MAT 168 Proof Writing 2

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1. An example follows.

   Given the dictionary:

$$\begin{aligned} \zeta &= 3 + x_1 + 2x_2 \\ x_3 &= 4 - x_1 - x_2 \end{aligned}$$

   We have non-basic variables x_1, x_2 and the basic variable x_3.

   We can let x_1 enter and x_3 leave to get:

$$\begin{aligned} \zeta &= 7 - x_3 + x_2 \\ x_1 &= 4 - x_3 - x_2 \end{aligned}$$

   And then let x_2 enter and x_1 leave to get:

$$\begin{aligned} \zeta &= 11 - 2x_3 - x_1 \\ x_2 &= 4 - x_3 - x_1 \end{aligned}$$

   So we have x_1 becoming basic in one iteration and non-basic in the next.

2. *Proof.* Given some dictionary:

$$\begin{aligned} \zeta &= \bar{\zeta} + \sum_{j \in N} \bar{c}_j x_j \\ x_i &= \bar{b}_i - \sum_{j \in N} \bar{c}_{ij} x_j \text{ for } i \in B \end{aligned}$$

If $x_k, k \in B$ is chosen to leave the basis and become non-basic, and $x_l, l \in N$ is chosen to enter the basis and become basic.

Then we know that $\overline{c}_l > 0$, otherwise x_l would not be chosen to enter the basis.

We also know that $x_l = \overline{b}_l - \ldots - \overline{c}_{lk} x_k - \ldots$.

When we go to pivot in the objective function ζ, we substitute this new value for x_l.

We end up with $\zeta = \overline{\zeta} + \ldots + \overline{c}_l \left(\overline{b}_l - \ldots - \overline{c}_{lk} x_k - \ldots \right) + \ldots$.

After simplification, and letting $\overline{c}'_{lk} = \overline{c}_l \overline{c}_{lk}$, we have

$\zeta = \overline{\zeta}' + \ldots - \overline{c}'_{lk} x_k + \ldots$

Since $-\overline{c}'_{lk} < 0$, it will not be chosen to become basic in the next iteration.

Since our choice of entering and leaving variables was arbitrary, this holds for any such entering and leaving variables.

Thus, we have shown that if a variable becomes non-basic in one iteration, then it cannot become basic in the next iteration. □

3. We're asked to show that:

   Given a linear program with all right hand sides equal to 0, either $x_j = 0$ is optimal for all j, or the problem is unbounded.

   We can rephrase this as:

   Given a linear program with all right hand sides equal to 0, either it's not the case that $x_j = 0$ is not optimal for all j, or the problem is unbounded.

   And this is equivalent to the rephrasing:

   Given a linear program with all right hand sides equal to 0, if $x_j = 0$ is not optimal for all j, then the problem is unbounded.

   Proof. Given some linear program with all right hand sides equal to 0.

   Assume $x_j = 0$ is not optimal for all j. Then we should be able to increase some x_j and arrive at an optimal solution.

   From the constraints, we see that there must exist some $a_{ij} \leq 0$, for each i. Otherwise $\sum_{j=1}^{n} a_{ij} x_j \leq 0$ would fail to hold for all i.

   If we choose an arbitrary $a_{kl} \leq 0$ and its corresponding x_l, then we can increase x_l while still keeping the constraint valid.

   What we find is that, increasing any arbitrary x_l does not invalidate any constraints. However, the objective value becomes larger. Since we can increase any arbitrary x_l to any amount, this problem is unbounded.

   Thus, (with some mental reformulation) we have shown that:

   Given a linear program with all right hand sides equal to 0, either $x_j = 0$ is optimal for all j, or the problem is unbounded. □

4. (a) We're asked to prove or disprove the feasible region of problem 3.4 is a convex cone.

The feasible region of 3.4 is the set $\{x_j \in \mathbb{R}^n | x_j \geq 0\}$.

Proof. Choose some $x, y \in C = \{x_j \in \mathbb{R}^n | x_j \geq 0\}$.
Choose some $\lambda, \mu \geq 0 \in \mathbb{R}$.
Then $\lambda x + \mu y \geq 0 + 0 \geq 0$, so $\lambda x + \mu y \in C$.
Since our choice of x, y, λ, μ were arbitrary, this result holds for any such values.
So C is a convex cone.
Thus we have shown that the feasible region of problem 3.4 is a convex cone. □

(b) We're asked to prove or disprove that every convex polyhedron is a convex cone.

We can represent a convex polyhedron as $\{x \in \mathbb{R}^n | Ax = b\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$.

Proof. Let $C = \{x \in \mathbb{R}^n | Ax = b\}$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, be a convex polyhedron.
Then choose some arbitrary $x, y \in C$ and $\lambda, \mu > 0 \in \mathbb{R}$.
Now $A(\lambda x + \mu y) = A\lambda x + A\mu y = \lambda Ax + \mu Ay = \lambda b + \mu b = (\lambda + \mu) b > b$.
So $\lambda x + \mu y \notin C$.
Since we chose arbitrary x, y, λ, μ, and $\lambda x + \mu y \notin C$, C is not a convex cone.
Then there is at least one convex polyhedron that is not a convex cone.
Thus, we have disproved that every convex polyhedron is a convex cone. □

(c) We're asked to prove or disprove that every convex cone is convex.

A set $S \subseteq \mathbb{R}^n$ is convex if for any $x, y \in S, \lambda \in [0, 1]$, then $\lambda x + (1 - \lambda) y \in S$.

Proof. Let $S \subseteq \mathbb{R}^n$ be some convex cone.
Then choose $\lambda \in [0, 1]$.
Let $\mu = 1 - \lambda$, then $\lambda, \mu \geq 0$.
Now, since S is a convex cone, we can choose $x, y \in S$. Then $\lambda x + \mu y \in S$.
But $\lambda x + \mu y = \lambda x + (1 - \lambda) y \in S$.
Since our choice of x, y, λ was arbitrary, this result holds for all such values.
So S is convex.
Now, since our choice of S was arbitrary, this result holds for all such convex cones.
Thus we have shown that all convex cones are convex. □