MAT 67 Homework 1

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1. Let a, b, c and d be fixed real numbers and consider the following system of linear equations in two real variables x_1 and x_2

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned}$$

Note that $x_1 = x_2 = 0$ is a solution of the above equations for any choice of a, b, c, and d.

Prove that if $ad - bc \neq 0$, then $x_1 = x_2 = 0$ is the only solution.

Proof. Assume $ad - bc \neq 0$

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned} \tag{1}$$

Let's multiply the first equation by d and the second equation by b.

$$\begin{aligned} adx_1 + bdx_2 &= 0 \\ bcx_1 + bdx_2 &= 0 \end{aligned}$$

Now let's subtract the second equation from the first.

$$adx_1 - bcx_1 = (ad - bc)x_1 = 0$$

We assumed that $ad - bc \neq 0$ so we can divide through by $ad - bc$.

$$x_1 = 0$$

Now we need x_2. Let's go back to the system in (1).

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0 \end{aligned} \tag{1}$$

Now let's multiply the first equation by c and the second by a.

$$\begin{aligned} acx_1 + bcx_2 &= 0 \\ acx_1 + adx_2 &= 0 \end{aligned}$$

Now let's subtract the first equation from the second.

$$adx_2 - bcx_2 = (ad - bc)x_2 = 0$$

Again, we assumed that $ad - bc \neq 0$ so we divide by $ad - bc$.

$$x_2 = 0$$

So $x_1 = x_2 = 0$. Thus if $ad - bc \neq 0$, then $x_1 = x_2 = 0$.

$\square$

2. Let $z, \omega \in \mathbb{C}$.

   Prove:
$$|z - \omega|^2 + |z + \omega|^2 = 2(|z|^2 + |\omega|^2)$$

   Proof. Given $z, \omega \in \mathbb{C}$,

   $\exists a, b, c, d \in \mathbb{R} : a + ib = z, c + id = \omega$

$$\begin{aligned}
|z - \omega|^2 + |z + \omega|^2 &= |(a + ib) - (c + id)|^2 + |(a + ib) + (c + id)|^2 \\
&= |(a - c) + i(b - d)|^2 + |(a + c) + i(b + d)|^2 \\
&= \left(\sqrt{(a - c)^2 + (b - d)^2}\right)^2 + \left(\sqrt{(a + c)^2 + (b + d)^2}\right)^2 \\
&= (a - c)^2 + (b - d)^2 + (a + c)^2 + (b + d)^2 \\
&= (a^2 - 2ac + c^2) + (b^2 - 2bd + d^2) + (a^2 + 2ac + c^2) + (b^2 + 2bd + d^2) \\
&= a^2 + a^2 + b^2 + b^2 + c^2 + c^2 + d^2 + d^2 + \cancel{2ac - 2ac} + \cancel{2bd - 2bd} \\
&= 2a^2 + 2b^2 + 2c^2 + 2d^2 \\
&= 2(a^2 + b^2 + c^2 + d^2) \\
&= 2\left[\left(\sqrt{a^2 + b^2}\right)^2 + \left(\sqrt{c^2 + d^2}\right)^2\right] \\
&= 2(|a + ib|^2 + |c + id|^2) \\
&= 2(|z|^2 + |\omega|^2)
\end{aligned}$$

   So $|z - \omega|^2 + |z + \omega|^2 = 2(|z|^2 + |\omega|^2)$. $\square$