16-311-Q Introduction to Robotics

Lecture 4:
(Non) Holonomic Robots
Wheeled Robots, Kinematics

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ROBOT "SPACES"

Representation and control

- Workpace
- Task space
- Configuration space

Robot design

- Components, Design, Geometry
- Stability, reliability
- Maneuverability
- Controllability
- Task efficacy

- Can the robot fully access its task space? ➔ ***Task efficacy***
- Is the robot able to move between two feasible poses/configurations without any restrictions? ➔ ***Maneuverability, Task efficacy***
- How difficult, reliable is to control robot motion? ➔ ***Controllability***

LOCOMOTION ➔ KINEMATICS VS. DYNAMICS

Locomotion:

Refers to the process of **moving** from one point to another, which requires the application of *forces*

Dynamics:

The study of motion (of a mass) through the direct modeling of the forces that cause it

Kinematics:

The study of motion without taking into consideration the forces that cause it. It is based on ***geometric relations, positions, velocities, and accelerations.***

Forward Kinematics:

Use of kinematic equations to determine / predict the final configuration/pose of a robot based on the specification of the values for the control variables (e.g., $\mathbf{v}, \boldsymbol{\omega}$)

Inverse Kinematics:

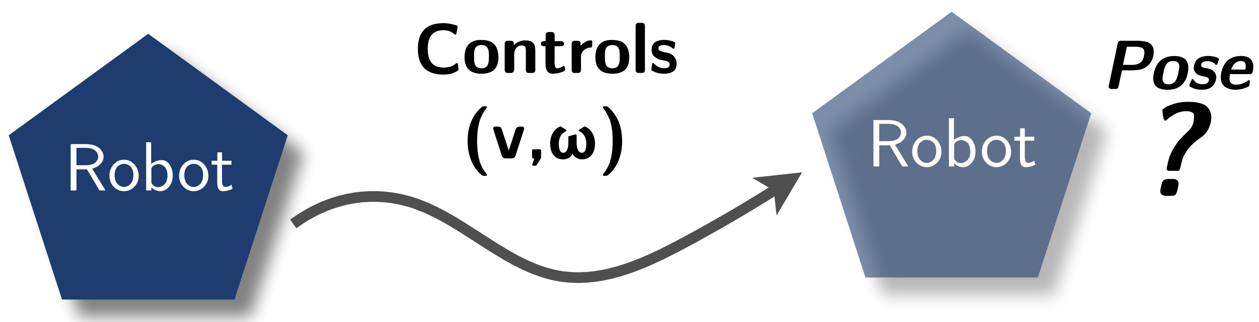
Given the desired final configuration (of the effectors/pose), make use of the kinematic equations to determine the values of the control variables that allow to achieve it.

Motion planning:

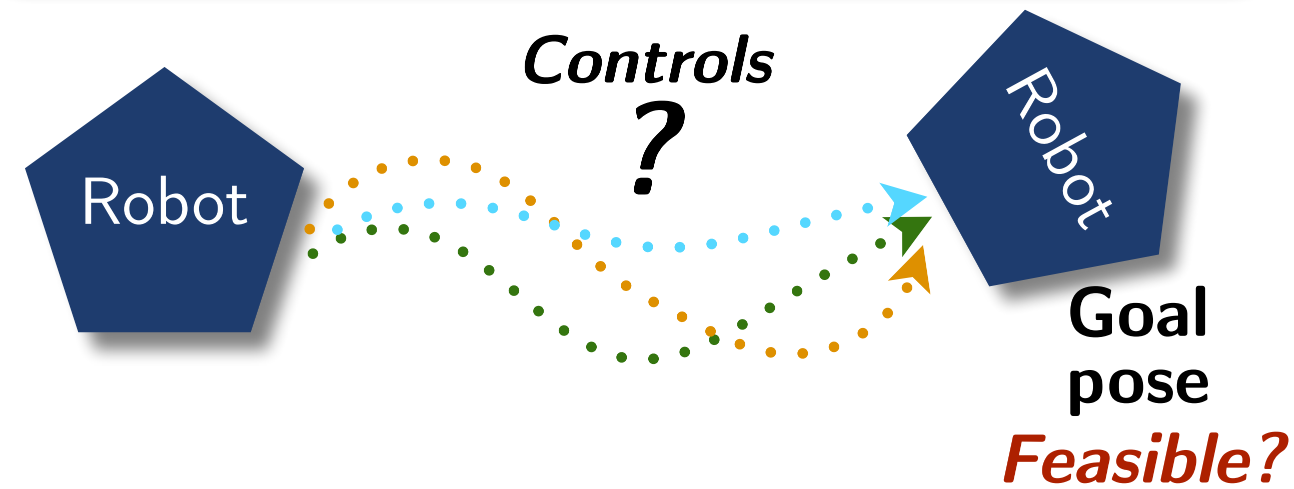
The specification of the *entire movement* of the robot in terms of its control variables to achieve the desired configurations in (s,t).

MOTION *CONTROL* AND MOTION *PREDICTION*

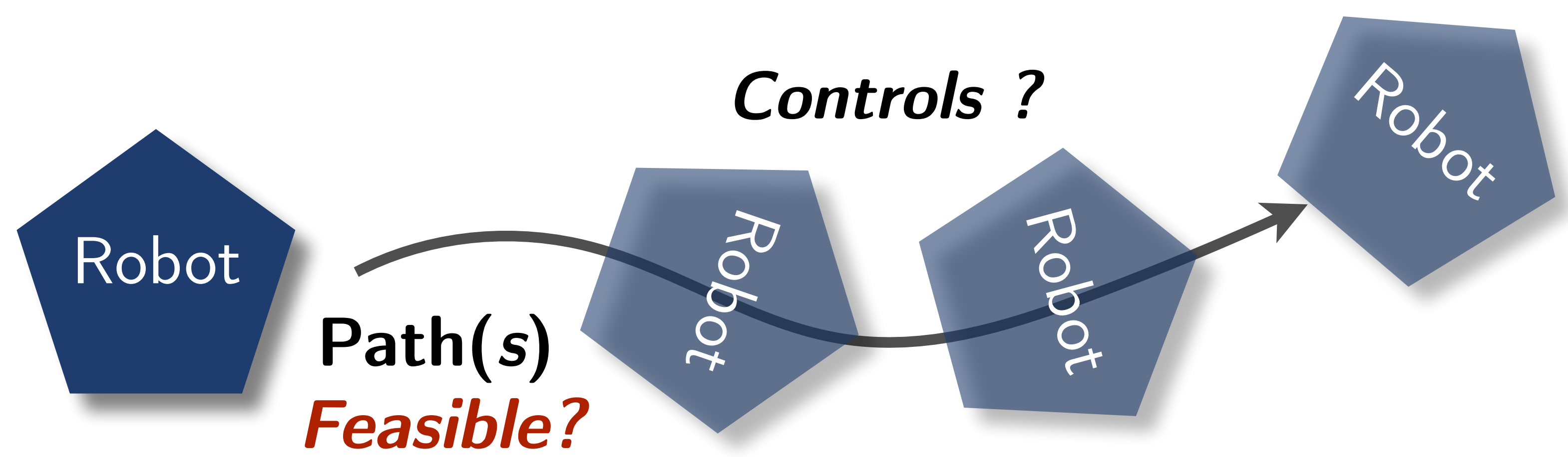
Posture prediction: *Forward Kinematics*



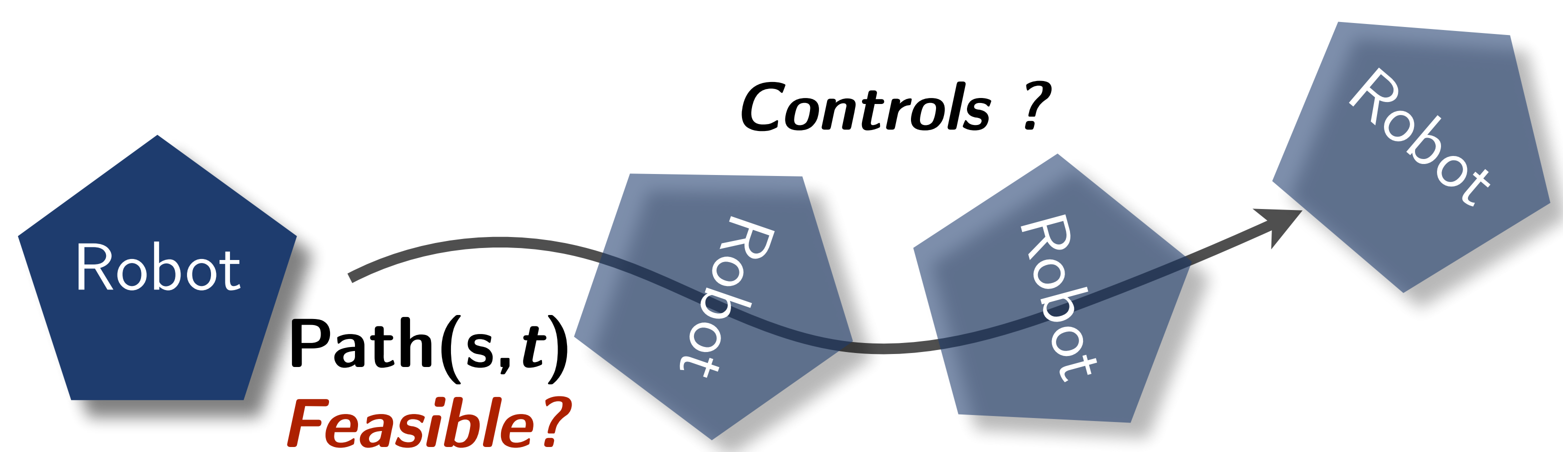
Posture regulation: *Inverse Kinematics*



Path following (geometry)

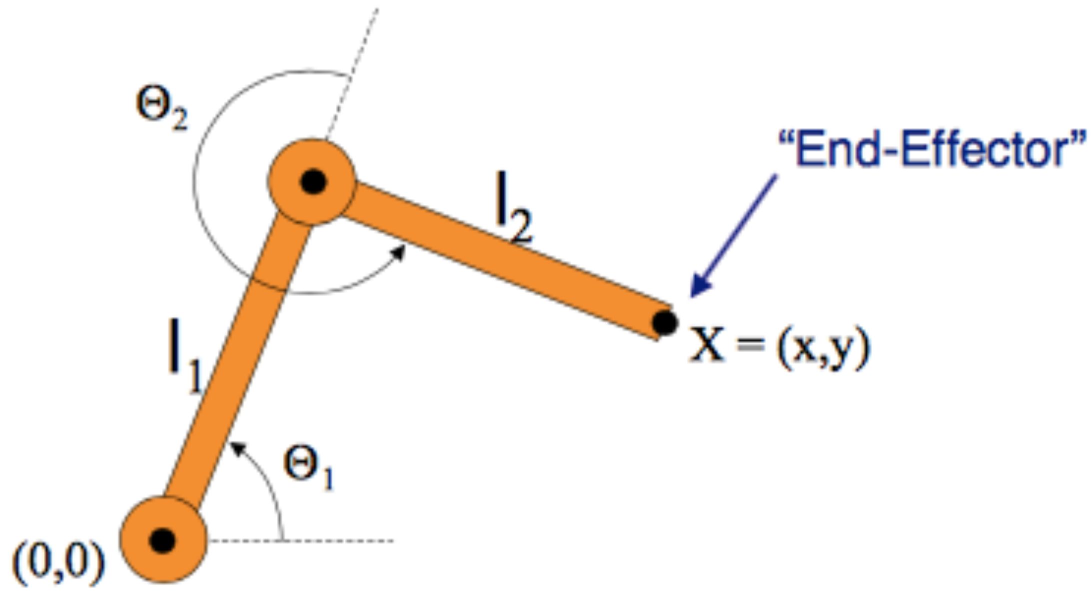


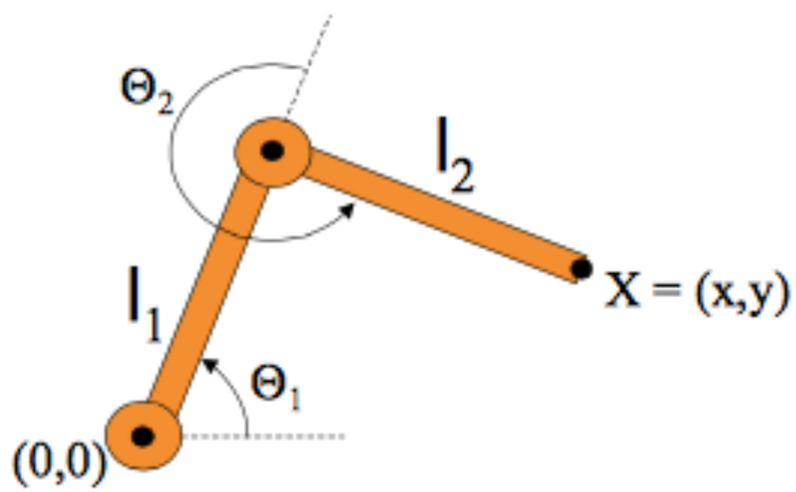
Trajectory following (kinematics, time)



FORWARD KINEMATICS FOR A ROBOT ARM

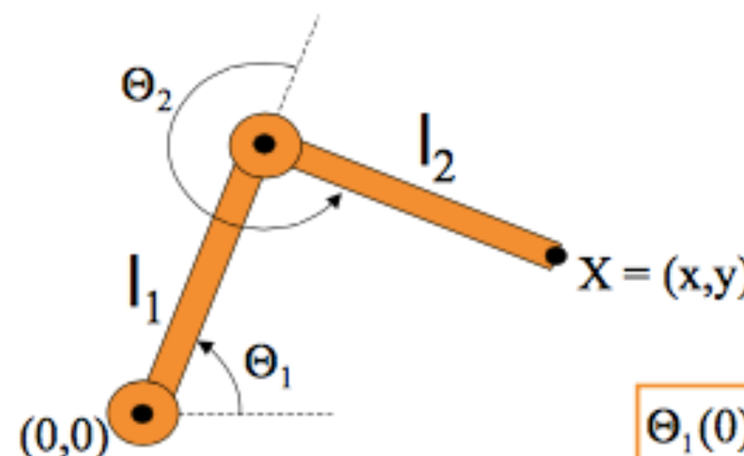
A robot arm with two links connected by revolute joints: determine the end-effector position, X





$$X = (l_1 \cos\Theta_1 + l_2 \cos(\Theta_1 + \Theta_2), l_1 \sin\Theta_1 + l_2 \sin(\Theta_1 + \Theta_2))$$

Specification of the two joint angles



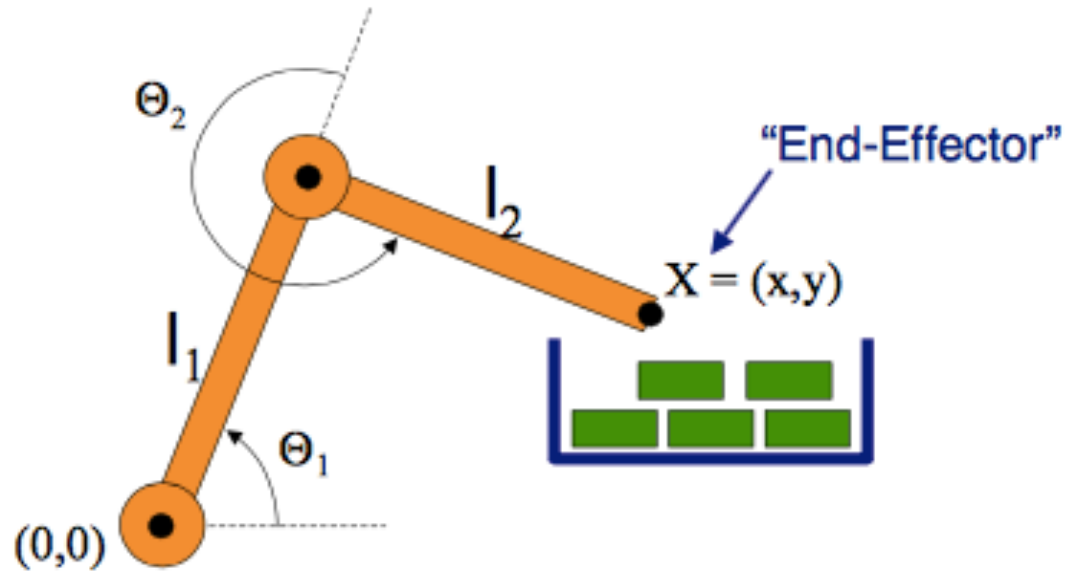
$$\Theta_1(0) = 60^\circ \qquad \Theta_2(0) = 250^\circ$$

