MAT 167 HW 1

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§ 1.4 2

$$\begin{bmatrix} 4 & 1 \\ 5 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

7 (a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

9 Assuming A has as many pivots as rows, we have the following results.

(a) a_{11}

(b) $l_{i1} = \frac{a_{i1}}{a_{11}}$

(c) $a_{ij} - a_{1j}\left(\frac{a_{i1}}{a_{11}}\right)$

(d) $a_{22} - a_{12}\left(\frac{a_{21}}{a_{11}}\right)$

10 (a) True.

(b) False.
AB may not even have three rows.
For example, let A be a 1×3 matrix and B be a 3×3 matrix.
Then AB is a 1×3 matrix, so it has no third row.

(c) True.

(d) False.
Let

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Then

$$\begin{aligned} (AB)^2 &= \left(\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right)^2 \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} A^2B^2 &= \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 3 & 6 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \end{aligned}$$

So $(AB)^2 \neq A^2B^2$.

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- Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \end{aligned}$$

- *Proof.* For any lower triangular matrices A, B with dimension $n \times n$, each entry ab_{ij} in AB it is computed by:

$$\sum_{k=1}^{n} a_{ik} b_{kj}$$

If $i < k, a_{ik} = 0$.
If $k < j, b_{kj} = 0$.

Each entry above the main diagonal has one of either $i < k$ or $k < j$.
So for each entry above the main diagonal of AB, we have a sum of products where at least one of the factors is 0.
So, each entry above the main diagonal is 0.
Thus, the product of any two lower triangular matrices is lower triangular. $\square$

13 (a) Let

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

$$\begin{aligned} A^2 &= \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} \right)^2 \\ &= \frac{1}{3} \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= -I \end{aligned}$$

(b) Let

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$$

$$B^2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

(c) Let

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$CD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$DC = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Let

$$E = F = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$$

$$EF = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

24 We want

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$
\begin{aligned}
M &= E_{32}E_{31}E_{21} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}
\end{aligned}
$$

$$
MA = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}
$$

42 (a) True.
Since matrix multiplication is only defined for matrices A, B with dimension $m \times n, n \times p$, respectively, $A^2 = A * A$ must have $m = n = p$. That is A must be a square matrix.

(b) False.
We can choose A, B with dimension $m \times n, n \times m$, respectively, where $m \neq n$.
So A and B are not square.
Then AB is defined, as well as BA, yet A and B are not square.

(c) True.
We can choose A, B with dimension $m \times n, n \times m$, respectively.
Then AB is defined, as well as BA.
These two products have dimension $m \times m$ and $n \times n$ respectively, so AB and BA are square.

(d) False.
Let $B = 0$.
Then $A0 = 0$ for all appropriate matrices, but A is not necessarily I.

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§ 1.5 11 Forward-substituting from $Lc = b$ gives

- $c_1 = 2$
- $2 + c_2 = 0 \implies c_2 = -2$
- $2 + c_3 = 2 \implies c_3 = 0$

Now Back-substituting from $Ux = c$ gives

- $w = 0$

- $v + 2(0) = -2 \implies v = -2$
- $2u + 4(-2) + 4(0) = 2 \implies 2u - 8 = 2 \implies u = 5$

So we have $u = 5, v = -2, w = 0$.

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- We could factor A into UL if we reduced the rows with the pivots below all 0's. That is, we'd want to form the lower triangular factor first, then perform elementary row operations to create the upper triangular factor.
- No, these two decompositions do not necessarily produce the same factors, as LU-decomposition in general is not unique.

18 We can solve the first two at the same time.

- Let
$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$
Then if we attempt to reduce A to an upper triangular we get
$$\begin{aligned} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$
But this matrix is singular, so it has no solutions.
- Let
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then if we attempt to reduce A to an upper triangular we get

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\
E_1 A &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A & = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\
E_2 E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A & = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \\
E_3 E_2 E_1 A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A & = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

Since E_1 is a permutation matrix, we have $E_1 = P \implies PA = LU$.
Now we have

$$
\begin{aligned}
L &= E_2^{-1} E_3^{-1} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}
\end{aligned}
$$

So A is nonsingular, in fact we have one unique solution.
We take $PAx = Pb = y$ and solve with our LU factors.

$$
y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
$$

We can compute the c matrix with $Lc = y$.

- $c_1 = 1$
- $c_2 = 1$
- $1 - 1 + c_3 = 1 \implies c_3 = 1$

We can compute the x matrix with $Ux = c$.

- $2w = 1 \implies w = \frac{1}{2}$
- $v + \frac{1}{2} = 1 \implies v = \frac{1}{2}$
- $u + \frac{1}{2} = 1 \implies u = \frac{1}{2}$

Finally, we have our solution: $u = \frac{1}{2}, v = \frac{1}{2}, w = \frac{1}{2}$.

