MAT 108 HW 4

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§1.6 6 (i) We want to show that there exists some $K \in \mathbb{N}$, such that for all $r \in \mathbb{R}$ if $r > K$, then $\frac{1}{r^2} < 0.01$.
It helps to rewrite the consequent a bit first:

$$\frac{1}{r^2} < 0.01 \implies \frac{1}{r^2} < \frac{1}{100} \implies 100 < r^2$$

Since we're only concerned with r greater than some natural number, we only need to concern ourself with $r > 0$.
So we can simplify the consequent a bit more:

$$100 < r^2 \implies 10 < r$$

Now it's fairly obvious that we need $K \geq 10$.

Proof. Choose $K = 10$.
K is a natural number.
Now, for any arbitrary real number r that is greater than K, we have:

$$10 < r \implies 100 < r^2 \implies \frac{1}{r^2} < \frac{1}{100} \implies \frac{1}{r^2} < 0.01$$

Thus we have shown,
there exists some natural number K, such that for any real number r, if r is greater than K, then r is less than 0.01. □

(j) We want to show that there exist integers L, G, such that for any real number x, if $L < x < G$, then $40 > 10 - 2x > 12$.
It helps to manipulate the consequent a bit:

$$40 > 10 - 2x > 12 \implies 30 > -2x > 2 \implies -15 < x < -1$$

So if we have $L = -15, G = -1$, then we can work backwards to the proof.

Proof. Choose $L = -15, G = -1$.
Then $L, G \in \mathbb{Z}$.
Now, for any real number x.
If $L < x < G$, then we have:

$$-15 < x < -1 \implies 30 > -2x > 2 \implies 40 > 10 - 2x > 12$$

So we have shown that there exist integers L, G, such that for any real number x, if $L < x < G$, then $40 > 10 - 2x > 12$. □

(n) We want to show for any positive real numbers x, y with $x < y$, there exists some natural M, such that if n is a natural and $n > M$, then $\frac{1}{n} < y - x$.
Let's manipulate the consequent a bit.

$$\frac{1}{n} < y - x \implies x + \frac{1}{n} < y$$

So if we can show that $x + \frac{1}{n}$ is less than y, we can prove this.
We have that $M < n$, so $\frac{1}{n} < \frac{1}{M}$.
So, if we choose our M carefully, we can prove this.

Proof. Given any two positive real numbers x, y with $x < y$, choose a natural M such that $x + \frac{1}{M} < y$.
Now, for any natural $n > M$ we have:

$$M < n \implies \frac{1}{n} < \frac{1}{M}$$

Then we have:

$$x + \frac{1}{M} < y \implies x + \frac{1}{n} < x + \frac{1}{M} < y \implies x + \frac{1}{n} < y \implies \frac{1}{n} < y - x$$

So we have shown that for any positive real numbers x, y with $x < y$, there exists some natural M, such that if n is a natural and $n > M$, then $\frac{1}{n} < y - x$. □

§2.1 14 (b) $\mathcal{P}(X) = \{\varnothing, \{S\}, \{\{S\}\}, \{S, \{S\}\}\}$

(d)

$$\begin{aligned}\mathcal{P}(X) = \{&\varnothing \\ &, \{1\}, \{\{\varnothing\}\}, \{\{2, \{3\}\}\} \\ &, \{1, \{\varnothing\}\}, \{1, \{2, \{3\}\}\}, \{\{\varnothing\}, \{2, \{3\}\}\} \\ &, \{1, \{\varnothing\}, \{2, \{3\}\}\} \\ &\}\end{aligned}$$

15 (b) True.

(c) False.

(d) True.

17 (a) True.

(b) True.

(d) True.

(f) True.

18 We want to prove, for any sets A, B, $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$.

We need to show both directions

Proof. • ($\Longrightarrow$)

Assume $A = B$.
Then, $\mathcal{P}(A) = \{X | X \subseteq A\} = \{X | X \subseteq B\} = \mathcal{P}(B)$.
Thus, if $A = B$, then $\mathcal{P}(A) = \mathcal{P}(B)$.

• ($\Longleftarrow$)

Assume $\mathcal{P}(A) = \mathcal{P}(B)$.
Then we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, and $\mathcal{P}(B) \subseteq \mathcal{P}(A)$
Now, since $A \subseteq A, A \in \mathcal{P}(A)$.
And since $\mathcal{P}(A) \subseteq \mathcal{P}(B), A \in \mathcal{P}(B)$.
This means $A \subseteq B$.
Additionally, since $B \subseteq B, B \in \mathcal{P}(B)$.
And since $\mathcal{P}(B) \subseteq \mathcal{P}(A), B \in \mathcal{P}(A)$.
This means $B \subseteq A$.
So we have $A \subseteq B$ and $B \subseteq A$.
Then $A = B$.
Thus, if $\mathcal{P}(A) = \mathcal{P}(B)$, then $A = B$.

Since we have shown both directions, we have shown for any sets A, B, $A = B$ if and only if $\mathcal{P}(A) = \mathcal{P}(B)$. □

§2.2 3 (e) Since $\mathbb{Z}^+ \cap \mathbb{Z}^- = \varnothing$, $\mathbb{Z}^+ - \mathbb{Z}^- = \mathbb{Z}+$

(f) Since $E \cap D = \varnothing$, and $E \cup D = \mathbb{Z}$, and $\mathbb{Z}$ is the universe, $E^C = D$.

(h) Since $(E \cap \mathbb{Z}^-)^C = E^C \cup (\mathbb{Z}^-)^C = D \cup \mathbb{Z}^+ \cup \{0\}$.
In words this is all of the integers except the negative evens.
Or $\{-1, -3, -5, \dots\} \cup \{0, 1, 2, \dots\}$

(i) $\varnothing^C = \mathbb{Z}$, which is the universe.

5 The only disjoint pairs of sets are C, D.