MAT 25 Homework 4

Hardy Jones
999397426
Professor Bae
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1. 1.4.10 Show that the set of all finite subsets of $\mathbb{N}$ is a countable set.

   Proof. In order to prove this, we simply need to construct an isomorphism from $\mathbb{N}$ to the set of all finite subsets of $\mathbb{N}$, hereafter referred to as $\mathcal{S}$.

   Let's take a look at some elements of $\mathcal{S}$. We have:

   $$S = \{\{\}, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}, ...\}$$

   What we see is that we can arbitrarily number these sets:

   $$\begin{array}{lccccccccc} \mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & ... \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & ... \\ \mathcal{S}: & \{\} & \{1\} & \{2\} & \{1,2\} & \{3\} & \{1,3\} & \{2,3\} & \{1,2,3\} & ... \end{array}$$

   And thus we have a mapping that is 1-1 and onto between $\mathbb{N}$ and $\mathcal{S}$.

   So $\mathcal{S}$ has the same cardinality as $\mathbb{N}$, from which it follows that $\mathcal{S}$ is countable. □

2. 1.4.12 A real number $x \in R$ is called algebraic if there exist integers $a_0, a_1, a_2, ..., a_n \in N$, not all zero, such that $a_n x^n + a_{n1} x^n 1 + ... + a_1 x + a_0 = 0$. Said another way, a real number is algebraic if it is the root of a polynomial with integer coefficients. Real numbers that are not algebraic are called transcendental numbers.

   (a) Show that $\sqrt{2}$, $\sqrt[3]{2}$, and $\sqrt{2} + \sqrt{3}$ are algebraic numbers.

   Proof. i. If we let $x = \sqrt{2}$, then we can manipulate this equation.

   $$\begin{aligned} x &= \sqrt{2} \\ x^2 &= 2 \\ x^2 - 2 &= 0 \end{aligned}$$

   So we have $a_0 = -2, a_2 = 1, a_1 = a_3 = a_4 = ... = a_n = 0$ Thus, $\sqrt{2}$ is an algebraic number.

   ii. Following similar logic. If we let $x = \sqrt[3]{2}$, then we can manipulate this equation.

   $$\begin{aligned} x &= \sqrt[3]{2} \\ x^3 &= 2 \\ x^3 - 2 &= 0 \end{aligned}$$

   So we have $a_0 = -2, a_3 = 1, a_1 = a_3 = a_4 = ... = a_n = 0$ Thus, $\sqrt[3]{2}$ is an algebraic number.

iii. This is a bit more complex.

$$\begin{aligned}
x &= \sqrt{2}+\sqrt{3} \\
x^2 &= 2+3+2\sqrt{6} \\
x^2-5 &= 2\sqrt{6} \\
x^4-10x^2+25 &= 24 \\
x^4-10x^2+1 &= 0
\end{aligned}$$

So we have $a_0 = 1, a_2 = -10, a_4 = 1, a_1 = a_3 = a_5 = a_6 = ... = a_n = 0$ Thus, $\sqrt{2}+\sqrt{3}$ is an algebraic number.

□

(b) Fix $n \in N$, and let $\mathcal{A}_n$ be the algebraic numbers obtained as roots of polynomials with integer coefficients that have degree n. Using the fact that every polynomial has a nite number of roots, show that $\mathcal{A}_n$ is countable.

Proof. Since every polynomial has a unique representation, we can say that each polynomial can be thought of as a set of integers.

Let's call this set $\mathcal{P} = \{a_0, a_1, ..., a_n | n \in \mathbb{N}\}$.

We've already shown that the set of all finite subsets of $\mathbb{N}$ is countable. And, since $\mathcal{P}$ is actually the set of all finite subsets of $\mathbb{N}$, $\mathcal{P}$ is countable.

Since each polynomial in $\mathcal{P}$ has a finite number of roots, only a finite number of algebraic numbers correspond to each polynomial. We can construct countable subsets of $\mathcal{A}_n$ comprised of these roots.

Now, since each polynomial has a corresponding countable subset in $\mathcal{A}_n$, we can take the union of all these subsets in $\mathcal{A}_n$. According to Theorem 1.4.13.i, we have the union of all these subsets in $\mathcal{A}_n$ is countable.

And of course, the union of all these subsets in $\mathcal{A}_n$ is just $\mathcal{A}_n$. So $\mathcal{A}_n$ is countable.

□

(c) Now, argue that the set of all algebraic numbers is countable. What may we conclude about the set of transcendental numbers?

Proof. From above we have that $\mathcal{A}_n$ is countable. If we look at all algebraic numbers, we can construct this set as

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

From Theorem 1.4.13.ii we know that this set is countable. So the algebraic numbers are countable.

