Calculating Heating by the Electron Beam In an Fe Foil

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1 Solving the Heat Equation Specific to the Hall A Møller Polarimeter

To calculate the heating of the Møller polarimeter iron foil we start with the heat equation. Given the geometry of the Møller foil where we have a circular10 μm thick foil with a beam heat source located at the center, we can assume this has no azimuthal or z-dependence and we are left with only a radial dependence:

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \nabla^2 T + \rho \alpha B_{flux} - \frac{2\sigma\epsilon}{\Delta z}\left(T^4 - T_0^4\right). \tag{1}$$

- $T(r,t)$ is the foil temperature in Kelvin,
- κ is the temperature dependent thermal conductivity of Fe which is approximately 0.8 W/(K cm) at room temperature,
- $\rho = 7.87$ g/cm^3 is the density of Fe,
- $\sigma = 5.67 \times 10^{-12}$ W/(K^4 cm^2) is the Stefan-Boltzmann constant,
- ϵ is the foil emissivity which depends on the polish and structure of the surface ranging from 0 (perfect polish) to 1 (perfect blackbody). Given the polish of the foil, something like 0.1 can be assumed.
- $T_0 = 294$ K, is the ambient temperature of the target ladder holding the foil at its boundary,
- $\Delta z = 10$ μm is the thickness of the foil,
- α is the collision stopping power for electrons in Fe. It is a function of electron energy and is 2.043 (MeV cm^2)/g=3.273$\times$10^{-13}(J cm^2)/g for a 10 GeV electron using ESTAR. The ESTAR data along with a 5-degree polynomial fit used to calculate α as a function of energy is shown in Fig. 1. Care should be exercised when extrapolating outside the 1-10 GeV range.
- $C_p = 0.45$ J/(g K) is the specific heat of Fe and,
- $B_{flux} = \frac{d^3 N_e}{dsdt}$ is the flux density of the beam in e^-/(cm^2 s).

