MAT 167 Midterm

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1. (1) True.

Proof. Assume $A = L_1U_1 = L_2U_2$ with L_1, L_2 lower triangular and unit diagonal, U_1, U_2 upper triangular with nonzero diagonal.
So we have inverses for L_1, L_2 since the diagonal is all 1 with 0's above the diagonal, and we have inverses for U_1, U_2 since the diagonal is non-zero with 0's below the diagonal.
Then we have:

$$\begin{aligned}
L_1U_1 &= L_2U_2 \\
L_2^{-1}(L_1U_1) &= L_2^{-1}(L_2U_2) \\
L_2^{-1}(L_1U_1) &= \left(L_2^{-1}L_2\right)U_2 \\
L_2^{-1}(L_1U_1) &= IU_2 \\
L_2^{-1}(L_1U_1) &= U_2 \\
L_2^{-1}(L_1U_1)U_1^{-1} &= U_2U_1^{-1} \\
L_2^{-1}L_1\left(U_1U_1^{-1}\right) &= U_2U_1^{-1} \\
L_2^{-1}L_1I &= U_2U_1^{-1} \\
L_2^{-1}L_1 &= U_2U_1^{-1}
\end{aligned}$$

Now, $L_2^{-1}L_1$ is a lower triangular matrix and $U_2U_1^{-1}$ is an upper triangular matrix. In order for these two to be equal they have to both be lower triangular and upper triangular at the same time. The only matrices with this property are diagonal matrices.
So, $L_2^{-1}L_1, U_2U_1^{-1}$ are diagonal matrices. And since $L_2^{-1}L_1 = U_2U_1^{-1}$ we must have the same entries on the diagonal. And since L_1, L_2 have all 1 on the diagonal, $L_2^{-1}L_1$ has all 1 on the diagonal.
So $L_2^{-1}L_1 = I$.
Then,

$$\begin{aligned}
L_2^{-1}L_1 &= I \\
L_2\left(L_2^{-1}L_1\right) &= L_2I \\
\left(L_2L_2^{-1}\right)L_1 &= L_2 \\
IL_1 &= L_2 \\
L_1 &= L_2
\end{aligned}$$

And since $L_2^{-1}L_1 = I = U_2U_1^{-1}$, we have

$$\begin{aligned} I &= U_2U_1^{-1} \\ IU_1 &= \left(U_2U_1^{-1}\right)U_1 \\ U_1 &= U_2\left(U_1^{-1}U_1\right) \\ U_1 &= U_2 I \\ U_1 &= U_2 \end{aligned}$$

So we have shown,
if $A = L_1U_1 = L_2U_2$ with L_1, L_2 lower triangular and unit diagonal, U_1, U_2 upper triangular with nonzero diagonal,
then $L_1 = L_2, U_1 = U_2$. $\square$

(2) True.

Proof. Assume $A^2 + A = I$

$$A(A+I) = A^2 + A = I = A^2 + A = (A+I)A$$

So $A + I$ is a left and right inverse of A, then $A^{-1} = A + I$. $\square$

(3) False.
Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
Then all the diagonal entries of A are zero, but $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$, so A is non-singular.

2. (1) The shape depends on A.
If $A = 0$, then the shape is a point.
If A has a row or column with all 0, then the shape is a line.
In any other case, then the region stays as a parallelogram. It can stay a square, become a rectangle, or a some other parallelogram, but it is always necessarily a parallelogram.

(2) The region is a square when $A = \begin{bmatrix} a\cos(\theta) & -b\sin(\theta) \\ b\sin(\theta) & a\cos(\theta) \end{bmatrix}$
for some $a, b \in \mathbb{R}, \theta \in [0, 2\pi)$, with not $a = b = 0$.

(3) The region is a line when A has exactly one row or column with all 0.

3. (1) We have that $rank(A) = rank(U) = 3$.

