MAT 108 HW 6

Hardy Jones
999397426
Professor Bandyopadhyay
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§2.5 1 (b) *Proof.* Let $S = \{n \in \mathbb{N} | n > 33 \wedge n = 4s + 5t \text{ for some } s, t \in \mathbb{N}, \text{ with } s \geq 3, t \geq 2\}$

Then we have:

- for $s = 6, t = 2$
$$4(6) + 5(2) = 24 + 10 = 34$$
- for $s = 5, t = 3$
$$4(5) + 5(3) = 20 + 15 = 35$$
- for $s = 4, t = 4$
$$4(4) + 5(4) = 16 + 20 = 36$$
- for $s = 3, t = 5$
$$4(3) + 5(5) = 12 + 25 = 37$$

So, $S = \{34, 35, 36, 37, \ldots, m - 1\}$ for some $m \in \mathbb{N}$
If $m > 37$, then $m - 4 \in S$.
Then we have for some $s, t \in \mathbb{N}$, with $s \geq 3, t \geq 2$

$$\begin{aligned} m - 4 &= 4s + 5t \\ m &= 4s + 5t + 4 \\ m &= 4s + 4 + 5t \\ m &= 4(s + 1) + 5t \end{aligned}$$

And since $s \geq 3, s + 1 \geq 3$. So $m \in S$.
Then by the Principle of Complete Induction, every natural number greater than 33 can be written as $4s + 5t$, for some $s, t \in \mathbb{N}$, with $s \geq 3, t \geq 2$. □

5 (d) *Proof.* We need to show the base cases and the inductive case.

- Base
  - n = 1
$$f_1 = 1 = 2 - 1 = f_3 - 1 = f_{1+2} - 1$$
  - n = 2
$$f_1 + f_2 = 1 + 1 = 2 = 3 - 1 = f_4 - 1 = f_{2+2} - 1$$

- Inductive
  Assume $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$
  Then we have:

$$\begin{aligned} f_1 + f_2 + \cdots + f_n + f_{n+1} &= (f_{n+2} - 1) + f_{n+1} \\ &= f_{n+2} + f_{n+1} - 1 \\ &= f_{n+3} - 1 \\ &= f_{(n+1)+2} - 1 \end{aligned}$$

  Thus for any $n+1$, $f_1 + f_2 + \cdots + f_{n+1} = f_{(n+1)+2} - 1$

Since we have shown both the base case and the inductive case, we have shown that for all natural numbers, $f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$ □

6 (b) *Proof.* Let $S = \{n \in \mathbb{N} | f_{n+6} = 4f_{n+3} + f_n\}$.
Then we have:

- for $n = 1$ $\quad f_{1+6} = f_7 = 13 = 12 + 1 = 4(3) + 1 = 4f_4 + f_1 = 4f_{1+3} + f_1$
- for $n = 2$ $\quad f_{2+6} = f_8 = 21 = 20 + 1 = 4(5) + 1 = 4f_5 + f_2 = 4f_{2+3} + f_2$
- for $n = 3$ $\quad f_{3+6} = f_9 = 34 = 32 + 2 = 4(8) + 2 = 4f_6 + f_3 = 4f_{3+3} + f_3$

So $S = \{1, 2, 3, \ldots, m-1\}$ for some $m \in \mathbb{N}$.
Then we have:

$$\begin{aligned} f_{(m-1)+6} &= 4f_{(m-1)+3} + f_{m-1} \\ f_{m+5} &= 4f_{m+2} + f_{m-1} \\ f_{m+5} + f_{m-2} &= 4f_{m+2} + f_{m-1} + f_{m-2} \\ &= 4f_{m+2} + f_m \\ f_{m+5} + f_{m-2} + 4f_{m+1} &= 4f_{m+2} + f_m + 4f_{m+1} \\ &= 4f_{m+2} + 4f_{m+1} + f_m \\ &= 4\left(f_{m+2} + f_{m+1}\right) + f_m \\ f_{m+5} + f_{m-2} + 4f_{m+1} &= 4f_{m+3} + f_m \end{aligned}$$

$$
\begin{aligned}
f_{m+5} + f_{m-2} + 4f_{m+1} &= 4f_{m+3} + f_m \\
f_{m+5} + f_{m-2} + (f_{m+1} + 3f_{m+1}) &= \\
f_{m+5} + f_{m-2} + (f_{m-1} + f_m) + 3f_{m+1} &= \\
f_{m+5} + (f_{m-2} + f_{m-1}) + f_m + 3f_{m+1} &= \\
f_{m+5} + f_m + f_m + 3f_{m+1} &= \\
f_{m+5} + f_m + f_m + (f_{m+1} + 2f_{m+1}) &= \\
f_{m+5} + f_m + (f_m + f_{m+1}) + 2f_{m+1} &= \\
f_{m+5} + f_m + f_{m+2} + 2f_{m+1} &= \\
f_{m+5} + f_m + f_{m+2} + (f_{m+1} + f_{m+1}) &= \\
f_{m+5} + f_m + (f_{m+2} + f_{m+1}) + f_{m+1} &= \\
f_{m+5} + f_m + f_{m+3} + f_{m+1} &= \\
f_{m+5} + f_{m+3} + (f_m + f_{m+1}) &= \\
f_{m+5} + (f_{m+3} + f_{m+2}) &= \\
f_{m+5} + f_{m+4} &= \\
f_{m+6} &= 4f_{m+3} + f_m
\end{aligned}
$$

