MAT 108 HW 7

Hardy Jones
999397426
Professor Bandyopadhyay
Spring 2015

§3.2 5 (b) *Proof.* We need to show reflexivity, symmetry, and transitivity.

- Choose any $m \in \mathbb{N}$.
  Then m has a digit $p \in \{0, 1, \ldots, 9\}$ in its tens place. Since m has the same digit p in its tens place as m, we have mRm.
  So R is reflexive on m
  Since the choice of m was arbitrary, this result holds for all $m \in \mathbb{N}$.
  Thus, R is reflexive on $\mathbb{N}$
- Choose any $m, n \in \mathbb{N}$ with $m\mathbb{N}n$.
  Then m has the same digit in the tens place as n. Then n has the same digit in the tens place as m. Then nRm.
  So $\mathbb{N}$ is symmetric on m, n.
  Since the choice of m, n was arbitrary, this result holds for all $m, n \in \mathbb{N}$.
  Thus, $\mathbb{N}$ is symmetric on $\mathbb{N}$
- Choose any $m, n, p \in \mathbb{N}$ with mRn and nRp.
  Then m has the same digit in the tens place as n. Call this digit q. Then n has the same digit in the tens place as p. Call this digit r.
  Since q and r are the tens digit of n, we have $q = r$ So we know that the tens place of p is q.
  This means that m has the same digit in the tens place as p. So mRp.
  So R is transitive on m, n, p.
  Since the choice of m, n, p was arbitrary,
  this result holds for all $m, n, p \in \mathbb{N}$.
  Thus, R is transitive on $\mathbb{N}$

$\square$

An element of $106/R$ less than 50 is 3, since the tens place of 106 is 0, the tens place of 3 is 0, and $3 < 50$.
An element of $106/R$ between 150 and 300 is 203, since the tens place of 106 is 0, the tens place of 203 is 0, and $150 < 203 < 300$.
An element of $106/R$ greater than 1000 is 2003, since the tens place of 106 is 0, the tens place of 2003 is 0, and $1000 < 2003$.
Three elements of $635/R$ are $30, 31$, and 32, since the tens place of 635 is 3, the tens place of 30 is 3, the tens place of 31 is 3, and the tens place of 32 is 3.

(c) *Proof.* We need to show reflexivity, symmetry, and transitivity.

- Choose any $x \in \mathbb{R}$.
  Then $x = x$, so xVx.
  So V is reflexive on x
  Since the choice of x was arbitrary, this result holds for all $x \in \mathbb{R}$.
  Thus, V is reflexive on $\mathbb{R}$
- Choose any $x, y \in \mathbb{R}$ with $x\mathbb{R}y$.
  Then either $x = y$ or $xy = 1$.
  This also means $y = x$ or $yx = 1$. So we have yVx.
  So $\mathbb{R}$ is symmetric on x, y.
  Since the choice of x, y was arbitrary, this result holds for all $x, y \in \mathbb{R}$.
  Thus, $\mathbb{R}$ is symmetric on $\mathbb{R}$
- Choose any $x, y, z \in \mathbb{R}$ with xVy and yVz.
  Then either $x = y$ or $xy = 1$ and either $y = z$ or $yz = 1$.
  - If $x = y$ and $y = z$, then $x = z$ so xVz.
  - If $x = y$ and $yz = 1$, then $xz = yz = 1$ so xVz.
  - If $xy = 1$ and $y = z$, then $xz = xy = 1$ so xVz.
  - If $xy = 1$ and $yz = 1$, then $\frac{1}{z} = y = \frac{1}{x} \implies \frac{1}{z} = \frac{1}{x} \implies x = z$ so xVz.

  So V is transitive on x, y, z.
  Since the choice of x, y, z was arbitrary,
  this result holds for all $x, y, z \in \mathbb{R}$.
  Thus, V is transitive on $\mathbb{R}$

□

The equivalence class of $3 = \{3, \frac{1}{3}\}$.
The equivalence class of $-\frac{2}{3} = \{-\frac{2}{3}, -\frac{3}{2}\}$.
The equivalence class of $0 = \{0\}$.

