MAT 125A HW 1

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Exercise 4.2.1 (a) We want to prove:

$$\lim_{x \to 2}(2x + 4) = 8$$

Proof. Given $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies |(2x + 4) - 8| < \epsilon$$

.

We can simplify the consequent a bit.

$$\begin{aligned}
|(2x + 4) - 8| &< \epsilon \\
|2x - 4| &< \\
2\,|x - 2| &< \\
|x - 2| &< \frac{\epsilon}{2}
\end{aligned}$$

If we notice, this is exactly the form of the antecedent, assuming $\delta = \frac{\epsilon}{2}$.
So, choose $\delta = \frac{\epsilon}{2}$.
Then we have

$$0 < |x - 2| < \delta \implies |(2x + 4) - 8| < \epsilon$$

as was to be shown. □

(b) We want to prove:

$$\lim_{x \to 0} x^3 = 0$$

Proof. Given $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \implies \left|x^3 - 0\right| < \epsilon$$

.

We can simplify the consequent a bit.

$$
\begin{aligned}
\left|x^3 - 0\right| &< \epsilon \\
\left|x^3\right| &< \\
|x|^3 &< \\
|x| &< \sqrt[3]{\epsilon} \\
|x - 0| &< \sqrt[3]{\epsilon}
\end{aligned}
$$

If we notice, this is exactly the form of the antecedent, assuming $\delta = \sqrt[3]{\epsilon}$.
So, choose $\delta = \sqrt[3]{\epsilon}$.
Then we have

$$0 < |x - 0| < \delta \implies \left|x^3 - 0\right| < \epsilon$$

as was to be shown. □

(c) We want to prove:

$$\lim_{x \to 2} x^3 = 8$$

Proof. Given $\epsilon > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \epsilon$$

.

Start by manipulating the consequent a bit.

$$
\begin{aligned}
\left|x^3 - 8\right| &< \epsilon \\
\left|(x - 2)(x^2 + 2x + 4)\right| &< \\
|x - 2| \left|x^2 + 2x + 4\right| &<
\end{aligned}
$$

As in example 4.2.2 (ii), we can control the size of $|x - 2|$ but not $|x^2 + 2x + 4|$.
We arbitrarily choose some upper bound for δ, say $\delta \leq 1$. This gives us a delta neighborhood between 1 and 3.
Since $|x^2 + 2x + 4|$ is strictly increasing in the delta neighborhood, we only need compute the upper bound.
So we have $\forall x \in V_\delta(2), |x^2 + 2x + 4| \leq |3^2 + 2(3) + 4| = 19$ as our upper bound.
Continuing with the method used in the example, we choose $\delta = \min\{1, \frac{\epsilon}{19}\}$.
So if $0 < |x - 2| < \delta$, then we have:

$$
\begin{aligned}
\left|x^3 - 8\right| &= |x - 2| \left|x^2 + 2x + 4\right| \\
&< \frac{\epsilon}{19}(19) \\
&= \epsilon
\end{aligned}
$$

So, choose $\delta = \min\{1, \frac{\epsilon}{19}\}$.
Then we have
$$0 < |x - 2| < \delta \implies \left|x^3 - 8\right| < \epsilon$$
as was to be shown. □

(d) We want to prove:
$$\lim_{x \to \pi} \lfloor x \rfloor = 3$$

Proof. Given $\epsilon > 0$, we want to find $\delta > 0$ such that
$$0 < |x - \pi| < \delta \implies |\lfloor x \rfloor - 3| < \epsilon$$
.
We begin by noting that if $\lfloor x \rfloor = 3$ then $\lfloor x \rfloor - 3 = 0 < \epsilon$ for any choice of ϵ.
So, we restrict δ to only produce x such that $\lfloor x \rfloor = 3$. This happens for any x in the interval $[3, 4)$. We can restrict this further to get an exact value of δ by choosing the neighborhood to be at π with a delta of the fractional part of π. That is, $\delta = \pi - 3$.
So, choose $\delta = \pi - 3$.
Then we have
$$0 < |x - \pi| < \delta \implies |\lfloor x \rfloor - 3| < \epsilon$$
as was to be shown. □

Exercise 4.2.2 Any δ_0 smaller than δ will suffice, as it implies a stronger statement. This is because if $0 < |x - c| < \delta_0$ is true, then the following is also true: $0 < |x - c| < \delta_0 < \delta$. From which it follows $|f(x) - L| < \epsilon$.

