Topology
Homework 1

1. Are the following functions metrics on given sets?
   (a) $\rho(x,y) = \min\{1, d(x,y)\}$, where $x, y \in (X,d)$ and (X,d) is a metric space.
   (b) $\rho_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$, where $x, y \in (X,d)$ and (X,d) is a metric space.
   (c) $\rho_2(n,m) = |\frac{1}{n} - \frac{1}{m}|$, where $n, m \in \mathbb{N}$

2. Consider the set $C([0,1]) = \{f : [0,1] \to \mathbb{R} : f \text{ is continuous}\}$ endowed with the supremum metric:
$$d(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$
   Which of the sets below are open in that space? Why/Why not?
   - $A = \{f \in C([0,1]) : f(x) > 0 \text{ for } x \in [0,1]\}$
   - $B = \{f \in C([0,1]) : \exists x \in [0,1] \;\; f(x) = 0\}$
   - $C = \{f \in C([0,1]) : \int_0^1 |f(x)|dx < 1\}$
   - $D = \{f \in C([0,1]) : f \text{ is strictly increasing}\}$

3. Consider the following metric on $C([0,1])$:
$$\rho(f,g) = \int_0^1 |f(x) - g(x)|dx.$$
   Which sets A, B, C, D from the previous exercise are open in the metric space $(C([0,1]), \rho)$?

4. Let $\mathbb{N}^{\mathbb{N}}$ be the set of all sequences of positive integers (we assume here that $0 \notin \mathbb{N}$). For $a = (n_1, n_2, \ldots)$ and $b = (m_1, m_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$ define
$$d(a,b) = \begin{cases} \dfrac{1}{\min\{i : n_i \neq m_i\}} & \text{if } \;\; a \neq b \\ 0 & \text{if } \;\; a = b \end{cases}$$
   (a) Prove that d is a metric on $\mathbb{N}^{\mathbb{N}}$ and $d(a,b) \leq \max\{d(a,c), d(b,c)\}$ for any $a, b, c \in \mathbb{N}^{\mathbb{N}}$.
   (b) Show that any two balls in the space $(\mathbb{N}^{\mathbb{N}}, d)$ are either disjoint or one of them is contained in the other.
   (c) Which of the following sets are open in $(\mathbb{N}^{\mathbb{N}}, d)$:
   - $A = \{(n_1, n_2, \ldots) : n_i = 1 \text{ for at least three indexes } i\}$
   - $B = \{(n_1, n_2, \ldots) : n_i = 1 \text{ for infinitely many } i\text{'s}\}$

   (d) Let $<$ be the lexicographical order on $\mathbb{N}^{\mathbb{N}}$ (i.e. $(n_1, n_2, \ldots) < (m_1, m_2, \ldots)$ if for some i, $n_i < m_i$ and $n_j = m_j$ for $j < i$). For $a < b$ set $(a,b) = \{x : a < x < b\}$. Show that intervals of the form (a,b) are open in $(\mathbb{N}^{\mathbb{N}}, d)$.

5. Show that the interval $[0,1]$ is not a union of intervals of the form (a,b). Show that $(0,1)$ is not an intersection of intervals of the form $[a,b]$.

6. Prove that $\overline{A} = \bigcap\{K \subseteq X : K \text{ is closed in } X\}$, i.e. the closure $\overline{A}$ of a set A in a space X is the minimal closed set in X containing A.

7. Prove that $\mathrm{Int}(A) = \bigcup\{U \subseteq A : U \text{ is open in } X\ \}$, i.e. $\mathrm{Int}(A)$ is the maximal open subset of A.

8. Show that the closure operator has the following properties:
$$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}, \qquad \overline{A} \setminus \overline{B} \subseteq \overline{A \setminus B}.$$
   Give examples showing that the opposite inclusions do not hold.

9. Prove that for a topological space X we have:
   (i) $\mathrm{Int}(X) = X$
   (ii) $\mathrm{Int}(A) \subseteq A$
   (iii) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$
   (iv) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$

10. Find the closure, interior and boundary of the following subsets of the real line:
$$\emptyset, \quad \mathbb{R}, \quad \mathbb{N}, \quad \mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{N}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad [3,8), \quad (0,\infty)$$