(2) The basis for $R(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 2 \end{bmatrix}^T, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \\ -1 \end{bmatrix}^T, \begin{bmatrix} 2 \\ -1 \\ 5 \\ -1 \\ 5 \end{bmatrix}^T \right\}$

(3) False.
row 1 + 2 · row 2 = row 3

(4) The basis for $C(A) = \left\{ \begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix} \right\}$

(5) We have $dim(N(A^T)) = 4 - rank(A) = 4 - 3 = 1$

(6) We can use the LU decomposition.
$Ax = 0 \implies Lc = 0, Ux = c$
So we have for $c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$:

- $c_1 = 0$
- $c_2 = 0$
- $c_1 + c_2 + c_3 = 0 \implies c_3 = 0$
- $2c_1 + c_2 + c_4 = 0 \implies c_4 = 0$

Which gives us for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

- $2x_5 = 0 \implies x_5 = 0$
- $x_3 - 3x_4 + 2x_5 = 0 \implies x_3 = 3x_4$
- $2x_1 - x_2 + 4x_3 + 2x_4 + x_5 = 0 \implies 2x_1 = x_2 - 4x_3 - 2x_4 \implies x_1 = \frac{1}{2}x_2 - 7x_4$

So we end up with the general solution:

$$x = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} t$$

for any $s, t \in \mathbb{R}$.

4. (1) We have $x_1 + 3x_2 - x_3 = 0 \implies x_3 = x_1 + 3x_2$, so

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} t$$

for any $s, t \in \mathbb{R}$.

Then a basis for $V = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$

And since $V^\perp = R(A)$, a basis for $V^\perp = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$

(2) We need $\left\| \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\| = \sqrt{1+9+1} = \sqrt{11}$

Then, an orthonormal basis for $V^\perp = \left\{ \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right\}$.

Now we can find P_1 as:

$$\begin{aligned}
P_1 &= \frac{\frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \left(\frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right)^T}{\left(\frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right)^T \frac{1}{\sqrt{11}} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}} \\
&= \frac{\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right)^T}{\left(\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right)^T \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}} \\
&= \frac{1}{11} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \right)^T \\
&= \frac{1}{11} \begin{bmatrix} 1 & 3 & -1 \\ 3 & 9 & -3 \\ -1 & -3 & 1 \end{bmatrix}
\end{aligned}$$

(3) We take the basis of V and construct a matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}$.

Now we can find P_2 as:

$$
\begin{aligned}
P_2 &= A\left(A^T A\right)^{-1} A^T \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix}^T \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \left(\frac{1}{20-9} \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \frac{1}{11} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \frac{1}{11} \begin{bmatrix} 10 & -3 \\ -3 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \\
&= \frac{1}{11} \begin{bmatrix} 10 & -3 & 1 \\ -3 & 2 & 3 \\ 1 & 3 & 10 \end{bmatrix}
\end{aligned}
$$

5. Let $E = \{A \in \mathbb{R}^{4\times 4} | S^{-1}AS \text{ is diagonal}\}$

(1) *Proof.* We need to show that the zero vector is in E, and E is closed under linear combinations.

- Choose $A = 0$.
  Then $S^{-1}0S = 0$, and since 0 is diagonal, $0 \in E$.
- Choose $c, d \in \mathbb{R}$, and $A, B \in \mathbb{R}$.
  Then $S^{-1}(cA + dB)S = (S^{-1}(cA) + S^{-1}(dB))S = S^{-1}(cA)S + S^{-1}(dB)S = c(S^{-1}AS) + d(S^{-1}BS)$.
  Now, since A, B are diagonal, and $S^{-1}AS, S^{-1}BS$ are diagonal, we know $c(S^{-1}AS), d(S^{-1}BS)$ are also diagonal, so their sum is diagonal as well.
  So $cA + dB \in E$.

Note that we chose arbitrary $c, d \in \mathbb{R}$, and $A, B \in E$ so this holds for all cases,
In particular the case where $d = 0, B = 0$, which gives a proof of closure under scalar multiplication.
And also the case where $c = d = 1$, which gives a proof of closure under vector addition.

$\square$

(2) When $S = I, S^{-1} = I$.

Then $E = \{A \in \mathbb{R}^{4\times 4} | A \text{ is diagonal}\}$, which is the set of all 4×4 diagonal matrices.

(3) Since S diagonalizes 4×4 matrices, Λ must have 4 non-zero eigenvalues.

Then the dimension must be 4.