Topology
Homework 6

1. Denote by $I(a,b)$ the closed interval in the Euclidean plane $\mathbb{R}^2$ joining points a and b. Which of the subsets of the Euclidean plane $\mathbb{R}^2$ given below are compact?
   (a) $X = \bigcup_{n=1}^{\infty} I(a_0, a_n) \cup \bigcup_{n=1}^{\infty} I(b_0, b_n)$, where $a_0 = (0,1), a_n = (\frac{1}{n}, 0), b_0 = (0,-1), b_n = (-\frac{1}{n}, 0)$.
   (b) $Y = X \cup (\{0\} \times \mathbb{R})$
   (c) $Z = X \cup (\{0\} \times [-1,1])$

2. For $A \subseteq (0, +\infty)$, let $X(A)$ be the union of closed intervals in the Euclidean plane $\mathbb{R}^2$ joining the point $(-1,0)$ with points $(a, \frac{1}{a})$, $a \in A$. Show that $X(A)$ is closed iff A is compact iff $X(A)$ is compact.

3. Let (X,d) be a compact metric space. Prove that if a continuous map $f : X \to X$ preserves distance between points (i.e. $\forall x, y \in X \;\; d(x,y) = d(f(x), f(y))$) then f is onto. *Hint:* Assume there is $a \notin f(X)$ and consider the sequence $a, f(a), f(f(a)), \ldots$

4. Let (X,d) be a metric space. Define the distance between disjoint sets $A, B \subseteq X$ by $dist(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$
   (a) Show that if A is closed in X and B is compact and does not intersect A then $dist(A,B) > 0$.
   (b) Find two disjoint closed subsets A, B of the Euclidean plane $\mathbb{R}^2$ with $dist(A,B) = 0$.

5. Let X be a Hausdorff space. Suppose A and B are compact disjoint subsets of X. Show that there are disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

6. Prove that a compact Hausdorff space is normal.

7. Let (X,d), (Y,ρ) be metric spaces. A map $f : X \to Y$ is *uniformly continuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $x, y \in X$ if $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.
   Prove that if X is compact, then any continuous map $f : X \to Y$ is uniformly continuous. *Hint:* Lebesgue number may help.