And since transcendental numbers are all other real numbers, it follows that the transcendentals make up the rest of the reals. So $\mathcal{T} = \mathbb{R} \backslash \mathcal{A}$, similar to the relation between the rationals and irrationals. Following this relation, since the reals are uncountable, and $\mathcal{A}$ is countable, we must have that $\mathcal{T}$ is uncountable. □

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined as

$$f(m, n) = (2m - 1) \cdot 2^{n-1}$$

(a) Show that f is 1-1.

Proof. We want to show

$$\forall a, b, c, d \in \mathbb{N}, f(a, b) = f(c, d) \implies a = c \text{ and } b = d$$

$$\begin{aligned} f(a, b) &= f(c, d) \\ (2a - 1) \cdot 2^{b-1} &= (2c - 1) \cdot 2^{d-1} \\ (2a - 1) \cdot 2^{b} \cdot \frac{1}{2} &= (2c - 1) \cdot 2^{d} \cdot \frac{1}{2} \\ (2a - 1) \cdot 2^{b} &= (2c - 1) \cdot 2^{d} \\ 2a - 1 &= (2c - 1) \cdot 2^{d-b} \end{aligned}$$

Now, the LHS is an odd number. Looking at the RHS, $2c - 1$ is an odd number, and based on The Trichotomy Law we have three cases to consider:

i. $b < d$ In this case, 2^{d-b} is an even number, so the RHS would have to be an even number. But the LHS (an odd number) cannot be an even number also. So, this case is impossible.

ii. $b > d$ In this case, 2^{d-b} is a fraction, so the RHS would also be a fraction. But LHS is a positive integer. So, this case is also impossible.

iii. $b = d$ This must be the case, and we needn't show anything to prove it. What this means however is that 2^{d-b} is 1.

$$\begin{aligned} 2a - 1 &= 2c - 1 \\ 2a &= 2c \\ a &= c \end{aligned}$$

Now, we have $a = c$ and $b = d$.
So we have shown that

$$\forall a, b, c, d \in \mathbb{N}, f(a, b) = f(c, d) \implies a = c \text{ and } b = d$$

Thus f is 1-1. $\square$

(b) Using (a), show that $\mathbb{Q}^+ = \{q \in \mathbb{Q} : q > 0\}$

Proof. We can use f to construct a map:

$$f : \mathbb{Q}^+ \to \mathbb{N}$$

If we take the components of each rational:

$$\mathbb{Q}^+ = \left\{ r, s \in \mathbb{N}, q \in \mathbb{Q} : \frac{r}{s} = q > 0 \right\}$$

Now we use f to map $\mathbb{Q}^+$ to $\mathbb{N}$. Since f is 1-1, this implies that $\mathbb{Q}^+ \sim \mathbb{N}$, or that $\mathbb{Q}^+$ is countable. □

(c) Show that $\mathbb{Q}$ is countable.

Proof. We can use the result of (b) to help us here.
We simply need another function that is 1-1 with f.
Let's try the function $g : \mathbb{Q} \to \mathbb{Q}^+$,

$$g(q) = \begin{cases} q & \text{if } q > 0 \\ \frac{-q+1}{2} & \text{if } q <= 0 \end{cases}$$

It should be easy to see that this is injective.
That is:

$$\forall r, s \in \mathbb{Q}, g(r) = g(s) \implies r = s$$

Let's show it quickly, we need to handle two cases:

i. $r > 0, s > 0$

$$\begin{aligned} g(r) &= g(s) \\ r &= s \end{aligned}$$

ii. $r < 0, s < 0$

$$\begin{aligned} g(r) &= g(s) \\ \frac{-r+1}{2} &= \frac{-s+1}{2} \\ -r+1 &= -s+1 \\ -r &= -s \\ r &= s \end{aligned}$$

So g is 1-1. And since g is 1-1, we can construct a mapping $f \circ g : \mathbb{Q} \to \mathbb{Q}^+ \to \mathbb{N}$.
So $\mathbb{Q}$ is countable. □

4. Let $\mathcal{S}$ be the set consisting of all sequences of 0's and 1's;

$$\mathcal{S} = \{(a_1, a_2, a_3, \cdots) : a_n \in \{0, 1\}\}.$$

Show that $\mathcal{S}$ is uncountable.

Proof. Construct sequences at random:

$$\begin{aligned}
s_1 &= (0,1,1,0,0,0,0,0,1,\dots)\\
s_2 &= (1,1,0,0,0,0,0,0,0,\dots)\\
s_3 &= (0,0,1,0,0,0,0,0,1,\dots)\\
s_4 &= (0,0,1,1,0,0,0,1,1,\dots)\\
s_5 &= (0,1,1,0,1,1,0,1,0,\dots)\\
s_6 &= (1,1,0,0,0,0,0,0,0,\dots)\\
s_7 &= (1,1,1,1,1,0,1,0,1,\dots)\\
s_8 &= (0,1,0,0,0,0,0,0,0,\dots)\\
s_9 &= (0,1,1,0,0,1,0,1,0,\dots)
\end{aligned}$$

Now we can map each one of these sequences to a number in $\mathbb{N}$. Thus we have exhausted all of the numbers in $\mathbb{N}$ with this mapping.

