Topology
Homework 4

1. Let $Z_0 = \mathbb{N}$, $\quad Z_1 = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \ldots\}$, $\quad Z_2 = Z_1 \cup \mathbb{N}$,
$Z_3 = \{0\} \cup \{\frac{1}{n} + \frac{1}{k} : n, k = 2, 3, \ldots, \ \frac{1}{k} < \frac{1}{n-1} - \frac{1}{n}\}$. Show that the spaces Z_0, Z_1, Z_2, Z_3 are pairwise nonhomeomorphic. *Hint:* Observe that a homeomorphism sends isolated points to isolated points.

2. Prove that:
   (a) X is normal iff for any two open sets U, V satisfying $U \cup V = X$, there are closed sets $F \subseteq U$ and $H \subseteq V$ such that $F \cup H = X$.
   (b) If X is normal and $f : X \to Y$ is a continuous closed surjection, then Y is normal. *Hint:* To prove normality of Y use (a), i.e. take arbitrary two open sets U, V as in (a); to find F and H, consider the cover $\{f^{-1}(U), f^{-1}(V)\}$ of X and apply the theorem on point finite open covers.

3. Prove the following Cantor-Bendixon theorem:
Any second-countable space X is a union $A \cup B$ of a closed dense in itself set A (such sets are called perfect) and a countable discrete set B.

4. Show that a second-countable space is Lindelöf. *Hint:* Fix a countable base $\mathcal{B}$ and an open cover $\mathcal{U}$. For $B \in \mathcal{B}$ take $U(B) \in \mathcal{U}$ such that $B \subseteq U(B)$, if such set exist. Otherwise set $U(B) = \emptyset$. Show that $\{U(B) : B \in \mathcal{B}\}$ covers the space.

5. Show that a closed subspace of a Lindelöf space is Lindelöf. Conclude that Niemytzki plane L and the square of Sorgenfrey line $S \times S$ are not Lindelöf.