MAT 108 HW 5

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Spring 2015

§2.2 13 (c) $A \times B$

$\{(\varnothing, (\varnothing, \{\varnothing\})), (\varnothing, \{\varnothing\}), (\varnothing, (\{\varnothing\}, \varnothing)), (\{\varnothing\}, (\varnothing, \{\varnothing\})), (\{\varnothing\}, \{\varnothing\}), (\{\varnothing\}, (\{\varnothing\}, \varnothing)), (\{\varnothing, \{\varnothing\}\}, (\varnothing, \{\varnothing\})), (\{\varnothing, \{\varnothing\}\}, \{\varnothing\}), (\{\varnothing, \{\varnothing\}\}, (\{\varnothing\}, \varnothing))\}$

$B \times A$

$\{((\varnothing, \{\varnothing\}), \varnothing), ((\varnothing, \{\varnothing\}), \{\varnothing\}), ((\varnothing, \{\varnothing\}), \{\varnothing, \{\varnothing\}\}), (\{\varnothing\}, \varnothing), (\{\varnothing\}, \{\varnothing\}), (\{\varnothing\}, \{\varnothing, \{\varnothing\}\}), ((\{\varnothing\}, \varnothing), \varnothing), ((\{\varnothing\}, \varnothing), \{\varnothing\}), ((\{\varnothing\}, \varnothing), \{\varnothing, \{\varnothing\}\})\}$

(d) $A \times B$

$\{((2, 4), (4, 1)), ((2, 4), (2, 3)), ((3, 1), (4, 1)), ((3, 1), (2, 3))\}$

$B \times A$

$\{((4, 1), (2, 4)), ((4, 1), (3, 1)), ((2, 3), (2, 4)), ((2, 3), (3, 1))\}$

17 We're asked to show that $(a, b) = (x, y)$ iff $a = x$ and $b = y$.

Proof.

$$\begin{aligned}
(a, b) = (x, y) &\iff \{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\} \\
&\iff (\{\{a\}, \{a, b\}\} \subseteq \{\{x\}, \{x, y\}\}) \\
&\qquad \wedge (\{\{x\}, \{x, y\}\} \subseteq \{\{a\}, \{a, b\}\}) \\
&\iff (\{a\} \in \{\{x\}, \{x, y\}\}) \\
&\qquad \wedge (\{a, b\} \in \{\{x\}, \{x, y\}\}) \\
&\qquad \wedge (\{x\} \in \{\{a\}, \{a, b\}\}) \\
&\qquad \wedge (\{x, y\} \in \{\{a\}, \{a, b\}\}) \\
&\iff (\{a\} = \{x\}) \wedge (\{a, b\} = \{x, y\}) \\
&\iff (a = x) \wedge (b = y)
\end{aligned}$$

Since we have connected both sides with a series of bi-conditional statements, we have proven that:
$(a, b) = (x, y)$ iff $a = x$ and $b = y$. □

18 (a) *Proof.*

$$A \Delta B = (A - B) \cup (B - A) = (B - A) \cup (A - B) = B \Delta A$$

□

(b) This proof is a bit longer than the others.

Proof.

$$\begin{aligned}
A\Delta B &= (A-B)\cup(B-A)\\
&= \{x|\,(x\in A\wedge x\notin B)\vee(x\in B\wedge x\notin A)\}\\
&= \{x|\,[(x\in A\wedge x\notin B)\vee x\in B]\wedge[(x\in A\wedge x\notin B)\vee x\notin A]\}\\
&= \{x|\,(x\in A\vee x\in B)\wedge(x\notin B\vee x\in B)\\
&\qquad\wedge(x\in A\vee x\notin A)\wedge(x\notin B\vee x\notin A)\}\\
&= \{x|\,(x\in A\vee x\in B)\wedge(x\notin B\vee x\notin A)\}\\
&= \{x|\,(x\in A\vee x\in B)\wedge(x\notin A\vee x\notin B)\}\\
&= \{x|\,(x\in A\vee x\in B)\wedge\sim(x\in A\wedge x\in B)\}\\
&= \{x|\,(x\in A\cup B)\wedge\sim(x\in A\cap B)\}\\
&= \{x|\,(x\in A\cup B)\wedge(x\notin A\cap B)\}\\
&= (A\cup B)-(A\cap B)
\end{aligned}$$

□

(c) *Proof.*

$$A\Delta A=(A-A)\cup(A-A)=\varnothing\cup\varnothing=\varnothing$$

□

(d) *Proof.*

$$A\Delta\varnothing=(A-\varnothing)\cup(\varnothing-A)=A\cup\varnothing=A$$

□

§2.3 1 (f)

$$\bigcup_{i=1}^{10}A_i=\{1,2,\ldots,19\},\bigcap_{i=1}^{10}A_i=\varnothing$$

(h)

$$\bigcup_{r\in(0,\infty)}A_r=[-\pi,\infty),\quad\bigcap_{r\in(0,\infty)}A_r=[-\pi,0)$$