However, we can still create more sequences. One way to construct a new sequence would be to take an element from each previous sequence and alternate the number. If it was 0 make the new element 1, if it was 1, make the new element 0. So from s_1 we take element 1, from s_2 we take element 2, and so on.

Visually:

$$\begin{aligned}
s_1 &= (\mathbf{0},1,1,0,0,0,0,0,1,\dots)\\
s_2 &= (1,\mathbf{1},0,0,0,0,0,0,0,\dots)\\
s_3 &= (0,0,\mathbf{1},0,0,0,0,0,1,\dots)\\
s_4 &= (0,0,1,\mathbf{1},0,0,0,1,1,\dots)\\
s_5 &= (0,1,1,0,\mathbf{1},1,0,1,0,\dots)\\
s_6 &= (1,1,0,0,0,\mathbf{0},0,0,0,\dots)\\
s_7 &= (1,1,1,1,1,0,\mathbf{1},0,1,\dots)\\
s_8 &= (0,1,0,0,0,0,0,\mathbf{0},0,\dots)\\
s_9 &= (0,1,1,0,0,1,0,1,\mathbf{0},\dots)
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
s_1 &= (\mathbf{1},1,1,0,0,0,0,0,1,\dots)\\
s_2 &= (1,\mathbf{0},0,0,0,0,0,0,0,\dots)\\
s_3 &= (0,0,\mathbf{0},0,0,0,0,0,1,\dots)\\
s_4 &= (0,0,1,\mathbf{0},0,0,0,1,1,\dots)\\
s_5 &= (0,1,1,0,\mathbf{0},1,0,1,0,\dots)\\
s_6 &= (1,1,0,0,0,\mathbf{1},0,0,0,\dots)\\
s_7 &= (1,1,1,1,1,0,\mathbf{0},0,1,\dots)\\
s_8 &= (0,1,0,0,0,0,0,\mathbf{1},0,\dots)\\
s_9 &= (0,1,1,0,0,1,0,1,\mathbf{1},\dots)
\end{aligned}$$

This new sequence, call it $s_0 = (1, 0, 0, 0, 0, 1, 0, 1, 1, \dots)$ consists of all different elements from any previous sequence, thus it is not in our original set $\mathcal{S}$. So our set has larger cardinality than $\mathbb{N}$, and thus is uncountable. $\square$

5. Show that for any $n \in \mathbb{N}$, $\sqrt{n-1} + \sqrt{n+1}$ is irrational.

Proof. Let's assume that is not the case.
That is, $\exists n \in \mathbb{N} : \exists p, q \in \mathbb{N} : \sqrt{n-1} + \sqrt{n+1} = \frac{p}{q}$, where p and q are coprime.

$$\begin{aligned}
\sqrt{n-1} + \sqrt{n+1} &= \frac{p}{q} \\
\left(\sqrt{n-1} + \sqrt{n+1}\right)^2 &= \left(\frac{p}{q}\right)^2 \\
n - 1 + n + 1 + 2\sqrt{n^2-1} &= \frac{p^2}{q^2} \\
2n + 2\sqrt{n^2-1} &= \frac{p^2}{q^2} \\
2\left(n + \sqrt{n^2-1}\right) &= \frac{p^2}{q^2} \\
2q^2 \cdot \left(n + \sqrt{n^2-1}\right) &= p^2
\end{aligned}$$

This implies that one of $\left\{2, q^2, \left(n + \sqrt{n^2-1}\right)\right\} \mid p^2$.
Since we assumed that p and q were coprime, $q^2 \nmid p^2$. Since $p \in \mathbb{N}, p^2 \in \mathbb{N}$ and as such $\left(n + \sqrt{n^2-1}\right) \nmid p^2$. So $2 \mid p^2$.

In similar fashion to the proof of irrationality of $\sqrt{2}$, this implies that $2 \mid p$.

Following the steps taken in that proof, we create some new integer: let $p = 2r$, and sub it into our equation.

$$\begin{aligned}
2q^2 \cdot \left(n + \sqrt{n^2-1}\right) &= p^2 \\
2q^2 \cdot \left(n + \sqrt{n^2-1}\right) &= (2r)^2 \\
2q^2 \cdot \left(n + \sqrt{n^2-1}\right) &= 4r^2 \\
q^2 \cdot \left(n + \sqrt{n^2-1}\right) &= 2r^2 \\
q^2 &= 2\left(\frac{r^2}{n + \sqrt{n^2-1}}\right)
\end{aligned}$$

This of course means $2 \mid q^2 \implies 2 \mid q$. But as with the other proof, this is a contradiction, since we assumed p and q coprime. So our original assumption was incorrect, and thus:

$$\forall n \in \mathbb{N}, \sqrt{n-1} + \sqrt{n+1} \text{ is irrational.}$$

$\square$