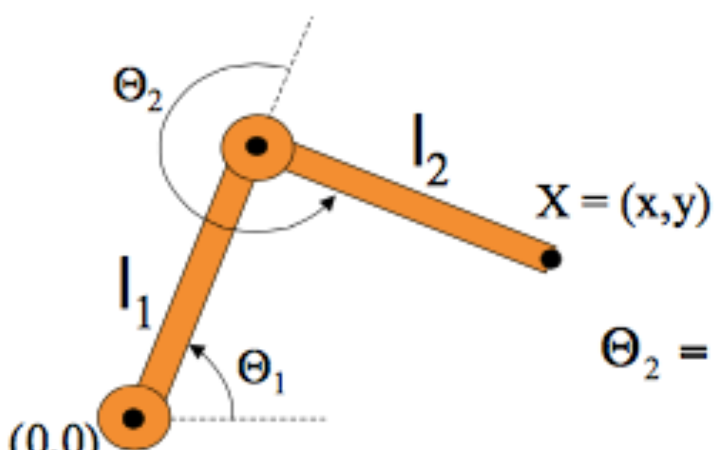
$$\frac{d\Theta_1}{dt} = 1.2 \qquad \frac{d\Theta_2}{dt} = -0.1$$

+ Specification of initial angles and velocities ➔ Integration of equations

INVERSE KINEMATICS FOR A ROBOT ARM

Given the desired position *X* of the end-effector, determine the values for the joint variables

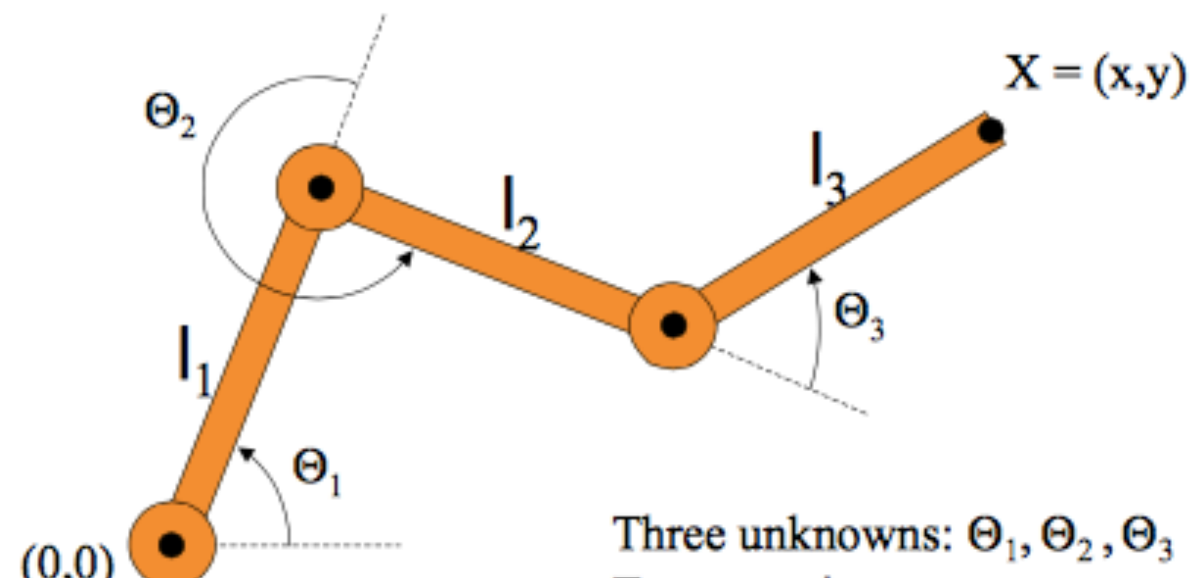




$$\Theta_2 = \cos^{-1}\left(\frac{x^2 + x^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

$$\Theta_1 = \frac{-(l_2 \sin(\Theta_2) x + (l_1 + l_2 \cos(\Theta_2)) y}{(l_2 \sin(\Theta_2)) y + (l_1 + l_2 \cos(\Theta_2)) x}$$

Solve the kinematic equations wrt the final pose



Three unknowns: $\Theta_1, \Theta_2, \Theta_3$
Two equations: x, y

Usually the problem admits multiple solutions

FORWARD KINEMATICS FOR A MOBILE ROBOT

Given:

the geometric parameters: number and type of wheels, wheel(s) radius, length of axes, ...

the initial conditions: pose and velocity

and assigned the spinning speeds of each wheel:

$$\dot{\varphi}_1, \dot{\varphi}_2$$

a forward kinematic model aims to predict

the robot's generalized velocity (rate of pose change) in the global reference frame:

$$\dot{\xi}_W = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T = f(l, r, \theta, \dot{\varphi}_1, \dot{\varphi}_2)$$

once integrated over time using the initial conditions, the new robot pose can be then computed (predicted)

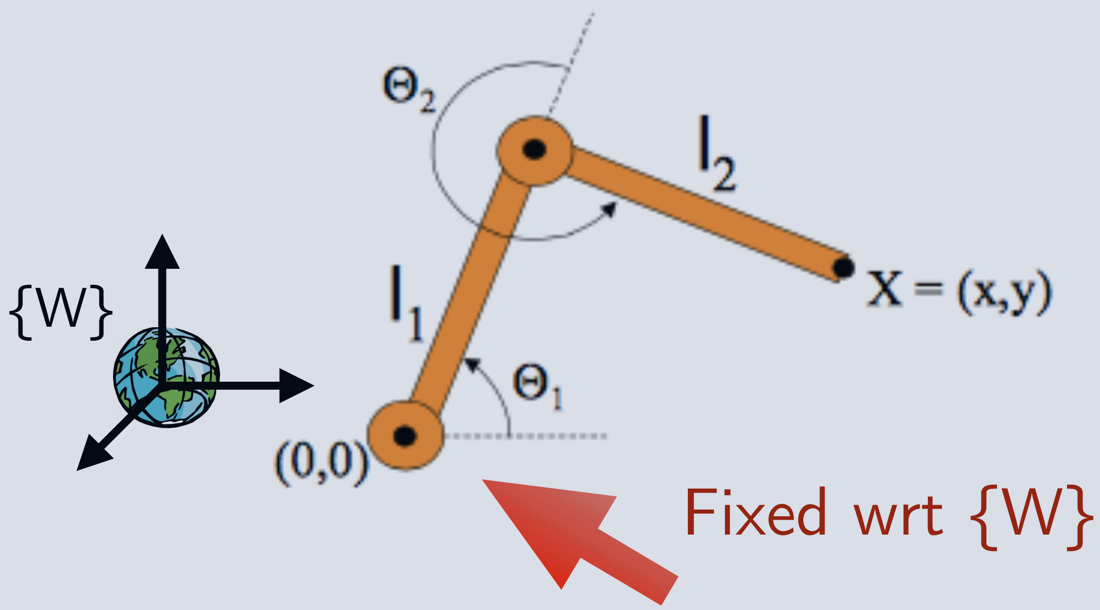
Strategy: compute the contribution to motion of each wheel, in the local reference frame and apply the transformations equations ....

MOTION CONTROL FOR MOBILE VS. ARM ROBOTS

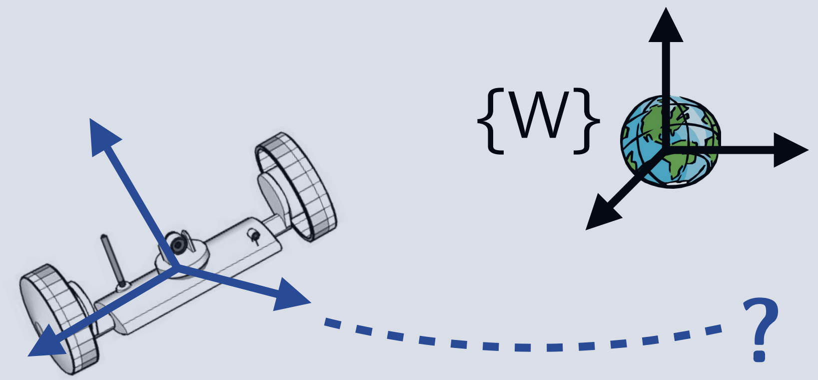
- Mobile robot's **controllability** (effectors + structure + constraints + mass) defines feasible paths and trajectories in the robot's *workspace*.

- **Difference between mobile and arm robots:** ***position estimation*** (of end effector, robot pose) in the world (inertial) reference frame {W}

- ***Arm*:** **Constrained workspace ➔** Measures of all intermediate joints + Kinematic equations

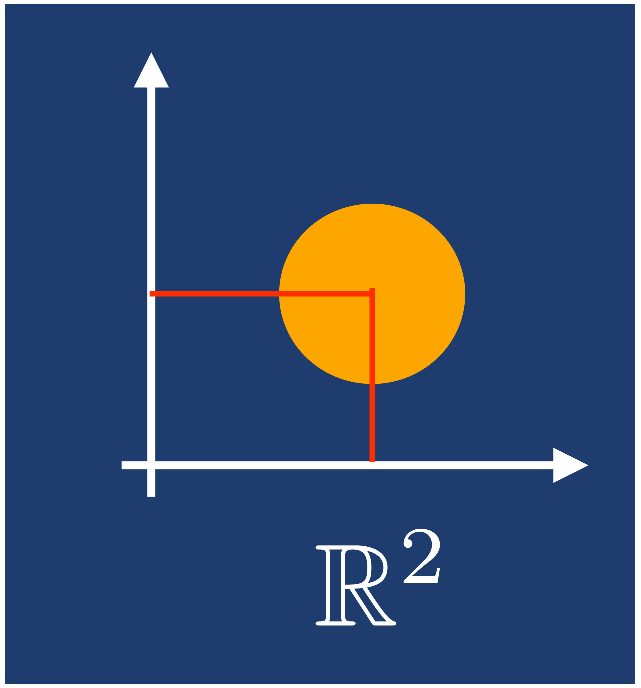


- ***Mobile*:** **It can span the entire environment**, no direct/obvious way to measure its position instantaneously/ exactly ➔ Integrate motion over time + include uncertainties and errors (e.g., due to wheels slippage)

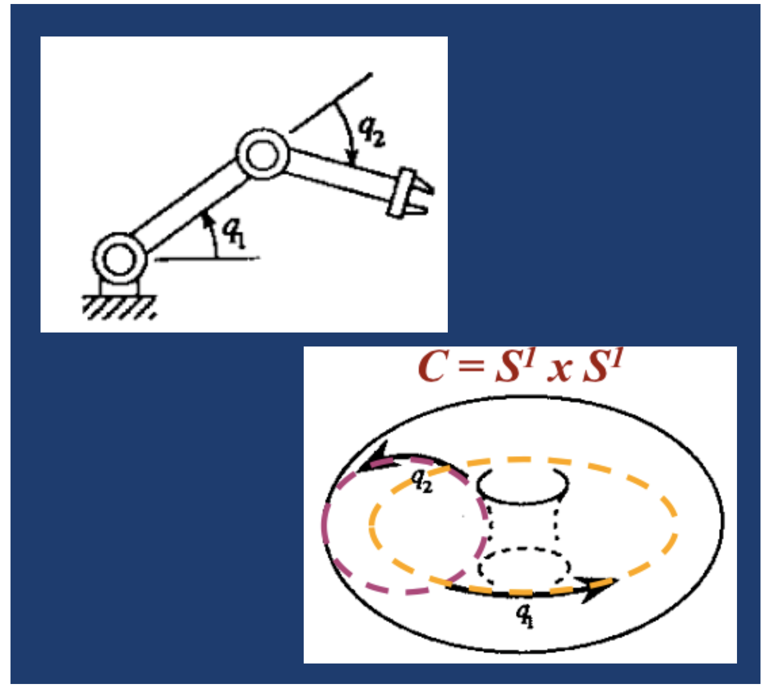


It is a much harder task!

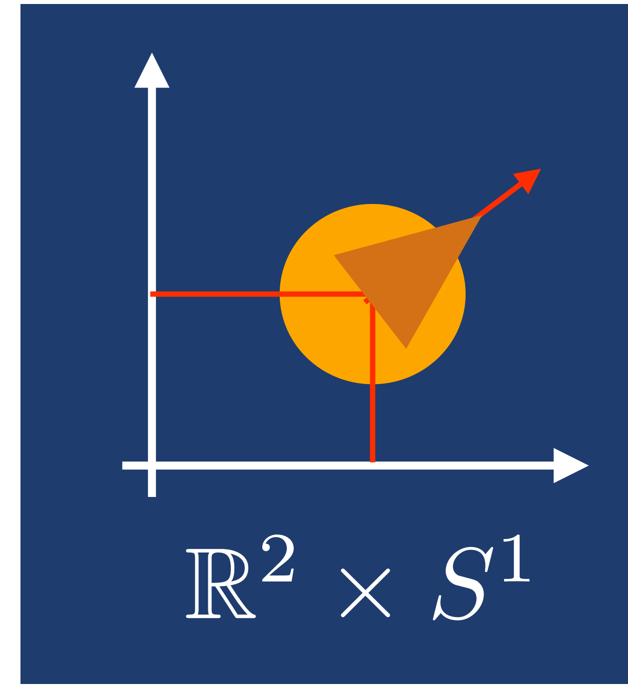
GENERALIZED COORDINATES, CONFIGURATION SPACE