(j)

$$\bigcup_{i=1}^{\infty}M_i=\mathbb{Z},\bigcap_{i=1}^{\infty}M_i=\{0\}$$

12 Let $A_n=(0,\frac{1}{n})$.

Then for any $m,n\in\mathbb{N}$

$$M_m\cap M_n=\begin{cases}\left(0,\frac{1}{m}\right), & \text{if } m<n\\ \left(0,\frac{1}{n}\right), & \text{otherwise}\end{cases}$$

But, $\bigcap_{i=1}^{\infty}M_i=\varnothing$

15 (e) *Proof.* Choose an arbitrary $x \in \bigcup_{i=1}^{k} A_i$.

Then there exists some $l \in \mathbb{N}$ such that $l \leq k$ and $x \in A_l$.

Now, since $l \leq k, l \leq m$, so $A_l \subseteq \bigcup_{i=1}^{m} A_i$, and $x \in \bigcup_{i=1}^{m} A_i$.

Since the choice of x was arbitrary, this works for all $x \in \bigcup_{i=1}^{k} A_i$.

Then every x contained in $\bigcup_{i=1}^{k} A_i$ is also in $\bigcup_{i=1}^{m} A_i$.

Thus $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{m} A_i$ □

(f) *Proof.* Choose an arbitrary $x \in \bigcap_{i=1}^{m} A_i$.

Then for all $l \in \{1, 2, \ldots, k, k+1, \ldots, m\}, x \in A_l$.

This implies that for all $l \in \{1, 2, \ldots, k\}, x \in A_l$.

Which means that $x \in \bigcap_{i=1}^{k} A_i$.

Since the choice of x was arbitrary, this works for all $x \in \bigcap_{i=1}^{m} A_i$.

Then every x contained in $\bigcap_{i=1}^{m} A_i$ is also in $\bigcap_{i=1}^{k} A_i$.

Thus, $\bigcap_{i=1}^{m} A_i \subseteq \bigcap_{i=1}^{k} A_i$. □

16 (a) *Proof.* We need to show both sides for any $k \in \mathbb{N}$.

First choose some arbitrary $k \in \mathbb{N}$.

- ($\subseteq$)

  Choose some $x \in \bigcap_{i=1}^{k} A_i$.

  Then for all $l \in \{1, 2, \ldots, k\}, x \in A_l$.

  This means that $x \in A_k$.

  Since the choice of x was arbitrary, this works for all $x \in \bigcap_{i=1}^{k} A_i$.

  Then every x contained in $\bigcap_{i=1}^{k} A_i$ is also in A_k.

  Thus, $\bigcap_{i=1}^{k} A_i \subseteq A_k$.

- ($\supseteq$)

  Choose some $x \in A_k$.

  Since $\mathcal{A}$ is a decreasing nested family of sets, for any $i \in \mathbb{N} \leq k, A_k \subseteq A_i$.

  Now, since x is an element of A_k, x is an element of all supersets of A_k.

  That is to say that $x \in A_{k-1} \land x \in A_{k-2} \land \cdots \land x \in A_1$.

  So $x \in \bigcap_{i=1}^{k} A_i$.

  Since the choice of x was arbitrary, this works for all $x \in A_k$.

Then every x contained in A_k is also in $\bigcap_{i=1}^{k} A_i$.

Thus, $A_k \subseteq \bigcap_{i=1}^{k} A_i$.

Since we have shown both $\bigcap_{i=1}^{k} A_i \subseteq A_k$, and $A_k \subseteq \bigcap_{i=1}^{k} A_i$, for any $k \in \mathbb{N}$.

We have shown that for all $k \in \mathbb{N}$, $\bigcap_{i=1}^{k} A_i = A_k$. □

(b) *Proof.* We need to show both sides.

