

# MAT 125A HW 5

Hardy Jones  
999397426  
Professor Slivken  
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- 6.2 1 Given:  $f_n(x) = \frac{nx}{1+nx^2}$   
(a)

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{nx}{1+nx^2} &= \lim_{n \rightarrow \infty} \frac{\frac{1}{x}(nx^2)}{1+nx^2} \\&= \lim_{n \rightarrow \infty} \frac{\frac{1}{x}(1-1+nx^2)}{1+nx^2} \\&= \lim_{n \rightarrow \infty} \frac{\frac{1}{x}(1+nx^2) - \frac{1}{x}}{1+nx^2} \\&= \lim_{n \rightarrow \infty} \left[ \frac{\frac{1}{x}(1+nx^2)}{1+nx^2} - \frac{\frac{1}{x}}{1+nx^2} \right] \\&= \lim_{n \rightarrow \infty} \left[ \frac{1}{x} - \frac{\frac{1}{x}}{1+nx^2} \right] \\&= \lim_{n \rightarrow \infty} \frac{1}{x} - \lim_{n \rightarrow \infty} \frac{\frac{1}{x}}{1+nx^2} \\&= \lim_{n \rightarrow \infty} \frac{1}{x} - 0 \\&= \lim_{n \rightarrow \infty} \frac{1}{x} \\&= \frac{1}{x}\end{aligned}$$

So the point-wise limit of  $(f_n)$  is  $\frac{1}{x}$

- (b) On  $(0, \infty)$  we check:

$$\begin{aligned}|f_n(x) - f(x)| &= \left| \frac{nx}{1+nx^2} - \frac{1}{x} \right| \\&= \left| \frac{nx^2 - (1+nx^2)}{x+nx^3} \right| \\&= \left| \frac{-1}{x+nx^3} \right| \\&= \frac{1}{x+nx^3}\end{aligned}$$

And we need to find some  $N \in \mathbb{N}$  for any  $\epsilon$ , such that, if  $n \geq N$ , then  $\frac{1}{x+nx^3} < \epsilon$   
 We can rewrite a bit:

$$\begin{aligned}\frac{1}{x+nx^3} &< \epsilon \\ \frac{1}{\epsilon} &< x+nx^3 \\ \frac{1}{\epsilon} - x &< nx^3 \\ \frac{1}{\epsilon x^3} - \frac{1}{x^2} &< n \\ \frac{1}{\epsilon x^3} - \frac{1}{x^2} &< N \leq n\end{aligned}$$

But when  $x \rightarrow 0$ ,  $\frac{1}{\epsilon x^3} - \frac{1}{x^2} \rightarrow \infty$ .

So there's no  $N$  we can choose for all  $\epsilon$  on  $(0, \infty)$ .

Thus, the convergence is not uniform on  $(0, \infty)$ .

(c) On  $(0, 1)$  we run into the same issues as previous.

So the convergence is not uniform on  $(0, 1)$ .

(d) On  $(1, \infty)$  we check:

The computation is the same up to  $\frac{1}{\epsilon x^3} - \frac{1}{x^2} < N \leq n$ .

For all  $x$  in  $(1, \infty)$ ,  $\frac{1}{\epsilon x^3} - \frac{1}{x^2} < \frac{1}{\epsilon} - 1$

So we can choose  $N > \frac{1}{\epsilon} - 1$ .

Then we have  $|f_n(x) - f(x)| < \epsilon$ .

So the convergence is uniform on  $(1, \infty)$ .

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$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{nx + \sin(nx)}{2n} &= \lim_{n \rightarrow \infty} \left( \frac{nx}{2n} + \frac{\sin(nx)}{2n} \right) \\ &= \lim_{n \rightarrow \infty} \left( \frac{x}{2} + \frac{\sin(nx)}{2n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{x}{2} + \lim_{n \rightarrow \infty} \frac{\sin(nx)}{2n} \\ &= \frac{x}{2} + 0 \\ &= \frac{x}{2}\end{aligned}$$

So the pointwise limit of  $(g_n)$  is  $\frac{x}{2}$

• On  $[-10, 10]$  we check:

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \frac{x}{2} - \frac{nx + \sin(nx)}{2n} \right| \\
&= \left| \frac{nx - nx + \sin(nx)}{2n} \right| \\
&= \left| \frac{\sin(nx)}{2n} \right|
\end{aligned}$$

Since  $-1 \leq \sin(nx) \leq 1$ , we have:

$$\begin{aligned}
|g_n(x) - g(x)| &= \left| \frac{\sin(nx)}{2n} \right| \\
&\leq \left| \frac{1}{2n} \right|
\end{aligned}$$

And we want  $\frac{1}{2n} < \epsilon \implies \frac{1}{2\epsilon} < n$ .

