MAT 167 HW 2

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§2.1

4 The smallest subspace containing both symmetric matrices and lower triangular matrices is the set of all 3 by 3 matrices. It must be the entire subspace since we have to be able to add symmetric matrices to lower triangular matrices. This combination would end up with matrices that have entries above and below the diagonal, though not necessarily symmetric or lower triangular.

The largest subspace in both the subspace of all symmetric matrices, let's call it $\mathcal{S}$, and the subspace of all lower triangular matrices, let's call it $\mathcal{L}$, would be the set of all diagonal matrices, since every diagonal matrix would be symmetric and would also be lower triangular.

8 The correct answer is (e). Since we have $Ax = 0$, the solutions x form the null space of A.

14 (a) The smallest subspace M that contains these two matrices is

$$\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} | a, b \in \mathbb{F} \right\}$$

For whatever field $\mathbb{F}$ the original matrices are defined over.

(b) The smallest subspace M that contains these two matrices is

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in \mathbb{F} \right\}$$

For whatever field $\mathbb{F}$ the original matrices are defined over.

(c) The smallest subspace M that contains these two matrices is

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a, b \in \mathbb{F} \right\}$$

For whatever field $\mathbb{F}$ the original matrices are defined over.

(d) The smallest subspace M that contains these two matrices is

$$\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a, b, c \in \mathbb{F} \right\}$$

For whatever field $\mathbb{F}$ the original matrices are defined over.

24 For which vectors (b_1, b_2, b_3) do these systems have a solution?

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{1}$$

and

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2}$$

Since the matrix in (1) is invertible, (b_1, b_2, b_3) is all of $\mathbf{R}^3$
For (2) we have solutions of the form $(x_1 + x_2, x_2, 0)$.

26 Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the column space of A is $\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\}$.
Meanwhile the column space of AB is $\{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\}$.
So $\text{col}(A) \neq \text{col}(AB)$

§2.2 6 We can do this by reducing the system:

$$\begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 2 & 3 & b_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 3 & -2b_1 + b_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & b_1 \\ 0 & 1 & b_2 \\ 0 & 0 & -2b_1 - 3b_2 + b_3 \end{bmatrix} \end{aligned}$$

So if we have $2b_1 + 3b_2 = b_3$, then the last equation is $0 = 0$
The rank is 2.
A particular solution is:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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- Using the trick explained in the book, We can read off some of the elements from the particular solution:

$$\begin{bmatrix} 1 & 0 & w_1 \\ 0 & 1 & w_2 \end{bmatrix}$$

All that is necessary now is to find w_1, w_2.
Which we can read from the null space solution. So our system is:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- It suffices to add another row so our sytem becomes

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

18 Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then the pivot columns for A are 1 and 3, while the pivot columns for A^T are 1 and 2.

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30 The steps are as follows.

Step 1 We reduce to

$$\begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 2 & 5 & 7 & 6 & b_2 \\ 2 & 3 & 5 & 2 & b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & -b_1 + b_2 \\ 0 & -1 & -1 & -2 & -b_1 + b_3 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 2 & 4 & 6 & 4 & b_1 \\ 0 & 1 & 1 & 2 & -b_1 + b_2 \\ 0 & 0 & 0 & 0 & -b_1 + b_2 + b_3 \end{bmatrix}$$

Step 2 So the solvability condition is $b_1 = b_2 + b_3$

Step 3 The column space is all linear combinations of the vectors $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \right\}$

Step 4 The special solutions have free variables x_3, x_4.

$$N = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Step 5

Step 6

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68 (a) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then the null space of A is spanned by $\begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix}$ for any $x_1 \in \mathbb{R}$.

Whereas the null space of A^T is spanned by $\begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix}$ for any $x_3 \in \mathbb{R}$.

(b) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then the free variables of A are x_1, x_3. Whereas the free variables of A^T are x_1, x_2.

