

Calculating Heating by the Electron Beam In an Fe Foil

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This technical note TargetHeating.tex, TargetHeating.pdf and the accompanying code FeFoil-Heating.C can be found in the following Github repository:
<https://github.com/jonesdc76/MollerPolarimetry/tree/master/TargetPolarization>

1 Solving the Heat Equation for Conditions Specific to the Hall A Møller Polarimeter

To calculate the heating of the Møller polarimeter iron foil we start with the heat equation. Given the geometry of the Møller foil where we have a circular $10\ \mu\text{m}$ thick foil with a beam heat source located at the center, we can assume this has no azimuthal or z-dependence and we are left with only a radial dependence:

$$\rho C_p \frac{\partial T}{\partial t} = \kappa \nabla^2 T + \rho \alpha B_{flux} - \frac{2\sigma\epsilon}{\Delta z} (T^4 - T_0^4). \quad (1)$$

- $T(r, t)$ is the foil temperature in Kelvin,
- κ is the temperature dependent thermal conductivity of Fe which is approximately $0.8\ \text{W}/(\text{K cm})$ at room temperature (see Fig. 1),
- $\rho = 7.87\ \text{g}/\text{cm}^3$ is the density of Fe,
- $\sigma = 5.67 \times 10^{-12}\ \text{W}/(\text{K}^4\ \text{cm}^2)$ is the Stefan-Boltzmann constant,
- ϵ is the foil emissivity which depends on the polish and structure of the surface ranging from 0 (perfect polish) to 1 (perfect blackbody). Given the polish of the foil, something like 0.1 can be assumed.
- $T_0 = 294\ \text{K}$, is the ambient temperature of the target ladder holding the foil at its boundary,
- $\Delta z = 10\ \mu\text{m}$ is the thickness of the foil,
- α is the collision stopping power for electrons in Fe. It is a function of electron energy and is $2.043\ (\text{MeV cm}^2)/\text{g} = 3.273 \times 10^{-13}\ (\text{J cm}^2)/\text{g}$ for a 10 GeV electron using ESTAR. The ESTAR data along with a 5-degree polynomial fit used to calculate α as a function of energy is shown in Fig. 2. Care should be exercised when extrapolating outside the 1-10 GeV range.

- $C_p = 0.45 \text{ J/(g K)}$ is the specific heat of Fe and,
- $B_{flux} = \frac{d^3 N_e}{dsdt}$ is the flux density of the beam in $e^-/(\text{cm}^2 \text{ s})$.

