MAT 67 Homework 5

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1. Let V be a vector space over $\mathbb{F}$, and suppose that the list $(v_1, v_2, ..., v_n)$ of vectors spans V, where each $v_i \in V$. Prove that the list

$$(v_1 - v_2, v_2 - v_3, v_3 - v_4, ..., v_{n-2} - v_{n-1}, v_{n-1} - v_n, v_n)$$

also spans V.

Proof. Each $v_j \in (v_1, v_2, ..., v_n)$ can be constructed from our new list.

$$\begin{aligned}
v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + ... + (v_{n-2} - v_{n-1}) + (v_{n-1} - v_n) + v_n \\
&= (v_1 - \cancel{v_2}) + (\cancel{v_2} - \cancel{v_3}) + (\cancel{v_3} - \cancel{v_4}) + ... + (\cancel{v_{n-2}} - \cancel{v_{n-1}}) + (\cancel{v_{n-1}} - \cancel{v_n}) + \cancel{v_n} \\
&= v_!\\
v_2 &= 0(v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + ... + (v_{n-2} - v_{n-1}) + (v_{n-1} - v_n) + v_n \\
&= (v_2 - \cancel{v_3}) + (\cancel{v_3} - \cancel{v_4}) + ... + (\cancel{v_{n-2}} - \cancel{v_{n-1}}) + (\cancel{v_{n-1}} - \cancel{v_n}) + \cancel{v_n} \\
&= v_2 \\
&\vdots \\
v_n &= 0(v_1 - v_2) + 0(v_2 - v_3) + 0(v_3 - v_4) + ... + 0(v_{n-2} - v_{n-1}) + 0(v_{n-1} - v_n) + v_n \\
&= v_n
\end{aligned}$$

Since we see that we can generate each one of these, we can generate the entire list $(v_!, v_2, ..., v_n)$, which spans V. So, $span(v_1 - v_2, v_2 - v_3, ..., v_{n-1} - v_n, v_n) = V$ $\square$

2. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $dim(V) = n$ for some $n \in \mathbb{Z}^+$. Prove that there are n one-dimensional subspaces $U_1, U_2, ..., U_n$ of V such that

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

Proof. Since V is a finite-dimensional vector space of dimension n, it has some basis $(v_1, v_2, ..., v_n)$.

Let each U_i be a one-dimensional subspace of V, where $i \in \mathbb{N}, 1 \leq i \leq n$, such that

$$\begin{aligned} U_1 &= \{v_1\} \\ U_2 &= \{v_2\} \\ &\vdots \\ U_n &= \{v_n\} \end{aligned}$$

Now, we can create any $v \in V$ by taking unique linear combinations of $v_1 + v_2 + ... + v_n$ with $v_1 \in U_1, v_2 \in U_2, ..., v_n \in U_n,$.

First, we show that v_i exists.

$$\begin{aligned} v_1 &= v_1 + 0v_2 + 0v_3 + ... + 0v_n = v_1 \\ v_2 &= 0v_1 + v_2 + 0v_3 + ... + 0v_n = v_2 \\ v_3 &= 0v_1 + 0v_2 + v_3 + ... + 0v_n = v_3 \\ &\vdots \\ v_n &= 0v_1 + 0v_2 + 0v_3 + ... + v_n = v_n \end{aligned}$$

Now, we show that v_i is unique.

Without loss of generality we examine v_1

$$v_1 = a_1v_1 + a_2v_2 + ... + a_nv_n = b_1v_1 + b_2v_2 + ... + b_nv_n$$
$$\forall a_j, b_j \in \mathbb{F}, j \in \mathbb{N}, i \leq j \leq n$$

Since v_1 comes from the basis of V, it is part of a linearly independent set of vectors. This means that v_1 does not have any components of the other vectors.

Symbolically,

$$\begin{aligned} a_1v_1 + 0v_2 + 0v_3 + ... + 0v_n &= b_1v_1 + 0v_2 + 0v_3 + ... + 0v_n \\ a_1v_1 &= b_1v_1 \\ a_1v_1 - b_1v_1 &= 0 \\ (a_1 - b_1)v_1 &= 0 \end{aligned}$$

Now, since we know that v_1 is a basis for U_1 and part of the basis for V, we know that $v_1 \neq 0$, so we must have:

$$\begin{aligned} (a_1 - b_1)v_1 &= 0 \\ a_1 - b_1 &= 0 \\ a_1 &= b_1 \end{aligned}$$

And so, there is only one unique way to create v_1.

Through similar reasoning, we can show that each v_i is unique.

Thus, since each $v_i \in V$ can be uniquely represented as $v_1 + v_2 + ... + v_n$, where $v_1 \in U_i, v_2 \in U_2, ..., v_n \in U_n$,

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

$\square$

3. Let $\mathbb{F}_m[z]$ denote the vector space of all polynomials with degree $\leq m \in \mathbb{Z}^+$ and having coefficient over $\mathbb{F}$, and suppose that $p_0, p_1, ..., p_m \in \mathbb{F}_m[z]$ satisfy $p_j(2) = 0$.

   Prove that $(p_0, p_1, ..., p_m)$ is a linearly dependent list of vectors in $\mathbb{F}_m[z]$.

   We need a bit more information first.

   By theorem 5.4.4.3,

   Given a vector space V of dimension n,

   if $(v_1, v_2, ..., v_n)$ is linearly independent, then $(v_1, v_2, ..., v_n)$ is a basis for V.

   this implies that $span(v_1, v_2, ..., v_n) = V$.

   So we just need to show that $(v_1, v_2, ..., v_n)$ does not span V.

   Proof. We know that $dim(\mathbb{F}_m[z]) = m + 1$

   and that there are $m + 1$ vectors in $(p_0, p_1, ..., p_m)$

   So we need to show that $(p_0, p_1, ..., p_m)$ does not span $\mathbb{F}_m[z]$

   Since we're assuming that $(p_0, p_1, ..., p_m)$ does not span $\mathbb{F}_m[z]$, there must be at least one vector in $\mathbb{F}_m[z]$ that cannot be represented as a linear combination of $(p_0, p_1, ..., p_m)$.

   Let's choose a constant polynomial $f(z) = c, c \in \mathbb{F}, c \neq 0$.

   What we see is that this polynomial cannot exist in $span(p_0, p_1, ..., p_m)$ because if we choose $z = 2$ and try to represent $f(2)$ as a linear combination of $(p_0, p_1, ..., p_m)$, we end up with the equation:

$$
\begin{aligned}
f(2) &= a_0 p_0(2) + a_1 p_1(2) + ... + a_m p_m(2) \\
c &= a_0(0) + a_1(0) + ... + a_m(0) \\
c &= 0
\end{aligned}
$$

   However, from our choice of this constant polynomial, we said that $c \neq 0$. This contradiction shows that there is at least one vector in $\mathbb{F}_m[z]$ which cannot be written as a linear combination of $(p_0, p_1, ..., p_m)$.

   So, $(p_0, p_1, ..., p_m)$ does not span $\mathbb{F}_m[z]$.

   So, $(p_0, p_1, ..., p_m)$ is not a basis for $\mathbb{F}_m[z]$.

   Thus, $(p_0, p_1, ..., p_m)$ is linearly dependent. $\square$