MAT 25 Homework 1

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1. Let x be a natural number.

   (a) Prove that x^2 is even if and only if x is even.

   Proof. We must show two statements are true:

   i. If x^2 is even, then x is even.
   ii. If x is even, then x^2 is even.

   i. We can show this by contraposition. So we must show:
   If x is not even, then x^2 is not even Assume x is odd.

   $$\exists r \in \mathbb{Z} : x = 2p + 1$$

   $$x^2 = x \cdot x = (2p+1)(2p+1) = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

   Since $\mathbb{Z}$ is closed under addition and multiplication, $2p^2 + 2p \in \mathbb{Z}$. So we can rename it q for clarity.

   $$2(2p^2 + 2p) + 1 = 2q + 1 = x^2$$

   So x^2 is odd. This is equivalent, by contraposition, to the statement:
   If x^2 is even, then x is even.

   ii. Assume x is even.

   $$\exists p \in \mathbb{Z} : x = 2r$$

   $$x^2 = x \cdot x = 2r \cdot 2r = 2 \cdot 2r^2$$

   Since $\mathbb{Z}$ is closed under multiplication, $2r^2 \in \mathbb{Z}$ so we can rename it s for clarity.

   $$2 \cdot 2r^2 = 2 \cdot s = x^2$$

   So x^2 is even.

   By i and ii, we have shown both cases and thus x^2 is even if and only if x is even.

   □

(b) Prove that x^2 is odd if and only if x is odd.

Proof. We can take a nearly identical approach to this as we did the previous. We must show two statements are true

i. If x^2 is odd, then x is odd.
ii. If x is odd, then x^2 is odd.

i. We can show this by contraposition. So we must show:
If x is not odd, then x^2 is not odd Assume x is even.

$$\exists r \in \mathbb{Z} : x = 2p$$

$$x^2 = x \cdot x = 2p \cdot 2p = 2 \cdot 2p^2$$

Since $\mathbb{Z}$ is closed under multiplication, $2p^2 \in \mathbb{Z}$. So we can rename it q for clarity.

$$2 \cdot 2p^2 = 2q = x^2$$

So x^2 is even. This is equivalent by contraposition to the statement:
If x^2 is odd, then x is odd.

ii. Assume x is odd.

$$\exists p \in \mathbb{Z} : x = 2r$$

$$x^2 = x \cdot x = 2r \cdot 2r = 2 \cdot 2r^2$$

Since $\mathbb{Z}$ is closed under multiplication, $2r^2 \in \mathbb{Z}$ so we can rename it s for clarity.

$$2 \cdot 2r^2 = 2 \cdot s = x^2$$

So x^2 is odd.

By i and ii, we have shown both cases and thus x^2 is even if and only if x is even.

□

(c) Prove that x^2 is divisible by 3 if and only if x is divisible by 3.

Proof. Again, we follow the same template We must show two statements are true

i. If x^2 is divisible by 3, then x is divisible by 3.
ii. If x is divisible by 3, then x^2 is divisible by 3.

i. Here we can prove by contraposition once again. That is: If x is not divisible by 3, then x^2 is not divisible by 3. Assume x is not divisible by 3.

$$\exists p \in \mathbb{Z} : x = 3p \pm 1$$

For the sake of clarity, we handle each instance of x separately.

A.
$$\exists p \in \mathbb{Z} : x = 3p + 1$$
$$x^2 = x \cdot x = (3p+1)(3p+1) = 9p^2 + 6p + 1 = 3(3p^2 + 2p) + 1$$
Since $\mathbb{Z}$ is is closed under addition and multiplication, $3p^2 + 2p \in \mathbb{Z}$ so we can rename it q for clarity.
$$3(3p^2 + 2p) + 1 = 3q + 1 = x^2$$
So x^2 is not divisible by 3.

B.
$$\exists p \in \mathbb{Z} : x = 3p - 1$$
$$x^2 = x \cdot x = (3p-1)(3p-1) = 9p^2 - 6p + 1 = 3(3p^2 - 2p) + 1$$
Since $\mathbb{Z}$ is is closed under addition and multiplication, $3p^2 - 2p \in \mathbb{Z}$ so we can rename it q for clarity.
$$3(3p^2 - 2p) + 1 = 3q + 1 = x^2$$
So x^2 is not divisible by 3.

From A and B we have shown that if x is not divisible by 3, then x^2 is not divisible by 3. By contraposition, this is equivalent to the statement: if x^2 is divisible by 3, then x is divisible by 3.

ii. Assume x is divisible by 3.

$$\exists r \in \mathbb{Z} : x = 3r$$
$$x^2 = x \cdot x = 3r \cdot 3r = 3 \cdot 3r^2$$

Since $\mathbb{Z}$ is closed under multiplication, $3r^2 \in \mathbb{Z}$ so we can rename it s for clarity.

$$3 \cdot 3r^2 = 3 \cdot s = x^2$$

So x^2 is divisible by 3.

From i and ii we have shown both cases and thus x^2 is divisible by 3 if and only if x is divisible by 3. □

2. Solve exercise 1.2.1.

   (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?

      - *Proof.* $\sqrt{3}$ is irrational.
        We can prove this by contradiction.

        Assume $\sqrt{3}$ is rational.

        $$\exists p, q \in \mathbb{Z} : \sqrt{3} = \frac{p}{q}, p \text{ and } q \text{ are relatively prime}$$

        $$\begin{aligned} \left(\sqrt{3}\right)^2 &= \left(\frac{p}{q}\right)^2 \\ 3 &= \frac{p^2}{q^2} \\ 3q^2 &= p^2 \qquad (1) \end{aligned}$$

        Since $\mathbb{Z}$ is closed under multiplication, $q^2 \in \mathbb{Z}$ we can rename it r for clarity.

        $$\begin{aligned} 3q^2 &= p^2 \\ 3r &= p^2 \qquad (2) \end{aligned}$$

        This shows that p^2 is divisible by 3. From 1.c, it follows that p is divisible by 3.
        We can substitute the value of p^2 from (2) into (1).

        $$\begin{aligned} 3q^2 &= p^2 \\ 3q^2 &= (3r)^2 \\ 3q^2 &= 9r^2 \\ q^2 &= 3r^2 \end{aligned}$$

        Since $\mathbb{Z}$ is closed under multiplication, $r^2 \in \mathbb{Z}$ we can rename it s for clarity.

        $$\begin{aligned} q^2 &= 3r^2 \\ q^2 &= 3s \end{aligned}$$

        This shows that q^2 is divisible by 3. From 1.c, it follows that q is divisible by 3.
        So we have that p and q are both divisible by 3. But this contradicts part of our assumption: p and q are relatively prime. So our assumption was false, thus $\sqrt{3}$ is not rational. $\square$

- Applying a similar proof to $\sqrt{6}$ would depend on the validity of the following statement:

  x^2 is divisible by 6 if and only if x is divisible by 6.

  Until that is proven, or disproven, we cannot say one way or the other with our method.

  However, a more useful result to prove would be a generalization that the square root of some number is irrational. This way we can know for certain whether the square root of a number is irrational or rational by its properties and not by performing a formal proof.

(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational.

The proof of Theorem 1.1.1 breaks down when we attempt to say x^2 is divisible by 4 if and only if x is divisible by 4. One such counter example is when $x^2 = 36$. In this case, x^2 is divisible by 4. However, x (which is 6) is not divisible by 4. Thus we cannot use Theorem 1.1.1 to prove $\sqrt{4}$ is irrational.