- ($\subseteq$)
  Choose some $x \in \bigcup_{i=1}^{\infty} A_i$.
  Then there exists some $l \in \mathbb{N}$ such that $x \in A_l$.
  Now, any $n \in \mathbb{N}$ is greater than or equal to 1.
  So $A_l \subseteq A_1$, since $1 \leq l$, and $\mathcal{A}$ is a decreasing nested family of sets.
  Then $A_l \subseteq A_1, x \in A_1$.
  Since the choice of x was arbitrary, this works for all $x \in \bigcup_{i=1}^{\infty} A_i$.
  Then every x contained in $\bigcup_{i=1}^{\infty} A_i$ is also in A_1.
  Thus, $\bigcup_{i=1}^{\infty} A_i \subseteq A_1$.
- ($\supseteq$)

$$\begin{aligned} A_1 &\subseteq A_1 \\ &\subseteq A_1 \cup A_2 \\ &\subseteq A_1 \cup A_2 \cup A_3 \\ &\vdots \\ &\subseteq \bigcup_{i=1}^{\infty} A_i \end{aligned}$$

Since we have shown both sides.
We have $\bigcup_{i=1}^{\infty} A_i = A_1$ □

17 (c) Let $A_i = \{0, 1\}$, then $\mathcal{A} = \{\{0, 1\}\}$, and $\bigcap_{i=1}^{\infty} A_i = \{0, 1\}$

(d) Let $A_i = \varnothing$, then $\mathcal{A} = \{\varnothing\}$, and $\bigcap_{i=1}^{\infty} A_i = \varnothing$

§2.4 6 (i) *Proof.* We show by PMI. $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$

- Base Case.
  Let $n = 1$.

$$\sum_{i=1}^{1} \frac{1}{(2i-1)(2i+1)} = \frac{1}{(2(1)-1)(2(1)+1)} = \frac{1}{3} = \frac{1}{2(1)+1}$$

- Inductive Case.
  Assume for some $n \in \mathbb{N}$, $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$.
  Then

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{(2i-1)(2i+1)} &= \sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} \\
&\quad + \frac{1}{(2(n+1)-1)(2(n+1)+1)} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+2-1)(2n+2+1)} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{n(2n+3)+1}{(2n+1)(2n+3)} \\
&= \frac{2n^2+3n+1}{(2n+1)(2n+3)} \\
&= \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} \\
&= \frac{n+1}{2n+3} \\
&= \frac{n+1}{2(n+1)+1}
\end{aligned}$$

- From the Base case and the inductive case, we use the PMI to state $\sum_{i=1}^{n} \frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}, \forall n \in \mathbb{N}$

$\square$

(k) *Proof.* We show by PMI. $\prod_{i=1}^{n} (2i-1) = \frac{(2n)!}{n!2^n}$

- Base Case.
  Let $n = 1$.

$$\prod_{i=1}^{1} (2i-1) = 2(1) - 1 = 2 - 1 = 1 = \frac{2}{2} = \frac{2(1)}{(1)2} = \frac{(2(1))!}{1!2^1}$$

- Inductive Case.
  Assume for some $n \in \mathbb{N}$, $\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!2^n}$.
  Then

$$
\begin{aligned}
\prod_{i=1}^{n+1}(2i-1) &= \prod_{i=1}^{n}(2i-1) \cdot (2(n+1)-1) \\
&= \frac{(2n)!}{n!2^n} \cdot (2(n+1)-1) \\
&= \frac{(2n)!}{n!2^n} \cdot (2n+2-1) \\
&= \frac{(2n)!}{n!2^n} \cdot (2n+1) \\
&= \frac{(2n+1)!}{n!2^n} \\
&= \frac{(2n+1)!}{n!2^n} \cdot \frac{2n+2}{2n+2} \\
&= \frac{(2n+2)!}{n!2^n\,(2n+2)} \\
&= \frac{(2n+2)!}{n!2^n\,(2\,(n+1))} \\
&= \frac{(2n+2)!}{n!2^{n+1}\,(n+1)} \\
&= \frac{(2n+2)!}{(n+1)!2^{n+1}} \\
&= \frac{(2\,(n+1))!}{(n+1)!2^{n+1}}
\end{aligned}
$$

- From the Base case and the inductive case, we use the PMI to state $\prod_{i=1}^{n}(2i-1) = \frac{(2n)!}{n!2^n}, \forall n \in \mathbb{N}$

$\square$

7 (l) *Proof.* We show by PMI. $\forall x > 0 \in \mathbb{R}, (1+x)^n \geq 1 + nx$

- Base Case.
  Let $n = 1$.

$$(1+x)^1 = 1 + x = 1 + (1)x \geq 1 + (1)x$$

- Inductive Case.
  Assume for some $n \in \mathbb{N}, \forall x > 0 \in \mathbb{R}, (1+x)^n \geq 1 + nx$.
  Then

$$
\begin{aligned}
(1+x)^{n+1} &= (1+x)^n (1+x) \\
&\geq (1+nx)(1+x) \\
&= 1+x+nx+nx^2 \\
&= 1+nx+x+nx^2 \\
&= 1+(n+1)x+nx^2 \\
&\geq 1+(n+1)x
\end{aligned}
$$