(c) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Whereas

$$\operatorname{rref}\left(A^T\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

§2.3 8 We can substitute:

$$\begin{aligned} c_1v_1 + c_2v_2 + c_3v_3 &= c_1(w_2 + w_3) + c_2(w_1 + w_3) + c_3(w_1 + w_2) \\ &= c_1w_2 + c_1w_3 + c_2w_1 + c_2w_3 + c_3w_1 + c_3w_2 \\ &= c_2w_1 + c_3w_1 + c_1w_2 + c_3w_2 + c_1w_3 + c_2w_3 \\ &= (c_2 + c_3)w_1 + (c_1 + c_3)w_2 + (c_1 + c_2)w_3 \end{aligned}$$

Since w_1, w_2, w_3 are linearly independent, this equation is 0 only when $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$

So we solve

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Reducing the matrix we get

$$\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 1 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

So $c_1 = c_2 = c_3 = 0$.

Thus v_1, v_2, v_3 are linearly independent.

17 The test is for how many rows are now all 0 rather than the columns. If there is a row with all zero entries, then that vector is a combination of the others. If no row has all zero entries, then the vectors are linearly independent.

30 The pivots are the second and third columns.

One basis is $\left\{ u_2 = \begin{bmatrix}5\\0\\0\\0\end{bmatrix}, u_3 = \begin{bmatrix}4\\2\\0\\0\end{bmatrix} \right\}$

We express column one as $u_1 = 0u_2 + 0u_3$. We express column four as $u_4 = \frac{1}{5}u_2 + \frac{1}{2}u_3$.

36 Since the rank is 11, and the matrix A is 64×17, there are 6 vectors in the nullspace. So there are 6 vectors satisfying $Ax = 0$.

Since the rank is 11, and the matrix A^T is 17×64, there are 53 vectors in the nullspace. So there are 53 vectors satisfying $A^T y = 0$.

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- One possible basis is $\{1, x, x^2, x^3\}$
- We need $p(1) = 0$ for all elements of the basis.
  It is easy to see that the basis vector 1 cannot be included.
  If we evaluate the polynomial at the basis vector x we get 1, so we'd need a new basis vector $x - 1$.
  If we evaluate the polynomial at the basis vector x^2 we get 1, so we'd need a new basis vector $x^2 - 1$.
  If we evaluate the polynomial at the basis vector x^3 we get 1, so we'd need a new basis vector $x^3 - 1$.
  Thus our new basis is: $\{x - 1, x^2 - 1, x^3 - 1\}$

§2.4 4

$$A^T = \begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$$

- $C(A)$ rank: 2 vectors: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$
- $N(A)$ rank: 1 vectors: $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$

- $R(A)$ rank: 2 vectors: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ $]\}\{[$
- $N(A^T)$ rank: 1 vectors: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

12 (a)

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$$

rank: 1

$$A = uv^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$$

rank: 1

$$A = uv^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix}$$

14 (a) The rank of this matrix is 2, which is the same as the number of rows, and it is a rectangular matrix, so it only has a right inverse, though there many inverses.
We can construct the “best” right inverse by $A^T(AA^T)^{-1}$

$$\begin{aligned} A^{-1} = A^T(AA^T)^{-1} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned}$$

(b) In this case, we have only one inverse.

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The rank of this matrix is 2, which is the same as the number of columns, and it is a rectangular matrix, so it only has a left inverse which is unique. We can use the fact that $M^T = A$ to use $(A^{-1})^T$ as the left inverse.

$$M^{-1} = (A^{-1})^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

(c)

$$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

The rank of this matrix is 2, which is the same as the number of rows and columns, so it has both a left and a right inverse which are the same.
This we can use the closed form to calculate $T^{-1} = \frac{1}{\det(T)}\mathrm{adj}(T)$

$$T^{-1} = \frac{1}{a^2}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$$

16 The problem is the assumption that $A^T A$ is left-invertible. This does not follow from any given information.

21 (a) Column space contains $\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right]$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Column space has basis $\left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$, nullspace has basis $\left[\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right]$.
Not possible since the rank would be 1, and the nullity 1, but it has 3 rows, and $3 - 1 \neq 1$

(c) Dimension of nullspace = 1 + dimension of left nullspace.