Exercise 4.2.3 (a) We have $f(x) = \frac{|x|}{x}$. It is helpful to enumerate some values of this function.

x	$f(x)$
-3	-1
-2	-1
-1	-1
0	$\frac{0}{0}$
1	1
2	1
3	1

So we can see that $f(0)$ is a problem. We'll need to construct two sequences that approach 0–so they have the same limit, but have different limits when f is applied to them element-wise.

Proof. Choose $x_n = \frac{1}{n}, y_n = -\frac{1}{n}$.
So $(x_n) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, and $(y_n) = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$.
Then $\lim(x_n) = 0$ and $\lim(y_n) = 0$.
Now, $f(x_n) = \{1, 1, 1, \dots\}$ and $f(y_n) = \{-1, -1, -1, \dots\}$.

So, $\lim f(x_n) = 1$ and $\lim f(y_n) = -1$.

Thus, we have our function $f(x) = \frac{|x|}{x}$, with $c = 0$. We have constructed two sequences $(x_n), (y_n)$ with $x_n \neq 0, y_n \neq 0, \lim(x_n) = \lim(y_n) = 0$, and $\lim f(x_n) \neq \lim f(y_n)$.

So we conclude by Corollary 4.2.5 that $\lim_{x \to 0} f(x)$ does not exist. □

(b) We have

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

.

We'll need to construct two sequences that approach 1–so they have the same limit, but have different limits when g is applied to them element-wise.

Proof. Choose $x_n = \frac{n+1}{n}, y_n = \frac{n+e}{n}$.

So $(x_n) = \{2, \frac{3}{2}, \frac{4}{3}, \dots\}$, and $(y_n) = \{1+e, \frac{2+e}{2}, \frac{3+e}{3}, \dots\}$.

Then $\lim(x_n) = 1$ and $\lim(y_n) = 1$.

If we look at each element of (x_n) we see that every element is in $\mathbb{Q}$, as $n+1 \in \mathbb{Q}, n \in \mathbb{Q}, n \neq 0$ and $\mathbb{Q}$ is closed under division where the quotient does not equal 0.

If we look at each element of (y_n) we see that every element is not in $\mathbb{Q}$, as $e \notin \mathbb{Q} \implies n + e \notin \mathbb{Q} \implies \frac{n+e}{n} \notin \mathbb{Q}$.

Now, $g(x_n) = \{1, 1, 1, \dots\}$ and $g(y_n) = \{0, 0, 0, \dots\}$.

So, $\lim g(x_n) = 1$ and $\lim g(y_n) = 0$.

Thus, we have our function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases},$$

with $c = 1$.

We have constructed two sequences $(x_n), (y_n)$ with $x_n \neq 1, y_n \neq 1, \lim(x_n) = \lim(y_n) = 1$, and $\lim g(x_n) \neq \lim g(y_n)$.

So we conclude by Corollary 4.2.5 that $\lim_{x \to 1} g(x)$ does not exist. □

Exercise 4.2.4 We have

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

.

(a) We can choose

- $x_n = \frac{n+1}{n}$,
  $(x_n) = \{2, \frac{3}{2}, \frac{4}{3}, \dots\}$

- $y_n = \frac{n+e}{n}$,
  $(y_n) = \{1+e, \frac{2+e}{2}, \frac{3+e}{3}, \dots\}$
- $z_n = \frac{n+e}{n+1}$,
  $(z_n) = \{\frac{1+e}{2}, \frac{2+e}{3}, \frac{3+e}{4}, \dots\}$

So all of these sequences converge to 1 and no sequence contains 1.

(b) Now we compute the limits.

- Taken element-wise $t(x_n) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.
  So, $\lim_{x\to 1} t(x_n) = 0$
- Taken element-wise $t(y_n) = \{0, 0, 0, \dots\}$.
  So, $\lim_{x\to 1} t(y_n) = 0$
- Taken element-wise $t(z_n) = \{0, 0, 0, \dots\}$.
  So, $\lim_{x\to 1} t(z_n) = 0$

(c) We conject that $\lim_{x\to 1} t(x) = 0$.

