MAT 168 Calculation 2

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2.8 We first start by rewriting as a dictionary, introducing the necessary slack variables:

$$
\begin{aligned}
\zeta &= \quad 3x_1 + 2x_2 \\
x_3 &= 1 - x_1 + 2x_2 \\
x_4 &= 2 - x_1 + x_2 \\
x_5 &= 6 - 2x_1 + x_2 \\
x_6 &= 5 - x_1 \\
x_7 &= 16 - 2x_1 - x_2 \\
x_8 &= 12 - x_1 - x_2 \\
x_9 &= 21 - x_1 - 2x_2 \\
x_{10} &= 10 \quad - x_2
\end{aligned}
$$

Then we can begin by entering with the first non-basic variable, x_1.

We look at the constraints and see:

- $x_3 \implies x_1 \leq 1$
- $x_4 \implies x_1 \leq 2$
- $x_5 \implies x_1 \leq 3$
- $x_6 \implies x_1 \leq 5$
- $x_7 \implies x_1 \leq 8$
- $x_8 \implies x_1 \leq 12$
- $x_9 \implies x_1 \leq 21$
- $x_{10} \implies x_1 \geq 0$

The more restrictive constraint is that $x_1 \leq 1$, so set $x_1 = 1$.

So we can let x_1 enter and x_3 leave.

$$x_1 = 1 + 2x_2 - x_3$$

This gives a new value for ζ. $\zeta = 3 + 8x_2 - 3x_3$

This gives a new value for x_4. $x_4 = 1 - 1x_2 + 1x_3$

This gives a new value for x_5. $x_5 = 4 - 3x_2 + 2x_3$

This gives a new value for x_6. $x_6 = 4 - 2x_2 + 1x_3$

This gives a new value for x_7. $x_7 = 14 - 5x_2 + 2x_3$

This gives a new value for x_8. $x_8 = 11 - 3x_2 + 1x_3$

This gives a new value for x_9. $x_9 = 20 - 4x_2 + 1x_3$

This gives a new value for x_{10}. $x_{10} = 10 - 1x_2$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= \ \ 3 + 8x_2 - 3x_3 \\
x_1 &= \ \ 3 + 2x_2 - 1x_3 \\
x_4 &= \ \ 1 - 1x_2 + 1x_3 \\
x_5 &= \ \ 4 - 3x_2 + 2x_3 \\
x_6 &= \ \ 4 - 2x_2 + 1x_3 \\
x_7 &= 14 - 5x_2 + 2x_3 \\
x_8 &= 11 - 3x_2 + 1x_3 \\
x_9 &= 20 - 4x_2 + 1x_3 \\
x_{10} &= 10 - 1x_2
\end{aligned}$$

Now, we can continue optimizing since x_1 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_1 \geq 0$
- $x_4 \implies x_1 \leq 1$
- $x_5 \implies x_1 \leq \frac{4}{3}$
- $x_6 \implies x_1 \leq 2$
- $x_7 \implies x_1 \leq \frac{14}{5}$
- $x_8 \implies x_1 \leq \frac{11}{3}$
- $x_9 \implies x_1 \leq 5$
- $x_{10} \implies x_1 \leq 10$

The more restrictive constraint is that $x_4 \leq 1$, so set $x_4 = 1$.

So we can let x_2 enter and x_4 leave.

$$x_2 = 1 + x_3 - x_4$$

This gives a new value for ζ. $\zeta = 11 + 5x_3 - 8x_4$

This gives a new value for x_1. $x_1 = 3 + 1x_3 - 2x_4$

This gives a new value for x_5. $x_5 = 1 - 1x_3 + 3x_4$

This gives a new value for x_6. $x_6 = 2 - 1x_3 + 2x_4$

This gives a new value for x_7. $x_7 = 9 - 3x_3 + 5x_4$

This gives a new value for x_8. $x_8 = 8 - 2x_3 + 3x_4$

This gives a new value for x_9. $x_9 = 16 - 3x_3 + 4x_4$

This gives a new value for x_{10}. $x_{10} = 9 - 1x_3 + 1x_4$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= 11 + 5x_3 - 8x_4 \\
x_1 &= \ \ 3 + 1x_3 - 2x_4 \\
x_2 &= \ \ 1 + 1x_3 - 1x_4 \\
x_5 &= \ \ 1 - 1x_3 + 3x_4 \\
x_6 &= \ \ 2 - 1x_3 + 2x_4 \\
x_7 &= \ \ 9 - 3x_3 + 5x_4 \\
x_8 &= \ \ 8 - 2x_3 + 3x_4 \\
x_9 &= 16 - 3x_3 + 4x_4 \\
x_{10} &= \ \ 9 - 1x_3 + 1x_4
\end{aligned}$$