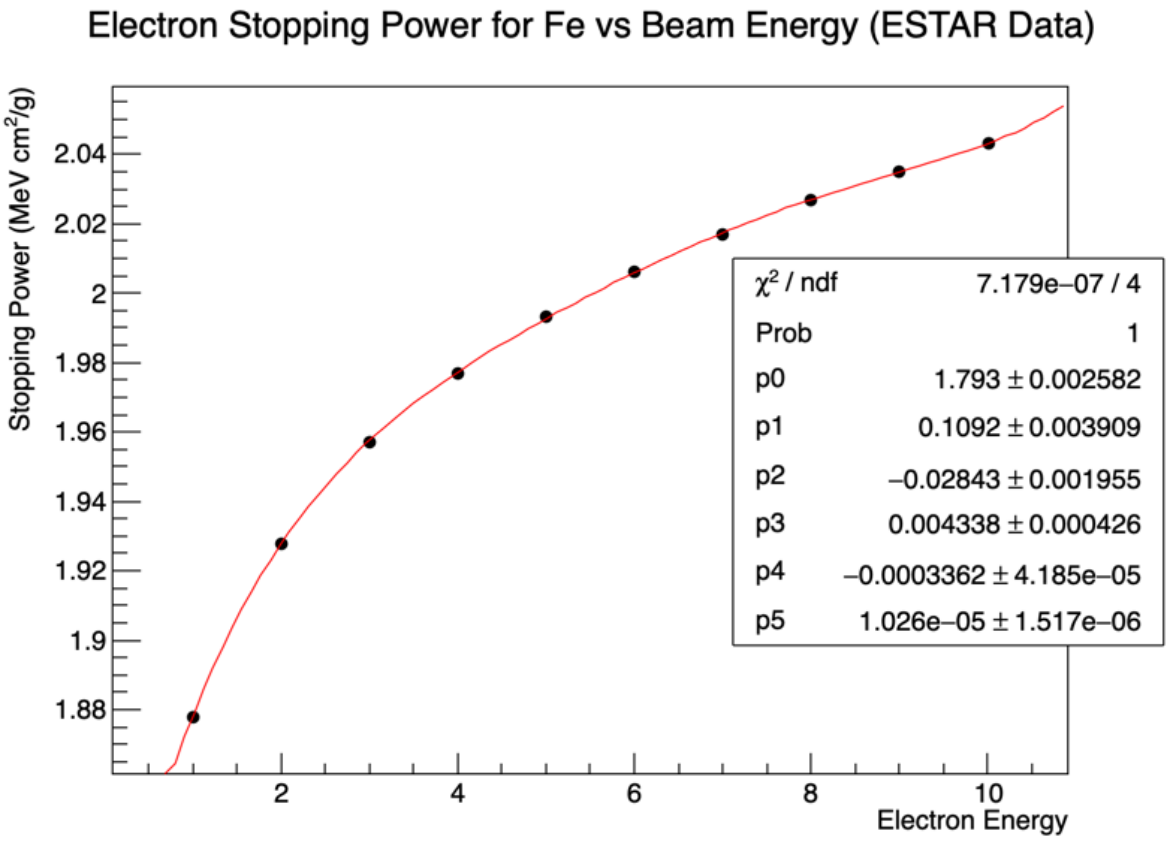

Figure 1: Stopping power for electrons as a function of energy in Fe. Data are from ESTAR and are fit to a 5-degree polynomial.

In principle T and B_{flux} are functions of position and time. However, we are interested in the temperature of the steady state which is presumably reached quite rapidly when the beam turns on. Setting $\frac{\partial T}{\partial t} = 0$ simplifies Eq. 1. The expected heat load on a 10 μm thick Fe foil in the electron beam is about 12 mW/μA. If the temperature increase with beam inside the beam flux is of 30 degrees Celsius or less, over a beam radius of 1 mm, then the radiated energy in this circular area is 0.13 mW or about 1% of the heat load. In this case, we can safely neglect the radiative cooling term. If we end up with a temperature increase greater than 30 degrees, then we will have to revisit this assumption. Under these assumptions, Eq. 1 simplifies to

$$\kappa \nabla^2 T = -\rho \alpha B_{flux} \tag{2}$$

$$\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\rho \alpha B_{flux} \tag{3}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{\rho \alpha}{\kappa} r B_{flux}. \tag{4}$$

The Hall A Møller polarimeter, does not typically take rastered beam, and it is thus reasonable to assume a Gaussian beam flux profile of radius r_b. Therefore, the Gaussian profiled electron flux B_{flux} from a beam current I in Amperes with a 1 σ radius of r_b becomes

$$B_{flux} = \frac{I}{1.6 \times 10^{-19}\,(2\pi r_b^2)} e^{-r^2/2r_b^2}. \tag{5}$$

Inserting this density profile for the electron beam heat source into Eq. 4 gives

$$\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\gamma r e^{-r^2/2r_b^2}, \tag{6}$$

where $\gamma \equiv \frac{I\rho\alpha}{1.6\times10^{-19}\kappa\left(2\pi r_b^2\right)}$. Integrating both sides of Eq. 6 w.r.t. r gives

$$r\frac{\partial T}{\partial r} = r_b^2\gamma e^{-r^2/2r_b^2} + C, \tag{7}$$

$$\frac{\partial T}{\partial r} = \frac{r_b^2\gamma}{r}e^{-r^2/2r_b^2} + \frac{C}{r} \tag{8}$$

where C is a constant of integration to be determined from boundary conditions in the steady state. To determine C, the total heat load from the beam is given by $I\alpha\rho\Delta z/1.6\times10^{-19} = 11.8\Delta z$ W/(μA cm). The heat flow through the boundary is the product of the conductivity κ, the cross sectional area of the foil along the foil perimeter $2\pi R_{foil}\Delta z$ and the temperature slope $\partial T/\partial r$, where length units are in cm. The perimeter of the foil at R_{foil} is assumed to be kept fixed at room temperature. The heat flow at the boundary has to equal the beam heat load in the steady state, so

$$\left(\kappa 2\pi R_{foil}\Delta z\right)\frac{\partial T}{\partial r}|_{r=R_{foil}} \approx -11.8\Delta z\left(\frac{\mathrm{W}}{\mu\mathrm{A\ cm}}\right) \approx \frac{\left(\kappa 2\pi R_{foil}\Delta z\right)C}{R_{foil}},$$

where the first term on the left side of Eq. 8) is not included since it is negligible at the boundary of the foil R_{foil}. The negative sign comes from the direction of heat flow towards higher radius making the temperature decrease with increasing r.

$$C \approx -\frac{11.8}{2\pi\kappa} = -2.50\left(\frac{\mathrm{K}}{\mu\mathrm{A}}\right),$$

where the value for Fe has been used κ =0.75 W/(K cm). Now to find the temperature difference between the outside perimeter of the foil at $r = R_{foil}$ and some $r < R_{foil}$ integrate both sides from R_{foil} to r yielding

$$\Delta T = \int_{R_{foil}}^{r}\left(\frac{r_b^2\gamma}{r'}e^{-r'^2/2r_b^2} + \frac{C}{r'}\right)dr'. \tag{9}$$

This can easily be integrated numerically as shown in Figures 2 and 3.

