Topology
Homework 9

1. Prove that the following conditions are equivalent for a topological space X:
   (i) X is connected.
   (ii) The only subsets of X that are closed and open at the same time (such sets are called clopen) are $\emptyset$ and X.
   (iii) X does not map surjectively onto two point discrete space $\{0, 1\}$.

2. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has *Darboux property* if for any pair of real numbers $s < t$ and any r lying between $f(s)$ and $f(t)$, there is $u \in [s, t]$ with $f(u) = r$.
   Show that if the graph $\{(x, f(x)) : x \in \mathbb{R}\}$ of a function $f : \mathbb{R} \to \mathbb{R}$ is a connected subset of the plane $\mathbb{R}^2$, then f has Darboux property.

3. Prove that a continuous function $f : [0, 1] \to [0, 1]$ has a fixed point, i.e. there is $x \in [0, 1]$ such that $f(x) = x$.

4. Show that every connected normal space that has more than one point has cardinality $\geq |\mathbb{R}|$. *Hint:* Use Urysohn's lemma.

5. Is the space obtaied by removing countably many points from the Euclidean plane $\mathbb{R}^2$ arcwise connected?

6. Consider the following subspace of the Euclidean plane: $A = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{R} \setminus \mathbb{Q}\}$. Is A connected?

7. Which of the following subspaces of the Eclidean plane $\mathbb{R}^2$ are/are not homeomorphic?
   (a) $X_1 = \{(x, y) : y = \sin \frac{1}{x} : x \neq 0\}$
   (b) $X_2 = X_1 \cup \{(0, y) : y \in [-1, 1]\}$
   (c) $X_3 = \{(x, y) : y = \sin \frac{1}{x} : x > 0\} \cup \{(0, y) : y \in [-1, 1]\}$
   (d) $X_4 = \{(x, y) : y = \sin \frac{1}{x} : x < 0\}$
   (e) $X_5 = X_4 \cup \{(0, -1), (0, 1)\}$
   (f) $X_6 = X_4 \cup \{(0, 0)\}$
   (g) $X_7 =$ The union of all lines through $(0, 0)$ with a rational slope
   (h) $X_8 =$ The union of all lines through $(0, 0)$ with a irrational slope

8. Let C be the Cantor set. We know that there is a continuous map from C onto $[0, 1]$. Is there a continuous map from $[0, 1]$ onto C?

9. Prove that every zero-dimensional space is Tychonoff.

10. Show that a Cartesian product $\prod_{t \in T} X_t$ of zero-dimensional spaces, is zero-dimensional.

11. Prove that if X is second-countable then every base $\mathcal{B}$ for X ($\mathcal{B}$ may not be countable) contains a countable family $\mathcal{B}_0 \subseteq \mathcal{B}$ which is a base for X. Conclude that if X is a second-countable, zero-dimensional space, then there is a countable base for X consisting of clopen sets.

12. Let X be a second countable zero-dimensional space. Prove that X can be embedded into the Cantor set $\{0, 1\}^{\mathbb{N}}$. *Hint:* Use the previous problem.