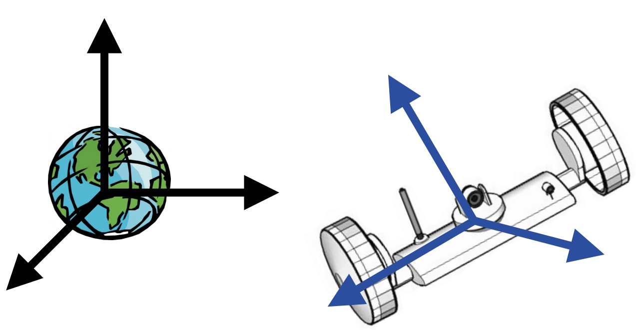
$\boldsymbol{q} = (x,y)$



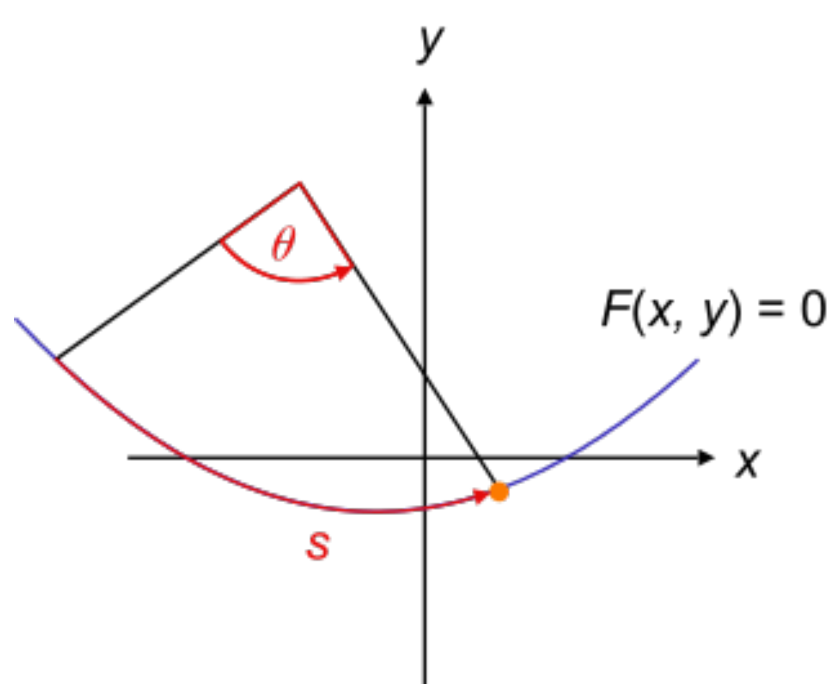
$\boldsymbol{q} = (\theta_1,\theta_2)$



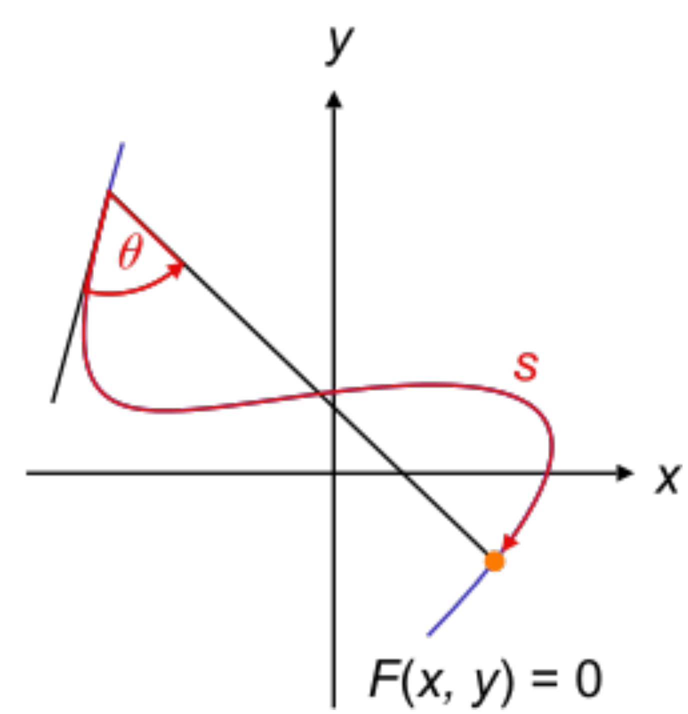
$\boldsymbol{q} = (x,y,\theta)$



$\boldsymbol{q} = (x,y,z,\ \phi,\theta,\Psi)$

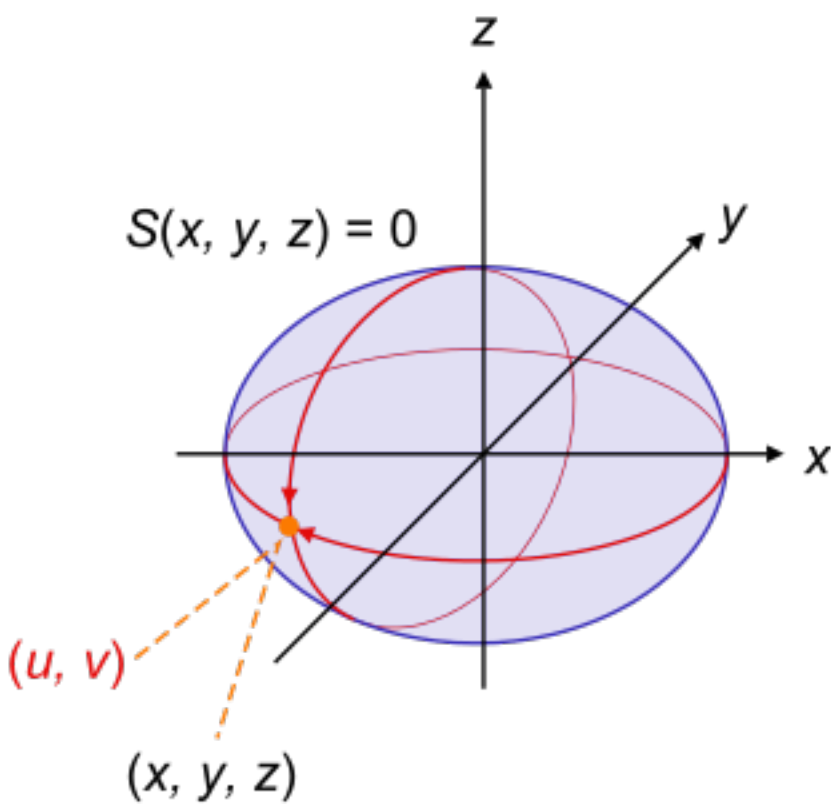


$\boldsymbol{q} = (\theta)$ $\boldsymbol{q} = (s)$

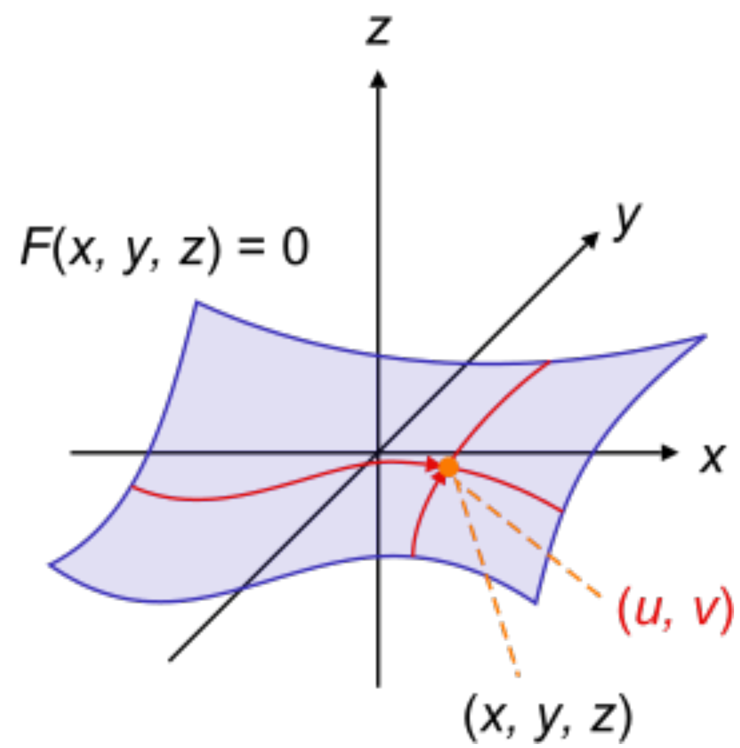


$\boldsymbol{q} = (s,\theta)$

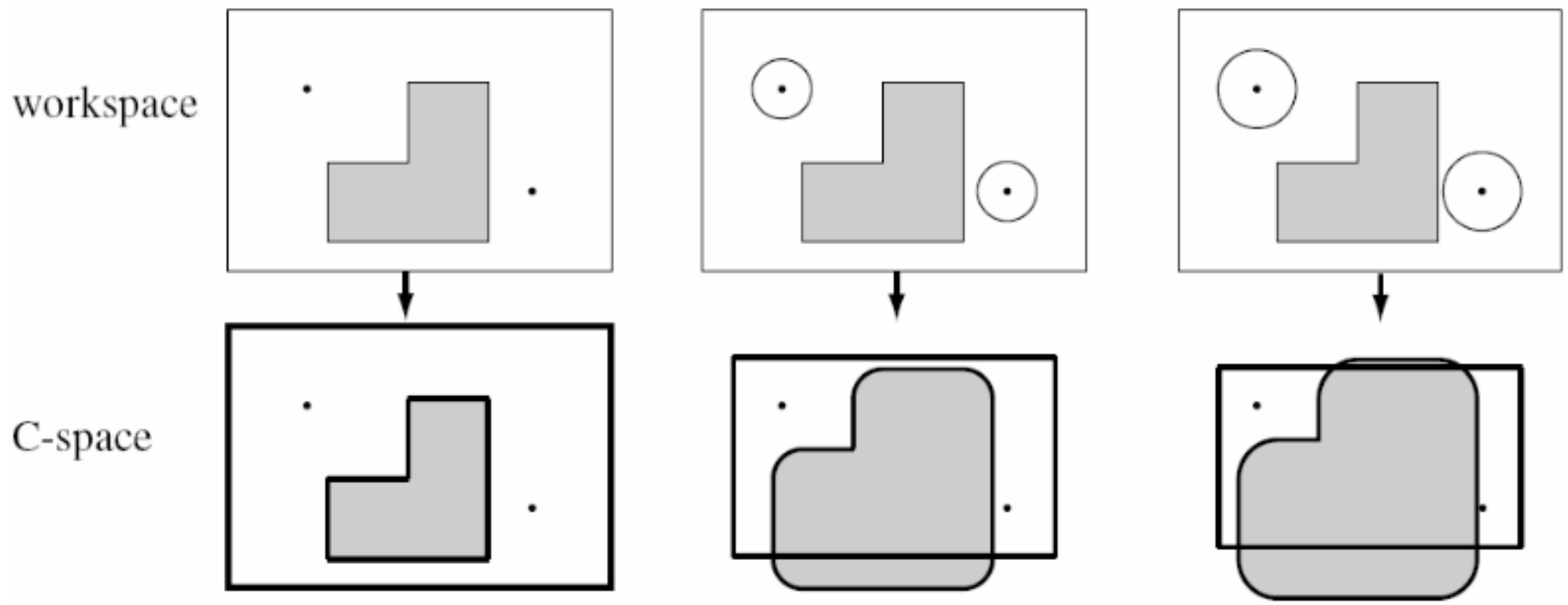
GENERALIZED COORDINATES, CONFIGURATION SPACE



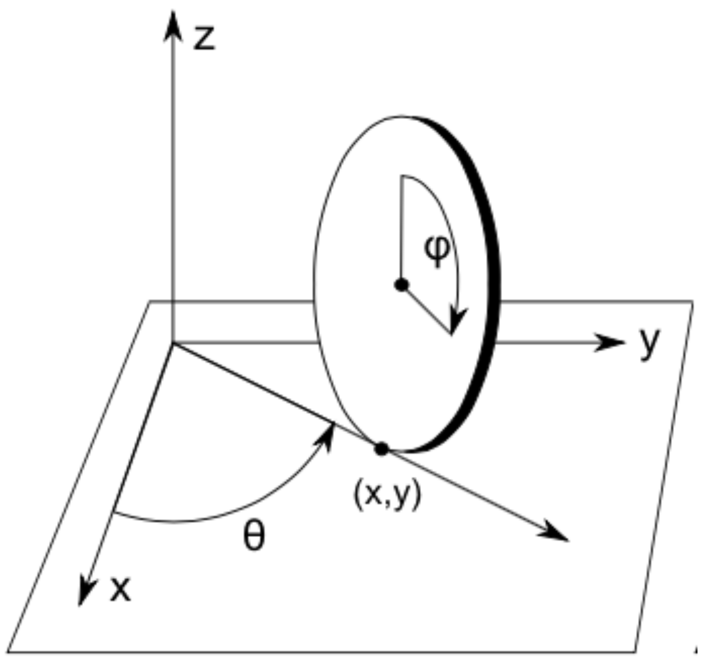
$$\boldsymbol{q} = (\mathrm{u,v})$$



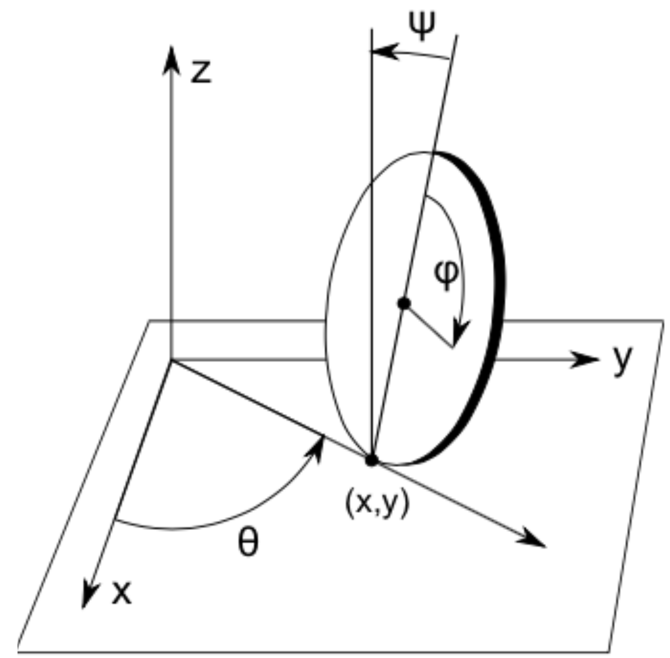
With obstacles



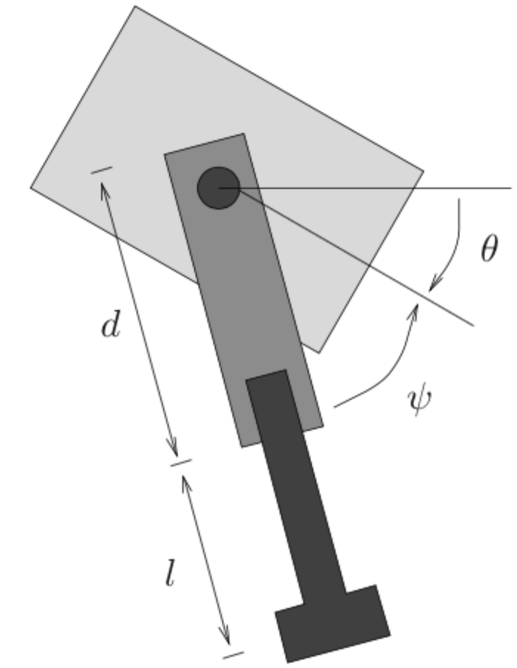
GENERALIZED COORDINATES, CONFIGURATION SPACE



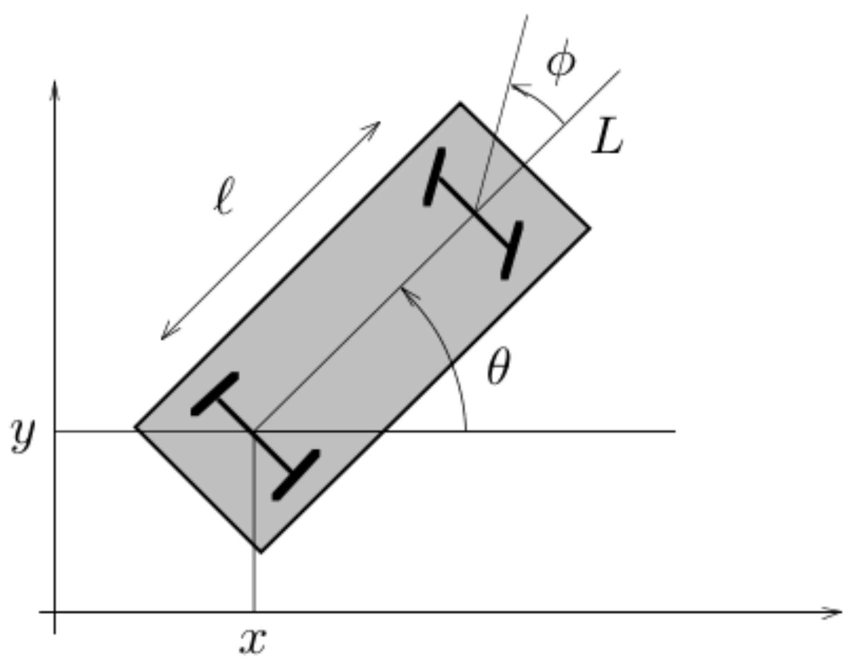
$\boldsymbol{q} = (x,y,\ \varphi,\theta)$



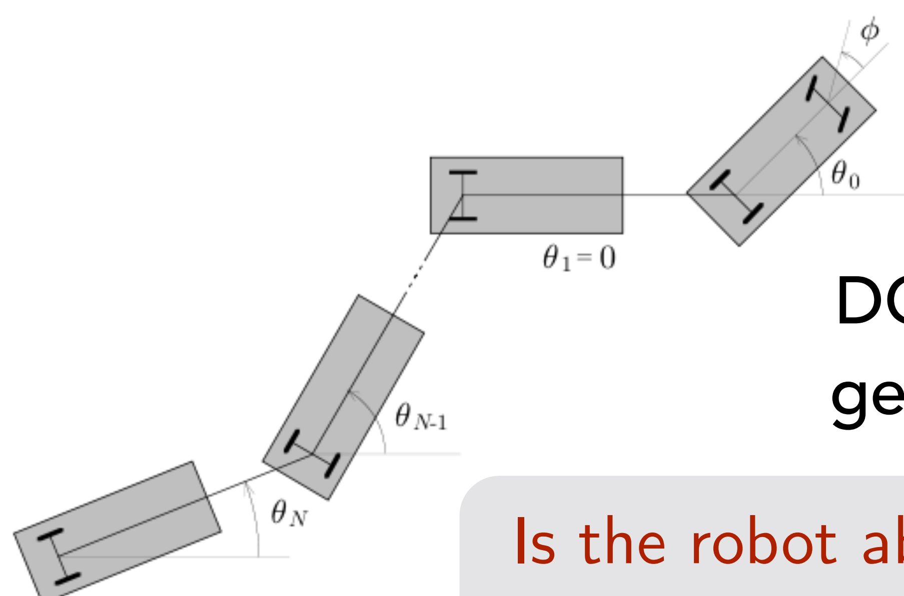
$\boldsymbol{q} = (x,y,\ \varphi,\theta,\Psi)$



$\boldsymbol{q} = (x,y,z,\ l,\ \theta,\Psi)$



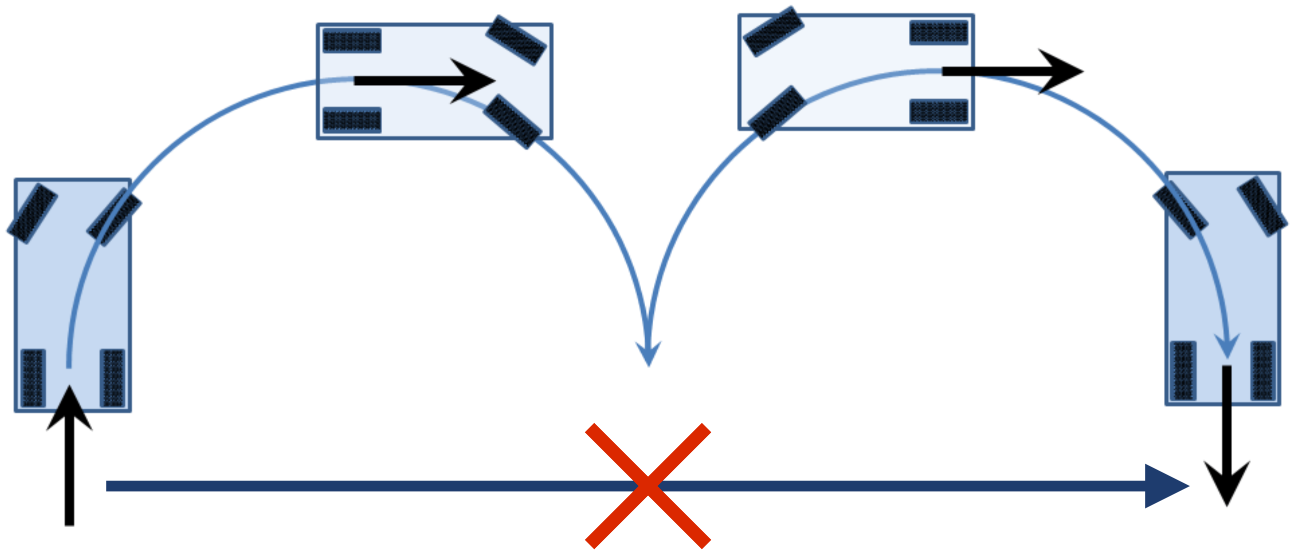
$\boldsymbol{q} = (x,y,\ \varphi,\theta)$