(d) *Proof.*
- Choose any $a \in \mathbb{N}$.
  Then the prime factorization of a has exactly as many 2s as the prime factorization of a.
  So aRa.
  So R is reflexive on a
  Since the choice of a was arbitrary, this result holds for all $a \in \mathbb{N}$.
  Thus, R is reflexive on $\mathbb{N}$
- Choose any $a, b \in \mathbb{N}$ with $a\mathbb{N}b$.
  Then the prime factorization of a has exactly as many 2s as the prime factorization of b.
  This means the prime factorization of b has exactly as many 2s as the prime factorization of a.
  So bRa.
  So $\mathbb{N}$ is symmetric on a, b.
  Since the choice of a, b was arbitrary, this result holds for all $a, b \in \mathbb{N}$.
  Thus, $\mathbb{N}$ is symmetric on $\mathbb{N}$
- Choose any $a, b, c \in \mathbb{N}$ with aRb and bRc.
  Then the prime factorization of a has exactly as many 2s as the prime

factorization of b. Call this number of 2s p.
And the prime factorization of b has exactly as many 2s as the prime factorization of c. Call this number of 2s q.
Then $p = q$. This means the prime factorization of a has exactly as many 2s as the prime factorization of c.
So aRc.
So R is transitive on a, b, c.
Since the choice of a, b, c was arbitrary,
this result holds for all $a, b, c \in \mathbb{N}$.
Thus, R is transitive on $\mathbb{N}$

□

Three elements in $1/R$ are $2 = 2^1, 6 = 2^1(3), 10 = 2^1(5)$.
Three elements in $4/R$ are $16 = 2^4, 48 = 2^4(3), 80 = 2^4(5)$.
Three elements in $72/R$ are
$4722366482869645213696 = 2^{72}$,
$14167099448608935641088 = 2^{72}(3)$,
$23611832414348226068480 = 2^{72}(5)$.

(e) *Proof.*
- Choose any $(a, b) \in \mathbb{R} \times \mathbb{R}$.
Then $a^2 + b^2 = a^2 + b^2$. So $(a, b)T(a, b)$.
So T is reflexive on (a, b)
Since the choice of (a, b) was arbitrary, this result holds for all $(a, b) \in \mathbb{R} \times \mathbb{R}$.
Thus, T is reflexive on $\mathbb{R} \times \mathbb{R}$
- Choose any $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$ with $(a, b)\mathbb{R} \times \mathbb{R}(c, d)$.
Then $a^2 + b^2 = c^2 + d^2 \implies c^2 + d^2 = a^2 + b^2$. So $(c, d)T(a, b)$.
So $\mathbb{R} \times \mathbb{R}$ is symmetric on $(a, b), (c, d)$.
Since the choice of $(a, b), (c, d)$ was arbitrary, this result holds for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$.
Thus, $\mathbb{R} \times \mathbb{R}$ is symmetric on $\mathbb{R} \times \mathbb{R}$
- Choose any $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$ with $(a, b)T(c, d)$ and $(c, d)T(e, f)$.
Then $a^2 + b^2 = c^2 + d^2$ and $c^2 + d^2 = e^2 + f^2$.
But this means $a^2 + b^2 = c^2 + d^2 = e^2 + f^2$.
So $a^2 + b^2 = e^2 + f^2$.
And $(a, b)T(e, f)$.
So T is transitive on $(a, b), (c, d), (e, f)$.
Since the choice of $(a, b), (c, d), (e, f)$ was arbitrary,
this result holds for all $(a, b), (c, d), (e, f) \in \mathbb{R} \times \mathbb{R}$.
Thus, T is transitive on $\mathbb{R} \times \mathbb{R}$

□

See Figures 1, 2.

(h) *Proof.* Call the set of all differentiable functions D.
- Choose any $f \in D$.
Then f has the same first derivative as f.
So fRf.