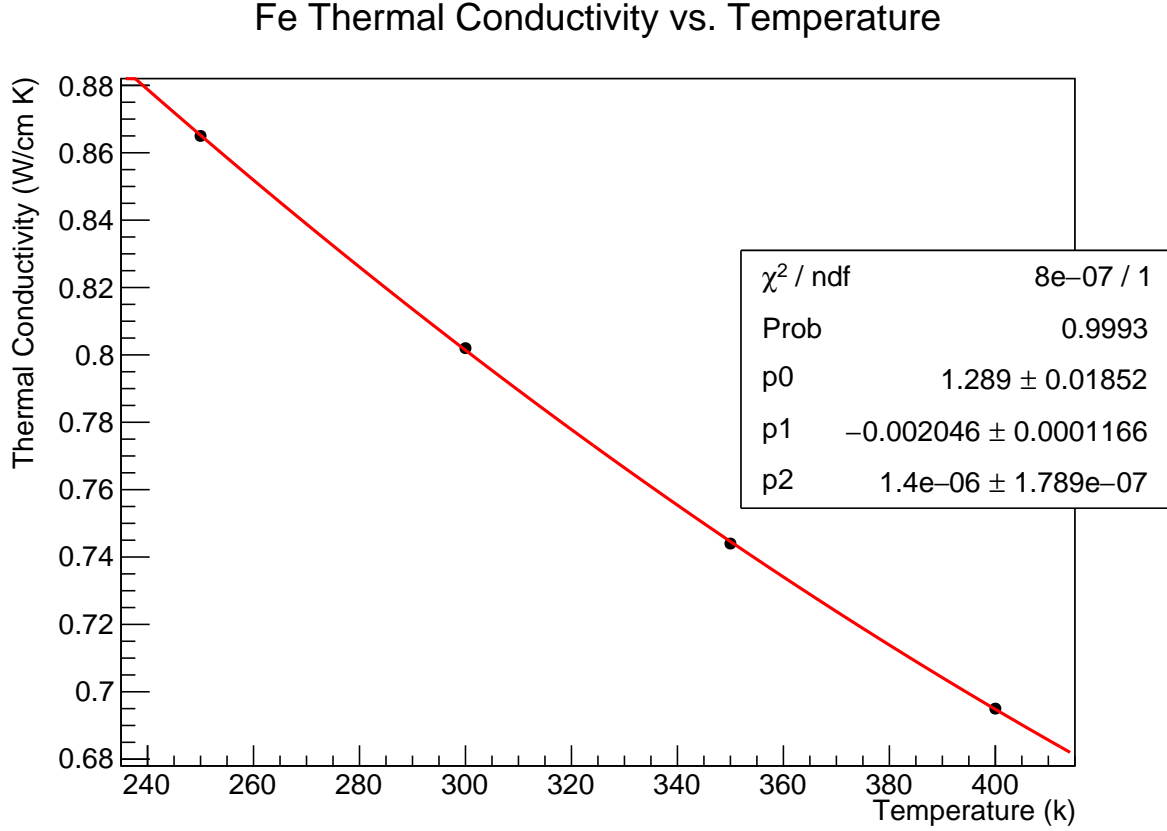


Figure 1: Fe thermal conductivity κ as a function of temperature. Data are from https://www.efunda.com/materials/elements/TC_Table.cfm?Element_ID=Fe and are fit to a 2nd degree polynomial.

In principle T and B_{flux} are functions of position and time. However, we are interested in the temperature of the steady state which is presumably reached quite rapidly when the beam turns on. Setting $\frac{\partial T}{\partial t} = 0$ simplifies Eq. 1. The expected heat load on a $10 \mu\text{m}$ thick Fe foil in the electron beam is about $12 \text{ mW}/\mu\text{A}$. If the temperature increase with beam inside the beam flux is of 30 degrees Celsius or less, over a beam radius of 1 mm, then the radiated energy in this circular area is 0.13 mW or about 1% of the heat load. In this case, we can safely neglect the radiative cooling term. If we end up with a temperature increase greater than 30 degrees, then we will have to revisit this assumption. Under these assumptions, Eq. 1 simplifies to

$$\kappa \nabla^2 T = -\rho \alpha B_{flux} \quad (2)$$

$$\frac{\kappa}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = -\rho \alpha B_{flux} \quad (3)$$

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = -\frac{\rho \alpha}{\kappa} r B_{flux}. \quad (4)$$

Electron Stopping Power for Fe vs Beam Energy (ESTAR Data)