Now, we can continue optimizing since x_3 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_1 \geq 0$
- $x_2 \implies x_1 \geq 0$
- $x_5 \implies x_1 \leq 1$
- $x_6 \implies x_1 \leq 2$
- $x_7 \implies x_1 \leq 3$
- $x_8 \implies x_1 \leq 4$
- $x_9 \implies x_1 \leq \frac{16}{3}$
- $x_{10} \implies x_1 \leq 9$

The more restrictive constraint is that $x_5 \leq 1$, so set $x_5 = 1$.

So we can let x_3 enter and x_5 leave.

$$x_3 = 1 + 3x_4 - 1x_5$$

This gives a new value for ζ. $\zeta = 16 + 7x_4 - 5x_5$

This gives a new value for x_1. $x_1 = 4 + 1x_4 - 1x_5$

This gives a new value for x_2. $x_2 = 2 + 2x_4 - 1x_5$

This gives a new value for x_6. $x_6 = 1 - 1x_4 + 1x_5$

This gives a new value for x_7. $x_7 = 6 - 4x_4 + 3x_5$

This gives a new value for x_8. $x_8 = 6 - 3x_4 + 2x_5$

This gives a new value for x_9. $x_9 = 13 - 5x_4 + 3x_5$

This gives a new value for x_{10}. $x_{10} = 8 - 2x_4 + 1x_5$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= 16 + 7x_4 - 5x_5 \\
x_1 &= 4 + 1x_4 - 1x_5 \\
x_2 &= 2 + 2x_4 - 1x_5 \\
x_3 &= 1 - 1x_4 + 1x_5 \\
x_6 &= 1 - 1x_4 + 1x_5 \\
x_7 &= 6 - 4x_4 + 3x_5 \\
x_8 &= 6 - 3x_4 + 2x_5 \\
x_9 &= 13 - 5x_4 + 3x_5 \\
x_{10} &= 8 - 2x_4 + 1x_5
\end{aligned}$$

Now, we can continue optimizing since x_4 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_1 \geq 0$
- $x_2 \implies x_1 \geq 0$
- $x_3 \implies x_1 \geq 0$
- $x_6 \implies x_1 \leq 1$
- $x_7 \implies x_1 \leq \frac{6}{4}$
- $x_8 \implies x_1 \leq 2$
- $x_9 \implies x_1 \leq \frac{13}{5}$

- $x_{10} \implies x_1 \leq 4$

The more restrictive constraint is that $x_6 \leq 1$, so set $x_6 = 1$.

So we can let x_4 enter and x_6 leave.

$$x_4 = 1 + 1x_5 - 1x_6$$

This gives a new value for ζ. $\zeta = 23 + 2x_5 - 7x_6$

This gives a new value for x_1. $x_1 = 5 + 0x_5 - 1x_6$

This gives a new value for x_2. $x_2 = 4 + 1x_5 - 2x_6$

This gives a new value for x_3. $x_3 = 4 + 2x_5 - 3x_6$

This gives a new value for x_7. $x_7 = 2 - 1x_5 + 4x_6$

This gives a new value for x_8. $x_8 = 3 - 1x_5 + 3x_6$

This gives a new value for x_9. $x_9 = 8 - 2x_5 + 5x_6$

This gives a new value for x_{10}. $x_{10} = 6 - 1x_5 + 2x_6$

$$\begin{aligned}
\zeta &= 23 + 2x_5 - 7x_6 \\
x_1 &= \ \ 5 + 0x_5 - 1x_6 \\
x_2 &= \ \ 4 + 1x_5 - 2x_6 \\
x_3 &= \ \ 4 + 2x_5 - 3x_6 \\
x_4 &= \ \ 1 + 1x_5 - 1x_6 \\
x_7 &= \ \ 2 - 1x_5 + 4x_6 \\
x_8 &= \ \ 3 - 1x_5 + 3x_6 \\
x_9 &= \ \ 8 - 2x_5 + 5x_6 \\
x_{10} &= \ \ 6 - 1x_5 + 2x_6
\end{aligned}$$