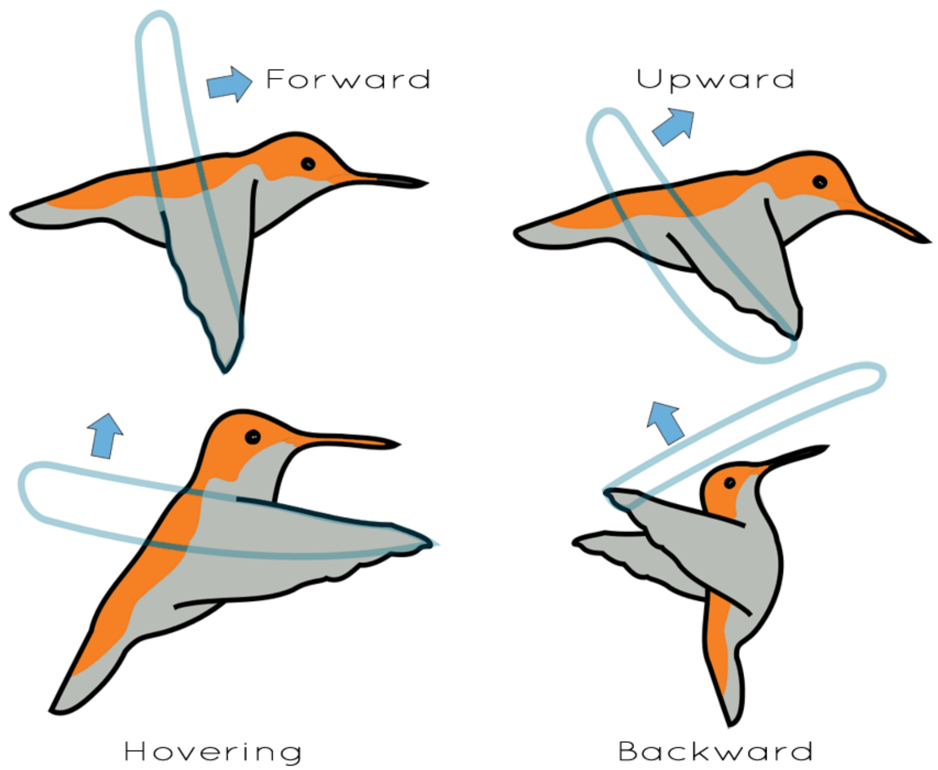
DOFs = #independent generalize coordinates

Is the robot able to move between two feasible configurations without any restrictions? ➔ ***Maneuverability***

MOTION ACTUATORS IMPOSE LIMITS

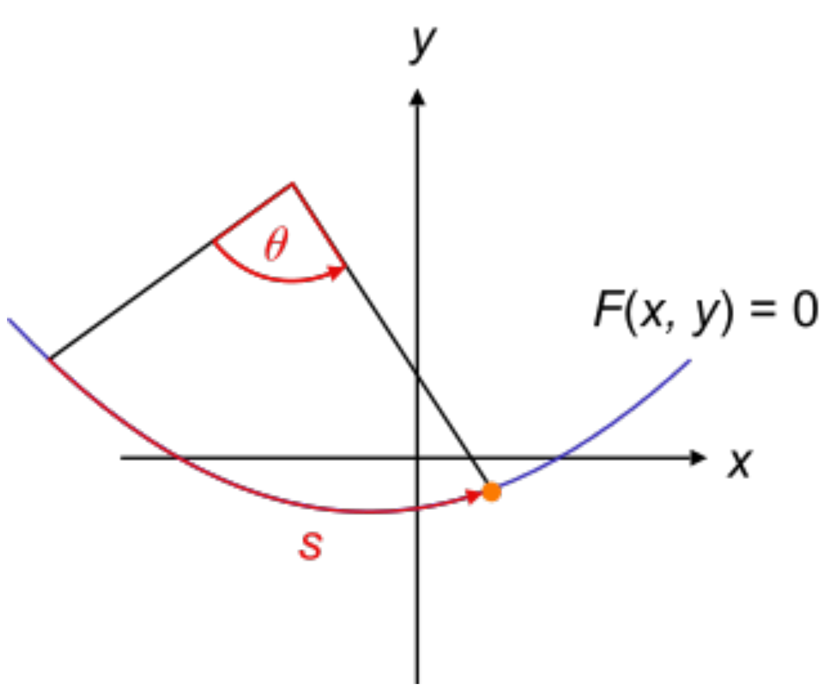


Two-moves car parking:
no side-way motion



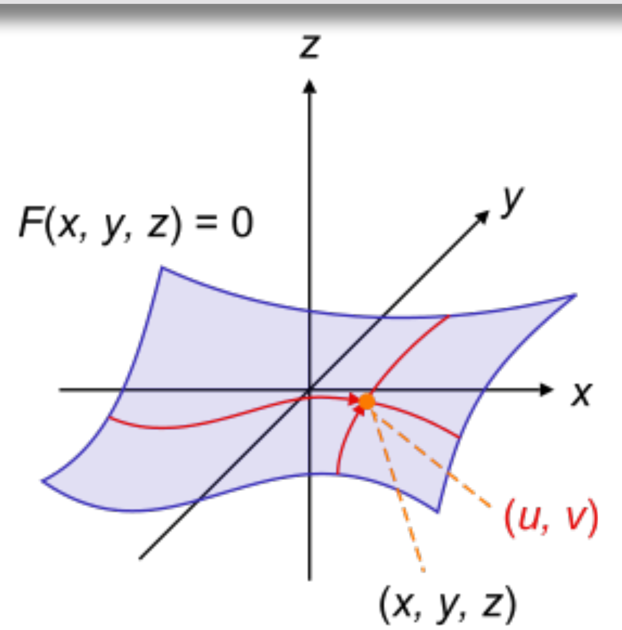
No easy side-way motion in 3D

A train robot moving forward/backward on a track can reach any point in its configuration space without limitations regarding the trajectories



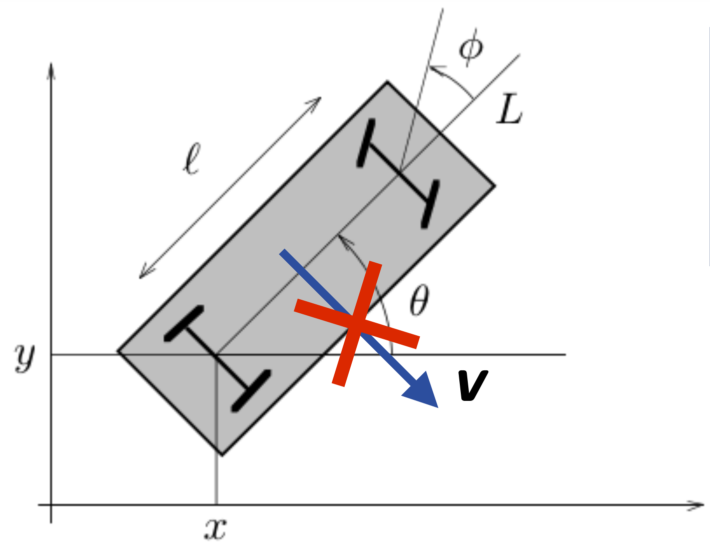
LET'S START FORMALIZING THESE LIMITATIONS…

A ***geometric constraint*** imposes restrictions on the achievable configurations of the robot.
It is based on a functional relation among (some subset of) the configuration variables



$f(\boldsymbol{\xi}, t) = 0$

A ***kinematic constraint*** imposes restrictions on the achievable velocities of the robot.
It is based on a functional relation among configuration variables and their derivatives



$g(\boldsymbol{\xi}, d\boldsymbol{\xi}/dt, t) = 0$

If g() is linear in the derivatives
the constraint is said *Pfaffian*

HOLONOMIC AND NON HOLONOMIC CONSTRAINTS

A *geometric constraint* is expressed through "positional" variables (e.g., (α, β, φ_1, φ_2, x, y, θ, …)) and is said ***holonomic***.
A holonomic constraint limits *the motion of the system to a manifold of the configuration space*, depending on the initial conditions

A kinematic constraint can be ***integrable***, meaning that it can be expressed in a form:

$$f\,(\boldsymbol{\xi}, t) = 0$$

where $\boldsymbol{\xi}$ is a vector of configuration variables, and it becomes a holonomic constraint.

A *kinematic constraint* which is not integrable is said an ***non holonomic*** constraint, meaning that it is expressed through "<u>derivatives of positional variables</u>" (and cannot be integrated to provide a constraint in terms of positional variables).

A non holonomic constraint does not limit the accessible configurations, but limits *the paths that can be followed to reach them.*

A car-like vehicle is an example of non holonomic vehicle: all poses can be achieved in the configuration space, but the paths to reach them can be complex (e.g., parallel parking is not allowed)
A sliding puzzle is also non holonomic!

NEED FOR WORKING IN THE VELOCITY SPACES

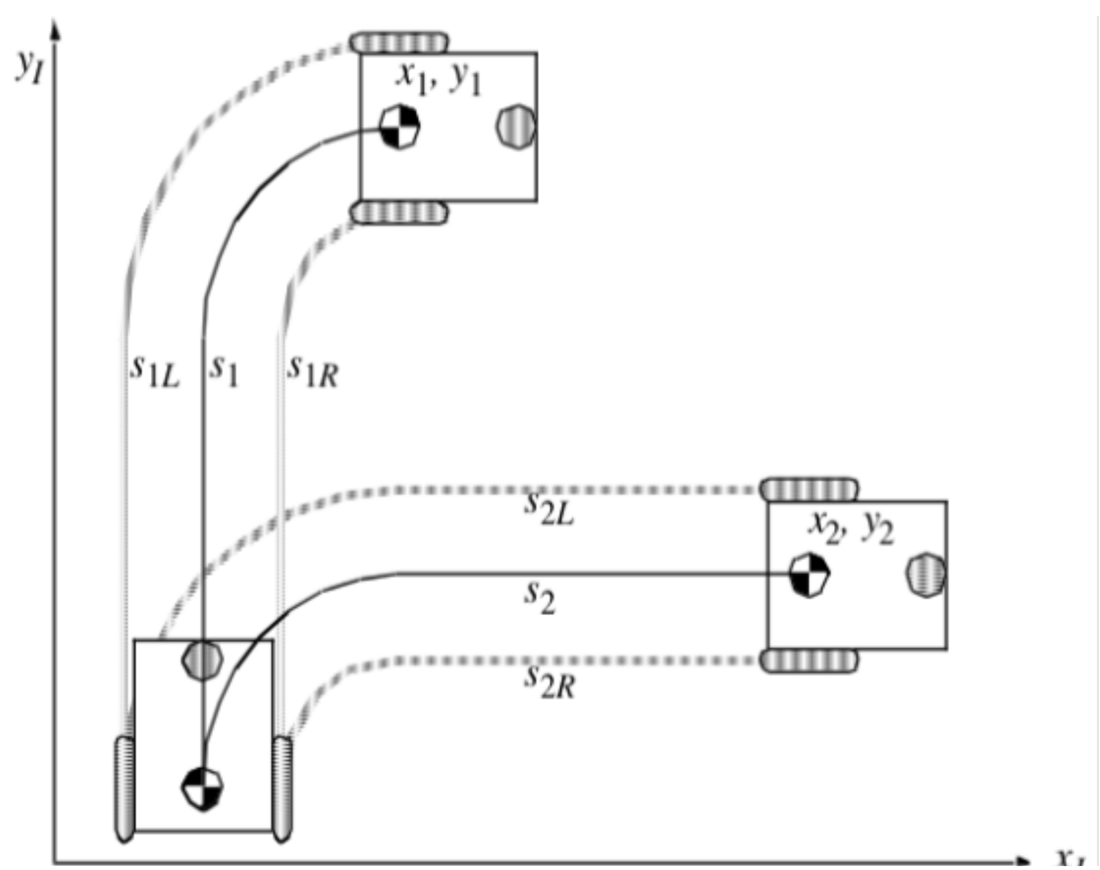
The presence of non holonomic constraints forces to work in the terms of transformations on velocities rather than on positions

➡ In presence of non holonomic constraints, the differential equations of motion are *not integrable to the final position.*

For instance, in a wheeled robot, the measure of the traveled distance of each wheel is not sufficient to calculate the final position of the robot.
One has also to know how this movement was executed as a function of time.

$$s_1 = s_2, s_{1R} = s_{2R}, s_{1L} = s_{2L}$$

$$x_1 \neq x_2, y_1 \neq y_2$$



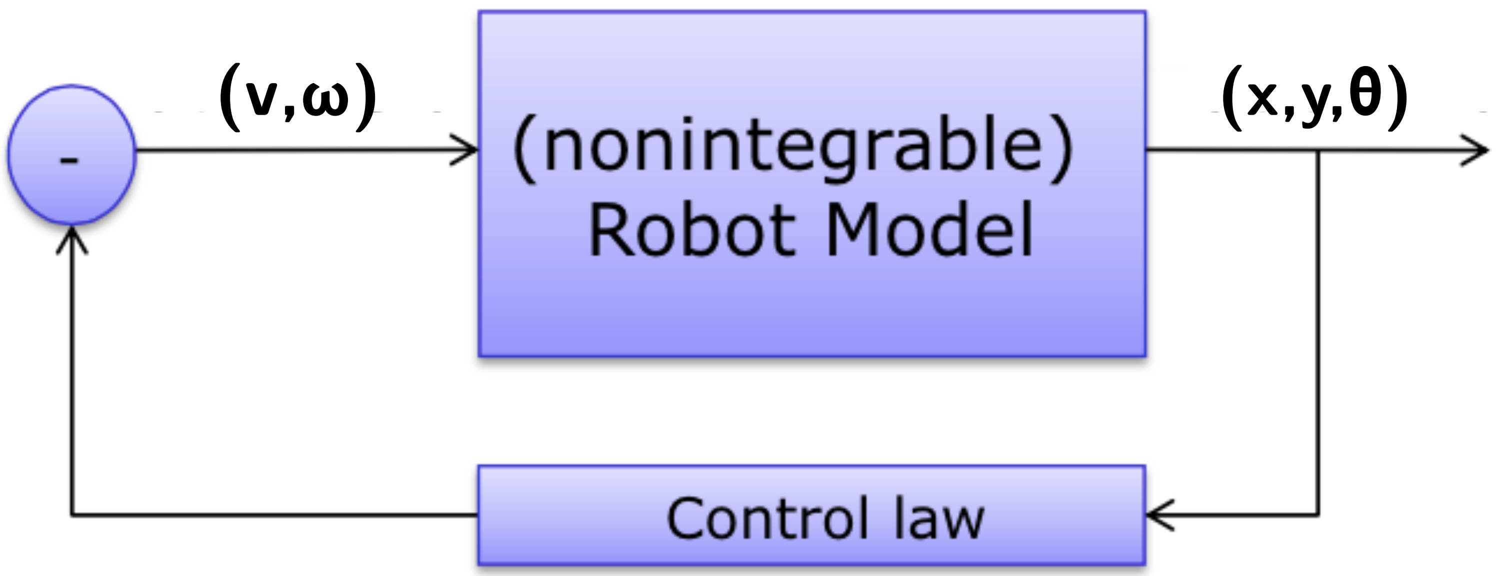
NEED FOR WORKING IN THE VELOCITY SPACES

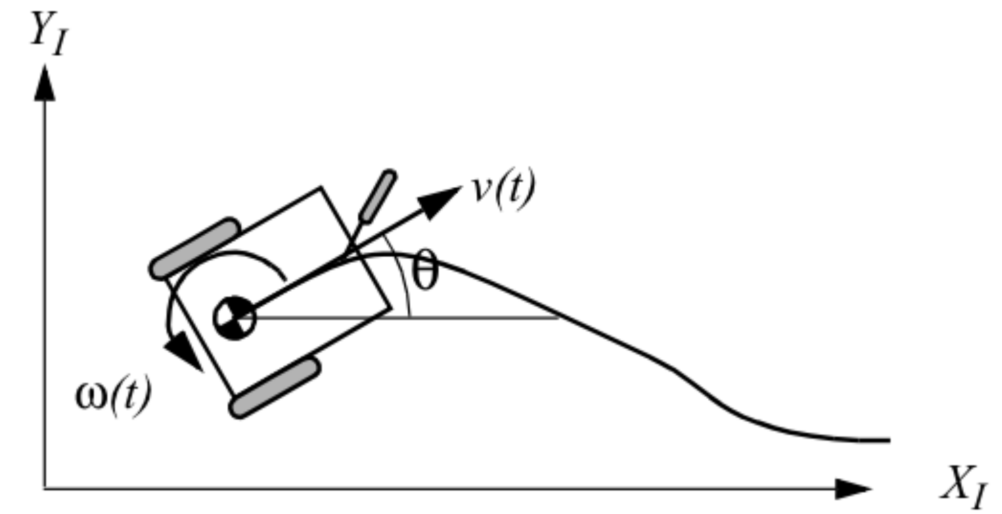
Forward kinematics: Transformation from configuration space to physical space