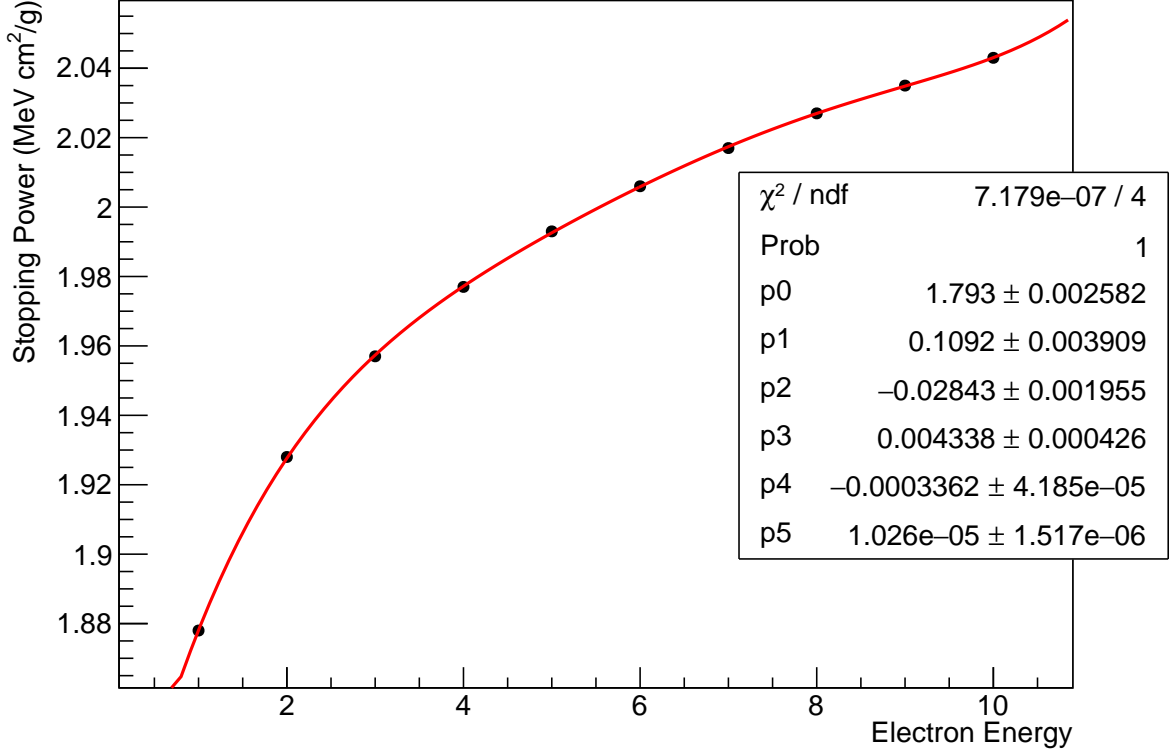


Figure 2: Stopping power for electrons as a function of energy in Fe. Data are from ESTAR and are fit to a 5-degree polynomial.

1.1 Solving for a heating from a Gaussian profile beam spot

The Hall A Möller polarimeter, does not typically take rastered beam, and it is thus reasonable to assume a Gaussian beam flux profile of radius r_b . Therefore, the Gaussian profiled electron flux B_{flux} from a beam current I in Amperes with a 1σ radius of r_b becomes

$$B_{flux} = \frac{I}{1.6 \times 10^{-19} (2\pi r_b^2)} e^{-r^2/2r_b^2}. \quad (5)$$

Inserting this density profile for the electron beam heat source into Eq. 4 gives

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = -\gamma r e^{-r^2/2r_b^2}, \quad (6)$$

where $\gamma \equiv \frac{I\rho\alpha}{1.6 \times 10^{-19} \kappa (2\pi r_b^2)}$. Integrating both sides of Eq. 6 w.r.t. r gives

$$r \frac{\partial T}{\partial r} = r_b^2 \gamma e^{-r^2/2r_b^2} + C, \quad (7)$$

$$\frac{\partial T}{\partial r} = \frac{r_b^2 \gamma}{r} e^{-r^2/2r_b^2} + \frac{C}{r} \quad (8)$$

where C is a constant of integration to be determined from boundary conditions in the steady state. To determine C , the total heat load from the beam is given by $I\alpha\rho\Delta z/1.6 \times 10^{-19} =$

$16.1\Delta z \text{ W}/(\mu\text{A cm})$. The heat flow through the boundary is the product of the conductivity κ , the cross sectional area of the foil along the foil perimeter $2\pi R_{foil}\Delta z$ and the temperature slope $\partial T/\partial r$, where length units are in cm. The perimeter of the foil at R_{foil} is assumed to be kept fixed at room temperature. The heat flow at the boundary has to equal the beam heat load in the steady state, so

$$(\kappa 2\pi R_{foil}\Delta z) \frac{\partial T}{\partial r}|_{r=R_{foil}} \approx -16.1\Delta z \left(\frac{\text{W}}{\mu\text{A cm}} \right) \approx \frac{(\kappa 2\pi R_{foil}\Delta z) C}{R_{foil}},$$

where the first term on the left side of Eq. 8) is not included since it is negligible at the boundary of the foil R_{foil} . The negative sign comes from the direction of heat flow towards higher radius making the temperature decrease with increasing r .

$$C \approx -\frac{16.1}{2\pi\kappa} = -3.20 \left(\frac{\text{K}}{\mu\text{A}} \right),$$

where the temperature dependent κ for Fe has been used (see Fig. 1). Now to find the temperature difference between the outside perimeter of the foil at $r = R_{foil}$ and some $r < R_{foil}$ integrate both sides from R_{foil} to r yielding

$$\Delta T = \int_{R_{foil}}^r \left(\frac{r_b^2 \gamma}{r'} e^{-r'^2/2r_b^2} + \frac{C}{r'} \right) dr'. \quad (9)$$

This can easily be integrated numerically as shown in Figures 3 and 4.

