MAT 67 Homework 3

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1. Let V be the set of all pairs (x, y) of real numbers and suppose vector addition and scalar multiplication are defined in the following way:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
$$a(x, y) = (ax, y)$$

for any scalar a in the field of real numbers.

Is the set V a vector space over the field $\mathbb{R}$?

The set V is not a vector space over the field $\mathbb{R}$ for it fails to hold for distributivity of scalar addition over scalar multiplication.

Proof. Let $\vec{u} \in V$ and $c, k \in \mathbb{R}$. Let's check distributivity of scalar addition over scalar multiplication. We should have $(c + k)\vec{u} = c\vec{u} + k\vec{u}$.

$$\begin{aligned}(c + k)\vec{u} &= ((c + k)u_1, u_2)\\ &= (cu_1 + ku_1, u_2)\end{aligned}$$

$$\begin{aligned}c\vec{u} + k\vec{u} &= (cu_1, u_2) + (ku_1, u_2)\\ &= (cu_1 + ku_1, u_2 + u_2)\\ &= (cu_1 + ku_1, 2u_2)\end{aligned}$$

But $(cu_1 + ku_1, u_2) \neq (cu_1 + ku_1, 2u_2)$, so V does not hold for distributivity of scalar addition over scalar multiplication.

Thus V is not a vector space. $\square$

2. Let W_1 and W_2 be subspaces of a vector space V such that their union $W_1 \cup W_2$ is also a subspace of V.

   Prove that either W_1 is contained in W_2 or vice versa.

   Proof. Without loss of generality, we arbitrarily choose to examine W_1 against W_2.

   Since W_1 is a subspace, $\forall w_1 \in W_1, -w_1 \in W_1$. And since W_2 is a subspace, $\forall w_2 \in W_2, -w_2 \in W_2$.

   Now since $W_1 \cup W_2$ is also a subspace, we should be able to take arbitrary vectors from this union and perform vector addition with them. Meaning, we can take one vector from W_1 and add it to another vector from W_2, for instance.

   So, we have $w_1 + w_2 \in W_1 \cup W_2$.

   By the definition of union we must have one of the two:

   (a) $w_1 + w_2 \in W_1$
   (b) $w_1 + w_2 \in W_2$

   (a) If $w_1 + w_2 \in W_1$, then $(w_1 + w_2) + (-w_1) \in W_1$. Which means that $w_2 \in W_1$. Now since we chose all w_2, we are saying that $\forall w_2 \in W_2, w_2 \in W_1$.

      So, it follows that $W_2 \subseteq W_1$.

   (b) If $w_1 + w_2 \in W_2$, then we have $(w_1 + w_2) + (-w_2) \in W_2$. Which means $w_1 \in W_2$. So $\forall w_1 \in W_1, w_1 \in W_2$ and thus $W_1 \subseteq W_2$.

   Thus either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

   Now, a similar argument can be used to show that if we choose to examine W_2 against W_1 we arrive at the same conclusion. □