

MAT 150A Homework 3

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1. To show that f is an automorphism, we need to show that f is an isomorphism from $GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$. To show that f is an isomorphism, we need to show that f is a homomorphism and bijective. To show that f is a homomorphism, we need to show that f is closed, and that it preserves the group operation.

$$GL_n(\mathbb{R}) := \{A_n | A \text{ is an } n \times n \text{ matrix, } |A| \neq 0\}$$

Proof. We begin by showing that f is a homomorphism.

- We first show closure.

Choose any $A \in GL_n(\mathbb{R})$. $f(A) = (A^T)^{-1}$.

The size of $f(A)$ has not changed. We know $|A^T| = |A|$ and $|A^{-1}| = |A|^{-1}$, so $|f(A)| = |(A^T)^{-1}| = |(A^T)|^{-1} = |A|^{-1} \neq 0$.

So, f maps $GL_n(\mathbb{R}) \mapsto GL_n(\mathbb{R})$

Thus, f is closed.

- Now we show that f preserves the group operation.

Choose any $A, B \in GL_n(\mathbb{R})$.

$$f(AB) = ((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1} (B^T)^{-1} = f(A)f(B)$$

So, f preserves the group operation.

Thus, $f : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ is a homomorphism.

Now we need to show that f is bijective.

- We first show that $\ker f = \{I_n\}$

So we want to find all $A \in GL_n(\mathbb{R})$ such that $f(A) = (A^T)^{-1} = I_n$.

But we know that for any group the identity is its own inverse, so $A^T = I_n$.

We also know that $I_n^T = I_n$, so $A = I_n$.

And since we know that the identity is unique, we have that $\ker f = \{I_n\}$.

- Now we show that $\text{im } f = GL_n(\mathbb{R})$

It suffices to show that f has an inverse. Namely, $f^{-1}(A) = A^T$.

So, we have shown that $f : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ is an isomorphism. And since f 's domain is its co-domain, f is an automorphism. \square

2. We want to show that $\forall \varphi : G \rightarrow G'$ that are group homomorphisms, $\ker \varphi \leq G$ and $\text{im } \varphi \leq G'$.

For both of these possible subgroups, it suffices to show two things:

- (a) The possible subgroup is non-empty.
- (b) For all a, b in the possible subgroup, ab^{-1} is also in the subgroup.

- $\ker \varphi \leq G$

Proof. We need to show the two conditions above.

- (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\ker \varphi$ is non-empty (as $e_G \in \ker \varphi$).
- (b) Choose $a, b \in \ker \varphi$.

So we have, $\varphi(a) = e_{G'}$ and $\varphi(b) = e_{G'}$, and since φ is a homomorphism.

$$\begin{aligned} \varphi(ab^{-1}) &= \varphi(a)\varphi(b^{-1}) \\ &= \varphi(a)\varphi(b)^{-1} \\ &= e_{G'}\varphi(b)^{-1} \\ &= e_{G'}e_{G'}^{-1} \\ &= e_{G'}e_{G'} \\ &= e_{G'} \end{aligned}$$

Thus, $\ker \varphi \leq G$. \square

- $\text{im } \varphi \leq G'$

Proof. We need to show the two conditions above.

- (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\text{im } \varphi$ is non-empty (as $e_G \in \text{im } \varphi$).
- (b) Choose $a, b \in \text{im } \varphi$.

This means $\exists a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b$.

Since φ is a homomorphism and $a'b'^{-1} \in G$.

$$\begin{aligned} \varphi(a'b'^{-1}) &= \varphi(a')\varphi(b'^{-1}) \\ &= \varphi(a')\varphi(b')^{-1} \\ &= a\varphi(b)^{-1} \\ &= ab^{-1} \in \text{im } \varphi \end{aligned}$$

Thus, $\text{im } \varphi \leq G'$. \square

3. The subgroups of S_3 are:

$$\begin{aligned} &\{id\} \\ &\{id, (1, 2)\}, \{id, (1, 3)\}, \{id, (2, 3)\} \\ &\{id, (1, 2, 3)\}, \{id, (1, 3, 2)\} \\ &\{id, (1, 2), (1, 3), (2, 3), (1, 2, 3), (1, 3, 2)\} \end{aligned}$$

The trivial subgroup and the group itself are normal.

