

Proof of variance and bias relationship [\[edit \]](#)

$$\begin{aligned}\text{MSE}(\hat{\theta}) &= \mathbb{E}_{\theta} \left[(\hat{\theta} - \theta)^2 \right] \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] + \mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 \right] \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right)^2 + 2 \left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right) \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right) + \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 \right] \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right)^2 \right] + \mathbb{E}_{\theta} \left[2 \left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right) \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right) \right] + \mathbb{E}_{\theta} \left[\left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 \right] \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right)^2 \right] + 2 \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right) \mathbb{E}_{\theta} \left[\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right] + \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 && \mathbb{E}_{\theta}[\hat{\theta}] - \theta = \text{const.} \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right)^2 \right] + 2 \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right) \left(\mathbb{E}_{\theta}[\hat{\theta}] - \mathbb{E}_{\theta}[\hat{\theta}] \right) + \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 && \mathbb{E}_{\theta}[\hat{\theta}] = \text{const.} \\&= \mathbb{E}_{\theta} \left[\left(\hat{\theta} - \mathbb{E}_{\theta}[\hat{\theta}] \right)^2 \right] + \left(\mathbb{E}_{\theta}[\hat{\theta}] - \theta \right)^2 \\&= \text{Var}_{\theta}(\hat{\theta}) + \text{Bias}_{\theta}(\hat{\theta}, \theta)^2\end{aligned}$$