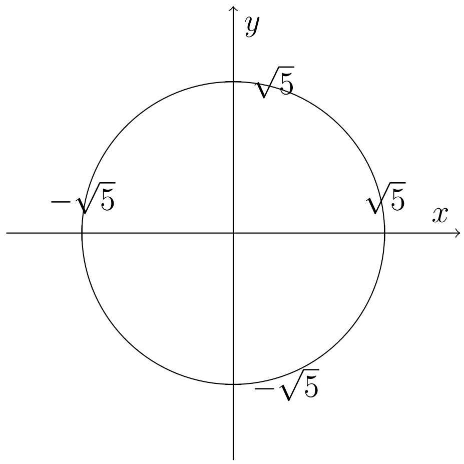

Figure 1: $(1,2)/(\mathbb{R}\times\mathbb{R})$

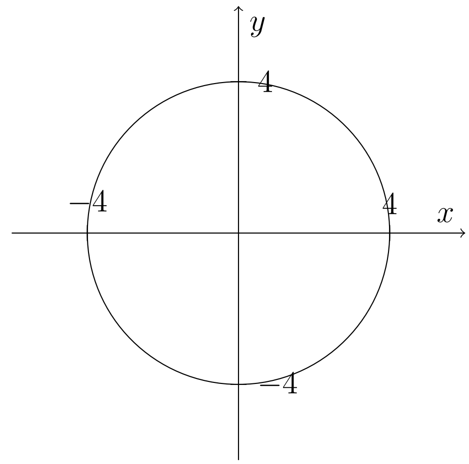

Figure 2: $(4,0)/(\mathbb{R}\times\mathbb{R})$

So R is reflexive on f
Since the choice of f was arbitrary, this result holds for all $f \in D$.
Thus, R is reflexive on D

- Choose any $f, g \in D$ with fDg.
  Then f has the same first derivative as g. This means that g has the same first derivative as f.
  So gRf.
  So D is symmetric on f, g.
  Since the choice of f, g was arbitrary, this result holds for all $f, g \in D$.
  Thus, D is symmetric on D
- Choose any $f, g, h \in D$ with fRg and gRh.
  Then f has the same first derivative as g. Call this derivative p. Then g has the same first derivative as h. Call this derivative q.
  Then $f' = g' = p = q = g' = h'$.
  So $f' = h'$ and fRh.
  So R is transitive on f, g, h.
  Since the choice of f, g, h was arbitrary,
  this result holds for all $f, g, h \in D$.
  Thus, R is transitive on D

$\square$

Three elements in x^2/R are: x^2, x^2+1, x^2+2.
Three elements in $(4x^2+10x)/R$ are: $4x^2+10x, 4x^2+10x+1, 4x^2+10x+2$.
x^3/R is the set of all functions whose first derivative is $3x^2$.
$7/R$ is the set of all functions whose first derivative is 0.

6 *Proof.*
- Choose any $\frac{a}{b} \in \mathbb{Q}$.

Then $ab = ba$ so $\frac{a}{b} R \frac{a}{b}$.
So R is reflexive on $\frac{a}{b}$
Since the choice of $\frac{a}{b}$ was arbitrary, this result holds for all $\frac{a}{b} \in \mathbb{Q}$.
Thus, R is reflexive on $\mathbb{Q}$

- Choose any $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ with $\frac{a}{b}\mathbb{Q}\frac{c}{d}$.
  Then $ad = bc$. But also $bc = ad \implies cb = ad \implies cb = da$.
  So $\frac{c}{d}R\frac{a}{b}$.
  So $\mathbb{Q}$ is symmetric on $\frac{a}{b}, \frac{c}{d}$.
  Since the choice of $\frac{a}{b}, \frac{c}{d}$ was arbitrary, this result holds for all $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.
  Thus, $\mathbb{Q}$ is symmetric on $\mathbb{Q}$
- Choose any $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$ with $\frac{a}{b}R\frac{c}{d}$ and $\frac{c}{d}R\frac{e}{f}$.
  Then $ad = bc$. Then $cf = de$.
  So $c = \frac{de}{f}$ and $ad = b\frac{de}{f} \implies a = \frac{be}{f} \implies af = be$.
  So $\frac{a}{b}R\frac{e}{f}$
  So R is transitive on $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.
  Since the choice of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ was arbitrary,
  this result holds for all $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in \mathbb{Q}$.
  Thus, R is transitive on $\mathbb{Q}$

$\square$

The ordered pairs in $\frac{2}{3}/R$ are pairs $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ such that $2q = 3p$.

