MAT 167 HW 1

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$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

7 (a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

9 Assuming A has as many pivots as rows, we have the following results.

(a) a_{11}

(b) $l_{i1} = \frac{a_{i1}}{a_{11}}$

(c) $a_{ij} - a_{1j}\left(\frac{a_{i1}}{a_{11}}\right)$

(d) $a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$

10 (a) True.

(b) False.
AB may not even have three rows.
For example, let A be a 1×3 matrix and B be a 3×3 matrix.
Then AB is a 1×3 matrix, so it has no third row.

(c) True.

(d) False.
Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} (AB)^2 &= \left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} A^2B^2 &= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \end{aligned}$$

So $(AB)^2 \neq A^2B^2$.

12 • Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \end{aligned}$$

• *Proof.* For any lower triangular matrices A, B with dimension $n \times n$, each entry ab_{ij} in AB it is computed by:

$$\sum_{k=1}^{n} a_{ik} b_{kj}$$

If $i < k, a_{ik} = 0$.
If $k < j, b_{kj} = 0$.

Each entry above the main diagonal has one of either $i < k$ or $k < j$.
So for each entry above the main diagonal of AB, we have a sum of products where at least one of the factors is 0.
So, each entry above the main diagonal is 0.
Thus, the product of any two lower triangular matrices is lower triangular. $\square$

13 (a) Let

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \right)^2 \\ &= \frac{1}{3} \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -I \end{aligned}$$

(b) Let

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$$

$$B^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(c) Let

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$CD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Let

$$E = F = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

$$EF = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

24 We want

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\begin{aligned}
M &= E_{32}E_{31}E_{21} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}
\end{aligned}$$

$$MA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

42 (a) True.
Since matrix multiplication is only defined for matrices A, B with dimension $m \times n, n \times p$, respectively, $A^2 = A * A$ must have $m = n = p$. That is A must be a square matrix.

(b) False.
We can choose A, B with dimension $m \times n, n \times m$, respectively, where $m \neq n$. So A and B are not square.
Then AB is defined, as well as BA, yet A and B are not square.

(c) True.
We can choose A, B with dimension $m \times n, n \times m$, respectively.
Then AB is defined, as well as BA.
These two products have dimension $m \times m$ and $n \times n$ respectively, so AB and BA are square.

(d) False.
Let $B = 0$.
Then $A0 = 0$ for all appropriate matrices, but A is not necessarily I.

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§ 1.5 11 Forward-substituting from $Lc = b$ gives

- $c_1 = 2$
- $2 + c_2 = 0 \implies c_2 = -2$
- $2 + c_3 = 2 \implies c_3 = 0$

Now Back-substituting from $Ux = c$ gives

- $w = 0$

- $v + 2(0) = -2 \implies v = -2$
- $2u + 4(-2) + 4(0) = 2 \implies 2u - 8 = 2 \implies u = 5$

So we have $u = 5, v = -2, w = 0$.

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- We could factor A into UL if we reduced the rows with the pivots below all 0's. That is, we'd want to form the lower triangular factor first, then perform elementary row operations to create the upper triangular factor.
- No, these two decompositions do not necessarily produce the same factors, as LU-decomposition in general is not unique.

18 We can solve the first two at the same time.

- Let
$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$
Then if we attempt to reduce A to an upper triangular we get
$$\begin{aligned} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$
But this matrix is singular, so it has no solutions.
- Let
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then if we attempt to reduce A to an upper triangular we get

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
E_1 A &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A && = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
E_2 E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A && = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\
E_3 E_2 E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A && = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

Since E_1 is a permutation matrix, we have $E_1 = P \implies PA = LU$.
Now we have

$$
\begin{aligned}
L &= E_2^{-1} E_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

So A is nonsingular, in fact we have one unique solution.
We take $PAx = Pb = y$ and solve with our LU factors.

$$
y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
$$

We can compute the c matrix with $Lc = y$.

- $c_1 = 1$
- $c_2 = 1$
- $1 - 1 + c_3 = 1 \implies c_3 = 1$

We can compute the x matrix with $Ux = c$.

- $2w = 1 \implies w = \frac{1}{2}$
- $v + \frac{1}{2} = 1 \implies v = \frac{1}{2}$
- $u + \frac{1}{2} = 1 \implies u = \frac{1}{2}$

Finally, we have our solution: $u = \frac{1}{2}, v = \frac{1}{2}, w = \frac{1}{2}$.

22 The elementary operations we performed were

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

- So

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

  We have

$$Lc = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

  and

$$Ux = c$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Plugging in the values for c we compute:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

  So this c solves the first system.
- We compute
  - $z = 2$
  - $y + 2(2) = 2 \implies y + 4 = 2 \implies y = -2$
  - $x + (-2) + 2 = 5 \implies x = 5$

  So the x that solves the second system is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}$$

$$E_{A0} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

$$E_{A0} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

So

$$L_A = E_{A0}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, D_A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, U_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
B &= \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix} \\
E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 0 \end{bmatrix} \\
E_{B1}E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \\
E_{B1}E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

So

$$L = E_{B0}^{-1}E_{B1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

For these symmetric matrices, $L = U^T$

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§ 1.6 2

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