Inverse kinematics: Transformation from physical space to configuration space

In mobile robotics, due to (pervasive presence of) non holonomic constraints, usually we need to work with ***differential (inverse) kinematics***:

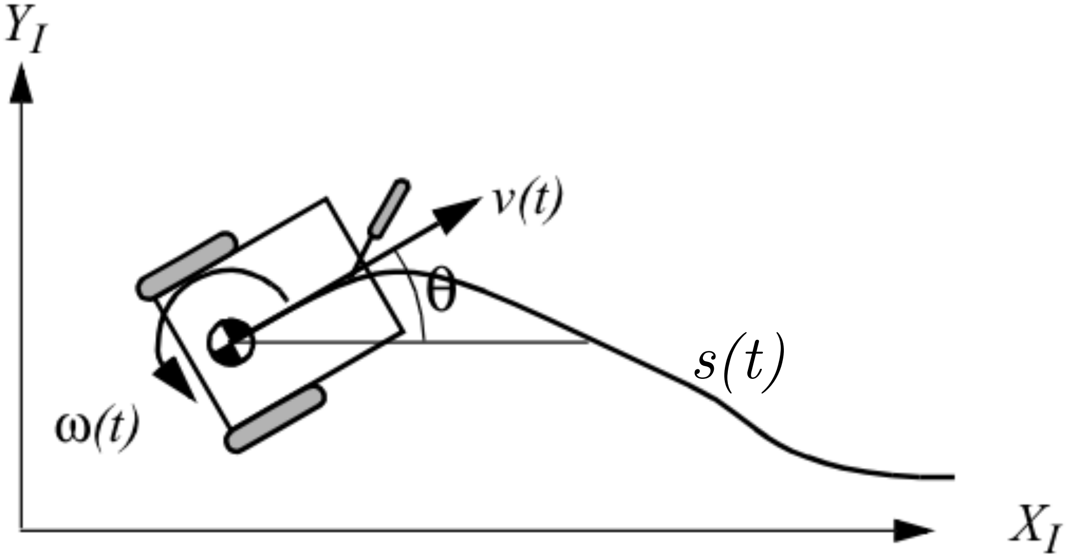
Transformation between velocities instead of positions





- Due to a lack of alternatives:
  - establish the robot speed $\dot{\xi} = \begin{bmatrix} \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^T$ as a function of the wheel speeds $\dot{\varphi}_i$, steering angles β_i, steering speeds $\dot{\beta}_i$ and the geometric parameters of the robot (*configuration coordinates*).
  - forward kinematics

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m)$$



  - Inverse kinematics

$$\begin{bmatrix} \dot{\varphi}_1 & \cdots & \dot{\varphi}_n & \beta_1 & \ldots & \beta_m & \dot{\beta}_1 & \ldots & \dot{\beta}_m \end{bmatrix}^T = f(\dot{x}, \dot{y}, \dot{\theta})$$

  - But generally not integrable into

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = f(\varphi_1, \ldots \varphi_n, \beta_1, \ldots \beta_m)$$

ROBOT POSE: REPRESENTATION AND EVOLUTION

- Inertial frame: $\{X_I, Y_I\}$
- Robot frame: $\{X_R, Y_R\}$
- Robot pose: $\xi_I = [x \quad y \quad \theta]^T$
- Mapping between the two frames

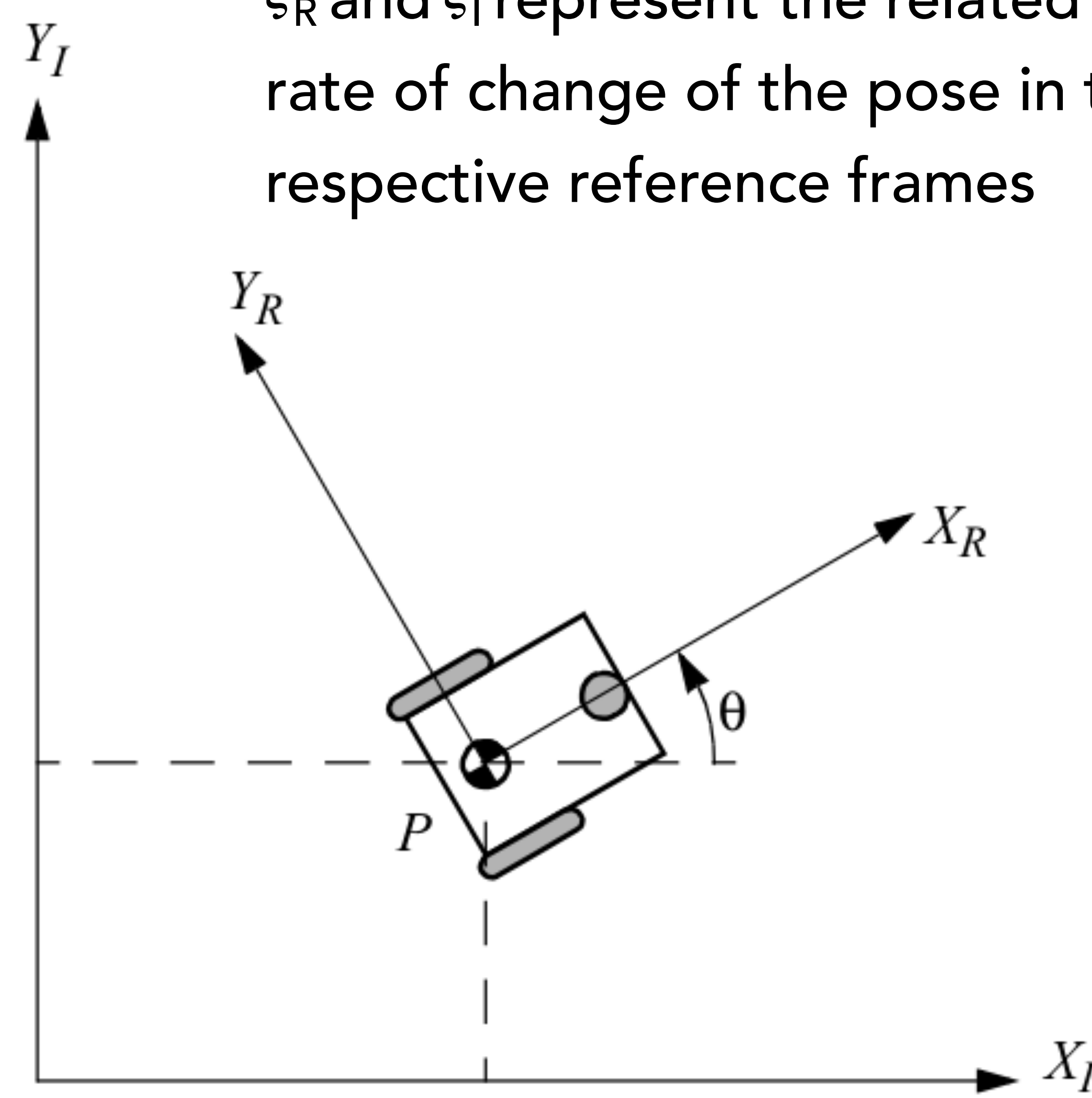
$$\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta) \cdot [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Instantaneous rotation matrix
(in the *dt* time for pose change calculation)

ξ_I represents the pose of the robot wrt the inertial global reference {I}, while ξ_R is the pose in the local robot reference frame {R}.

$\dot{\xi}_R$ and $\dot{\xi}_I$ represent the related <u>velocities</u>: rate of change of the pose in the respective reference frames

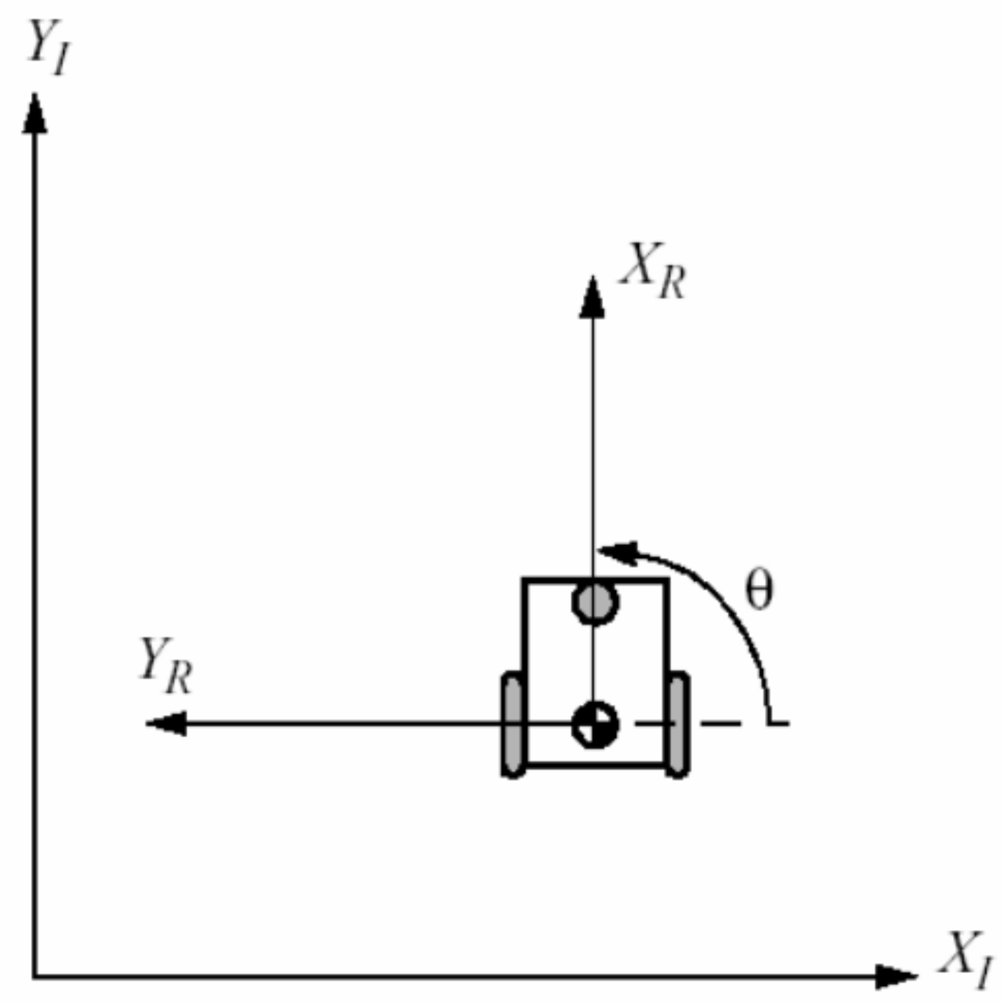


<u>Note</u>: in previous lecture the notation R(**θ**) was used for the inverse of this matrix

EXAMPLE OF POSE TRANSFORMATION

The robot is aligned with Y_I

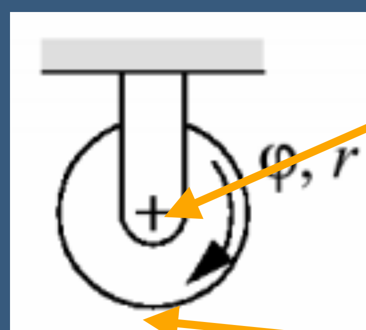
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$

Note: It's more useful to compute $\dot{\xi}_I$, the motion (pose velocity) in the global frame {*I*} from the motion in the local frame {R} (which is what the robot can directly control)

To derive kinematic equations, we will add motion constraints due to the physical characteristics of the wheels ...

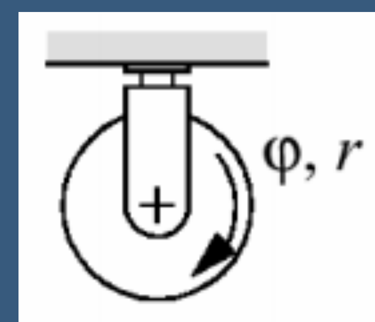


Wheel axle

Contact point

Fixed standard

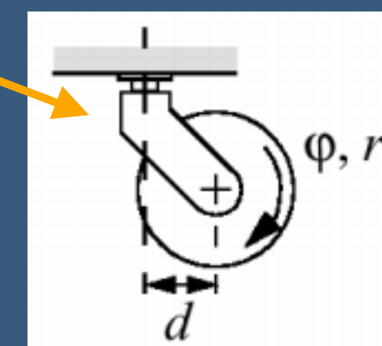
2 DoF



Steered standard

3 DoF

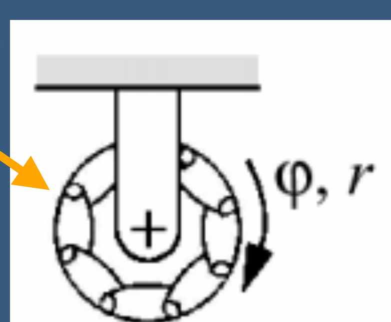
Castor axle



Castor

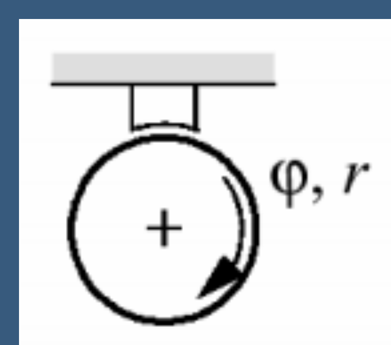
3 DoF

Rollers



Swedish

3 DoF

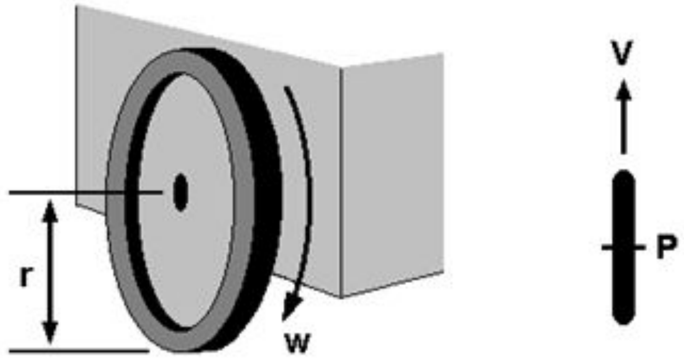


Spherical

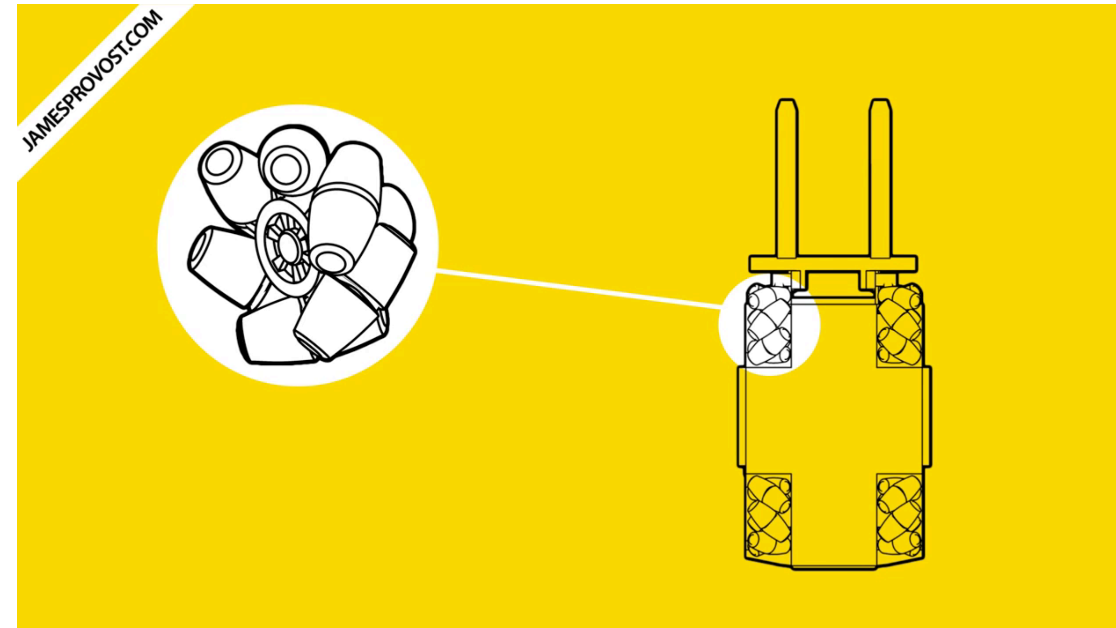
3 DoF

...