So $m \in S$.
Then by the Principle of Complete Induction, $S = \mathbb{N}$. □

(c) *Proof.* Let $S = \{n \in \mathbb{N} | \text{ for any } a \in \mathbb{N}, f_a f_n + f_{a+1} f_{n+1} = f_{a+n+1}\}$
Then we have, for any $a \in \mathbb{N}$:

- for $n = 1$

$$f_a f_1 + f_{a+1} f_{1+1} = f_a(1) + f_{a+1}(1) = f_a + f_{a+1} = f_{a+2} = f_{a+1+1}$$

- for $n = 2$

$$
\begin{aligned}
f_a f_2 + f_{a+1} f_{2+1} &= f_a(1) + f_{a+1}(2) = f_a + f_{a+1} + f_{a+1} = f_{a+2} + f_{a+1} = f_{a+3} \\
&= f_{a+2+1}
\end{aligned}
$$

So $S = \{1, 2, \ldots, m-1\}$ for some $m \in \mathbb{N}$.
Then we have, for any $a \in \mathbb{N}$:

$$
\begin{aligned}
f_a (f_{m-1} + f_{m-2}) + f_{a+1} (f_m + f_{m-1}) &= (f_a f_{m-1} + f_a f_{m-2}) + (f_{a+1} f_m + f_{a+1} f_{m-1}) \\
&= (f_a f_{m-1} + f_{a+1} f_m) + (f_a f_{m-2} + f_{a+1} f_{m-1}) \\
&= (f_a f_{m-1} + f_{a+1} f_m) + (f_a f_{m-2} + f_{a+1} f_{m-1}) \\
&= (f_{a+m}) + (f_{a+m-1}) \\
&= f_{a+m+1}
\end{aligned}
$$

So $m \in \mathbb{N}$.
Thus, by the Principle of Complete Induction, $S = \mathbb{N}$. □

(d) *Proof.* Let $S = \left\{ n \in \mathbb{N} | f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \right\}$.

Then we have:

- for $n = 1$

$$f_1 = 1 = \frac{\alpha - \beta}{\alpha - \beta} = \frac{\alpha^1 - \beta^1}{\alpha - \beta}$$

- for $n = 2$

$$f_2 = 1 = \frac{1}{2} + \frac{1}{2} = \frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} = \alpha + \beta = \frac{\alpha - \beta}{\alpha - \beta}(\alpha + \beta) = \frac{\alpha^2 - \beta^2}{\alpha - \beta}$$

So $S = \{1, 2, \ldots, m - 1\}$, for some $m \in \mathbb{N}$.

We should note that

$$\begin{aligned}
\alpha^2 &= \alpha + 1 \\
\alpha^{m-2}\alpha^2 &= \alpha^{m-2}(\alpha + 1) \\
\alpha^m &= \alpha^{m-1} + \alpha^{m-2} \\
\alpha^{m-1} &= \alpha^m - \alpha^{m-2}
\end{aligned}$$

$$\begin{aligned}
\beta^2 &= \beta + 1 \\
\beta^{m-2}\beta^2 &= \beta^{m-2}(\beta + 1) \\
\beta^m &= \beta^{m-1} + \beta^{m-2} \\
\beta^{m-1} &= \beta^m - \beta^{m-2}
\end{aligned}$$

Then we have:

$$\begin{aligned}
f_{m-1} &= \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \\
&= \frac{(\alpha^m - \alpha^{m-2}) - (\beta^m - \beta^{m-2})}{\alpha - \beta} \\
&= \frac{(\alpha^m - \beta^m) - (\alpha^{m-2} - \beta^{m-2})}{\alpha - \beta} \\
&= \frac{\alpha^m - \beta^m}{\alpha - \beta} - \frac{\alpha^{m-2} - \beta^{m-2}}{\alpha - \beta} \\
&= \frac{\alpha^m - \beta^m}{\alpha - \beta} - f_{m-2} \\
f_{m-1} + f_{m-2} &= \frac{\alpha^m - \beta^m}{\alpha - \beta} \\
f_m &= \frac{\alpha^m - \beta^m}{\alpha - \beta}
\end{aligned}$$

So $m \in \mathbb{N}$.