- From the Base case and the inductive case, we use the PMI to state $\forall x > 0 \in \mathbb{R}, (1+x)^n \geq 1+nx, \forall n \in \mathbb{N}$

$\square$

(m) *Proof.* We show by PMI. $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{N}$

- Base Case.
  Let $n = 1$.

$$\frac{1^3}{3} + \frac{1^5}{5} + \frac{7(1)}{15} = \frac{1}{3} + \frac{1}{5} + \frac{7}{15} = \frac{5}{15} + \frac{3}{15} + \frac{7}{15} = \frac{15}{15} = 1$$

  And $1 \in \mathbb{N}$

- Inductive Case.
  Assume for some $n \in \mathbb{N}$, $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{N}$.
  Then

$$
\begin{aligned}
\frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15} &= \frac{n^3+3n^2+3n+1}{3} \\
&\quad + \frac{n^5+5n^4+10n^3+10n^2+5n+1}{5} \\
&\quad + \frac{7n+7}{15} \\
&= \frac{n^3}{3} + \frac{3n^2+3n}{3} + \frac{1}{3} \\
&\quad + \frac{n^5}{5} + \frac{5n^4+10n^3+10n^2+5n}{5} + \frac{1}{5} \\
&\quad + \frac{7n}{15} + \frac{7}{15} \\
&= \left(\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15}\right) + \left(\frac{1}{3} + \frac{1}{5} + \frac{7}{15}\right) \\
&\quad + n^2 + n + n^4 + 2n^3 + 2n^2 + n
\end{aligned}
$$

  Now, since we assumed $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{N}$, and $\frac{1}{3} + \frac{1}{5} + \frac{7}{15} = 1 \in \mathbb{N}$, and $n^2 + n + n^4 + 2n^3 + 2n^2 + n \in \mathbb{N}$,

we have $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} + 1 + n^2 + n + n^4 + 2n^3 + 2n^2 + n \in \mathbb{N}$.
Thus $\frac{(n+1)^3}{3} + \frac{(n+1)^5}{5} + \frac{7(n+1)}{15} \in \mathbb{N}$

- From the Base case and the inductive case, we use the PMI to state $\frac{n^3}{3} + \frac{n^5}{5} + \frac{7n}{15} \in \mathbb{N}, \forall n \in \mathbb{N}$

$\square$

8 (h) *Proof.* We show by the Generalized PMI. $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for $n \geq 2$

- Base Case.
  Let $n = 2$.

$$\begin{aligned} 1 &< \sqrt{2} \\ 2 &< \sqrt{2} + 1 \\ \sqrt{2}\left(\sqrt{2}\right) &< \sqrt{2}\left(1 + \frac{1}{\sqrt{2}}\right) \\ \sqrt{2} &< 1 + \frac{1}{\sqrt{2}} \\ \sqrt{2} &< \frac{1}{1} + \frac{1}{\sqrt{2}} \\ \sqrt{2} &< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} \end{aligned}$$

- Inductive Case.
  Assume for some $n \geq 2 \in \mathbb{N}, \sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$.
  Then

$$\begin{aligned}
\sqrt{n} &< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \\
\sqrt{n}\left(\sqrt{n}\right) &< \sqrt{n}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) \\
n &< \sqrt{n}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) \\
n &< \sqrt{n+1}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) \\
n+1 &< \sqrt{n+1}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) + \frac{\sqrt{n+1}}{\sqrt{n+1}} \\
n+1 &< \sqrt{n+1}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right) \\
\sqrt{n+1}\left(\sqrt{n+1}\right) &< \sqrt{n+1}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}\right) \\
\sqrt{n+1} &< \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}}
\end{aligned}$$

- From the Base case and the inductive case, we use the Generalized PMI to state:
  $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for $n \geq 2$.

□

12 (b) *Proof.* We show by PMI. Every n-player tournament has a top player.

- Base Case.
  Let $n = 1$. Then this tournament has a top player vacuously.
- Inductive Case.
  Assume for some $n \in \mathbb{N}$, the n-player tournament has a top player x. Now if we add a new player, y, then this tournament is now an $n+1$-player tournament. y will play all other n players, and three outcomes are possible.

  A. If y beats x, then y also beats a player that beats all other players. So y is also a top player.
  B. If y does not beat x, but beats a player z that beats x, then for every other player w, y beats a player that beats w. So y is also a top player.
  C. If y does not beat x, nor does y beat a player z that beats x, then y is not a top player. However, x still remains a top player.

  In any of the outcomes, there is always a top player.
- From the Base case and the inductive case, we use the PMI to state $\forall n \in \mathbb{N}$, every n-player tournament has a top player.

□