Exercise 4.2.6 *Proof.* Since $f(x)$ is bounded by some $M > 0$ such that $\forall x \in A, |f(x)| \leq M$, we know that $|f(x)| > 0$.

If $\lim_{x\to c} g(x) = 0$, then we know that for any $\epsilon > 0$, there exists some $\delta > 0$, such that $0 < |x - c| < \delta \implies |g(x) - 0| = |g(x)| < \epsilon$.

We can choose $\epsilon_0 = \frac{\epsilon}{M}$, and we have some δ_0 such that:

$0 < |x - c| < \delta_0 \implies |g(x)| < \frac{\epsilon_0}{M}$.

From this we can show:

$$\begin{aligned} |g(x)| &< \frac{\epsilon}{M} \\ |g(x)|\,|f(x)| &< \left(\frac{\epsilon}{M}\right) M \\ |g(x)|\,|f(x)| &< \epsilon \\ |g(x)f(x)| &< \epsilon \\ |g(x)f(x) - 0| &< \epsilon \end{aligned}$$

So, $\lim_{x\to c} g(x)f(x) = 0$.

Thus, if $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} g(x)f(x) = 0$ as well. □

Exercise 4.2.7 (a) Let $f : A \to R$, and let c be a limit point of the domain A. We say that $\lim_{x\to c} f(x) = \infty$ provided that, for all arbitrarily large ϵ, there exists a $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

Proof. We have $f(x) = \frac{1}{x^2}, c = 0$.
Given an arbitrarily large ϵ, we want to find $\delta > 0$ such that

$$0 < |x - 0| = |x| < \delta \implies \frac{1}{x^2} > \epsilon$$

.

We can simplify the consequent to:

$$\frac{1}{x^2} > \epsilon$$
$$\frac{1}{\epsilon} > x^2$$
$$\frac{1}{\sqrt{\epsilon}} > |x|$$

So, choose $\delta = \frac{1}{\sqrt{\epsilon}}$.
Then we have

$$0 < |x| < \delta \implies \frac{1}{x^2} > \epsilon$$

as was to be shown.

□

(b) Let $f : A \to R$. We say that $\lim_{x\to\infty} f(x) = L$ provided that, for all $\epsilon > 0$, there exists an arbitrarily large δ such that whenever $x > \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.

Proof. We have $f(x) = \frac{1}{x}$.
Given an $\epsilon > 0$, we want to find an arbitrarily large δ such that

$$x > \delta \implies \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$$

.

Since we know $x > 0$, we can simplify the consequent to:

$$\left|\frac{1}{x}\right| < \epsilon$$
$$\frac{1}{x} < \epsilon$$
$$\frac{1}{\epsilon} < x$$

So, choose $\delta = \frac{1}{\epsilon}$.
Then we have

$$x > \delta \implies \left|\frac{1}{x} - 0\right| < \epsilon$$

as was to be shown.

□

(c) Let $f : A \to R$.

We say that $\lim_{x\to\infty} f(x) = \infty$ provided that, for all arbitrarily large ϵ, there exists an arbitrarily large δ such that whenever $x > \delta$ (and $x \in A$) it follows that $f(x) > \epsilon$.

An example of such a limit is

$$\lim_{x\to\infty} x = \infty$$

Proof. We have $f(x) = x$.

Given an arbitrarily large ϵ, we want to find an arbitrarily large δ such that

$$x > \delta \implies x > \epsilon$$

.

So, choose $\delta = \epsilon$.

Then we have

$$x > \delta \implies x > \epsilon$$

as was to be shown.

□

Exercise 4.2.8 *Proof.* Choose some sequence (x_n) such that $(x_n) \to c$ and $x_n \neq c$.

Then we have $f(x_n) \geq g(x_n)$ for all $x \in A$.

Let $\lim_{x\to c} f(x_n) = L$ and $\lim_{x\to c} g(x_n) = M$.

Now the Order Limit Theorem states:

If $f(x_n) \geq g(x_n)$ for all n, then $L \geq M$.

Thus we have

$$\lim_{x\to c} f(x) \geq \lim_{x\to c} g(x)$$

as was to be shown. □

Exercise 4.2.9