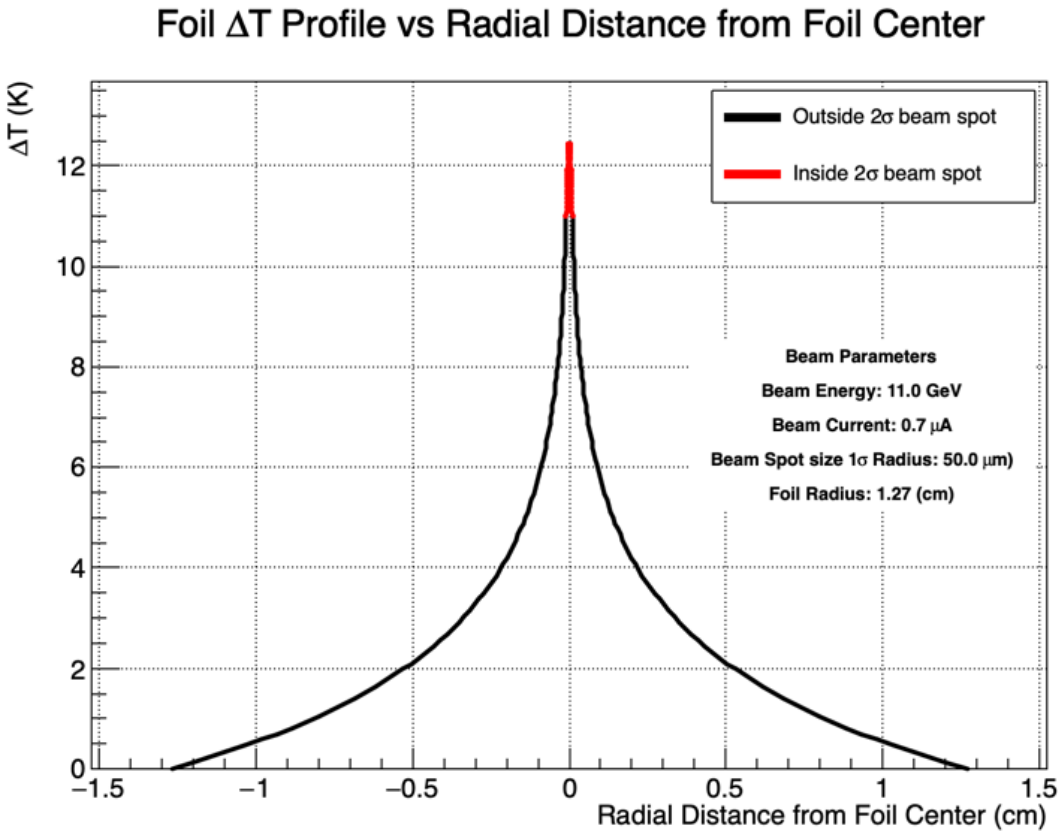

Figure 2: Fe foil ΔT profile from integrating Eq. 9 with beam spot size, and energy given. For this example, the foil diameter was 0.50 inches, the size of the target foils used during PREX/CREX.

