Topology
Homework 7

1. Define a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Similarly, define a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$.

2. Prove that in the unit interval $[0,1]$ there are continuum many pairwise disjoint Cantor sets (i.e. there is a pairwise disjoint family $\mathcal{C}$ of subsets of $[0,1]$ consisting of sets homeomorphic to the Cantor set C and such that $|\mathcal{C}| = |\mathbb{R}|$).

3. Show that no point in the Cantor middle-third set C is isolated.

4. Check that the metric σ on the Cantor set $\{0,1\}^{\mathbb{N}}$, defined by letting
$$\sigma(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{10^n}, \quad \text{where } x = (x_n)_{n\in\mathbb{N}}, \quad y = (y_n)_{n\in\mathbb{N}}$$
induces the product topology on $\{0,1\}^{\mathbb{N}}$.

5. Prove that for every non-empty closed subset A of $\{0,1\}^{\mathbb{N}}$, there is a continuous surjection $r : \{0,1\}^{\mathbb{N}} \to A$ such that $r(x) = x$, for every $x \in A$ (a function with this property is called a retraction). *Hint:* Let σ be the metric defined in the previous exercise. Observe that if $\sigma(x,y) = \sigma(x,z)$, then $y = z$ and deduce that for every $x \in \{0,1\}^{\mathbb{N}}$, there is exactly one point $a \in A$ such that $\sigma(x,a) = \sigma_A(x)$ (the function σ_A is defined in Problem set 3, Exercise 1).

6. Use the previous exercise to show that the Cantor set $\{0,1\}^{\mathbb{N}}$ maps continuously onto any second countable metrizable space.

7. The unit circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is compact. Give two different justifications of that fact.

8. Prove Dini's theorem:
Let $f_1 \geq f_2 \geq f_3 \geq \ldots \geq 0$ be a decreasing sequence of continuous functions $f_i : X \to [0,\infty)$ defined on a compact space X. Suppose that $\lim_{n\to\infty} f_n(x) = 0$, for each x. Then for every $\varepsilon > 0$, there is k such that $f_k(x) < \varepsilon$, for $x \in X$ (and hence $f_n \rightrightarrows 0$, i.e the sequence converges uniformly to the zero function). *Hint:* Observe that the sets $f_i^{-1}([0,\varepsilon))$, $i = 1,2,\ldots$ cover X.

9. Let K be a compact space. Prove that the metric given by the supremum norm on $C(K)$ is complete, where $C(K) = \{f : K \to \mathbb{R} \mid f \text{ is continuous}\}$ and the supremum norm is $\|f\| = \sup\{f(x) : x \in K\}$. Why we need to assume compactness here?

10. Show that the Hilbert space ℓ_2 is separable and complete.