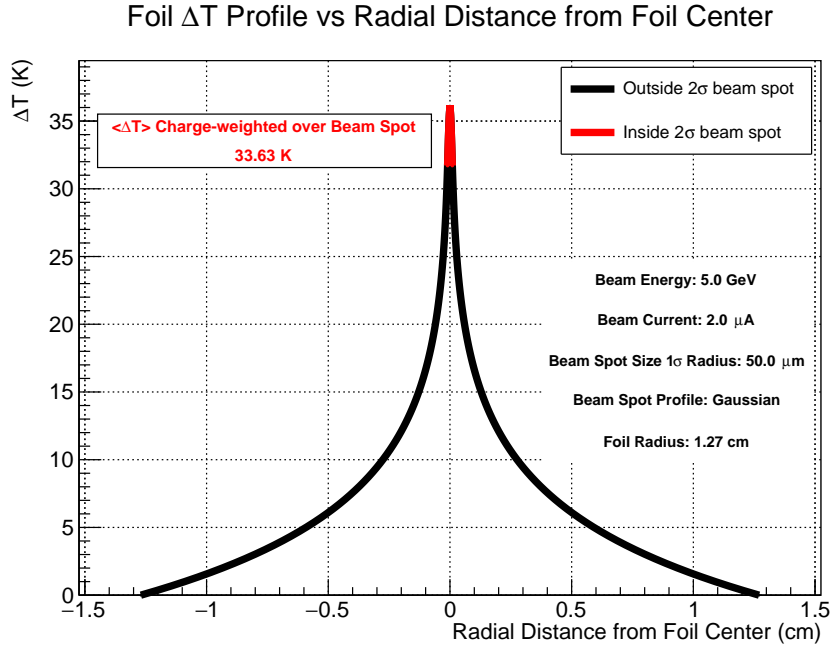


Figure 3: Fe foil ΔT profile from integrating Eq. 9 with beam spot size, and energy given.

1.2 Solving for heating from a uniform circular distribution

In the case where a beam is rastered, the charge distribution can be considered to be uniform. Let's solve for the case of a uniform circular raster pattern of radius r_{rast} centered on the foil. In

Foil Temperature Profile vs Radial Distance from Foil Center

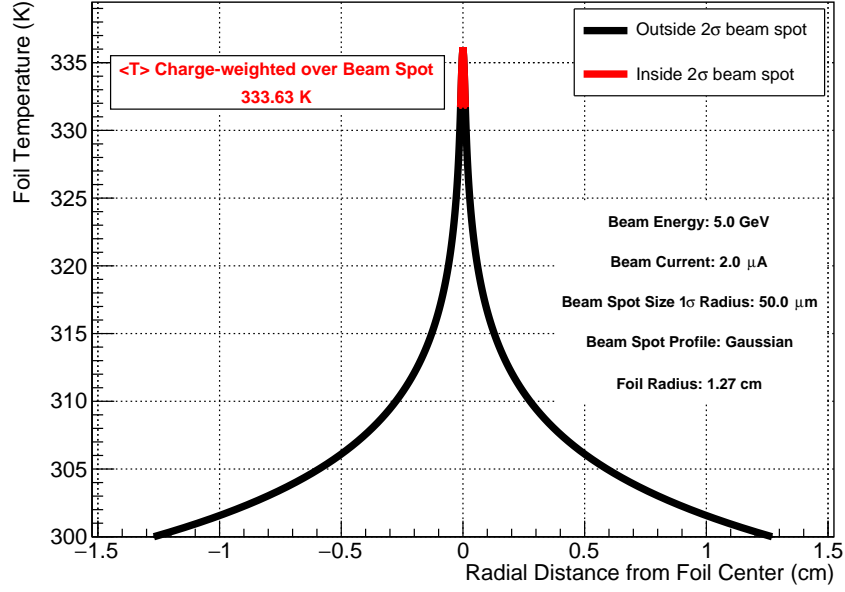


Figure 4: Fe foil temperature profile from integrating Eq. 9 with beam spot size, and energy given.

