

MAT 150A Homework 1

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1.

$$X = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$$

(1)

$$XY = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+d & b+e+af \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix}$$

Since \mathbb{F} is a field, it is closed under addition and multiplication.

So, $a+d, b+e+af, c+f \in \mathbb{F}$.

Thus, $XY \in H(F)$.

(2) Given some matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] &= \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \\ &= \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \\ &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

So, we assume our inverse is

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

Since \mathbb{F} is a field, it has additive inverses, is closed under addition and multiplication.

So, $-a, ac - b, -c \in \mathbb{F}$, and $A^{-1} \in H(F)$.

We need to check that A^{-1} is the inverse by showing that $AA^{-1} = I = A^{-1}A$

$$AA^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the closed form of the inverse is given by

$$A^{-1} = \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

(3) Given

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} \in H(F)$$

We want to show that $A(BC) = (AB)C$.

$$\begin{aligned} A(BC) &= A \left(\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= A \left(\begin{bmatrix} 1 & d+g & e+h+id \\ 0 & 1 & f+i \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d+g & e+h+id \\ 0 & 1 & f+i \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+d+g & af+ai+b+e+h+id \\ 0 & 1 & c+f+i \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
(AB)C &= \left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \right) C \\
&= \left(\begin{bmatrix} 1 & a+d & af+b+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \right) C \\
&= \begin{bmatrix} 1 & a+d & af+b+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a+d+g & af+ai+b+e+h+id \\ 0 & 1 & c+f+i \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

So, $A(BC) = (AB)C$.

Thus, $H(F)$ is associative under matrix multiplication.

- (4) The elements of \mathbb{F}_2 are $\{0, 1\}$, with $+$ and \cdot defined:

$+$	0	1
0	0	1
1	1	0

\cdot	0	1
0	0	0
1	0	1

The elements of $H(\mathbb{F}_2)$ are:

$$H(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

For simplicity we enumerate these as $\{H_0, H_1, \dots, H_7\}$

We see that H_0 has order 1.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } H_1 \text{ has order 2.}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } H_2 \text{ has order 2.}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } H_3 \text{ has order 2.}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } H_4 \text{ has order 2.}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so}$$

H_5 has order 3.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so } H_6 \text{ has order 2.}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ so}$$

H_7 has order 3.

2. Let's look at a few cases first.

$n = 2$,

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$n = 3$,

$$\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} = 4$$

$n = 4$,

$$\begin{vmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 5$$

Looks like the determinate of the $n \times n$ matrix is $n + 1$

Proof. Base Case $n = 2$

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 2(2) - (-1)(-1) = 4 - 1 = 3$$

and $n + 1 = 3$.

So our base case holds.

Inductive Case

Assume determinate of an $n \times n$ matrix is $n + 1$. Need to show, the determinate of an $(n + 1) \times (n + 1)$ matrix is $(n + 1) + 1 = n + 2$

$$\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix}$$

We first expand along the first column.

$$\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} \\
+ 0 |\dots| - 0 |\dots| + \dots \\
= 2 \begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & 2 & -1 & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

Looking at the first matrix expansion, we see that it is an $n \times n$ matrix of the same form. By our assumption, it has determinate of $n + 1$.

$$\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1) + \begin{vmatrix} -1 & 2 & -1 & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

We expand the remaining determinate across the first column.

$$\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1) + (-1) \begin{vmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 \end{vmatrix} \\
- 0 |\dots| + 0 |\dots| + \dots \\
= 2(n+1) - \begin{vmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 \end{vmatrix}$$

This is now an $(n-1) \times (n-1)$ matrix of the same form. Again, by our assumption, the determinate is $(n-1) + 1 = n$.

$$\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & \ddots & \ddots \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1) - n = 2n+2 - n = n+2$$

So, we have shown by induction, that the determinate of an $n \times n$ matrix of this form is $n+1$. □