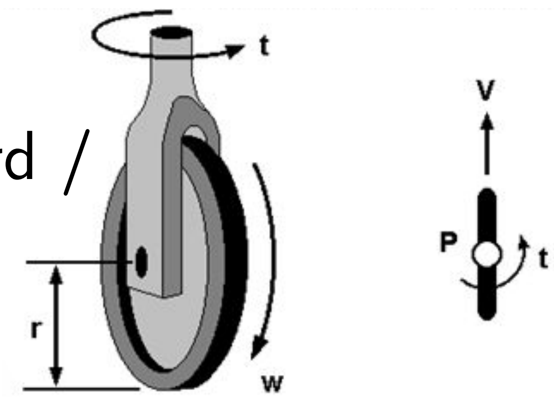
TYPES OF WHEELS



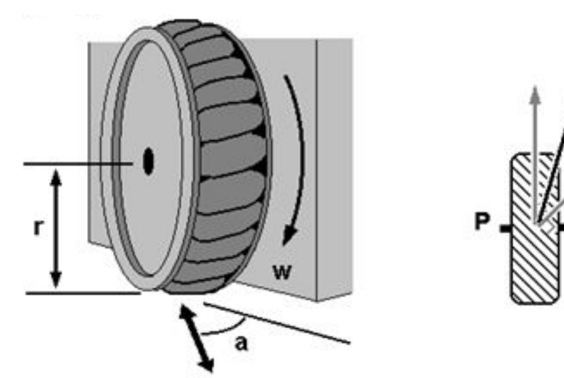
Fixed standard

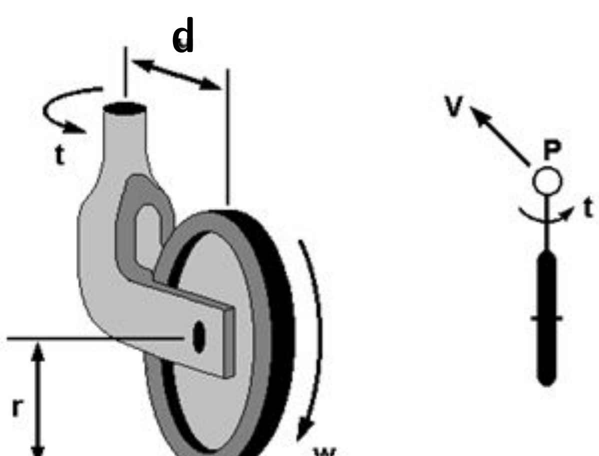


Steered standard / Orientable

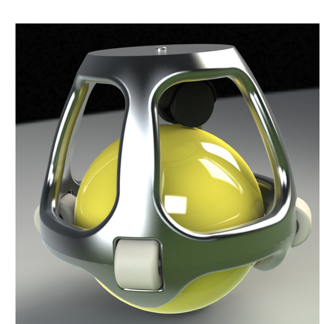


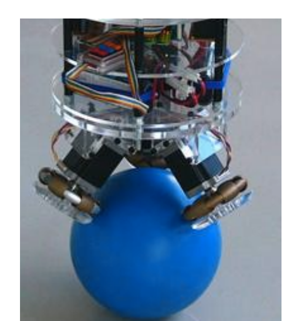
Mecanum/Swedish





Castor / Off-centered orientable



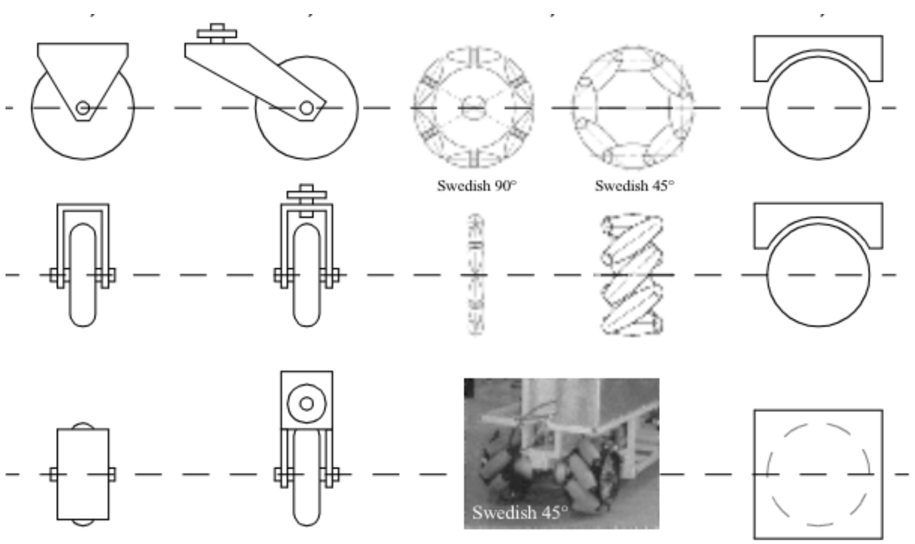


Spherical

LOCOMOTION (EFFECTORS) IS MORE THAN WHEELS…

Type of motion	Resistance to motion	Basic kinematics of motion
Flow in a Channel	Hydrodynamic forces	Eddies
Crawl	Friction forces	Longitudinal vibration
Sliding	Friction forces	Transverse vibration
Running	Loss of kinetic energy	Oscillatory movement of a multi-link pendulum
Jumping	Loss of kinetic energy	Oscillatory movement of a multi-link pendulum
Walking	Gravitational forces	Rolling of a polygon (see figure 2.2)

Locomotion modalities in natural systems



Locomotion using wheels (man-made)

- Walk
- Run
- Fly
- Swim
- Dive
- Drive
- Jump
- Crawl
- Roll
- Slide
- Flow
- ….

Usually easier to control compared to other effectors!

Wheels are the most appropriate solution for most applications

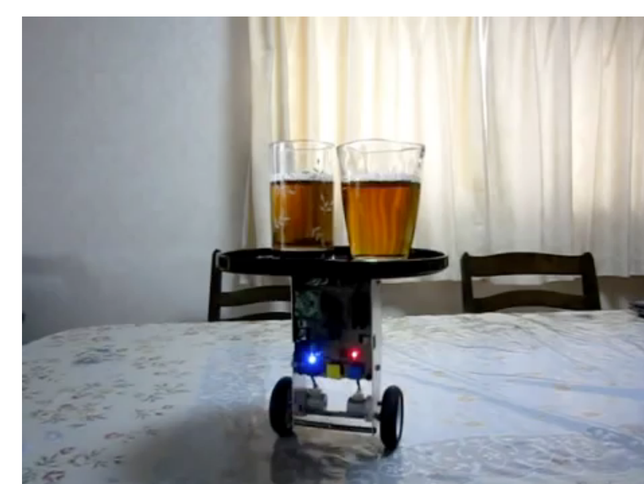
Three wheels are sufficient to guarantee stability

With less than three wheels stability is an issue

With more than three wheels an appropriate suspension is required

Selection of wheels depends on the application

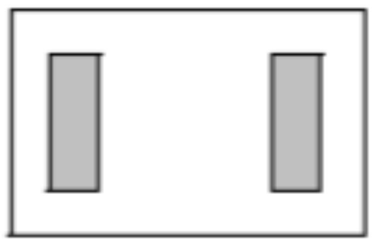
Most arrangements are *non holonomic*



ARRANGEMENTS OF WHEELS

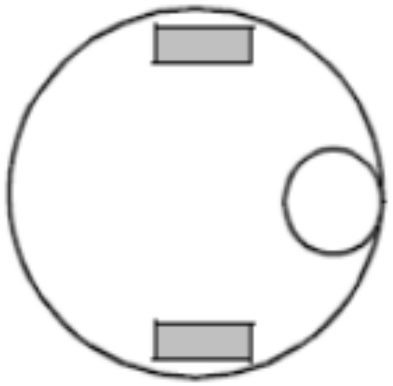
Two wheels

Bicycle

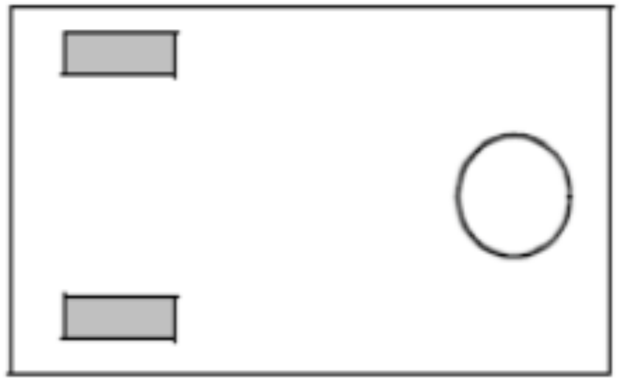


Differential drive

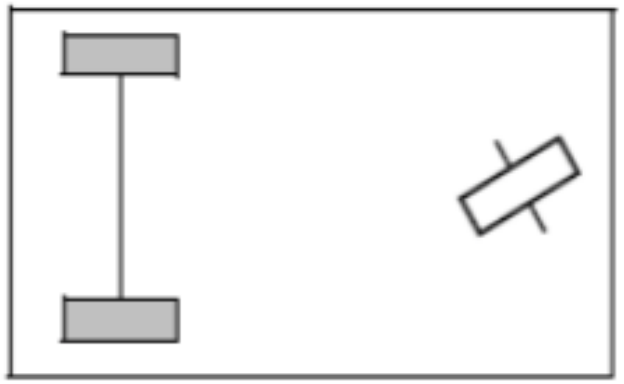
Three wheels



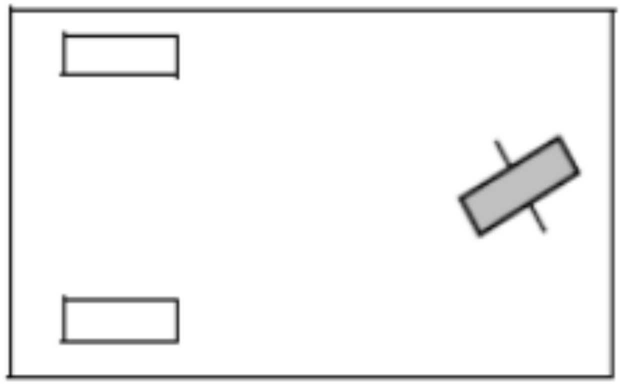
Differential with castor



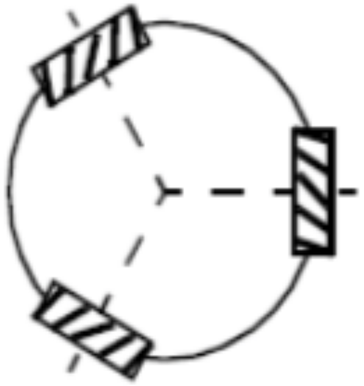
Differential cart
with castor



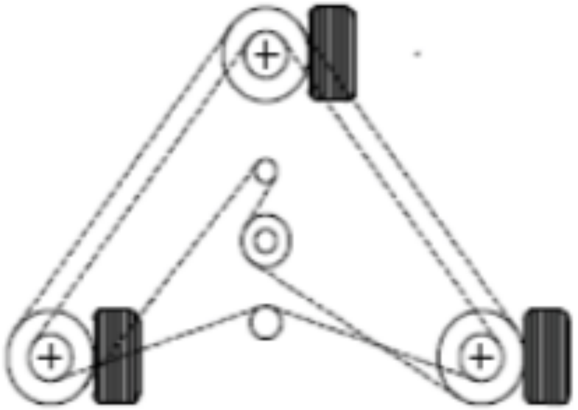
Tricycle



Tricycle - Horse buggy

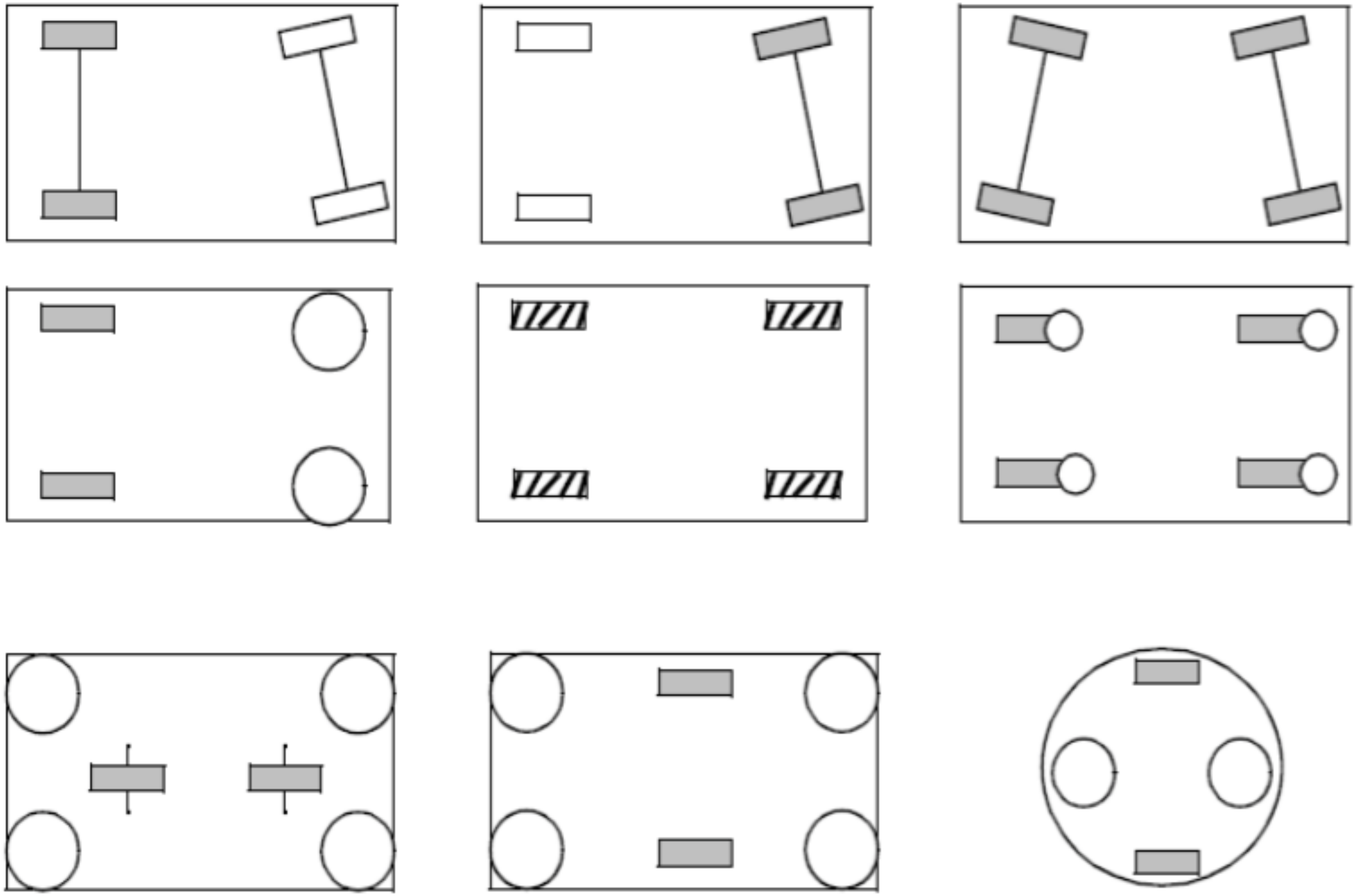


Omni drive



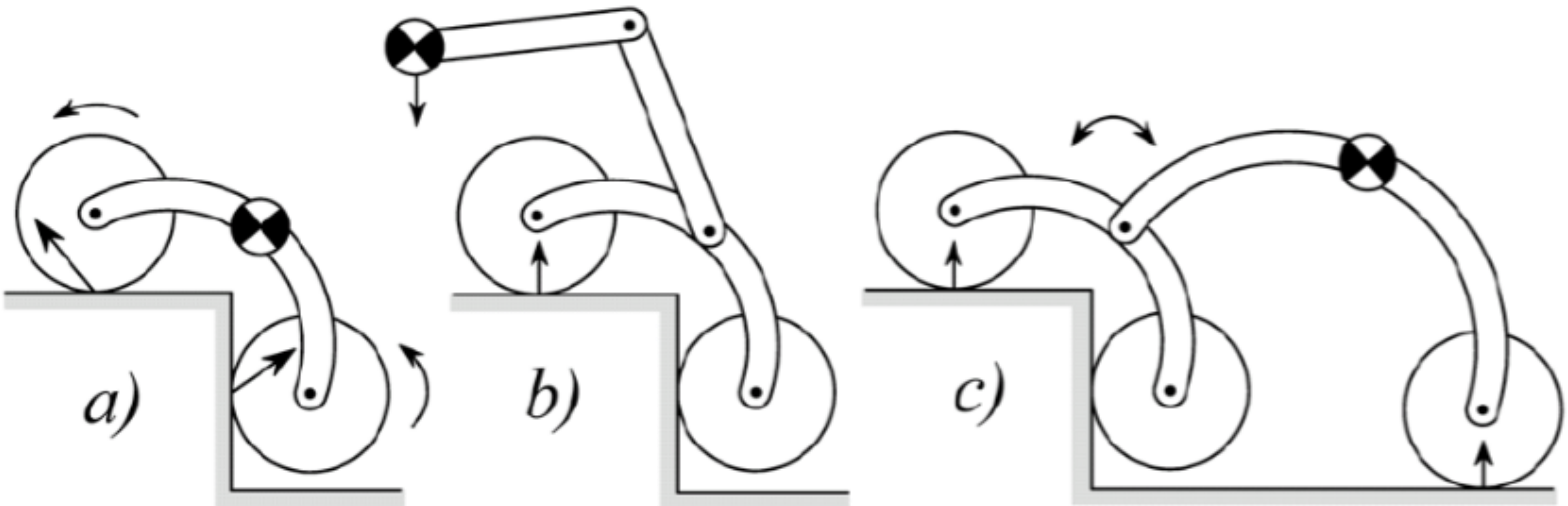
Synchro drive

Four wheels



ARRANGEMENTS OF WHEELS

Rovers for climbing



Purely friction based

Change of center of gravity (CoG)

