Topology
Homework 2

1. Prove that for any open interval $(a,b) \subseteq \mathbb{R}$ there is a rational number q and an irrational number t with $q,t \in (a,b)$ (i.e. the sets $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$).

2. Let (X,d) be a metric space. Suppose that $A \subseteq X$ is dense in (X,d). Show that the family $\left\{B(a,\frac{1}{n}) : a \in A, n = 1,2,\ldots\right\}$ is a base for $\tau(d)$.

3. A subset of a topological space is called *clopen* if it is both open and closed. Let X be a topological space. Show that the sets $\emptyset$ and X are clopen. Prove that a set A is clopen if and only if $\mathrm{bd}(A) = \emptyset$

4. Recall that $w(X) = \min\{\kappa \geq \aleph_0 : X \text{ has a base of cardinality } \kappa\}$ and $d(X) = \min\{\kappa \geq \aleph_0 : X \text{ has a dense subset of cardinality } \kappa\}$.
   Show that for any topological space X we have $d(X) \leq w(X)$. Prove that if (X,ρ) is a metric space then $d(X) = w(X)$.

5. Let $A \subseteq X$ be a dense subset of X. Show that for any open set $U \subseteq X$ we have $\overline{U} = \overline{U \cap A}$.

6. Let $f : X \to Y$ be a map between topological spaces. Show that the following conditions are equivalent:
   (i) f is continuous
   (ii) There is a base $\mathcal{B}$ in Y such that $f^{-1}(U)$ is open in X for every $U \in \mathcal{B}$
   (iii) There is a subbase $\mathcal{S}$ in Y such that $f^{-1}(U)$ is open in X for every $U \in \mathcal{S}$
   (iv) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any $B \subseteq Y$
   (v) $f^{-1}(\mathrm{Int}_Y(B)) \subseteq \mathrm{Int}_X(f^{-1}(B))$ for any $B \subseteq Y$

7. Let X be a topological space. Given two mappings $f,g : X \to \mathbb{R}$ and $\lambda \in \mathbb{R}$, show that:
   (i) If f and g are continuous then $f+g$ and $f \cdot g$ are continuous
   (ii) If f and g are continuous and $g(x) \neq 0$ for any $x \in X$ then $\frac{f}{g}$ is continuous
   (iii) If f is continuous then λf is continuous
   (iv) If f and g are continuous then $\max(f,g)$ and $\min(f,g)$ are continuous

8. Suppose that X is a topological space and $f_n : X \to \mathbb{R}$ is continuous for any $n \in \mathbb{N}$. Prove that if $f_n \rightrightarrows f$ (i.e. the sequence (f_n) uniformly converges to f) then f is continuous.

9. Let $f : X \to Y$ be a continuous surjection (X and Y are topological spaces). Show that $d(Y) \leq d(X)$.

10. Show that each of the following expresses a topological property (i.e. a property preserved by homeomorphisms) of X:
    (a) X has cardinality κ
    (b) The topology of X has cardinality κ
    (c) $w(X) = \kappa$
    (d) $d(X) = \kappa$

11. Suppose that $f : X \to Y$ is a continuous. If $x \in A^d$, where A is a subset of X, is it necessarily true that $f(x) \in f(A)^d$?

12. Show that for functions $f : \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity is equivalent to the open set definition.

13. Let τ and τ' be two topologies on a set X. Let $id : (X,\tau) \to (X,\tau')$ be the identity function.
    (a) Show that id is continuous iff τ is finer than τ' (a topology τ is finer than τ' is it has more open sets, i.e. if $\tau \supseteq \tau'$)
    (b) Show that id is a homeomorphism iff $\tau = \tau'$

14. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.