Now, we can continue optimizing since x_5 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_1 \geq 0$
- $x_2 \implies x_1 \geq 0$
- $x_3 \implies x_1 \geq 0$
- $x_4 \implies x_1 \geq 0$
- $x_7 \implies x_1 \leq 2$
- $x_8 \implies x_1 \leq 3$
- $x_9 \implies x_1 \leq 4$

- $x_{10} \implies x_1 \leq 6$

The more restrictive constraint is that $x_7 \leq 2$, so set $x_7 = 2$.

So we can let x_5 enter and x_7 leave.

$$x_5 = 2 + 4x_6 - 1x_7$$

This gives a new value for ζ. $\zeta = 27 + 1x_6 - 2x_7$

This gives a new value for x_1. $x_1 = 5 - 1x_6 - 0x_7$

This gives a new value for x_2. $x_2 = 6 + 2x_6 - 1x_7$

This gives a new value for x_3. $x_3 = 6 + 5x_6 - 2x_7$

This gives a new value for x_4. $x_4 = 3 + 3x_6 - 1x_7$

This gives a new value for x_8. $x_8 = 1 - 1x_6 + 1x_7$

This gives a new value for x_9. $x_9 = 4 - 3x_6 + 2x_7$

This gives a new value for x_{10}. $x_{10} = 4 - 2x_6 + 1x_7$

$$\begin{aligned}
\zeta &= 27 + 1x_6 - 2x_7 \\
x_1 &= \ \ 5 - 1x_6 - 0x_7 \\
x_2 &= \ \ 6 + 2x_6 - 1x_7 \\
x_3 &= \ \ 6 + 5x_6 - 2x_7 \\
x_4 &= \ \ 3 + 3x_6 - 1x_7 \\
x_5 &= \ \ 2 + 4x_6 - 1x_7 \\
x_8 &= \ \ 1 - 1x_6 + 1x_7 \\
x_9 &= \ \ 4 - 3x_6 + 2x_7 \\
x_{10} &= \ \ 4 - 2x_6 + 1x_7
\end{aligned}$$

Now, we can continue optimizing since x_6 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_1 \geq 0$
- $x_2 \implies x_1 \geq 0$
- $x_3 \implies x_1 \geq 0$
- $x_4 \implies x_1 \geq 0$
- $x_5 \implies x_1 \geq 0$
- $x_8 \implies x_1 \leq 1$
- $x_9 \implies x_1 \leq \frac{4}{3}$

- $x_{10} \implies x_1 \leq 2$

The more restrictive constraint is that $x_8 \leq 1$, so set $x_8 = 1$.

So we can let x_6 enter and x_8 leave.

$$x_6 = 1 + 1x_7 - 1x_8$$

This gives a new value for ζ. $\zeta = 28 - 1x_7 - 1x_8$

This gives a new value for x_1. $x_1 = 4 - 1x_7 + 1x_8$

This gives a new value for x_2. $x_2 = 8 + 1x_7 - 2x_8$

This gives a new value for x_3. $x_3 = 13 + 3x_7 - 5x_8$

This gives a new value for x_4. $x_4 = 6 + 2x_7 - 3x_8$

This gives a new value for x_5. $x_5 = 6 - 3x_7 - 4x_8$

This gives a new value for x_9. $x_9 = 1 - 1x_7 + 3x_8$

This gives a new value for x_{10}. $x_{10} = 2 - 1x_7 + 2x_8$

$$\begin{aligned}
\zeta &= 28 - 1x_7 - 1x_8 \\
x_1 &= 4 - 1x_7 + 1x_8 \\
x_2 &= 8 + 1x_7 - 2x_8 \\
x_3 &= 13 + 3x_7 - 5x_8 \\
x_4 &= 6 + 2x_7 - 3x_8 \\
x_5 &= 6 - 3x_7 - 4x_8 \\
x_6 &= 1 + 1x_7 - 1x_8 \\
x_9 &= 1 - 1x_7 + 3x_8 \\
x_{10} &= 2 - 1x_7 + 2x_8
\end{aligned}$$

Since we have no more optimizable variables (all variable coefficients of ζ are non-positive), we can no longer maximize ζ.

