Topology
Homework 3

1. Let (X, d) be a metric space. For a nonempty subset $A \subseteq X$ we define a function $d_A : X \to \mathbb{R}$ by the formula
$$d_A(x) = \inf\{d(x, z) : z \in A\},$$
i.e. the function d_A measures distance from a point $x \in X$ to the set A.
Show that $|d_A(x) - d_A(y)| \leq d(x, y)$. Conclude that d_A is continuous.

2. Let X and Y be topological spaces and let $A \subseteq X$, $B \subseteq Y$. Prove that:
   (a) $\mathrm{Int}(A \times B) = \mathrm{Int}(A) \times \mathrm{Int}(B)$. Show (give an example) that analogous equality fails for infinite products.
   (b) $\mathrm{bd}(A \times B) = (\overline{A} \times \mathrm{bd}(B)) \cup (\mathrm{bd}(A) \times \overline{B})$

3. Let X_α be a (nonempty) topological space and $A_\alpha \subseteq X_\alpha$ for each $\alpha \in A$. Show that:
   (a) $\overline{\prod_{\alpha \in A} A_\alpha} = \prod_{\alpha \in A} \overline{A_\alpha}$
   (b) $\prod_{\alpha \in A} A_\alpha$ is dense in $\prod_{\alpha \in A} X_\alpha$ iff A_α is dense in X_α for each $\alpha \in A$.

4. Let X be a topological space. Prove that if two sets of indexes I and J have the same cardinality, then the products X^I and X^J are homeomorphic ($X^A = \prod_{\alpha \in A} X_\alpha$, where $X_\alpha = X$ for every $\alpha \in A$).

5. Give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ which is not closed. Similarly, give an example of a continuous map $f : \mathbb{R} \to \mathbb{R}$ which is not open.

6. We say that a continuous map $f : X \to Y$ is $\mathbb{R}$-*quotient* if for every $g : Y \to \mathbb{R}$, the map g is continuous if $g \circ f$ is continuous. Prove that every quotient map is $\mathbb{R}$-quotient.

7. Show that the projection $p_1 : \mathbb{R}^2 \to \mathbb{R}$ onto the first factor is not closed.

8. Prove that for any pair f, g of continuous mappings of a space X into a Hausdorff space Y, the set $\{x \in X : f(x) = g(x)\}$ is closed in X. Conclude that if f and g agree on a dense subset of X then $f = g$.

9. Let $f : X \to Y$ be a continuous map between topological spaces. Let $G(f) = \{(x, f(x) : x \in X)\} \subseteq X \times Y$ be the graph of f (considered as a subspace of the product $X \times Y$). Prove that:
   (a) Spaces X and $G(f)$ are homeomorphic.
   (b) If Y is Hausdorff then $G(f)$ is a closed subset of $X \times Y$.

10. Let X be an infinite set and let $\tau = \{U \subseteq X : X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Show that the family τ is a topology on X. Prove that (X, τ) is T_1 but not T_2.

11. Prove that X is Hausdorff iff the diagonal $\Delta = \{(x, x) \in X \times X : x \in X\}$ is closed in $X \times X$.

12. Show that the Sorgenfrey line S is normal but its square $S \times S$ is not normal.
Hint: To prove that S is normal, for any two disjoint closed sets A, B consider open sets $U(A) = \bigcup_{x \in A} [x, x + \delta(x))$, where $\delta(x) > 0$ and $[x, x + \delta(x)) \cap B = \emptyset$, the set $U(B)$ is defined analogously. To prove that $S \times S$ is not normal apply Jones' lemma.