this case, the electron flux density B is given by

$$B = \frac{I\Theta(r_{rast} - r)}{1.6 \times 10^{-19} \pi r_{rast}^2}, \quad (10)$$

where $\Theta(r_{rast} - r)$ is the Heaviside function which is unity for $r < r_{rast}$ and zero for $r > r_{rast}$. Inserting Eq. 10 into Eq. 4 yields

$$\frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) = -\frac{\rho\alpha}{\kappa} \frac{I\Theta(r_{rast} - r)}{1.6 \times 10^{-19} \pi r_{rast}^2} r \quad (11)$$

$$= -\gamma\Theta(r_{rast} - r)r, \quad (12)$$

where $\gamma \equiv \frac{\rho\alpha I}{1.602 \times 10^{-19} \kappa \pi r_{rast}^2}$. Integrating both sides with respect to r gives

$$r \frac{\partial T}{\partial r} = \begin{cases} -\frac{\gamma r^2}{2} + C, & r < r_{rast} \\ -\frac{\gamma r_{rast}^2}{2} + C, & r \geq r_{rast}. \end{cases}$$

This becomes

$$\frac{\partial T}{\partial r} = \begin{cases} -\frac{\gamma r}{2} + \frac{C}{r}, & r < r_{rast} \\ -\frac{\gamma r_{rast}}{2} + \frac{C}{r}, & r \geq r_{rast}. \end{cases}$$

Similar to before, the heat flow through the foil thickness at $r \geq r_{rast}$ has to equal the beam heat load in the steady state, so let's solve at $r = r_{rast}$:

$$(\kappa 2\pi r_{rast} \Delta z) \frac{\partial T}{\partial r} \Big|_{r=R_{foil}} \approx \frac{\rho\alpha I}{1.602 \times 10^{-19}} \Delta z \left(\frac{\text{W}}{\mu\text{A cm}} \right) \approx \left(-\frac{\gamma r_{rast}^2}{2} + C \right) 2\pi\kappa\Delta z.$$

Solving gives $C = 0$, so we now have

$$\frac{\partial T}{\partial r} = \begin{cases} -\frac{\gamma r}{2}, & r < r_{rast} \\ -\frac{\gamma r_{rast}}{2}, & r \geq r_{rast}. \end{cases} \quad (13)$$

Integrating both sides with respect to r in reverse direction from $r = R_{foil}$ to $r \leq r_{rast}$ gives ΔT which can be solved analytically:

$$\Delta T = -\frac{\gamma r_{rast}^2}{2} \int_{R_{foil}}^{r_{rast}} \frac{dr}{r} - \frac{\gamma}{2} \int_{r_{rast}}^r r' dr' \quad (14)$$

$$= \frac{\gamma r_{rast}^2}{2} \ln \left(\frac{R_{foil}}{r_{rast}} \right) + \frac{\gamma}{4} (r_{rast}^2 - r^2) . \quad (15)$$

Figures 6 and 6 give plots of ΔT and T respectively for a uniformly rastered beam for the parameters given on the plots.

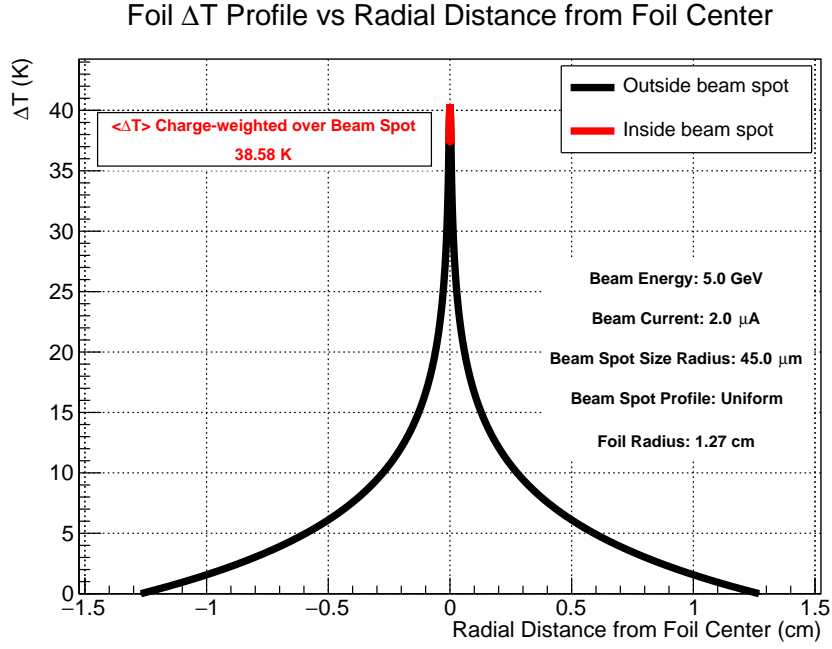


Figure 5: Fe foil ΔT profile from using Eq. 15 with beam spot size, and energy given.