Adapted suspension mechanism with passive or active joints

LET'S RECAP OUR ORIGINAL GOAL …

Forward kinematics: Transformation from robot's configuration space (e.g., linear and angular wheels' velocities) to physical space (e.g., pose, velocity in the world frame)

Inverse kinematics: Transformation from physical space to configuration space

Kinematic equations

Kinematic constraints imposed by wheels (characteristics and arrangements)

Non holonomic constraints

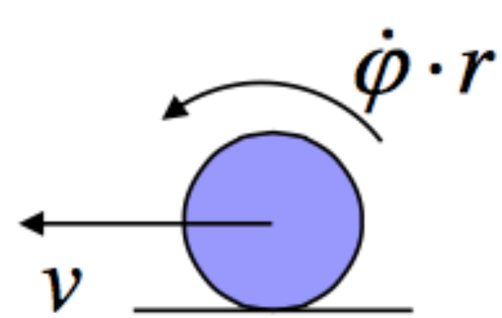
Geometric constraints (imposed by robot structure and task)

Holonomic constraints

ASSUMPTIONS ON WHEELS (TO EASE KINEMATICS)

- To create a kinematic model express constraints on the motions of individual wheels
- These motions are combined to compute motion for the whole robot
- Assumptions:
  - The wheel plane must remain vertical at all times
  - There is one single point of contact between the wheel and ground
  - There is no sliding at the single point of contact
  - Movement on a horizontal plane
  - Wheels not deformable
  - Pure rolling (v = 0 at contact point)
  - No slipping, skidding or sliding
  - No friction for rotation around contact point
  - Steering axes orthogonal to the surface
  - Wheels connected by rigid frame (chassis)
- Constraints
  - The wheel must roll when motion takes place in the opposite direction
  - The wheel must not slide orthogonal to the wheel plane

Applicable to standard wheels



The ***kinematic mobility (maneuverability)*** of a robot chassis is its ability to directly move in the environment, which is the result of:

1. The rule that every standard wheel must satisfy its no sliding and rolling constraints ($\leftrightarrow$Each wheel imposes zero or more constraints on the motion)

   No-motion line through the ICC/ICR

2. The additional freedom contributed by steering and spinning the steerable wheels

Mathematically, **maneuverability** is defined as the sum of:

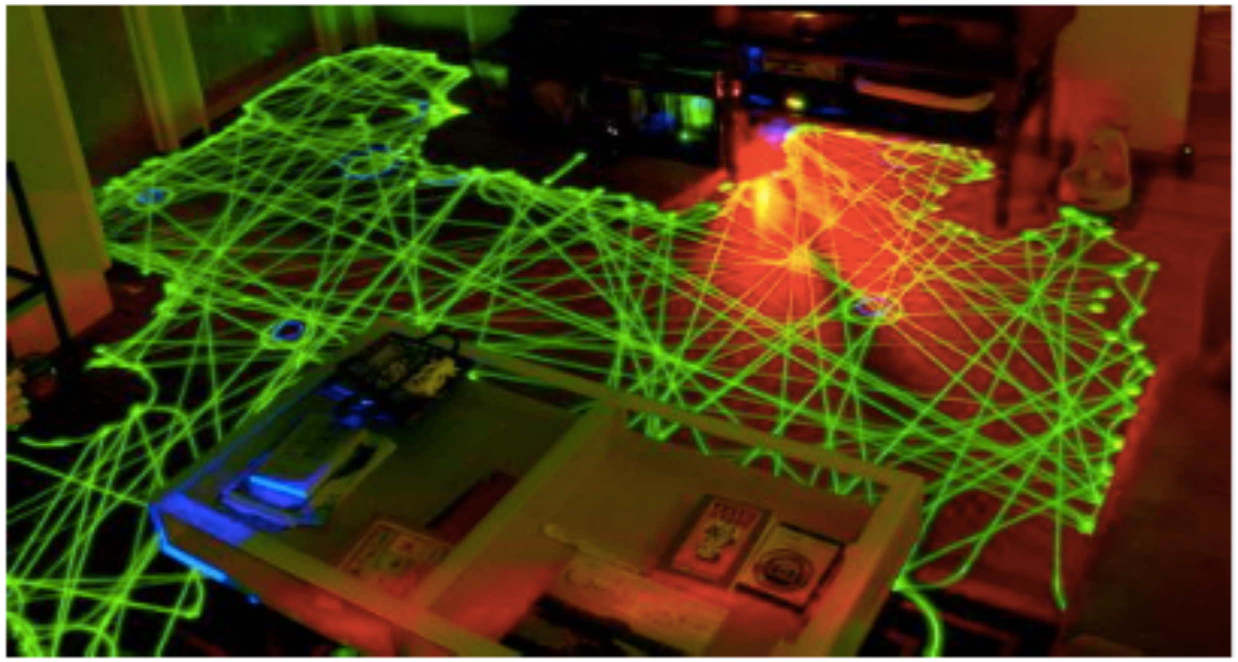
Degree of mobility, δ_m

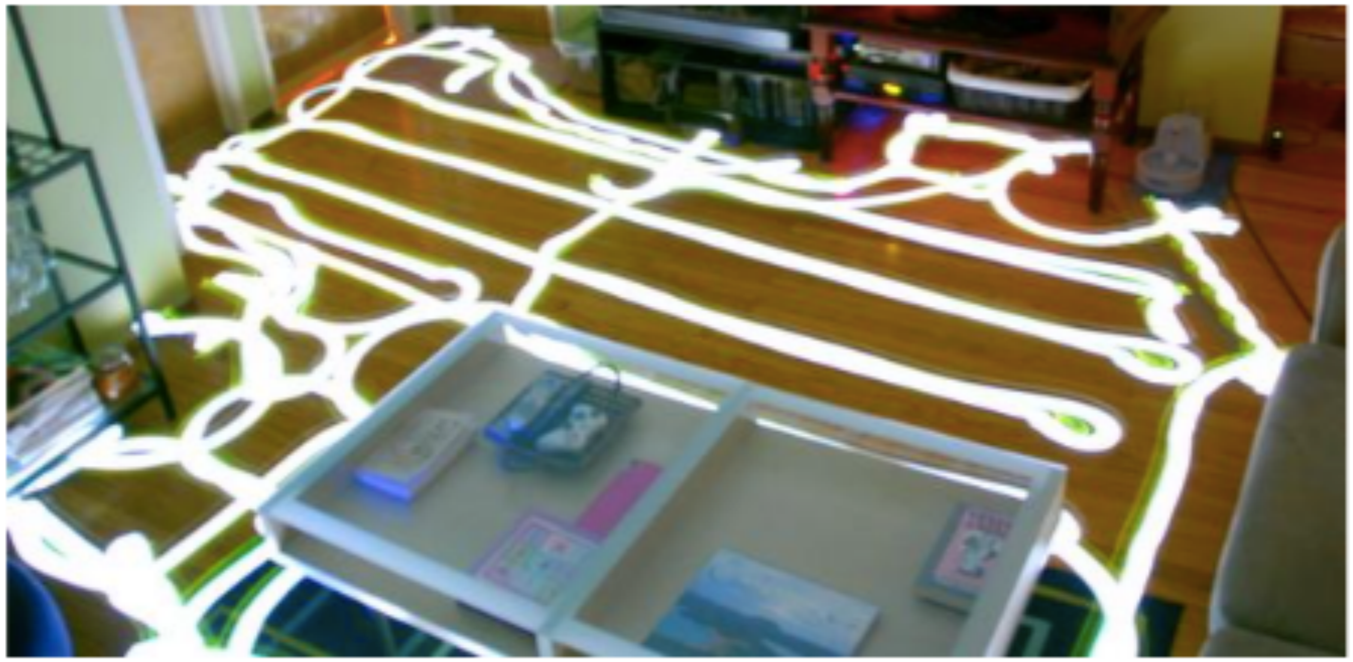
Degree of steerability, δ_s

HOLONOMIC DRIVE ROBOT IN ACTION

https://www.youtube.com/watch?v=2JxNjgKE8HQ

HOLONOMIC DRIVE VS NON HOLONOMIC IN ACTION





MOBILE ROBOT MANEUVERABILITY AND ICC/ICR

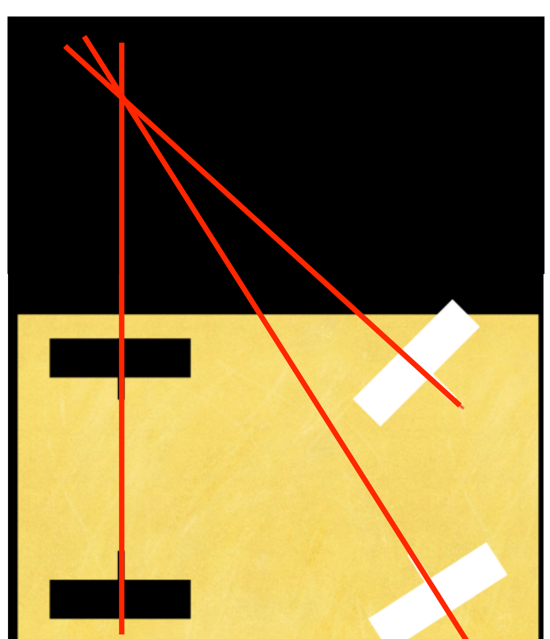
The ***degree of mobility*** quantifies the controllable degrees of freedom of a mobile robot based on the changes applied to wheel velocities

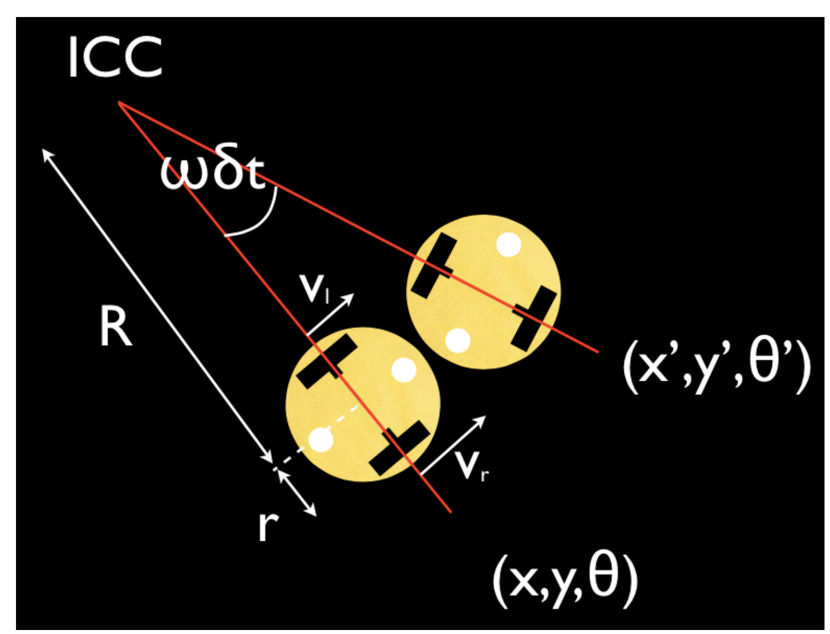
Holonomic: If the controllable degrees of freedom is equal to total degrees of freedom, then the robot is said to be Holonomic.

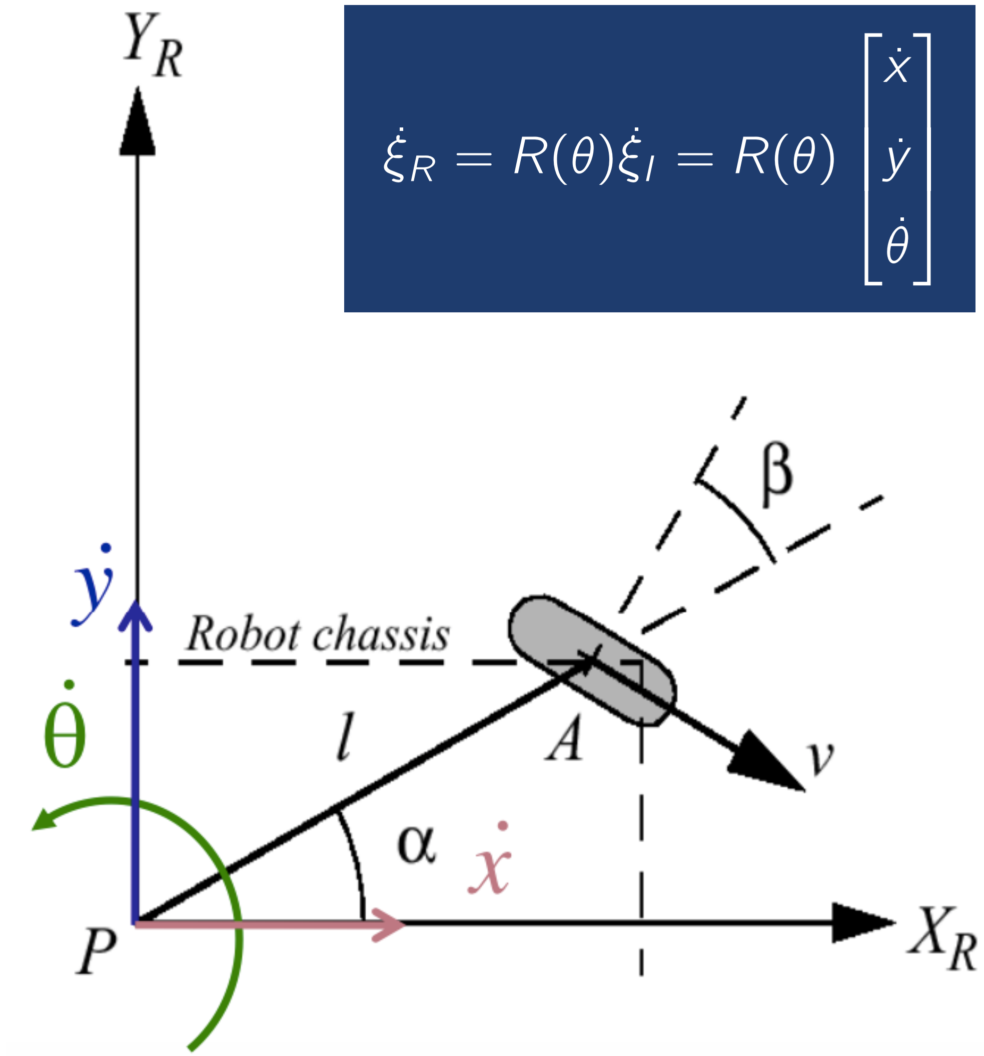
The kinematic constraints of a robot with respect to the degree of mobility can be demonstrated geometrically using the:

Instantaneous center of rotation (ICR) / Instantaneous center of curvature (ICC)