8 (d) The equivalence classes for the relation congruence modulo 7 are:

$$\begin{aligned}
\overline{0} &= \{\ldots, -14, -7, 0, 7, 14, \ldots\} \\
\overline{1} &= \{\ldots, -13, -6, 1, 8, 15, \ldots\} \\
\overline{2} &= \{\ldots, -12, -5, 2, 9, 16, \ldots\} \\
\overline{3} &= \{\ldots, -11, -4, 3, 10, 17, \ldots\} \\
\overline{4} &= \{\ldots, -10, -3, 4, 11, 18, \ldots\} \\
\overline{5} &= \{\ldots, -9, -2, 5, 12, 19, \ldots\} \\
\overline{6} &= \{\ldots, -8, -1, 6, 13, 20, \ldots\}
\end{aligned}$$

9 We use the equivalent definition that $a \equiv b (mod m)$ is the same as $a - b = km$ for some $k \in \mathbb{Z}$.

(d) We want to find an a such that $a - 3 = 4k$ and $a - 3 = 5l$.
We can solve this algebraically by substitution.

$$\begin{aligned}
a - 3 &= 4k \\
(5l + 3) - 3 &= 4k && \text{since } a = 5l + 3 \\
5l &= 4k \\
l &= \frac{4}{5}k
\end{aligned}$$

Since $k, l \in \mathbb{Z}$ we know that k must be a multiple of 5.

Choose $k = 5$. then we have $l = \frac{4}{5}5 = 4$, and we end up with

$$\begin{aligned} a - 3 &= 5l \\ &= 5(4) \\ &= 20 \\ a &= 23 \end{aligned}$$

Choose $k = -5$. then we have $l = \frac{4}{5}(-5) = -4$, and we end up with

$$\begin{aligned} a - 3 &= 5l \\ &= 5(-4) \\ &= -20 \\ a &= -17 \end{aligned}$$

So a positive integer that is congruent to 3 (mod 4) and congruent to 3 (mod 5) is 23.
And a negative integer that is congruent to 3 (mod 4) and congruent to 3 (mod 5) is -17.

(e) We want to find an a such that $a - 1 = 3k$ and $a - 1 = 7l$.
We can solve this algebraically by substitution.

$$\begin{aligned} a - 1 &= 3k \\ (7l + 1) - 1 &= 3k && \text{since } a = 7l + 1 \\ 7l &= 3k \\ l &= \frac{3}{7}k \end{aligned}$$

Since $k, l \in \mathbb{Z}$ we know that k must be a multiple of 7.
Choose $k = 7$. then we have $l = \frac{3}{7}7 = 3$, and we end up with

$$\begin{aligned} a - 1 &= 7l \\ &= 7(3) \\ &= 21 \\ a &= 22 \end{aligned}$$

Choose $k = -7$. then we have $l = \frac{3}{7}(-7) = -3$, and we end up with

$$\begin{aligned} a - 1 &= 7l \\ &= 7(-3) \\ &= -21 \\ a &= -20 \end{aligned}$$

So a positive integer that is congruent to 1 (mod 3) and congruent to 1 (mod 7) is 22.
And a negative integer that is congruent to 1 (mod 3) and congruent to 1 (mod 7) is -20.

11 *Proof.* We can show that S is not an equivalence relation by showing that S is not transitive.
Choose $x = 1, y = 2, z = 4$. Then $x, y, z, w \in \mathbb{N}$ and $1+2 = 3 = 3(1), 2+4 = 6 = 3(2)$, so xSy and ySz.
Now, if S were transitive, we'd have xSz. However, $1 + 4 = 5$ and 3 does not divide 5.
So S is not transitive.
Thus, S is not an equivalence relation. □

§3.3 3 (a) The relation R creates equivalence classes where each x in each class has the same fractional part.
The following are examples of these equivalence classes:

$$\mathbb{Z}, \{\dots, -1.5, 0.5, 1.5, \dots\}, \{\dots, e-3, e-2, e-1, e, e+1, \dots\}$$

Then we have the partition is the family of sets of the form:

$$A_r = \{x \in \mathbb{R} : x - \lfloor x \rfloor = r\}, \text{ where } r \in [0, 1)$$

(d) For any $x, y \in \mathbb{R}$, x and y are in the same equivalence class iff $x = \pm y$.
The following are examples of these equivalence classes:

$$\{0\}, \{-1, 1\}, \left\{-\frac{3}{7}, \frac{3}{7}\right\}, \{-e, e\}$$

Then we have the partition as the family of sets of the form:

$$A_r = \{r, -r\}, \text{ where } r \in \mathbb{R}$$

4 Let's first enumerate all of the relations.

$$\{(i,i), (i,-i), (-i,i), (-i,-i), (1,1), (1,-1), (-1,1), (-1,-1)\}$$

From this, we can see the equivalence classes:
$i/C = -i/C = \{i, -i\}$,
$1/C = -1/C = \{1, -1\}$
So the partition $\mathcal{P} = \{\{i, -i\}, \{1, -1\}\}$