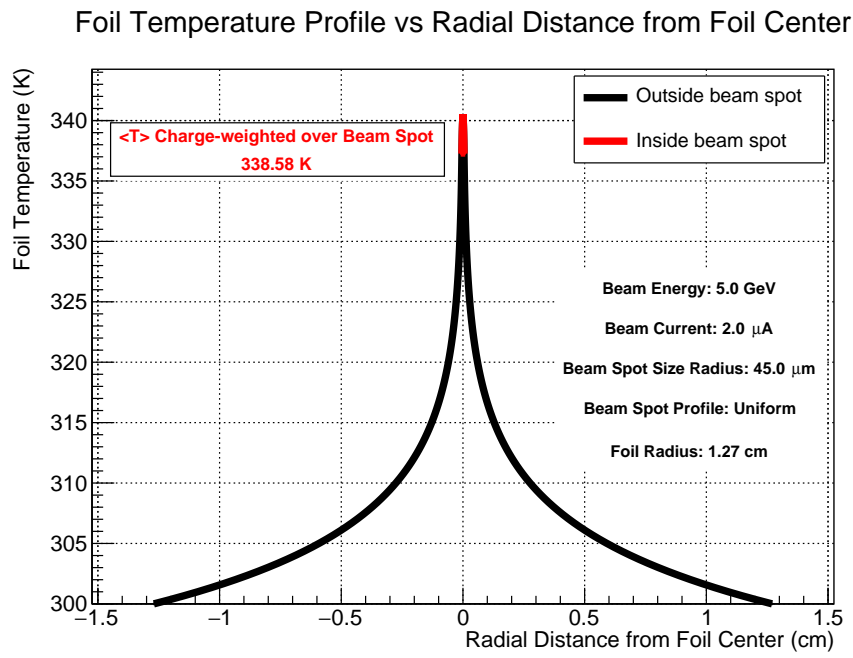


Figure 6: Fe foil temperature profile from integrating Eq. 15 with beam spot size, and energy given.

2 C++/ROOT Code for Numerically Integrating Eq. 9

The following ROOT macro uses Eq. 9 to calculate the foil heating for a circular Fe foil in a Gaussian profile electron beam.

```
#include "TF1.h"
#include <iostream>
#include "TGraph.h"
#include "TLegend.h"
#include "TAxis.h"
#include "TPad.h"
#include "TCanvas.h"
#include "TStyle.h"
#include "TPaveText.h"
#include "TString.h"

////////////////////
//Donald C. Jones//
//Nov. 2021      //
////////////////////

////////////////////////////////////
//FeFoilHeating() calculates and graphs the temperature difference
//in a thin circular Fe foil between its edge held at a fixed
//temperature T0 and inside a circular Gaussian-distributed
//electron beam.
//
//
//Arguments:
// beam_cur: beam current in Amperes
// beam_r:   1 sigma beam spot size radius in cm
// beam_E:   beam energy in GeV
// T0:       ambient (Hall) temperature in Kelvin taken as foil
//           boundary temperature
//
//Returns the foil temperature difference in degrees K between T0
//at the foil edge and the temperature at the 1-sigma beam
//radius r_beam.
//
//NOTE: it is helpful to recall that for a 2D circular Gaussian
//distribution the volume between r=0 and the n-sigma points
//are as follows:
//1sigma = 39.35%, 2sigma = 86.47%, 3sigma = 98.89%, 4sigma = 99.97%
//Therefore, the temperature should be averaged over at least 3 sigma.
////////////////////////////////////

double FeFoilHeating(double beam_cur = 1e-6, double beam_r=5e-3,
                    double beam_E = 11, double T0 = 294){
    gStyle->SetStatY(0.7);
    gStyle->SetStatH(0.2);
```



```

gStyle->SetOptFit(1111);
gStyle->SetTitleW(0.95);

const double rho = 7.87; //density of Fe
const double sigma = 5.670e-12; //Stefan Boltzman constant W/(cm^2 K^4)
const double Cp = 0.45; //Fe specific heat capacity in J/(g K)
const double echarge = 1.602e-19; //Coulombs per electron
const double R_foil = 0.5*2.54/2.0; //radius of Fe foil in cm
const double PI = 3.1415927; //pi obviously

//Use ESTAR data to estimate energy loss as a function of electron energy
//-----
TCanvas *c = new TCanvas("c","c",0,0,800,600);
double beam_en[10]={1,2,3,4,5,6,7,8,9,10}; //beam energy in GeV
double stop_en[10]={1.878,1.928,1.957,1.977,1.993, //collision stopping power
2.006,2.017,2.027,2.035,2.043}; //in (MeV cm^2/g) using ESTAR
TGraph *grStop = new TGraph(10,beam_en,stop_en);
grStop->SetTitle("Electron Stopping Power for Fe vs Beam Energy (ESTAR Data)");
grStop->SetMarkerStyle(8);
grStop->Draw("ap");
grStop->GetXaxis()->SetTitle("Electron Energy");
grStop->GetYaxis()->SetTitle("Stopping Power (MeV cm^{2}/g)");
gPad->Update();
TF1 *fStop = new TF1("fStop","pol5",0,1); //use fit to give continuous function
grStop->Fit(fStop);
double alpha = echarge*fStop->Eval(beam_E)*1e6; //Collision stopping power in
(Jcm^2/g)
cout<<"Stopping power "<<alpha<<" (J cm^2/g)"<<endl;
c->SaveAs("StoppingPower.png");

//Calculate the energy dependent thermal conductivity of Fe using data either from
//https://www.efunda.com/materials/elements/TC_Table.cfm?Element_ID=Fe
//or
//https://www.engineeringtoolbox.com/thermal-conductivity-metals-d_858.html
//-----
bool data_efunda = 1;
TCanvas *ct = new TCanvas("ct","ct",0,0,800,600);
double temp[4] = {250,300,350,400};
double cond[4] = {0.865,0.802,0.744,0.695}; //www.efunda.com
TGraph *grC = new TGraph(4,temp,cond);
grC->SetTitle("Fe Thermal Conductivity vs. Temperature");
grC->SetMarkerStyle(8);
grC->Draw("ap");
grC->GetXaxis()->SetTitle("Temperature (k)");
grC->GetYaxis()->SetTitle("Thermal Conductivity (W/cm K)");

