MAT 150A Homework 1

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1.

$$X = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, Y = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}$$

(1)

$$XY = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+d & b+e+af \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix}$$

Since $\mathbb{F}$ is a field, it is closed under addition and multiplication.
So, $a+d, b+e+af, c+f \in \mathbb{F}$.
Thus, $XY \in H(F)$.

(2) Given some matrix

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] &= \left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \\ &= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{aligned}$$

So, we assume our inverse is

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

.

Since $\mathbb{F}$ is a field, it has additive inverses, is closed under addition and multiplication.

So, $-a, ac-b, -c \in \mathbb{F}$, and $A^{-1} \in H(F)$.
We need to check that A^{-1} is the inverse by showing that $AA^{-1} = I = A^{-1}A$

$$AA^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, the closed form of the inverse is given by

$$A^{-1} = \begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

.

(3) Given

$$A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} \in H(F)$$

.

We want to show that $A(BC) = (AB)C$.

$$\begin{aligned} A(BC) &= A\left(\begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= A\left(\begin{bmatrix} 1 & d+g & e+h+id \\ 0 & 1 & f+i \\ 0 & 0 & 1 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d+g & e+h+id \\ 0 & 1 & f+i \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+d+g & af+ai+b+e+h+id \\ 0 & 1 & c+f+i \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
(AB)C &= \left(\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \right) C \\
&= \left(\begin{bmatrix} 1 & a+d & af+b+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \right) C \\
&= \begin{bmatrix} 1 & a+d & af+b+e \\ 0 & 1 & c+f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & g & h \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a+d+g & af+ai+b+e+h+id \\ 0 & 1 & c+f+i \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

So, $A(BC) = (AB)C$.

Thus, $H(F)$ is associative under matrix multiplication.

(4) The elements of $\mathbb{F}_2$ are $\{0, 1\}$, with $+$ and $\cdot$ defined:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

The elements of $H(\mathbb{F}_2)$ are:

$$
H(\mathbb{F}_2) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}
$$

For simplicity we enumerate these as $\{H_0, H_1, \ldots, H_7\}$

We see that H_0 has order 1.

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so H_1 has order 2.

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so H_2 has order 2.

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so H_3 has order 2.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so H_4 has order 2.

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \text{ and } \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \text{ so}$$

H_5 has order 3.

$$\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \text{ so } H_6 \text{ has order 2.}$$

$$\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \text{ and } \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \text{ so}$$

H_7 has order 3.

2. Let's look at a few cases first.

$n = 2$,

$$\begin{vmatrix}2&-1\\-1&2\end{vmatrix}=3$$

$n = 3$,

$$\begin{vmatrix}2&-1&\\-1&2&-1\\&-1&2\end{vmatrix}=4$$

$n = 4$,

$$\begin{vmatrix}2&-1&&\\-1&2&-1&\\&-1&2&-1\\&&-1&2\end{vmatrix}=5$$

Looks like the determinate of the $n \times n$ matrix is $n + 1$

Proof. **Base Case** $n = 2$

$$\begin{vmatrix}2&-1\\-1&2\end{vmatrix}=2(2)-(-1)(-1)=4-1=3$$

and $n + 1 = 3$.

So our base case holds.

Inductive Case

Assume determinate of an $n \times n$ matrix is $n + 1$. Need to show, the determinate of an $(n + 1) \times (n + 1)$ matrix is $(n + 1) + 1 = n + 2$

$$\begin{vmatrix}2&-1&&&&\\-1&2&-1&&&\\&-1&2&-1&&\\&&-1&\ddots&\ddots&\\&&&\ddots&2&-1\\&&&&-1&2\end{vmatrix}$$

We first expand along the first column.

$$\begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

$$+ 0\left|\ldots\right| - 0\left|\ldots\right| + \ldots$$

$$= 2\begin{vmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} + \begin{vmatrix} -1 & & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

Looking at the first matrix expansion, we see that it is an $n \times n$ matrix of the same form. By our assumption, it has determinate of $n + 1$.

$$\begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1) + \begin{vmatrix} -1 & & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

We expand the remaining determinate across the first column.

$$\begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1) + (-1)\begin{vmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 2 \end{vmatrix}$$

$$- 0\left|\ldots\right| + 0\left|\ldots\right| + \ldots$$

$$= 2(n+1) - \begin{vmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 2 \end{vmatrix}$$

This is now an $(n-1)\times(n-1)$ matrix of the same form. Again, by our assumption, the determinate is $(n-1)+1=n$.

$$\begin{vmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix} = 2(n+1)-n = 2n+2-n = n+2$$

So, we have shown by induction, that the determinate of an $n\times n$ matrix of this form is $n+1$. $\square$

3. Given the permutation $(1,3,4,2)$.

   (a)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

   (b)

$$\begin{aligned} p &= (1,2)(1,4)(1,3) \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{aligned}$$

   (c) Since p can be written as the product of three transpositions, the sign is odd.

4. Given some $n\times n$ permutation matrix P, by definition, each row and column has exactly one 1 and the rest 0. This means that P^T also has every row and column with exactly one 1 and the rest 0.

   Without loss of generality, we can compute the product of these matrices by:

$$PP^T = \begin{bmatrix} {PP^T}_{11} & {PP^T}_{12} & \cdots & {PP^T}_{1n} \\ {PP^T}_{21} & {PP^T}_{22} & \cdots & {PP^T}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ {PP^T}_{n1} & {PP^T}_{n2} & \cdots & {PP^T}_{nn} \end{bmatrix}$$

Where each ${PP^T}_{ij}$ is the inner product between row i of P and column j of P^T.

By definition of transposition, when $i = j$ the inner product is between row i of P and the transpose of row i of P.

In other words:

$$\begin{bmatrix} 0, 0, \cdots, 1_i, \cdots, 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1_i \\ \vdots \\ 0 \end{bmatrix}$$

where 1_i is in the same row/column number.

The inner product of this is 1.

However, when $i \neq j$ the inner product is between a row and a column which are orthogonal to each other.

In other words:

$$\begin{bmatrix} 0, 0, \cdots, 1_i, \cdots, 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1_j \\ \vdots \\ 0 \end{bmatrix}$$

where 1_i and 1_j are in different row/column numbers.

Since these vectors are orthogonal, the inner product is 0.

So, for each ${PP^T}_{ij}$, if $i = j$, the value is 1, and if $i \neq j$, the value is 0.

In other words, it produces all 0's except on the main diagonal. This is the Identity matrix.

So, $PP^T = I$.

A similar argument holds for $P^T P = I$.

Thus, the transpose of a permutation matrix is its inverse.