Thus by the Principle of Complete Induction, $S = \mathbb{N}$. $\square$

§3.1 2 (f) Dom(W) = (-2, 2), Rng(W) = $\{3\}$.

(g) Dom(W) = $\mathbb{R}$, Rng(W) = $\mathbb{R}$.

(h) Dom(W) = $\mathbb{R}$, Rng(W) = $\mathbb{R}$.

4 (b)

$$x = -5y + 2 \implies 5y = -x + 2 \implies y = \frac{-x+2}{5}$$
$$R_2^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} | y = \frac{-x+2}{5}\right\}$$

(d)

$$x = y^2 + 2 \implies y^2 = x - 2 \implies y = \sqrt{x-2}$$
$$R_4^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} | y = \sqrt{x-2}\right\}$$

(h)

$$x = \frac{2y}{y-2} \implies xy - 2x = 2y \implies y(x-2) = 2x \implies y = \frac{2x}{x-2}$$
$$R_4^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} | y = \frac{2x}{x-2}\right\}$$

5 (a) $R \circ S = \{(3, 5), (5, 2)\}$

(c) $T \circ S = \{(2, 1), (3, 1), (3, 4)\}$

(e) $S \circ R = \{(1, 5), (2, 4), (5, 4)\}$

9 For $R \subseteq (A \times B), S \subseteq (B \times C)$

(a) *Proof.*

$$\begin{aligned}
\text{Dom}(S \circ R) &= \{x \in A | \exists z \in A \times C : (x, z) \in S \circ R\} \\
&= \{x \in A | \exists y \in B, z \in C : (x, y) \in R \wedge (y, z) \in S\} \\
&= \{x \in A | \exists y \in B : (x, y) \in R\} \\
&\quad \cap \{x \in A | \exists y \in B : (x, y) \in R \wedge \exists z \in C : (y, z) \in S\} \\
&\subseteq \{x \in A | \exists y \in B : (x, y) \in R\} \\
&= \text{Dom}(R)
\end{aligned}$$

So Dom($S \circ R$) $\subseteq$ Dom(R). □

(b) Let $R = \{(a, b)\}, S = \varnothing$.
Then $S \circ R = \varnothing$, Dom(R) = $\{a\}$, Dom($S \circ R$) = $\varnothing$.
So $\varnothing \subseteq \{a\}$, but $\varnothing \not\supseteq \{a\}$.
So Dom($S \circ R$) $\neq$ Dom(R).

(c) *Proof.*

$$\begin{aligned}
\text{Rng}(S \circ R) &= \{z \in C | \exists x \in A \times C : (x, z) \in S \circ R\} \\
&= \{z \in C | \exists x \in A, y \in B : (x, y) \in R \wedge (y, z) \in S\} \\
&= \{z \in C | \exists x \in A : (x, y) \in R \wedge \exists y \in B : (y, z) \in S\} \\
&\quad \cap \{z \in C | \exists y \in B : (y, z) \in S\} \\
&\subseteq \{z \in C | \exists y \in B : (y, z) \in S\} \\
&= \text{Rng}(S)
\end{aligned}$$

So $\text{Rng}(S \circ R) \subseteq \text{Rng}(S)$. □

A counter example is:

Let $R = \varnothing, S = \{(a, b)\}$.

Then $S \circ R = \varnothing$, $\text{Rng}(S) = \{b\}$, $\text{Rng}(S \circ R) = \varnothing$.

So $\varnothing \subseteq \{b\}$, but $\varnothing \not\supseteq \{b\}$.

So $\text{Rng}(S) \not\subseteq \text{Rng}(S \circ R)$.

12 *Proof.* Given a set A with m elements and a set B with n elements.

We can form $A \times B$ as a new set.

We can count the number of elements in $A \times B$ as:

$$A \times B = \left\{ \overbrace{\underbrace{(a_1, b_1), \ldots, (a_1, b_n)}_{n \text{ pairs}}, \underbrace{(a_2, b_1), \ldots, (a_2, b_n)}_{n \text{ pairs}}, \ldots, \underbrace{(a_m, b_1), \ldots, (a_m, b_n)}_{n \text{ pairs}}}^{m \text{ times}} \right\}$$

So there are $m \cdot n$ elements in $A \times B$.

We can enumerate all subsets by taking the power set of $A \times B$.

And we have proved that for any set S of size k there are 2^k subsets in $\mathcal{P}(S)$.

So we have 2^{mn} subsets in $\mathcal{P}(A \times B)$.

Since each subset in $\mathcal{P}(A \times B)$ is a subset of $A \times B$, every one is a relation from A to B.

So there are 2^{mn} relations from A to B. □