MAT 150A Homework 7

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1. *Proof.* Assume $O_n \trianglelefteq M_n$ is a normal subgroup.

   Choose $g = t_a\rho_\theta r, g^{-1} = t_{-a}\rho_{-\theta} r \in M_n, h = \rho_\psi r \in O_n$.

   Then we should have $ghg^{-1} \in O_n$.

$$
\begin{aligned}
ghg^{-1} &= t_a\rho_\theta(r\rho_\psi)t_{-a}\rho_{-\theta}r \\
&= t_a\rho_\theta\rho_{-\psi}(rt_{-a})\rho_{-\theta}r \\
&= t_a\rho_\theta\rho_{-\psi}t_{-a'}(r\rho_{-\theta})r \\
&= t_a\rho_\theta\rho_{-\psi}t_{-a'}\rho_\theta r^2 \\
&= t_a\rho_\theta(\rho_{-\psi}t_{-a'})\rho_\theta \\
&= t_a(\rho_\theta t_{-a''})\rho_{-\psi}\rho_\theta \\
&= (t_a t_{-a'''})(\rho_\theta\rho_{-\psi}\rho_\theta) \\
&= t_{a-a'''}\rho_{\theta-\psi+\theta}
\end{aligned}
$$

   However, there's no guarantee that $a - a''' = 0$, so this may be a translation, which is not an orthogonal matrix.

   So, $ghg^{-1} \notin O_n$, and our assumption is wrong.

   Thus, O_n is not a normal subgroup of M_n. $\square$

2. *Proof.* Let $m = t_a\rho_\theta r$

$$\begin{aligned}
m^2 &= (t_a\rho_\theta r)(t_a\rho_\theta r) \\
&= t_a\rho_\theta(rt_a)\rho_\theta r \\
&= t_a\rho_\theta(t_{a'}r)\rho_\theta r \\
&= t_a\rho_\theta t_{a'}(r\rho_\theta)r \\
&= t_a\rho_\theta t_{a'}(\rho_{-\theta}r)r \\
&= t_a\rho_\theta t_{a'}\rho_{-\theta}r^2 \\
&= t_a(\rho_\theta t_{a'})\rho_{-\theta} \\
&= t_a(t_{a''}\rho_\theta)\rho_{-\theta} \\
&= t_a t_{a''}(\rho_\theta\rho_{-\theta}) \\
&= t_a t_{a''}(\rho_{\theta-\theta}) \\
&= t_a t_{a''} \\
&= t_{a+a''}
\end{aligned}$$

So m^2 is just a translation by $a + a''$, where $a'' = \rho(r(a))$ □

3. To show this, we need to show first that $SM \leq M$ is a subgroup, and then $SM \trianglelefteq M$ is normal.

Where $SM = \{t_a\rho_\theta r| \, |\rho_\theta r| = 1\}, M = \{t_a\rho_\theta r| \, |\rho_\theta r| = \pm 1\}$.

And

$$\rho_\theta(x) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Proof. We first note that $\forall a = t_a\rho_\theta r \in SM, |r| = 1$ implies that there is no reflection. So for the sake of brevity, it is not included unless necessary.

This also implies that $|\rho_\theta| = 1$ for all elements of SM.

- **Closure**

  Choose $a = t_a\rho_\theta, b = t_b\rho_\psi \in SM$

  $$ab = t_a\rho_\theta t_b\rho_\psi = t_a t_b\rho_\theta\rho_\psi = t_{a+b}\rho_{\theta+\psi}$$

  So we want to find

  $$\begin{vmatrix} \cos\theta+\psi & -\sin\theta+\psi \, \sin\theta+\psi & \cos\theta+\psi \end{vmatrix} = \cos^2(\theta+\psi) + \sin^2(\theta+\psi) = 1$$

  Thus SM has closure.

- **Identity**

  Choose $a = 0 \in \mathbb{R}^n, \theta = 0 \in \mathbb{R}$, so $e = t_0\rho_0$ and

$$\begin{vmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Now, choose $b = t_b\rho_\psi \in SM$.

$$be = t_b\rho_\psi t_0\rho_0 = t_b t_0 \rho_\psi \rho_0 = t_{b+0}\rho_{\psi+0} = t_b\rho_\psi = t_{0+b}\rho_{0+\psi} = t_0 t_b \rho_0 \rho_\psi = t_0\rho_0 t_b \rho_\psi = eb$$

Thus, SM has the identity.

- **Inverse**

  Choose $a = t_a\rho_\theta \in SM$. Since $-a \in \mathbb{R}^n, -\theta \in \mathbb{R}$ exist, $a^{-1} = t_{-a}\rho_{-\theta} \in SM$.

$$\begin{bmatrix} \cos -\theta & -\sin -\theta \\ \sin -\theta & \cos -\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} aa^{-1} &= t_a\rho_\theta t_{-a}\rho_{-\theta} \\ &= t_a t_{-a}\rho_\theta\rho_{-\theta} \\ &= t_{a-a}\rho_{\theta-\theta} \\ &= t_0\rho_0 \\ &= e \\ &= t_{(-a)+a}\rho_{(-\theta)+\theta} \\ &= t_{-a}t_a\rho_{-\theta}\rho_\theta \\ &= t_{-a}\rho_{-\theta}t_a\rho_\theta \\ &= a^{-1}a \end{aligned}$$

So, $SM \leq M$ is a subgroup.

Now we need to show that $SM \trianglelefteq M$ is normal.

Choose an arbitrary $g = t_g\rho_\theta \in SM, h = t_h\rho_\psi r \in M$.

$$\begin{aligned} ghg^{-1} &= t_g\rho_\theta(t_h\rho_\psi r)t_{-g}\rho_{-\theta} \\ &= t_g(\rho_\theta t_h)\rho_\psi r t_{-g}\rho_{-\theta} \\ &= t_g(t'_h\rho_\theta)\rho_\psi r t_{-g}\rho_{-\theta} \\ &= (t_g t'_h)(\rho_\theta\rho_\psi) r t_{-g}\rho_{-\theta} \\ &= t_{g+h'}\rho_{\theta+\psi} r t_{-g}\rho_{-\theta} \\ &= t_{g+h'}\rho_{\theta+\psi}(t_{-g'}r)\rho_{-\theta} \\ &= t_{g+h'}\rho_{\theta+\psi}t_{-g'}(\rho_\theta r) \\ &= t_{g+h'}\rho_{\theta+\psi}t_{-g'}(\rho_\theta r) \end{aligned}$$

□

4. We choose a homomorphism

$$\varphi : SM \to GL_2(\mathbb{C})$$
$$\varphi(t_a\rho_\theta) \mapsto \begin{bmatrix} e^{i\theta} & a_1 + ia_2 \\ 0 & 1 \end{bmatrix}$$

We need to show that φ preserves the group operation.

Choose $a = t_a\rho_\theta, b = t_b\rho_\psi \in SM$.

$$\begin{aligned} \varphi(a)\varphi(b) &= \begin{bmatrix} e^{i\theta} & a_1 + ia_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\psi} & b_1 + ib_2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} e^{i\theta} & a_1 + ia_2 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

5.

6. (a)

(b)

(c)