So choose  $N > \frac{1}{2\epsilon}$ , then we have  $|g_n(x) - g(x)| < \epsilon$ .

So the convergence is uniform on  $[-10, 10]$ .

- Since  $x$  does not affect our decision of  $N$ , the same reasoning can be used to choose  $N > \frac{1}{2\epsilon}$  on all of  $\mathbb{R}$ .  
So the convergence is uniform on  $\mathbb{R}$ .

3 (a) We have to consider three cases:

- $0 \leq x < 1$
- $x = 1$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + 1^n} = \lim_{n \rightarrow \infty} \frac{1}{2} = \frac{1}{2}$$

- $1 < x$

$$\lim_{n \rightarrow \infty} \frac{x}{1 + x^n} = 0$$

- (b) Since there is a jump discontinuity on  $h_n(x)$  when  $x = 1$ , and when  $x > 1$ ,  $h_n(x)$  is not continuous. And from Theorem 6.2.6,  $h_n$  is not continuous. Since  $h_n$  is not continuous, it cannot have convergence on  $[0, \infty)$
- (c) Choose  $(1, \infty)$ .  
Then we check:

$$\begin{aligned}
|h_n(x) - h(x)| &= \left| \frac{x}{1+x^n} - 0 \right| \\
&= \left| \frac{x}{1+x^n} \right| \\
&= \frac{x}{1+x^n} \\
&< 1
\end{aligned}$$

Then we can choose  $N = 1$ .

And for any  $\epsilon$

4 We take some derivatives to find maxima and minima.

$$\begin{aligned}
f'_n(x) &= \frac{(1+nx^2) \cdot 1 - x(2nx)}{(1+nx^2)^2} \\
&= \frac{1+nx^2-2nx^2}{(1+nx^2)^2} \\
&= \frac{1-nx^2}{(1+nx^2)^2}
\end{aligned}$$

$$\begin{aligned}
f''_n(x) &= \frac{(1+2nx^2+n^2x^4)(-2nx) - (4nx+4n^2x^3)(1-nx^2)}{(1+nx^2)^4} \\
&= \frac{-6nx-4n^2x^3+2n^3x^5}{(1+nx^2)^4}
\end{aligned}$$

Setting the first derivative to zero, we can find possible maxima and minima.

$$\begin{aligned}
\frac{1-nx^2}{(1+nx^2)^2} &= 0 \\
1-nx^2 &= 0 \\
1 &= nx^2 \\
\frac{1}{n} &= x^2 \\
\pm \frac{1}{\sqrt{n}} &= x
\end{aligned}$$

Now, we can plug into the second derivative.

$$\begin{aligned}
f_n''\left(\frac{1}{\sqrt{n}}\right) &= \frac{-6n\left(\frac{1}{\sqrt{n}}\right) - 4n^2\left(\frac{1}{\sqrt{n}}\right)^3 + 2n^3\left(\frac{1}{\sqrt{n}}\right)^5}{\left(1 + n\left(\frac{1}{\sqrt{n}}\right)^2\right)^4} \\
&= \frac{-6\sqrt{n} - 4n^{2-\frac{3}{2}} + 2n^{3-\frac{5}{2}}}{(1+1)^4} \\
&= \frac{-6\sqrt{n} - 4\sqrt{n} + 2\sqrt{n}}{16} \\
&= \frac{-8\sqrt{n}}{16} \\
&= \frac{-\sqrt{n}}{2} \\
&< 0
\end{aligned}$$

$$\begin{aligned}
f_n''\left(\frac{-1}{\sqrt{n}}\right) &= \frac{-6n\left(\frac{-1}{\sqrt{n}}\right) - 4n^2\left(\frac{-1}{\sqrt{n}}\right)^3 + 2n^3\left(\frac{-1}{\sqrt{n}}\right)^5}{\left(1 + n\left(\frac{-1}{\sqrt{n}}\right)^2\right)^4} \\
&= \frac{6\sqrt{n} + 4n^{2-\frac{3}{2}} - 2n^{3-\frac{5}{2}}}{(1+1)^4} \\
&= \frac{6\sqrt{n} + 4\sqrt{n} - 2\sqrt{n}}{16} \\
&= \frac{8\sqrt{n}}{16} \\
&= \frac{\sqrt{n}}{2} \\
&> 0
\end{aligned}$$

So the maximum occurs at  $\frac{-1}{\sqrt{n}}$  and the minimum occurs at  $\frac{1}{\sqrt{n}}$

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