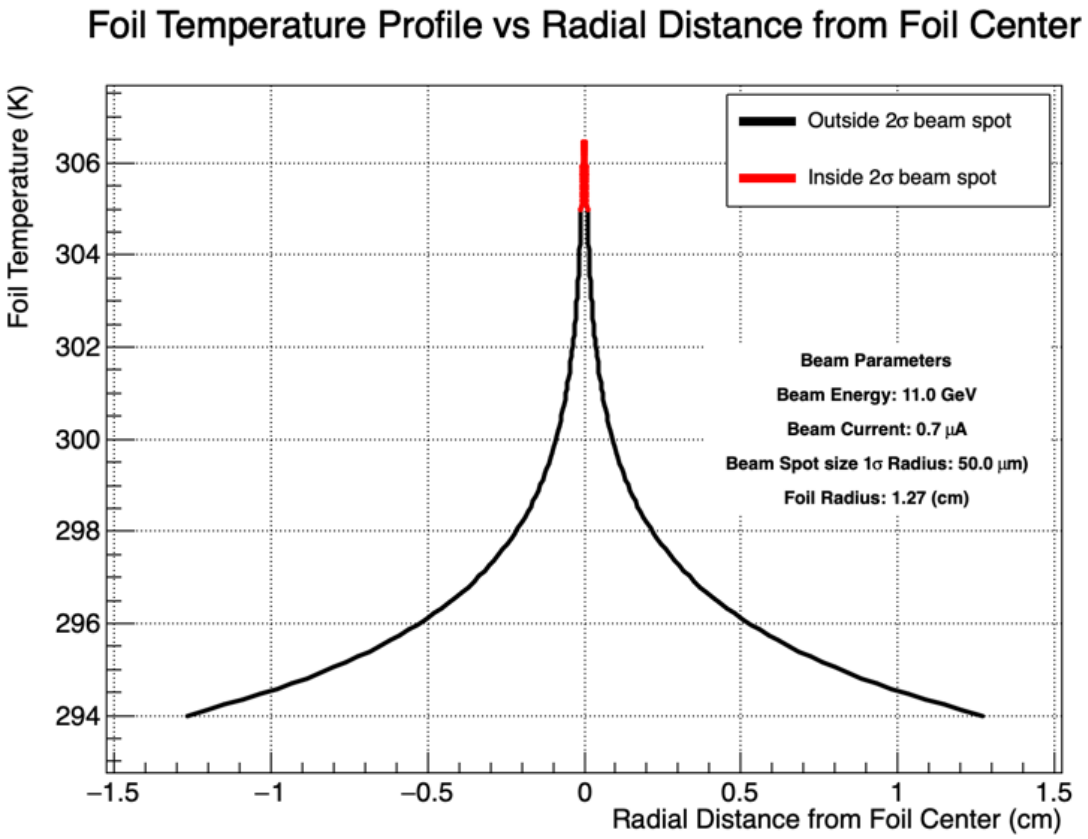

Figure 3: Fe foil temperature profile from integrating Eq. 9 with beam spot size, and energy given. For this example, the foil diameter was 0.50 inches, the size of the target foils used during PREX/CREX.