22 The elementary operations we performed were

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

- So

  $$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

  We have

  $$Lc = b$$

  $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

  and

  $$Ux = c$$

  $$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

- Plugging in the values for c we compute:

  $$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 11 \end{bmatrix}$$

  So this c solves the first system.
- We compute
  - $z = 2$
  - $y + 2(2) = 2 \implies y + 4 = 2 \implies y = -2$
  - $x + (-2) + 2 = 5 \implies x = 5$

  So the x that solves the second system is:

  $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 4 \\ 4 & 11 \end{bmatrix}$$

$$E_{A0} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

$$E_{A0} A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

So

$$L_A = E_{A0}^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, D_A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, U_A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
B &= \begin{bmatrix} 1 & 4 & 0 \\ 4 & 12 & 4 \\ 0 & 4 & 0 \end{bmatrix} \\
E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 4 & 0 \end{bmatrix} \\
E_{B1}E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 4 & 0 \\ 0 & -4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \\
E_{B1}E_{B0}B &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B & &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

So

$$L = E_{B0}^{-1}E_{B1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, U = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

For these symmetric matrices, $L = U^T$

33 We find c

- $c_1 = 4$
- $4 + c_2 = 5 \implies c_2 = 1$
- $4 + 1 + c_3 = 6 \implies c_3 = 1$

So we have

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

Now we find x

- $x_3 = 1$
- $x_2 + 1 = 1 \implies x_2 = 0$
- $x_1 + 0 + 1 = 4 \implies x_1 = 3$

So we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

Finally,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

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- Choose

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$P_1 P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P_2 P_1$$

- Choose

$$P_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Then

$$P_3 P_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = P_4 P_3$$

§ 1.6 2 i. For all permutation matrices P, $P^{-1} = P^T$.
So the first P^{-1} is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The second P^{-1} is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

ii. Intuitively we can think of P^T as performing the inverse permutation of P.

4 i.

$$AB = AC \implies A^{-1}AB = A^{-1}AC \implies B = C$$

ii. Choose

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = AC$$

But $B \neq C$.

5 *Proof.* Let $(A^2)^{-1} = B$

$$\begin{aligned}
(A^2)^{-1} &= B \\
(AA)^{-1} &= \\
A^{-1}A^{-1} &= \\
A^{-1} &= AB
\end{aligned}$$

Thus, the inverse of A is AB. $\square$

10 (a)

$$\left[\begin{array}{cccc|cccc} 0&0&0&1&1&0&0&0\\ 0&0&2&0&0&1&0&0\\ 0&3&0&0&0&0&1&0\\ 4&0&0&0&0&0&0&1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 4&0&0&0&0&0&0&1\\ 0&3&0&0&0&0&1&0\\ 0&0&2&0&0&1&0&0\\ 0&0&0&1&1&0&0&0 \end{array}\right]$$
$$= \left[\begin{array}{cccc|cccc} 1&0&0&0&\frac14&0&0&0\\ 0&1&0&0&0&\frac13&0&0\\ 0&0&1&0&0&0&\frac12&0\\ 0&0&0&1&0&0&0&\frac11 \end{array}\right]$$

So

$$A_1^{-1} = \begin{bmatrix} \frac14&0&0&0\\ 0&\frac13&0&0\\ 0&0&\frac12&0\\ 0&0&0&\frac11 \end{bmatrix}$$

(b)

$$\left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ -\frac12&1&0&0&0&1&0&0\\ 0&-\frac23&1&0&0&0&1&0\\ 0&0&-\frac34&1&0&0&0&1 \end{array}\right] = \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&2&1&0&0\\ 0&-\frac23&1&0&0&0&1&0\\ 0&0&-\frac34&1&0&0&0&1 \end{array}\right]$$
$$= \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&2&1&0&0\\ 0&0&1&0&3&\frac32&1&0\\ 0&0&-\frac34&1&0&0&0&1 \end{array}\right]$$
$$= \left[\begin{array}{cccc|cccc} 1&0&0&0&1&0&0&0\\ 0&1&0&0&2&1&0&0\\ 0&0&1&0&3&\frac32&1&0\\ 0&0&0&1&\frac94&\frac98&\frac34&1 \end{array}\right]$$

So

$$A_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & \frac{3}{2} & 1 & 0 \\ \frac{9}{4} & \frac{9}{8} & \frac{3}{4} & 1 \end{bmatrix}$$