4. Want to show $\varphi(x) = \varphi(y) \iff xy^{-1} \in \ker \varphi$

Proof. • (\Rightarrow)

Since φ is a homomorphism and $\varphi(x) = \varphi(y)$.

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(x)^{-1} = \varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(e_G) = e_{G'}$$

By the definition of the kernel, $xy^{-1} \in \ker \varphi$.

• (\Leftarrow)

Since φ is a homomorphism and $xy^{-1} \in \ker \varphi$.

$$\begin{aligned} \varphi(xy^{-1}) &= e_{G'} \\ \varphi(x)\varphi(y^{-1}) &= e_{G'} \\ \varphi(x)\varphi(y)^{-1} &= e_{G'} \\ \varphi(x)\varphi(y)^{-1}\varphi(y) &= e_{G'}\varphi(y) \\ \varphi(x)e_{G'} &= e_{G'}\varphi(y) \\ \varphi(x) &= e_{G'}\varphi(y) \\ \varphi(x) &= \varphi(y) \end{aligned}$$

So, $\varphi(x) = \varphi(y)$

Thus, we have shown both directions and $\varphi(x) = \varphi(y) \iff xy^{-1} \in \ker \varphi$. □

5. (a) We need to show that gHg^{-1} is closed, has an identity and has inverses.

Proof. • Choose $x = ghg^{-1}, y = gh'g^{-1} \in ghg^{-1}$.

$$xy = (ghg^{-1})(gh'g^{-1}) = gh(g^{-1}g)h'g^{-1} = gheh'g^{-1} = gh h' g^{-1}$$

Now, since $hh' \in H$, we have that $gh h' g^{-1} = xy \in H$.

Thus, gHg^{-1} is closed.

- Choose $e \in G$ as the identity.

$$ghg^{-1}e = ghg^{-1} = eghg^{-1}$$

Thus, gHg^{-1} has an identity.

- $\forall x = ghg^{-1} \in gHg^{-1}$, choose $x^{-1} = gh^{-1}g^{-1}$

$$(ghg^{-1})(gh^{-1}g^{-1}) = gh(g^{-1}g)h^{-1}g^{-1} = gheh^{-1}g^{-1} = g(hh^{-1})g^{-1} = geg^{-1} = gg^{-1} = e$$

Thus, gHg^{-1} has inverses.

From the three results shown, $gHg^{-1} \leq G$ is a subgroup. □

(b) We want to prove:

$$H \leq G \text{ is normal} \iff \forall g \in G, gHg^{-1} = H$$

Proof. We need to show both sides of this equivalence.

(\Rightarrow) Choose $h \in H$, we know that $\forall g \in G, gH = Hg$, since $H \leq G$ is normal.
So, choose $g \in G$, we have:

$$\begin{aligned} gh &= hg \\ ghg^{-1} &= hgg^{-1} \\ ghg^{-1} &= he \\ ghg^{-1} &= h \end{aligned}$$

Since our choices for g, h were arbitrary, we have that this result holds for all $g \in G, h \in H$.

Thus we have that $gHg^{-1} = H$.

(\Leftarrow) Choose $g \in G$, we know $\forall g \in G, gHg^{-1} = H$.
So, choose $h \in H$, we have:

$$\begin{aligned} ghg^{-1} &= h \\ ghg^{-1}g &= hg \\ ghg^{-1}g &= hg \\ gh(g^{-1}g) &= hg \\ ghe &= hg \\ gh &= hg \end{aligned}$$

Since our choices for g, h were arbitrary, we have that this result holds for all $g \in G, h \in H$.

Thus we have that $gH = Hg$, in other words, $H \leq G$ is normal. □