MAT 125A HW 1

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Exercise 4.2.1 (a) We want to prove:

$$\lim_{x \to 2} (2x + 4) = 8$$

Proof. Choose $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies |(2x + 4) - 8| < \epsilon$$

.

We can simplify the consequent a bit.

$$\begin{aligned} |(2x + 4) - 8| &< \epsilon \\ |2x - 4| &< \\ 2\,|x - 2| &< \\ |x - 2| &< \frac{\epsilon}{2} \end{aligned}$$

If we notice, this is exactly the form of the antecedent, assuming $\delta = \frac{\epsilon}{2}$.
So, choose $\delta = \frac{\epsilon}{2}$.
Then we have

$$0 < |x - 2| < \delta \implies |(2x + 4) - 8| < \epsilon$$

as was to be shown. □

(b) We want to prove:

$$\lim_{x \to 0} x^3 = 0$$

Proof. Choose $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \implies \left|x^3 - 0\right| < \epsilon$$

.

We can simplify the consequent a bit.

$$\begin{aligned} \left|x^3 - 0\right| &< \epsilon \\ \left|x^3\right| &< \\ |x|^3 &< \\ |x| &< \sqrt[3]{\epsilon} \\ |x - 0| &< \sqrt[3]{\epsilon} \end{aligned}$$

If we notice, this is exactly the form of the antecedent, assuming $\delta = \sqrt[3]{\epsilon}$.
So, choose $\delta = \sqrt[3]{\epsilon}$.
Then we have

$$0 < |x - 0| < \delta \implies \left|x^3 - 0\right| < \epsilon$$

as was to be shown. □

(c) We want to prove:

$$\lim_{x \to 2} x^3 = 8$$

Proof. Choose $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \epsilon$$

.

We can simplify the consequent a bit.

$$\begin{aligned} \left|x^3 - 8\right| &< \epsilon \\ \left|(x-2)(x^2 + 2x + 4)\right| &< \end{aligned}$$

If we notice, this is exactly the form of the antecedent, assuming $\delta = \sqrt[3]{\epsilon}$.
So, choose $\delta = \sqrt[3]{\epsilon}$.
Then we have

$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \epsilon$$

as was to be shown. □

(d) We want to prove:

$$\lim_{x \to \pi} \lfloor x \rfloor = 3$$

Proof. Choose $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - \pi| < \delta \implies |\lfloor x \rfloor - 3| < \epsilon$$

.

We can simplify the consequent a bit.

If we notice, this is exactly the form of the antecedent, assuming $\delta = 0$.

So, choose $\delta = 0$.

Then we have

$$0 < |x - \pi| < \delta \implies |\lfloor x \rfloor - 3| < \epsilon$$

as was to be shown. $\square$

Exercise 4.2.2 Any δ_0 smaller than δ will suffice, as it implies a stronger statement. This is because if $0 < |x - c| < \delta_0$ is true, then the following is also true: $0 < |x - c| < \delta_0 < \delta$. From which it follows $|f(x) - L| < \epsilon$.

Exercise 4.2.3 (a) We have $f(x) = \frac{|x|}{x}$. It is helpful to enumerate some values of this function.

x	$f(x)$
-3	-1
-2	-1
-1	-1
0	$\frac{0}{0}$
1	1
2	1
3	1

So we can see that $f(0)$ is a problem. We'll need to construct two sequences that approach 0–so they have the same limit, but have different limits when f is applied to them element-wise.

Proof. Choose $x_n = \frac{1}{n}, y_n = -\frac{1}{n}$.

So $(x_n) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, and $(y_n) = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$.

Then $\lim(x_n) = 0$ and $\lim(y_n) = 0$.

Now, $f(x_n) = \{1, 1, 1, \dots\}$ and $f(y_n) = \{-1, -1, -1, \dots\}$.

So, $\lim f(x_n) = 1$ and $\lim f(y_n) = -1$.

Thus, we have our function $f(x) = \frac{|x|}{x}$, with $c = 0$. We have constructed two sequences $(x_n), (y_n)$ with $x_n \neq 0, y_n \neq 0, \lim(x_n) = \lim(y_n) = 0$, and $\lim f(x_n) \neq \lim f(y_n)$.

So we conclude by Corollary 4.2.5 that $\lim_{x \to 0} f(x)$ does not exist. $\square$

(b) We have

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

.

We'll need to construct two sequences that approach 1–so they have the same limit, but have different limits when g is applied to them element-wise.

Proof. Choose $x_n = \frac{n+1}{n}, y_n = \frac{n+e}{n}$.
So $(x_n) = \{2, \frac{3}{2}, \frac{4}{3}, \dots\}$, and $(y_n) = \{1+e, \frac{2+e}{2}, \frac{3+e}{3}, \dots\}$.
Then $\lim(x_n) = 1$ and $\lim(y_n) = 1$.

If we look at each element of (x_n) we see that every element is in $\mathbb{Q}$, as $n+1 \in \mathbb{Q}, n \in \mathbb{Q}, n \neq 0$ and $\mathbb{Q}$ is closed under division where the quotient does not equal 0.

If we look at each element of (y_n) we see that every element is not in $\mathbb{Q}$, as $e \notin \mathbb{Q} \implies n+e \notin \mathbb{Q} \implies \frac{n+e}{n} \notin \mathbb{Q}$.

Now, $g(x_n) = \{1, 1, 1, \dots\}$ and $g(y_n) = \{0, 0, 0, \dots\}$.

So, $\lim g(x_n) = 1$ and $\lim g(y_n) = 0$.

Thus, we have our function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases},$$

with $c = 1$.

We have constructed two sequences $(x_n), (y_n)$ with $x_n \neq 1, y_n \neq 1, \lim(x_n) = \lim(y_n) = 1$, and $\lim g(x_n) \neq \lim g(y_n)$.

So we conclude by Corollary 4.2.5 that $\lim_{x \to 1} g(x)$ does not exist. $\square$

Exercise 4.2.4

Exercise 4.2.6

Exercise 4.2.7

Exercise 4.2.8

Exercise 4.2.9