(c)

$$\left[\begin{array}{cccc|cccc} a & b & 0 & 0 & 1 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{cccc|cccc} a & b & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{ad-bc}{a} & 0 & 0 & -\frac{c}{a} & 1 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 0 & 0 & -\frac{c}{a} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} a & 0 & 0 & 0 & \frac{ad}{ad-bc} & -\frac{ab}{ad-bc} & 0 & 0 \\ 0 & \frac{ad-bc}{a} & 0 & 0 & -\frac{c}{a} & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & \frac{ad}{ad-bc} & -\frac{ab}{ad-bc} \\ 0 & 0 & 0 & \frac{ad-bc}{a} & 0 & 0 & -\frac{c}{a} & 1 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

So

$$A_3^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b & 0 & 0 \\ -c & a & 0 & 0 \\ 0 & 0 & d & -b \\ 0 & 0 & -c & a \end{bmatrix}$$

17 (a)

$$\begin{aligned} A &= A \\ L_2 D_2 U_2 &= L_1 D_1 U_1 \\ L_1^{-1} L_2 D_2 U_2 &= D_1 U_1 \\ L_1^{-1} L_2 D_2 &= D_1 U_1 U_2^{-1} \end{aligned}$$

Inverting a lower triangular matrix gives another lower triangular matrix. Inverting an upper triangular matrix gives another upper triangular matrix. Multiplication of one type of triangular matrix by a triangular matrix of the same type gives another triangular matrix of the same type.
Since diagonal matrices are trivially both upper and lower triangular, the left side is a lower triangular matrix while the right side is an upper triangular matrix.

(b) For the equation $L_1^{-1}L_2D_2 = D_1U_1U_2^{-1}$ to be true, both sides must be diagonal matrices. This is because the left side is a lower triangular matrix and the right side is an upper triangular matrix. The only possible matrix of this type is a diagonal matrix.
The main diagonal of the left hand side must be the same as the diagonal of D_2, as the triangular matrix on the left hand side has all 1's on the main diagonal. The main diagonal of the right hand side must be the same as the diagonal of D_1, as the triangular matrix on the right hand side has all 1's on the main diagonal.
The elements off the diagonal must be all 0 since both sides are diagonal matrices.
So we have that the left side is equal to D_2 and the right side is equal to D_1, thus $D_2 = D_1$.
Since the left side is equal to D_2, we have:

$$L_1^{-1}L_2D_2 = D_2 \implies L_1^{-1}L_2 = I \implies L_2 = L_1$$

Since the left side is equal to D_1, we have:

$$D_1U_1U_2^{-1} = D_1 \implies U_1U_2^{-1} = I \implies U_1 = U_2$$

So we have shown if $A = L_1D_1U_1$ and $A = L_2D_2U_2$, $L_1 = L_2$, $D_1 = D_2$, $U_1 = U_2$.
Thus the factorization is unique.

21 *Proof.* Let A, B be square matrices with $I - AB$ invertible.

$$\begin{aligned}
B(I - AB) &= (I - BA)B \\
B &= (I - BA)B(I - AB)^{-1} \\
I &= (I - BA)B(I - AB)^{-1}B^{-1} \\
(I - BA)^{-1} &= B(I - AB)^{-1}B^{-1}
\end{aligned}$$

Then $(I - BA)$ is invertible. □

40 (a) True. The matrix has less pivots than 4, and so is singular.

(b) False.
Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

A has 1's down the main diagonal, but is singular and so not invertible.

(c) True. The inverse of A^{-1} is A.

(d) True.

Proof. Given A^T is invertible we have:

$$
\begin{aligned}
A^{-1}A &= \left(A^{T^T}\right)^{-1}\left(A^{T^T}\right) \\
&= \left(\left(A^T\right)^{-1}\right)^T\left(A^{T^T}\right) \\
&= \left(A^T\left(A^T\right)^{-1}\right)^T \\
&= I^T \\
&= I
\end{aligned}
$$

and

$$
\begin{aligned}
AA^{-1} &= \left(A^{T^T}\right)\left(A^{T^T}\right)^{-1} \\
&= \left(A^{T^T}\right)\left(\left(A^T\right)^{-1}\right)^T \\
&= \left(\left(A^T\right)^{-1}A^T\right)^T \\
&= I^T \\
&= I
\end{aligned}
$$

So A is invertible. $\square$

49 i.

$$A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1(3) - 9(0)}\begin{bmatrix} -1 & 9 \\ 0 & -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 & 9 \\ 0 & -3 \end{bmatrix}$$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = \frac{1}{3}\begin{bmatrix} -1 & 0 \\ 9 & -3 \end{bmatrix}$$

ii.

$$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{1(0) - c(c)}\begin{bmatrix} -1 & c \\ c & -0 \end{bmatrix} = \frac{1}{c^2}\begin{bmatrix} -1 & c \\ c & -0 \end{bmatrix}, \text{ when } c \neq 0$$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1} = \frac{1}{c^2}\begin{bmatrix} -1 & c \\ c & -0 \end{bmatrix}, \text{ when } c \neq 0$$