Then we have an optimal solution with $x_1 = 4, x_2 = 8, x_3 = 13, x_4 = x_5 = 6, x_6 = x_9 = 1, x_{10} = 2, x_7 = x_8 = 0$, and value 28.

2.16 The first thing to do is to convert the dictionary back to a system of inequalities. At the moment we're only concerned with the constraints, so let's just convert those.

$$
\begin{aligned}
1x_7 - 1x_8 &\leq 4\\
-1x_7 + 2x_8 &\leq 8\\
-3x_7 + 5x_8 &\leq 13\\
-2x_7 + 3x_8 &\leq 6\\
3x_7 + 4x_8 &\leq 6\\
-1x_7 + 1x_8 &\leq 1\\
1x_7 - 3x_8 &\leq 1\\
1x_7 - 2x_8 &\leq 2
\end{aligned}
$$

Then we can plot these lines and find their intersection on the x_7, x_8 plane.

See Figure 1.

However, Figure 1 is very cluttered. Some of the constraints aren't actually helpful, and seeing all of them at once doesn't give more information. And the hyperplanes denoting how to optimize are not strictly necessary as we've already optimized the program.

We also omit the normal lines

Let's zoom in closer to the point (0, 0) in Figure 2.

We can see the feasible region is constrained by the equations

$$
\begin{aligned}
3x_7 + 4x_8 &\leq 6\\
1x_7 - 3x_8 &\leq 1\\
-1x_7 + 1x_8 &\leq 1\\
1x_7 \qquad &\geq 0\\
x_8 &\geq 0
\end{aligned}
$$

So we outline the feasible region (it probably wont show up when printed in black and white).

Now we can also see that the optimal solution is at (0, 0).

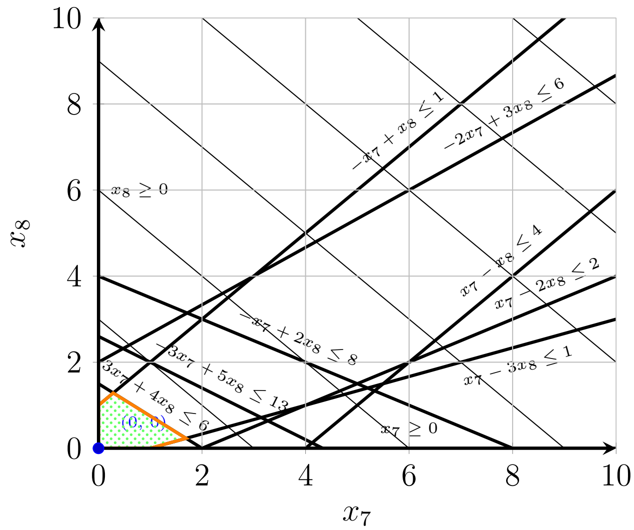

Figure 1: All constraints

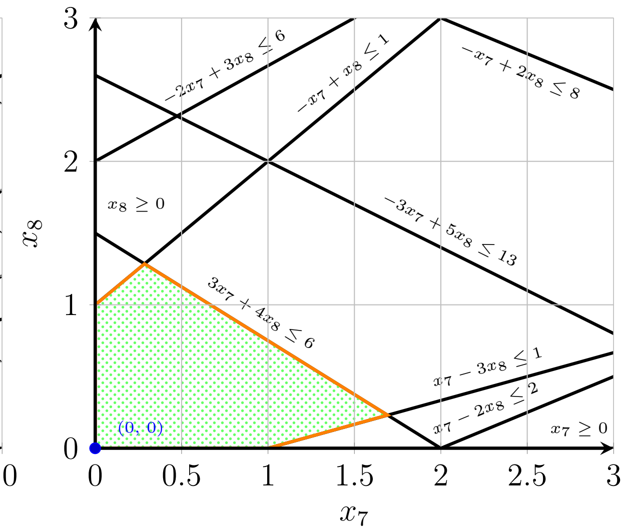

Figure 2: Closeup of feasible region