```

```

TF1 *fCond = new TF1("fCond","pol2",0,1);
grC->Fit(fCond);
gPad->Update();
if(!data_efunda)//www.engineeringtoolbox.com
    fCond = new TF1("fCond","0.835-0.001102*(x-273)",0,1);
double guessTemp = T0+15*beam_cur/1e-6;//starting guess for final foil temperature
double kappa = fCond->Eval(guessTemp);
cout<<"Conductivity at "<<guessTemp<<" K is "<<kappa<<endl;
ct->SaveAs("FeThermalCond.png");

//Integral of f(r) gives delta T. Create the integrand f(r)
//-----
double gam = beam_cur/echarge*rho*alpha/kappa/2./PI/pow(beam_r,2);
double C = -beam_cur/echarge*alpha*rho/2.0/PI/kappa;
TF1 *f = new TF1("f",Form("%e/x*exp(-x*x/%e)+%e/x",
    beam_r*beam_r*gam,2*beam_r*beam_r,C),0,R_foil);

//Improve thermal conductivity estimate using the calculated temperature.
//Temperature at 1.3*beam_r is a good estimate of the average temperature
//weighted by the beam spot charge distribution.
//-----
guessTemp = f->Integral(R_foil,1.3*beam_r)+T0;
kappa = fCond->Eval(guessTemp);
gam = beam_cur/echarge*rho*alpha/kappa/2./PI/pow(beam_r,2);
C = -beam_cur/echarge*alpha*rho/2.0/PI/kappa;
cout<<"Conductivity re-calculated at "<<guessTemp<<" K is "<<kappa<<endl;
f = new
    TF1("f",Form("%e/x*exp(-x*x/%e)+%e/x",beam_r*beam_r*gam,2*beam_r*beam_r,C),0,R_foil);

//Graph resulting temperature profile by integrating f(r)dr. Make points red inside
//2 sigma beam spot size radius.
//-----
const int N=500;
double r[N], T[N], dT[N],ri[N],Ti[N], dTi[N];
int n=0, ni=0;
double rp = R_foil;
for(int i=0;i<N/2;++i){
    r[i]=rp;
    dT[i] = f->Integral(R_foil,rp);
    T[i] = dT[i]+T0;
    if(rp<2*beam_r){
        ri[ni]=rp;
        Ti[ni]=T[i];
        dTi[ni]=dT[i];
    }
}