2 C++/ROOT Code for Numerically Integrating Eq. 9

The following ROOT macro uses Eq. 9 to calculate the foil heating for a circular Fe foil in a Gaussian profile electron beam.

```
#include "TF1.h"
#include <iostream>
#include "TGraph.h"
#include "TLegend.h"
#include "TAxis.h"
#include "TPad.h"
#include "TCanvas.h"
#include "TStyle.h"
#include "TPaveText.h"
#include "TString.h"

//////////////////
//Donald C. Jones//
//Nov. 2021      //
//////////////////

////////////////////////////////////////////////////////////////////////
//FeFoilHeating() calculates and graphs the temperature difference
//in a thin circular Fe foil between its edge held at a fixed
//temperature T0 and inside a circular Gaussian-distributed
//electron beam.
//
//
//Arguments:
//  beam_cur: beam current in Amperes
//  beam_r:   1 sigma beam spot size radius in cm
//  beam_E:   beam energy in GeV
//  T0:       ambient (Hall) temperature in Kelvin taken as foil
//            boundary temperature
//
//Returns the foil temperature difference in degrees K between T0
//at the foil edge and the temperature at the 1-sigma beam
//radius r_beam.
//
//NOTE: it is helpful to recall that for a 2D circular Gaussian
//distribution the volume between r=0 and the n-sigma points
//are as follows:
//1sigma = 39.35%, 2sigma = 86.47%, 3sigma = 98.89%, 4sigma = 99.97%
//Therefore, the temperature should be averaged over at least 3 sigma.
////////////////////////////////////////////////////////////////////////

double FeFoilHeating(double beam_cur = 1e-6, double beam_r=5e-3, double beam_E = 11,
    double T0 = 294){
  gStyle->SetStatY(0.7);
  gStyle->SetStatH(0.2);
```

```
gStyle->SetOptFit(1111);
gStyle->SetTitleW(0.95);


const double rho = 7.87;//density of Fe
const double sigma = 5.670e-12;//Stefan Boltzman constant W/(cm^2 K^4)
const double Cp = 0.45;//Fe specific heat capacity in J/(g K)
const double echarge = 1.602e-19;//Coulombs per electron
const double R_foil=2*0.5*2.54/2.0;//radius of Fe foil in cm
const double PI = 3.1415927;//pi obviously



//Use ESTAR data to estimate energy loss as a function of electron energy
//------------------------------------------------------------------------------
TCanvas *c = new TCanvas("c","c",0,0,800,600);
double beam_en[10]={1,2,3,4,5,6,7,8,9,10};//beam energy in GeV
double stop_en[10]={1.878,1.928,1.957,1.977,1.993, //collision stopping power
         2.006,2.017,2.027,2.035,2.043};//in (MeV cm^2/g) using ESTAR
TGraph *grStop = new TGraph(10,beam_en,stop_en);
grStop->SetTitle("Electron Stopping Power for Fe vs Beam Energy (ESTAR Data)");
grStop->SetMarkerStyle(8);
grStop->Draw("ap");
grStop->GetXaxis()->SetTitle("Electron Energy");
grStop->GetYaxis()->SetTitle("Stopping Power (MeV cm^{2}/g)");
gPad->Update();
TF1 *fStop = new TF1("fStop","pol5",0,1);//use fit to give continuous function
grStop->Fit(fStop);
double alpha = echarge*fStop->Eval(beam_E)*1e6;//Collision stopping power in
    (Jcm^2/g)
cout<<"Stopping power "<<alpha<<" (J cm^2/g)"<<endl;
c->SaveAs("StoppingPower.png");



//Calculate the energy dependent thermal conductivity of Fe using data either from
//https://www.efunda.com/materials/elements/TC_Table.cfm?Element_ID=Fe
//or
//https://www.engineeringtoolbox.com/thermal-conductivity-metals-d_858.html
//------------------------------------------------------------------------------
bool data_efunda = 1;
TCanvas *ct = new TCanvas("ct","ct",0,0,800,600);
double temp[4] = {250,300,350,400};
double cond[4] = {0.865,0.802,0.744,0.695};//www.efunda.com
TGraph *grC = new TGraph(4,temp,cond);
grC->SetTitle("Fe Thermal Conductivity vs. Temperature");
grC->SetMarkerStyle(8);
grC->Draw("ap");
grC->GetXaxis()->SetTitle("Temperature (k)");
grC->GetYaxis()->SetTitle("Thermal Conductivity (W/cm K)");
```

```
TF1 *fCond = new TF1("fCond","pol2",0,1);
grC->Fit(fCond);
gPad->Update();
if(!data_efunda)//www.engineeringtoolbox.com
  fCond = new TF1("fCond","0.835-0.001102*(x-273)",0,1);
double guessTemp = T0+15*beam_cur/1e-6;//starting guess for final foil temperature
double kappa = fCond->Eval(guessTemp);
cout<<"Conductivity at "<<guessTemp<<" K is "<<kappa<<endl;
ct->SaveAs("FeThermalCond.png");



//Integral of f(r) gives delta T. Create the integrand f(r)
//-------------------------------------------------------------------------------