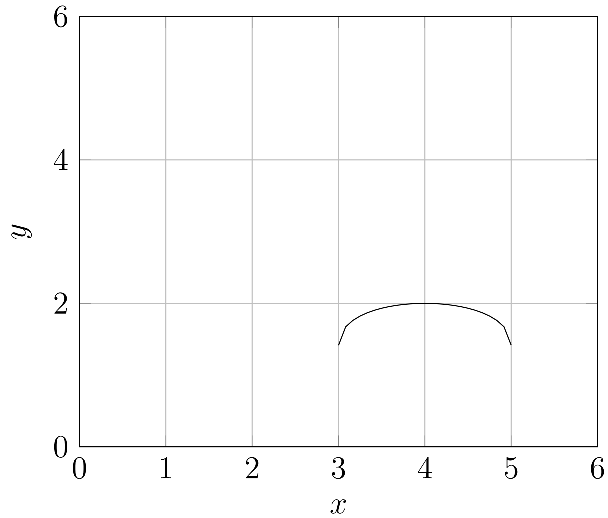

Figure 3: $f(x) = \sqrt{5-x} + \sqrt{x-3}$

6 (a) For $m, n \in \mathbb{N}$, mRn iff $10^p < m \leq n < 10^{p+1}$ for $p \in \{0, 1, 2, \dots\}$.

(e) For $m, n \in \mathbb{Z}$, mRn iff either both $m, n < 3$ or both $m, n \geq 3$.

§4.1 1 (e) If we look at some of the elements of the relation we see:

$$\{(1,1), (1,2), (1,3), \dots\}$$

Since 1 is mapped to many values, this relation is not a function.

(f) If we look at some of the elements of the relation we see:

$$\{(0,0), (1,1), (4,2), (4,-2), (9,3), (9,-3), \dots\}$$

Since 4 is mapped to many values, this relation is not a function.

3 (b) The domain of the mapping is $\mathbb{R}$.
The range of the mapping is $\{x \in \mathbb{R} : x \geq 5\}$.
Another possible codomain is $\mathbb{R}$.

(e) Let's expand the definition of $\mathcal{X}_{\mathbb{N}}(x)$.

$$\mathcal{X}_{\mathbb{N}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x \notin \mathbb{N} \end{cases}$$

So we see, the domain of the mapping is $\mathbb{N}$.
The range of the mapping is $\{0, 1\}$.
Another possible codomain is $\mathbb{R}$.

(f) This mapping is the $\cosh(x)$.
The domain of the mapping is $\mathbb{R}$.
The range of the mapping is $[0, \infty)$.
Another possible codomain is $\mathbb{R}$.

4 (e) $f(x)$ is only defined when $x \leq 5$ and $x \geq 3$.
So the domain of f is $[3, 5]$.
Looking at Figure 3, the range of f is $[\sqrt{2}, 2]$

(f) $f(x)$ is only defined when $x \geq -2$ and $x \leq -2$.
So the domain of f is $\{-2\}$.
The range of f is $\{0\}$

5 (b) *Proof.* We enumerate all possibilities:

- $x = 1, y = 1$
  $3x + y = 3(1) + 1 = 3 + 1 = 4$
  4 is not prime, so $(1, 1) \notin R$.
- $x = 1, y = 2$
  $3x + y = 3(1) + 2 = 3 + 2 = 5$
  5 is prime, so $(1, 2) \in R$.
- $x = 1, y = 3$
  $3x + y = 3(1) + 3 = 3 + 3 = 6$
  6 is not prime, so $(1, 3) \notin R$.
- $x = 2, y = 1$
  $3x + y = 3(2) + 1 = 6 + 1 = 7$
  7 is prime, so $(2, 1) \in R$.
- $x = 2, y = 2$
  $3x + y = 3(2) + 2 = 6 + 2 = 8$
  8 is not prime, so $(2, 2) \notin R$.
- $x = 2, y = 3$
  $3x + y = 3(2) + 3 = 6 + 3 = 9$
  9 is not prime, so $(2, 3) \notin R$.
- $x = 3, y = 1$
  $3x + y = 3(3) + 1 = 9 + 1 = 10$
  10 is not prime, so $(3, 1) \notin R$.
- $x = 3, y = 2$
  $3x + y = 3(3) + 2 = 9 + 2 = 11$
  11 is prime, so $(3, 2) \in R$.
- $x = 3, y = 3$
  $3x + y = 3(3) + 3 = 9 + 3 = 12$
  12 is not prime, so $(3, 3) \notin R$.