```

```

    ++ni;
}
rp*=0.95;
++n;
if(rp<0.00001)break;
}
for(int i=0;i<n;++i){
    r[i+n]=-r[n-i-1];
    dT[i+n] = dT[n-i-1];
    T[i+n] = T[n-i-1];
}
for(int i=0;i<ni;++i){
    ri[i+ni]=-ri[ni-i-1];
    dTi[i+ni] = dTi[ni-i-1];
    Ti[i+ni] = Ti[ni-i-1];
}
TCanvas *c1 = new TCanvas("c1","c1",0,0,800,600);
TGraph *grdT = new TGraph(2*n,r,dT);
grdT->SetMarkerStyle(8);
grdT->SetLineWidth(6);
grdT->SetMarkerSize(0.3);
grdT->Draw("acp");
grdT->SetTitle(Form("Foil #DeltaT Profile vs Radial Distance from Foil Center"));
grdT->GetXaxis()->SetTitle("Radial Distance from Foil Center (cm)");
grdT->GetYaxis()->SetTitle("#DeltaT (K)");
TGraph *gridT = new TGraph(2*ni,ri,dTi);
gridT->SetMarkerStyle(8);
gridT->SetMarkerColor(kRed);
gridT->SetLineColor(kRed);
gridT->SetLineWidth(6);
gridT->SetMarkerSize(0.4);
gridT->Draw("samecp");
gPad->SetGrid();
TPaveText *pt = new TPaveText(0.6,0.4,0.89,0.6,"ndc");
pt->SetFillColor(0);
pt->SetShadowColor(0);
pt->SetBorderSize(0);
pt->AddText("Beam Parameters");
pt->AddText(Form("Beam Energy: %0.1f GeV",beam_E));
pt->AddText(Form("Beam Current: %0.1f #muA", beam_cur*1e6));
pt->AddText(Form("Beam Spot size 1#sigma Radius: %0.1f #mum",beam_r*1e4));
pt->AddText(Form("Foil Radius: %0.2f (cm)",R_foil));
pt->Draw();
TLegend *lg = new TLegend(0.62,0.76,0.89,0.89);
lg->AddEntry(grdT,"Outside 2#sigma beam spot","lp");
lg->AddEntry(gridT,"Inside 2#sigma beam spot","lp");
lg->Draw();
c1->SaveAs("FoilHeatingdT.png");
TCanvas *c2 = new TCanvas("c2","c2",0,0,800,600);
TGraph *gr = new TGraph(2*n,r,T);

```

```

gr->SetMarkerStyle(8);
gr->SetLineWidth(6);
gr->SetMarkerSize(0.3);
gr->Draw("acp");
gr->SetTitle(Form("Foil Temperature Profile vs Radial Distance from Foil Center"));
gr->GetYaxis()->SetTitle("Foil Temperature (K)");
gr->GetXaxis()->SetTitle("Radial Distance from Foil Center (cm)");
TGraph *gri = new TGraph(2*ni,ri,Ti);
gri->SetMarkerStyle(8);
gri->SetMarkerColor(kRed);
gri->SetLineColor(kRed);
gri->SetLineWidth(6);
gri->SetMarkerSize(0.4);
gri->Draw("samecp");
gPad->SetGrid();
lg->Draw();
pt->Draw();
c2->SaveAs("FoilHeatingT.png");

//Integrate f(r) weighted by the beam charge distribution to find average delta T
//-----
gStyle->SetOptFit(0);
TF1 *fGaus = new TF1("fGaus","[0]*exp(-x*x/(2*[1]*[1]))+[2]",-2*beam_r,2*beam_r);
fGaus->SetParameters(guessTemp/2.,beam_r,T0);
gr->Fit(fGaus,"r");
TString func = Form("(%e*exp(-x*x/(2*%e))+%e)*x*exp(-x*x/2./%e)/%e",
    fGaus->GetParameter(0),pow(fGaus->GetParameter(1),2),
    fGaus->GetParameter(2),beam_r*beam_r,beam_r*beam_r);
TF1 *fAvgT = new TF1("fAvgT",func.Data(),0,1);
fAvgT->SetNpx(1000);
//fAvgT->Draw();
cout<<"dT at 1.3 sigma is "<<f->Integral(R_foil,beam_r*1.3)<<endl;

//Return average temperature, weighted by the beam spot charge distribution.
//-----
return fAvgT->Integral(0,10*beam_r);
}

```