double gam = beam_cur/echarge*rho*alpha/kappa/2./PI/pow(beam_r,2);
double C = -beam_cur/echarge*alpha*rho/2.0/PI/kappa;
TF1 *f = new TF1("f",Form("%e/x*exp(-x*x/%e)+%e/x",
          beam_r*beam_r*gam,2*beam_r*beam_r,C),0,R_foil);



//Improve thermal conductivity estimate using the calculated temperature.
//Temperature at 1.3*beam_r is a good estimate of the average temperature
//weighted by the beam spot charge distribution.
//-------------------------------------------------------------------------------
guessTemp = f->Integral(R_foil,1.3*beam_r)+T0;
kappa = fCond->Eval(guessTemp);
gam = beam_cur/echarge*rho*alpha/kappa/2./PI/pow(beam_r,2);
C = -beam_cur/echarge*alpha*rho/2.0/PI/kappa;
cout<<"Conductivity re-calculated at "<<guessTemp<<" K is "<<kappa<<endl;
f = new
    TF1("f",Form("%e/x*exp(-x*x/%e)+%e/x",beam_r*beam_r*gam,2*beam_r*beam_r,C),0,R_foil);



//Graph resulting temperature profile by integrating f(r)dr. Make points red inside
//2 sigma beam spot size radius.
//-------------------------------------------------------------------------------
const int N=500;
double r[N], T[N], dT[N],ri[N],Ti[N], dTi[N];
int n=0, ni=0;
double rp = R_foil;
for(int i=0;i<N/2;++i){
  r[i]=rp;
  dT[i] = f->Integral(R_foil,rp);
  T[i] = dT[i]+T0;
  if(rp<2*beam_r){
    ri[ni]=rp;
    Ti[ni]=T[i];
    dTi[ni]=dT[i];
```

```
    ++ni;
  }
  rp*=0.95;
  ++n;
  if(rp<0.00001)break;
}
for(int i=0;i<n;++i){
  r[i+n]=-r[n-i-1];
  dT[i+n] = dT[n-i-1];
  T[i+n] = T[n-i-1];
}
for(int i=0;i<ni;++i){
  ri[i+ni]=-ri[ni-i-1];
  dTi[i+ni] = dTi[ni-i-1];
  Ti[i+ni] = Ti[ni-i-1];
}
TCanvas *c1 = new TCanvas("c1","c1",0,0,800,600);
TGraph *grdT = new TGraph(2*n,r,dT);
grdT->SetMarkerStyle(8);
grdT->SetLineWidth(6);
grdT->SetMarkerSize(0.3);
grdT->Draw("acp");
grdT->SetTitle(Form("Foil #DeltaT Profile vs Radial Distance from Foil Center"));
grdT->GetXaxis()->SetTitle("Radial Distance from Foil Center (cm)");
grdT->GetYaxis()->SetTitle("#DeltaT (K)");
TGraph *gridT = new TGraph(2*ni,ri,dTi);
gridT->SetMarkerStyle(8);
gridT->SetMarkerColor(kRed);
gridT->SetLineColor(kRed);
gridT->SetLineWidth(6);
gridT->SetMarkerSize(0.4);
gridT->Draw("samecp");
gPad->SetGrid();
TPaveText *pt = new TPaveText(0.6,0.4,0.89,0.6,"ndc");
pt->SetFillColor(0);
pt->SetShadowColor(0);
pt->SetBorderSize(0);
pt->AddText("Beam Parameters");
pt->AddText(Form("Beam Energy: %0.1f GeV",beam_E));
pt->AddText(Form("Beam Current: %0.1f #muA", beam_cur*1e6));
pt->AddText(Form("Beam Spot size 1#sigma Radius: %0.1f #mum)",beam_r*1e4));
pt->AddText(Form("Foil Radius: %0.2f (cm)",R_foil));
pt->Draw();
TLegend *lg = new TLegend(0.62,0.76,0.89,0.89);
lg->AddEntry(grdT,"Outside 2#sigma beam spot","lp");
lg->AddEntry(gridT,"Inside 2#sigma beam spot","lp");
lg->Draw();
c1->SaveAs("FoilHeatingdT.png");
TCanvas *c2 = new TCanvas("c2","c2",0,0,800,600);
TGraph *gr = new TGraph(2*n,r,T);
```

```
  gr->SetMarkerStyle(8);
  gr->SetLineWidth(6);
  gr->SetMarkerSize(0.3);
  gr->Draw("acp");
  gr->SetTitle(Form("Foil Temperature Profile vs Radial Distance from Foil Center"));
  gr->GetYaxis()->SetTitle("Foil Temperature (K)");
  gr->GetXaxis()->SetTitle("Radial Distance from Foil Center (cm)");
  TGraph *gri = new TGraph(2*ni,ri,Ti);
  gri->SetMarkerStyle(8);
  gri->SetMarkerColor(kRed);
  gri->SetLineColor(kRed);
  gri->SetLineWidth(6);
  gri->SetMarkerSize(0.4);
  gri->Draw("samecp");
  gPad->SetGrid();
  lg->Draw();
  pt->Draw();
  c2->SaveAs("FoilHeatingT.png");



  //Integrate f(r) weighted by the beam charge distribution to find average delta T
  //-------------------------------------------------------------------------------
  gStyle->SetOptFit(0);
  TF1 *fGaus = new TF1("fGaus","[0]*exp(-x*x/(2*[1]*[1]))+[2]",-2*beam_r,2*beam_r);
  fGaus->SetParameters(guessTemp/2.,beam_r,T0);
  gr->Fit(fGaus,"r");
  TString func = Form("(%e*exp(-x*x/(2*%e))+%e)*x*exp(-x*x/2./%e)/%e",
           fGaus->GetParameter(0),pow(fGaus->GetParameter(1),2),
           fGaus->GetParameter(2),beam_r*beam_r,beam_r*beam_r);
  TF1 *fAvgT = new TF1("fAvgT",func.Data(),0,1);
  fAvgT->SetNpx(1000);
   //fAvgT->Draw();
  cout<<"dT at 1.3 sigma is "<<f->Integral(R_foil,beam_r*1.3)<<endl;



  //Return average temperature, weighted by the beam spot charge distribution.
  //-------------------------------------------------------------------------------
  return fAvgT->Integral(0,10*beam_r);
}
```