So we have:

$$R = \{(1, 2), (2, 1), (3, 2)\}$$

Now, $\text{dom}(R) = A$.
If we choose $(1, y), (1, z) \in R$, then $y = z = 2$. If we choose $(2, y), (2, z) \in R$, then $y = z = 1$. If we choose $(3, y), (3, z) \in R$, then $y = z = 2$.
From this we have shown that R is a function with domain A. $\square$

(c) *Proof.* Choose any $x \in \mathbb{Z}$.

Then $x \cdot x - 2 = -y \implies x^2 + y = 2$, so for any $x \in \mathbb{Z}$, we have $(x, 2 - x^2) \in R$.
Then the domain of R is $\mathbb{Z}$.
Now, choose $(x, y), (x, z) \in R$.
We have $x^2 + y = 2 \implies y = 2 - x^2 = z$, so $y = z$.
Thus we have shown that R is a function on $\mathbb{Z}$. $\square$

8 (b) $\{x \in U : \mathcal{X}_A(x) = 0\} = A^C$

(c) $\{x \in U : \mathcal{X}_A(x) = 2\} = \varnothing$, since $\text{Rng}(\mathcal{X}_A(x)) = \{0, 1\}$

9 (b) $x_n = \frac{(-1)^n}{n}$

(d) $x_n = n(\text{mod } 3)$

10 (b) $f(6) = \overline{6} = \overline{0} = \{\dots, -12, -6, 0, 6, 12, \dots\}$

(d) All pre-images of $\overline{1} = \{\dots, -11, -5, 1, 5, 11, \dots\}$

11 (c) f is a function.

(d) We have $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ with $f(\overline{x}) = [2x + 1]$.
The equivalence classes of $\mathbb{Z}_4$ are $\overline{0}, \overline{1}, \overline{2}, \overline{3}$.
The equivalence classes of $\mathbb{Z}_6$ are $[0], [1], [2], [3], [4], [5]$.
So $f\left(\overline{0}\right) = [1], f\left(\overline{1}\right) = [3], f\left(\overline{2}\right) = [5], f\left(\overline{3}\right) = [7] = [1]$
Then we have $f\left(\overline{4}\right) = f\left(\overline{0}\right) = [1]$, but $f\left(\overline{4}\right) = [9] = [3]$, and $[1] \neq [3]$.
So f is not a function because it is not well defined.

17 (a) In order for the relation from A to B to be a function we need a few things.
We need the relation to have the domain A.
So we need that the relation must have exactly m relations. We also need that the relation must have each $a \in A$ in the relation exactly once. I.e. $\{(a_1, b_1), (a_1, b_2), \dots\}$ is not a function.
We can enumerate all possibilities:

$$
\begin{aligned}
\{&\{(a_1, b_1), (a_2, b_1), \dots, (a_m, b_1)\},\\
&\{(a_1, b_2), (a_2, b_1), \dots, (a_m, b_1)\},\\
&\qquad\vdots\\
&\{(a_1, b_n), (a_2, b_1), \dots, (a_m, b_1)\},\\
&\{(a_1, b_1), (a_2, b_2), \dots, (a_m, b_1)\},\\
&\{(a_1, b_2), (a_2, b_2), \dots, (a_m, b_1)\},\\
&\qquad\vdots\\
&\{(a_1, b_1), (a_2, b_n), \dots, (a_m, b_n)\},\\
&\{(a_1, b_2), (a_2, b_n), \dots, (a_m, b_n)\},\\
&\qquad\vdots\\
&\{(a_1, b_n), (a_2, b_n), \dots, (a_m, b_n)\},\\
\}&
\end{aligned}
$$

After careful counting, we see that there are exactly mn relations that are functions.

(b) For each $a_i \in A$, there are exactly n mappings to some $b_j \in B$. Since there are m possible a_i, there are exactly mn relations that are functions.