We will come back to this …



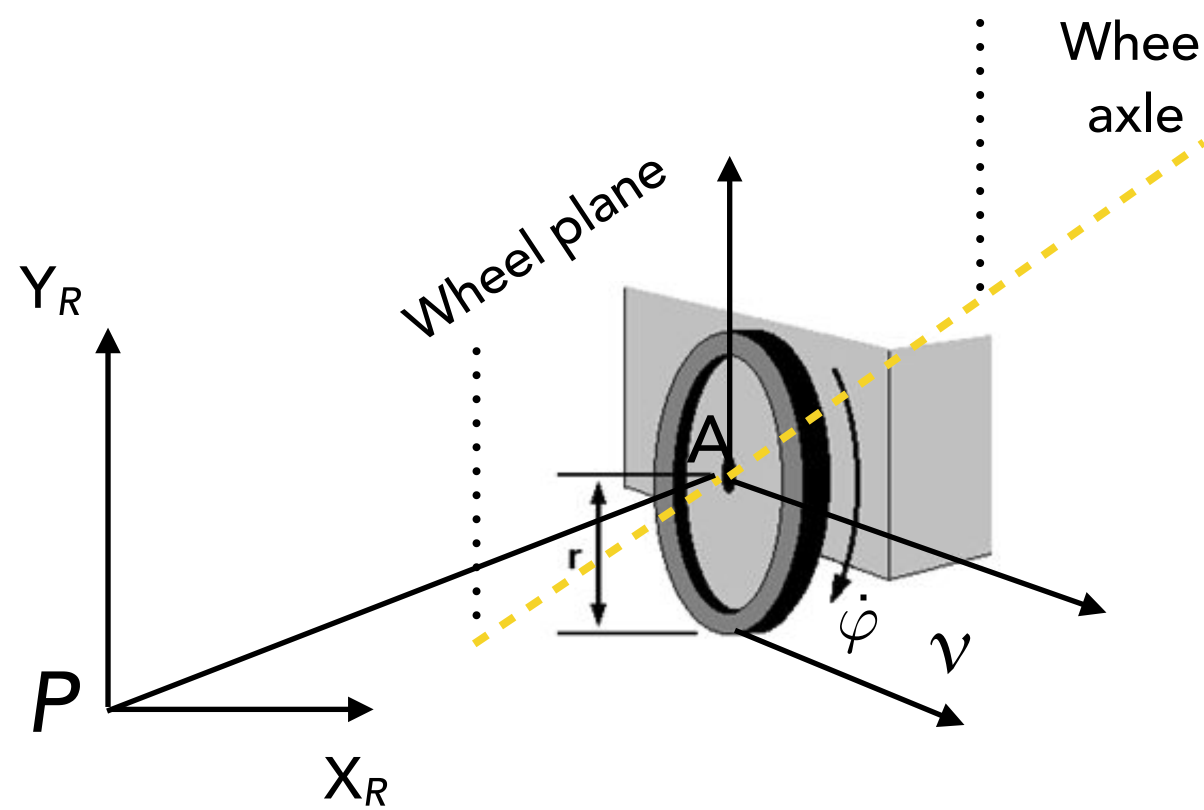




$$\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta)\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix}$$

Reference wheel point A (on the axle) is in *polar coordinates*: A(l, α)

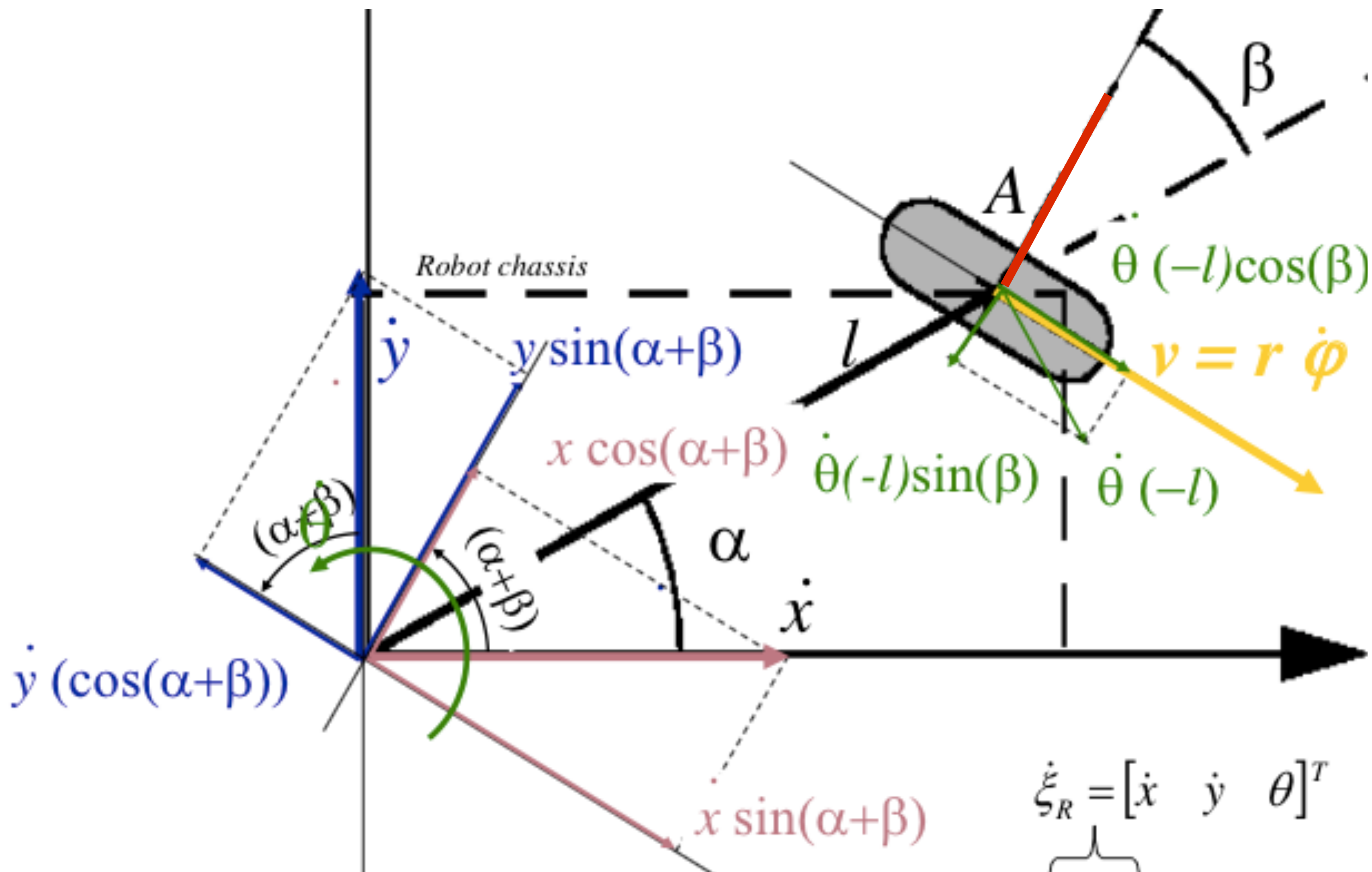
β: angle of wheel plane wrt chassis



- **Rolling constraint** (pure rolling at the contact point): All motion along the direction of the wheel plane is determined by wheel spin

- **Sliding constraint:** The component of the wheel's motion orthogonal to the wheel plane must be *zero*

FIXED STANDARD WHEEL



$$\dot{\xi}_R = [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$$

$$[\sin(\alpha+\beta) \ -\cos(\alpha+\beta) \ (-l)\cos\beta] R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$[\cos(\alpha+\beta) \ \sin(\alpha+\beta) \ l\sin\beta] R(\theta)\dot{\xi}_I = 0$$

3 component projection vectors

$\boldsymbol{\alpha}$, β, l are parameters

in the local {R} frame

The wheel, of radius r, **spins** over time such that its rotational position around the horizontal axle is a function of time:

$$\dot{\varphi}$$

and **linear velocity** is $r\dot{\varphi}$

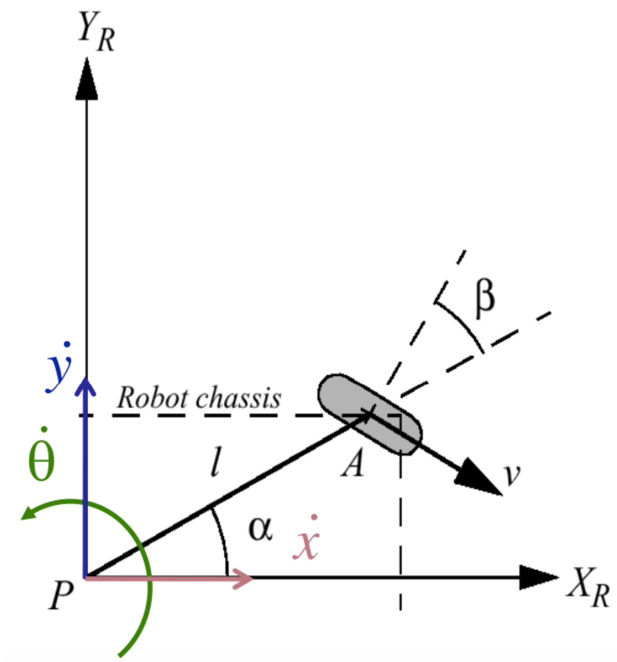
Rolling constraint:

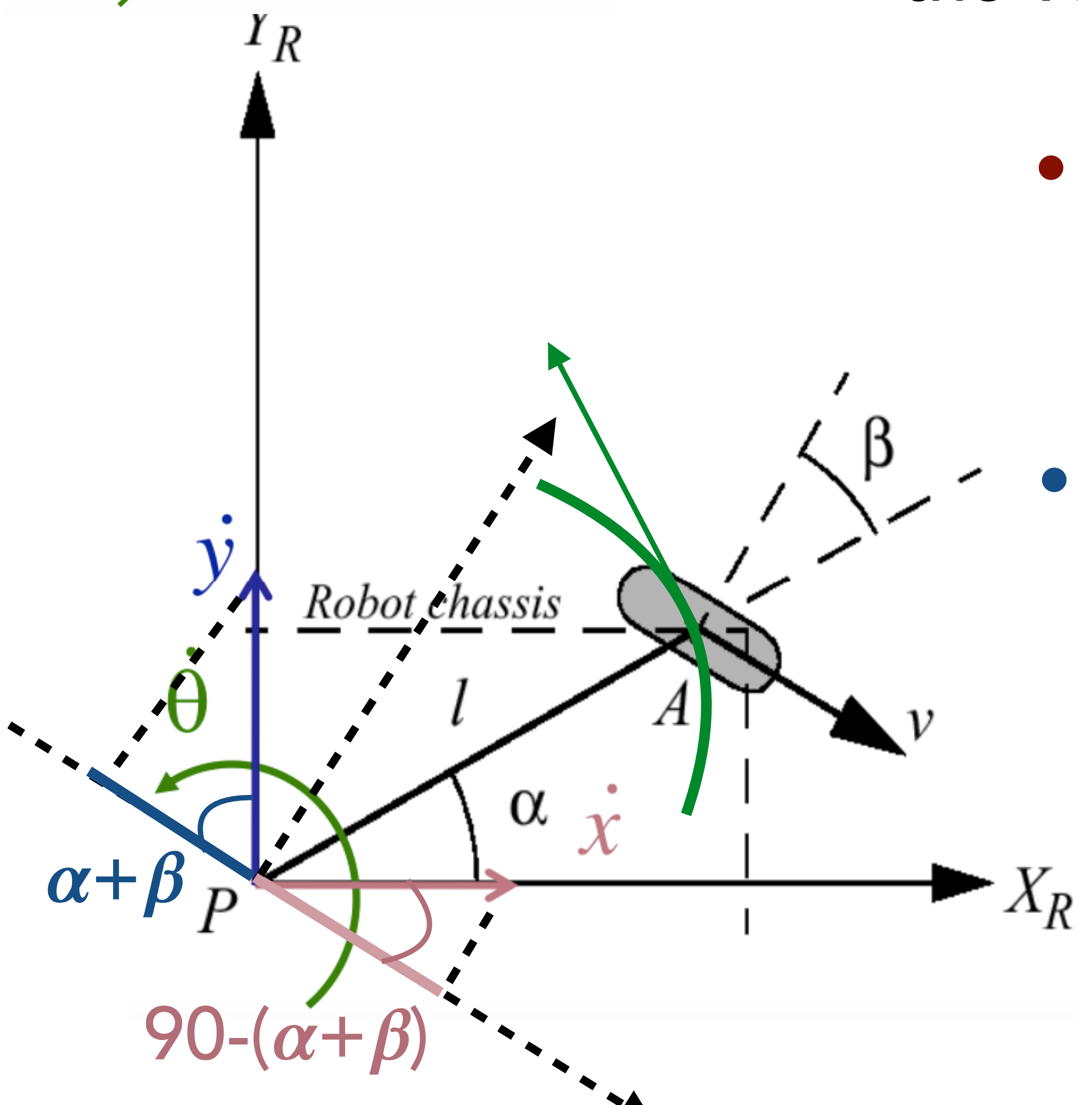
projections of ${}^R[\dot{x} \ \dot{y} \ \dot{\theta}]$ along wheel plane must equal linear velocity

Sliding constraint:

projections of ${}^R[\dot{x} \ \dot{y} \ \dot{\theta}]$ orthogonal to the wheel plane must be zero

FIXED STANDARD WHEEL: ROLLING COSTRAINT





- [$\dot{x}\,\dot{y}\,\dot{\theta}$] are the components of $\dot{\xi}$, the pose velocity vector in the coordinate frame {R} fixed to the robot in the reference point *P*.
- Projections of all robot's velocities (linear and angular) on the wheel's velocity plane must equal the velocity implied by the wheel's spinning

- Projection of $\dot{x}$ along wheel velocity plane:

  $\dot{x}$ cos(90-($\alpha+\beta$)) → $\dot{x}$ sin($\alpha+\beta$)

- Projection of $\dot{y}$ along wheel velocity plane:

  $\dot{y}$ (-cos($\alpha+\beta$))

- Projection of the robot angular velocity $\dot{\theta}$ (-*l*) along wheel velocity plane:

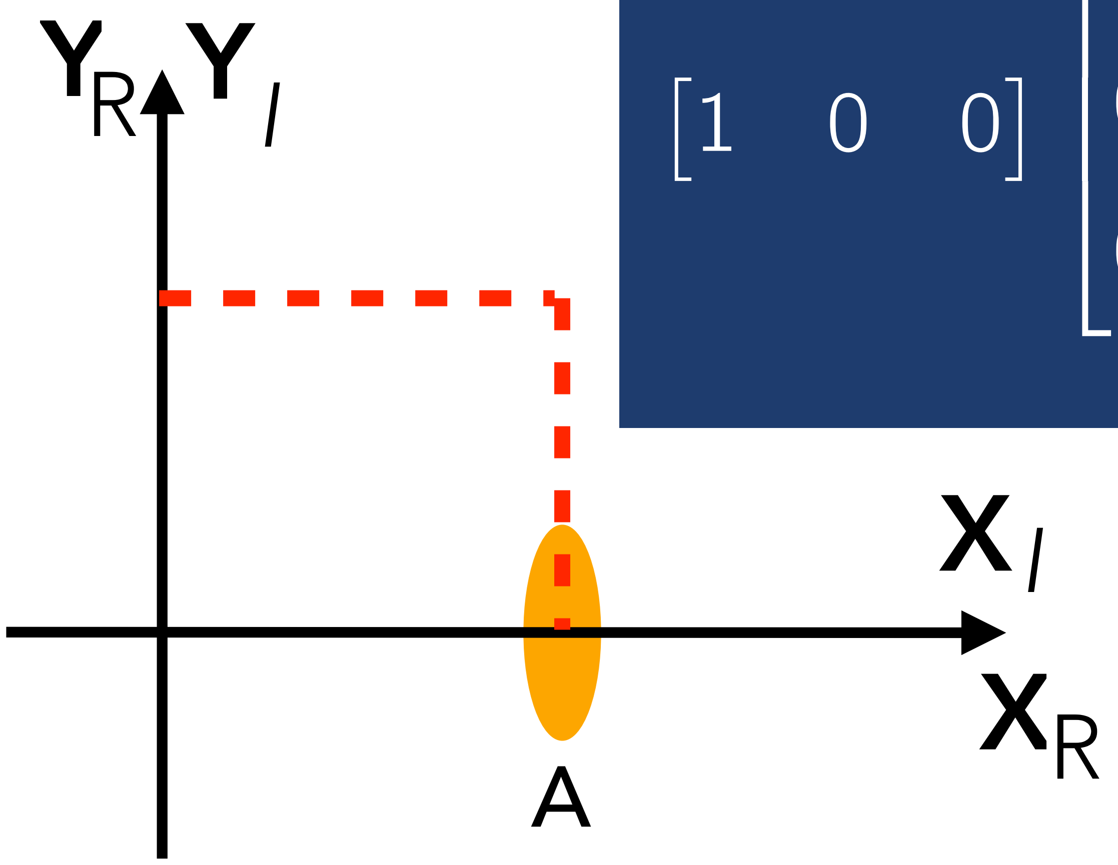
  $\dot{\theta}$ (-*l*)cos(β)

NUMERIC EXAMPLE

Wheel A is in position such that:

$\alpha = 0, \beta = 0, \theta = 0$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$



No instantaneous motion is possible along the x axis of the inertial frame

To be continued …