MAT 125A HW 2

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Exercise 4.3.6 (a) *Proof.* We have Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This is a function from $\mathbb{R} \to \mathbb{R}$.

Choose 0 as a limit point in $\mathbb{R}$.

Now we construct a sequence from $x_n = \frac{e}{n}$. So $(x_n) \to 0$, by The Algebraic Limit Theorem.

Now, $f(x_n) \to 0$, but $f(0) = 1$.

So, by Corollary 4.3.3, Dirichlet's function is not continuous at 0.

We can extend this to any $c \in \mathbb{Q}$ by constructing a new sequence $y_n = x_n + c$.

Following similar arguments, it can be shown that Dirichlet's function is not continuous at any point in $\mathbb{Q}$.

A similar argument holds for showing that Dirichlet's function is not continuous on $\mathbb{I}$. We choose some sequence of rationals such that for any limit point c, $(z_n) \to c$. Then we have that $f(z_n) \to 1$, but $f(c) = 0$. So Dirichlet's function is not continuous on $\mathbb{I}$ either.

Thus, Dirichlet's function is not continuous on $\mathbb{R}$. $\square$

(b) *Proof.* We have

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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Choose some rational number c in lowest terms $\frac{m}{n}$.

Now construct any sequence (x_n) from $\mathbb{I}$ such that, $(x_n) \to c$.

So, $f(x_n) \to 0$, but $f(c) = \frac{1}{n}$.

Now, since $f : \mathbb{R} \to \mathbb{R}$, c is a limit point in $\mathbb{R}$, $(x_n) \to c$, but $f(x_n) \neq f(c)$, we conclude that Thomae's function is not continuous at any rational point. $\square$

(c) *Proof.* We have

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

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Choose some $c \in \mathbb{I}$. Then we know that $f(c) = 0$, since $c \notin \mathbb{Q}$.
Now given, some $\epsilon > 0$, choose $\delta = \epsilon$.
So for $x \in \mathbb{I}$ where $|x - c| < \delta$, we have $|f(x) - f(c)| = |0 - 0| = 0 < \epsilon$.
Thus, by Theorem 4.3.2, f is continuous on $\mathbb{I}$. $\square$