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

(d) Left nullspace contains $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$.

$$\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

(e) Row space = column space, nullspace $\neq$ left nullspace.
Not possible since row space = column space implies a square matrix, and for square matrices, nullspace = left nullspace.

33 The combination is spelled out in the right hand side.

1 row 3 - 2 row 2 + 1 row 1 = the zero row.

Which vectors are in the nullspace of A^T and which are in the nullspace of A? The same vectors are in both spaces: scalar multiples of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

§2.5 1 We have

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \operatorname{rref}\left(A^T\right) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So we have complete solutions

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

4 Given

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

We have

$$\begin{aligned} A^T A &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \end{aligned}$$

Now $A^T A = \left(A^T A\right)^T$ by inspection. If we attempt to reduce it we get:

$$\begin{aligned}\begin{bmatrix}2 & 1 & -1\\ 1 & 2 & 1\\ -1 & 1 & 2\end{bmatrix} &= \begin{bmatrix}1 & 2 & 1\\ 2 & 1 & -1\\ -1 & 1 & 2\end{bmatrix}\\ &= \begin{bmatrix}1 & 2 & 1\\ 0 & -3 & -3\\ 0 & 3 & 3\end{bmatrix}\\ &= \begin{bmatrix}1 & 2 & 1\\ 0 & 3 & 3\\ 0 & 0 & 0\end{bmatrix}\end{aligned}$$

Since there is a row of all 0, A^TA is singular.
We get the complete solution

$$x = \begin{bmatrix}1\\ 3\\ 0\end{bmatrix} + x_3\begin{bmatrix}0\\ -1\\ 1\end{bmatrix}$$

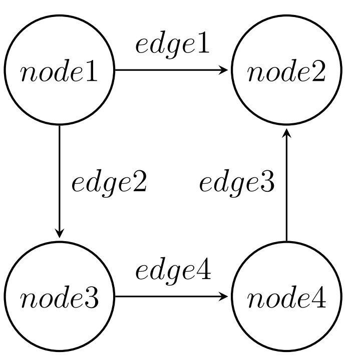
The 2×2 matrix is

$$\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$$

We can reduce to

$$\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Since this is the identity matrix, it is not singular.



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If we reduce A we get

$$\begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

So we have a row of all 0, thus the rows are not independent.
If we remove the last edge we have

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

which reduces to

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Since there is no row of all 0, this graph is a spanning tree.

§2.6 8

14 If T is linear we have, then we have for some $cx, dy \in \mathbb{R}^3$:

$$\begin{aligned} T^2(cx+dy) &= (T \circ T)(cx+dy) \\ &= T(T(cx+dy)) \\ &= T(cTx+dTy) \\ &= cTTx+dTTy \\ &= cT^2x+dT^2y \end{aligned}$$

So T^2 is linear.

22 (d) is not a linear transformation.
Let $x, y \in \operatorname{dom}(T)$
Now

$$\begin{aligned} T(cx+dy) &= T\left((cx_1, cx_2)+(dx_1, dx_2)\right) \\ &= T\left((cx_1+dx_1, cx_2+dx_2)\right) \\ &= (0,1) \end{aligned}$$

but

$$\begin{aligned} cTx+dTy &= cT(x_1,x_2)+dT(y_1,y_2) \\ &= c(0,1)+d(0,1) \\ &= (0,c)+(0,d) \\ &= (0,c+d) \end{aligned}$$

Which is not true for any c, d.

29 (a) The range of T is a line, not $\mathbb{R}^2$.

(b) The range of T is a plane, not $\mathbb{R}^3$.

(c) The kernel of T has infinitely many vectors of the form $(0, v_2)$, not just 0.

50 (a) Assuming A is non trivial, the shape is a parallelogram.

(b) A is a square if A is of the form

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} \text{ or } \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix}$$

(c) A is a line if A has only one non-zero entry.

(d) The area is still 1 if $\det(A) = 1$.