MAT 150A Homework 3

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1. To show that f is an automorphism, we need to show that f is an isomorphism from $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$. To show that f is an isomorphism, we need to show that f is a homomorphism and bijective. To show that f is a homomorphism, we need to show that f is closed, and that it preserves the group operation.

$$GL_n(\mathbb{R}) := \{A_n | A \text{ is an } n \times n \text{ matrix}, |A| \neq 0\}$$

Proof. We begin by showing that f is a homomorphism.

- We first show closure.
  Choose any $A \in GL_n(\mathbb{R})$. $f(A) = (A^T)^{-1}$.
  The size of $f(A)$ has not changed. We know $|A^T| = |A|$ and $|A^{-1}| = |A|^{-1}$, so $|f(A)| = |(A^T)^{-1}| = |(A^T)|^{-1} = |A|^{-1} \neq 0$.
  So, f maps $GL_n(\mathbb{R}) \mapsto GL_n(\mathbb{R})$
  Thus, f is closed.
- Now we show that f preserves the group operation.
  Choose any $A, B \in GL_n(\mathbb{R})$.

$$f(AB) = ((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = f(A)f(B)$$

  So, f preserves the group operation.
  Thus, $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a homomorphism.

Now we need to show that f is bijective.

- We first show that $\ker f = \{I_n\}$
  So we want to find all $A \in GL_n(\mathbb{R})$ such that $f(A) = (A^T)^{-1} = I_n$.
  But we know that for any group the identity is its own inverse, so $A^T = I_n$.
  We also know that $I_n^T = I_n$, so $A = I_n$.
  And since we know that the identity is unique, we have that $\ker f = \{I_n\}$.
- Now we show that $\text{im } f = GL_n(\mathbb{R})$
  It suffices to show that f has an inverse. Namely, $f^{-1}(A) = A^T$.

So, we have shown that $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is an isomorphism. And since f's domain is its co-domain, f is an automorphism. $\square$

2. We want to show that $\forall \varphi : G- > G'$ that are group homomorphisms, $\ker \varphi \leq G$ and $\text{im } \varphi \leq G$.

   For both of these possible subgroups, it suffices to show two things:

   (a) The possible subgroup is non-empty.

   (b) For all a, b in the possible subgroup, ab^{-1} is also in the subgroup.

   - $\ker \varphi \leq G$

     Proof. We need to show the two conditions above.

     (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\ker \varphi$ is non-empty (as $e_G \in \ker \varphi$).

     (b) Choose $a, b \in \ker \varphi$.
     So we have, $\varphi(a) = e_{G'}$ and $\varphi(b) = e_{G'}$, and since φ is a homomorphism.

$$\begin{aligned}
\varphi(ab^{-1}) &= \varphi(a)\varphi(b^{-1}) \\
&= \varphi(a)\varphi(b)^{-1} \\
&= e_{G'}\varphi(b)^{-1} \\
&= e_{G'}e_{G'}^{-1} \\
&= e_{G'}e_{G'} \\
&= e_{G'}
\end{aligned}$$

     Thus, $\ker \varphi \leq G$. $\square$

   - $\text{im } \varphi \leq G$

     Proof. We need to show the two conditions above.

     (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\text{im } \varphi$ is non-empty (as $e_G \in \text{im } \varphi$).

     (b) Choose $a, b \in \text{im } \varphi$.
     This means $\exists a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b$.
     Since φ is a homomorphism and $a'b'^{-1} \in G$.

$$\begin{aligned}
\varphi(a'b'^{-1}) &= \varphi(a')\varphi(b'^{-1}) \\
&= \varphi(a')\varphi(b')^{-1} \\
&= a\varphi(b)^{-1} \\
&= ab^{-1} \in \text{im } \varphi
\end{aligned}$$

     Thus, $\text{im } \varphi \leq G$. $\square$

3. The subgroups of S_3 are:

$$\begin{aligned}
&\{id\}\\
&\{id,(1,2)\},\{id,(1,3)\},\{id,(2,3)\}\\
&\{id,(1,2,3)\},\{id,(1,3,2)\}\\
&\{id,(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}
\end{aligned}$$

The trivial subgroup and the group itself are normal.

4. Want to show $\varphi(x)=\varphi(y) \iff xy^{-1}\in \ker\,\varphi$

Proof.

- $(\Rightarrow)$
  Since φ is a homomorphism and $\varphi(x)=\varphi(y)$.

  $$\varphi(xy^{-1})=\varphi(x)\varphi(y^{-1})=\varphi(x)\varphi(y)^{-1}=\varphi(x)\varphi(x)^{-1}=\varphi(x)\varphi(x^{-1})=\varphi(xx^{-1})=\varphi(e_G)=e_{G'}$$

  By the definition of the kernel, $xy^{-1}\in \ker\,\varphi$.

- $(\Leftarrow)$
  Since φ is a homomorphism and $xy^{-1}\in \ker\,\varphi$.

  $$\begin{aligned}
  \varphi(xy^{-1})&=e_{G'}\\
  \varphi(x)\varphi(y^{-1})&=e_{G'}\\
  \varphi(x)\varphi(y)^{-1}&=e_{G'}\\
  \varphi(x)\varphi(y)^{-1}\varphi(y)&=e_{G'}\varphi(y)\\
  \varphi(x)e_{G'}&=e_{G'}\varphi(y)\\
  \varphi(x)&=e_{G'}\varphi(y)\\
  \varphi(x)&=\varphi(y)
  \end{aligned}$$

  So, $\varphi(x)=\varphi(y)$

Thus, we have shown both directions and $\varphi(x)=\varphi(y) \iff xy^{-1}\in \ker\,\varphi$. □