in X.

5. Show that the set of irrational numbers is not a countable union of sets that are closed in $\mathbb{R}$.

6. We say that a topological space is *completely metrizable* if there is a metric d on X which generates the topology of X and such that the metric space (X, d) is complete. Prove that:
   (A) The space $\mathbb{Q}$ of rational numbers with the subspace topology inherited from the real line is not completely metrizable.
   (B) If X is completely metrizable and F is a closed subset of X, then $F \subseteq X$ endowed with the subspace topology is completely metrizable.
   (C) Which of the following subspaces of the Euclidean plane $\mathbb{R}^2$ are completely metrizable?
      (a) $X_1 = \mathbb{Q} \times \mathbb{R}$
      (b) $X_2 = \mathbb{Q} \times (\mathbb{R} \setminus \mathbb{Q})$
      (c) $X_3 = \bigcup_{n=1}^{\infty} (\{\frac{1}{n}\} \times [0, \frac{1}{n}])$
      (d) $X_4 = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$
      (e) $X_5 = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$
      (f) $X_6 = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} \in \mathbb{Q}\}$

7. Let $A \subseteq \mathbb{R}^n$ be a countable subset of the Euclidean space $\mathbb{R}^n$. Let $F_k \subseteq \mathbb{R}$ be a closed nowhere dense subset of the real line, for $k = 1, 2, \ldots$. Show that there is a point $c \in \mathbb{R}^n$ such that $d_e(c, a) \notin \bigcup_{k=1}^{\infty} F_k$, for every $a \in A$ (here d_e is a Euclidean distance in $\mathbb{R}^n$).

8. Let $F \subseteq \mathbb{R}$ be a closed nowhere dense subset of $\mathbb{R}$. Show that there is a point (a, b) on the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ such that for all $q \in \mathbb{Q}$ and for all $c \in F$ we have $b \neq qa + c$.

9. Let A be a G_δ subset of a topological space X. Show if B is G_δ in A then B is G_δ in X.

10. Let X be a metrizable space. Prove that any closed subset of X is G_δ.

11. Show that a totally bounded metric space (X, d) is separable. In particular any compact metric space is separable.