MAT 25 Homework 5

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1. 2.2.1

   (a) $\lim \frac{1}{6n^2+1} = 0$
   We need to show

$$\begin{aligned}\frac{1}{6n^2+1} &< \epsilon \\ \frac{1}{\epsilon} &< 6n^2+1 \\ \frac{1}{\epsilon} - 1 &< 6n^2 \\ \frac{1-\epsilon}{\epsilon} &< 6n^2 \\ \frac{1-\epsilon}{6\epsilon} &< n^2 \\ \sqrt{\frac{1-\epsilon}{6\epsilon}} &< n\end{aligned}$$

   Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \sqrt{\frac{1-\epsilon}{6\epsilon}}$.
   Let $n \geq N$. So, $n \geq N > \sqrt{\frac{1-\epsilon}{6\epsilon}} \implies \frac{1}{6n^2+1} < \epsilon$
   Thus $|a_n - 0| < \epsilon$.

   (b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$
   We need to show

$$\begin{aligned}\frac{3n+1}{2n+5} &< \epsilon \\ 3n+1 &< 2n\epsilon + 5\epsilon \\ 1 - 5\epsilon &< (2\epsilon - 3)n \\ \frac{1-5\epsilon}{2\epsilon - 3} &< n\end{aligned}$$

   Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \frac{1-5\epsilon}{2\epsilon-3}$.
   Let $n \geq N$. So, $n \geq N > \frac{1-5\epsilon}{2\epsilon-3} \implies \frac{3n+1}{2n+5} < \epsilon$.
   Thus $|a_n - \frac{3}{2}| < \epsilon$.

(c) $\lim \frac{2}{\sqrt{n+3}} = 0$
We need to show

$$\begin{aligned} \frac{2}{\sqrt{n+3}} &< \epsilon \\ \frac{2}{\epsilon} &< \sqrt{n+3} \\ \frac{4}{\epsilon^2} &< n+3 \\ \frac{4}{\epsilon^2} - 3 &< n \end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \frac{4}{\epsilon^2} - 3$.
Let $n \geq N$. So, $n \geq N > \frac{4}{\epsilon^2} - 3 \implies \frac{2}{\sqrt{n+3}} < \epsilon$.
Thus $|a_n - 0| < \epsilon$.

2. 2.2.5

(a) $a_n = \left\lfloor \frac{1}{n} \right\rfloor$
It is easy to see that after the first element in the sequence, all values are 0.
$\lim a_n = 0$

Proof. Let $\epsilon > 0$. Choose $N > 1$.
Let $n \geq N$. So, $n \geq N > 1 \implies \left\lfloor \frac{1}{n} \right\rfloor = 0 < \epsilon$.
Thus, $|a_n - 0| < \epsilon$. □

(b) $a_n = \left\lfloor \frac{10+n}{2n} \right\rfloor$
Again we see that after some elements all values are 0.
$\lim a_n = 0$

Proof. Let $\epsilon > 0$. Choose $N > 10$.
Let $n \geq N$. So, $n \geq N > 10 \implies \left\lfloor \frac{10+n}{2n} \right\rfloor = 0 < \epsilon$.
Thus, $|a_n - 0| < \epsilon$. □

3. 2.2.7

(a) A sequence (a_n) diverges to ∞ if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n| > \epsilon$
$\lim \sqrt{n} = \infty$

Proof. Let $\epsilon > 0$.
We want to show

$$\begin{aligned} \sqrt{n} &> \epsilon \\ n &> \epsilon^2 \end{aligned}$$

Choose $N > \epsilon^2$.
Let $n \geq N$. So, $n \geq N > \epsilon^2 \implies \sqrt{n} > \epsilon$.
Thus, $|a_n| > \epsilon$. $\square$

(b) It states that this particular sequence does not diverge to ∞. The reason being, if you choose some $\epsilon > 0$, and any $N \in \mathbb{N}$, then $\exists n \geq N|$ either $n = 0$ or $n + 1 = 0$. So, $|a_n| \not\geq \epsilon, \forall n$.

4. 2.3.4 Using the Algebraic Limit Theorem, if $\lim a_n = l_1$ and $\lim a_n = l_2$, then

$$\begin{aligned} \lim(a_n - a_n) &= 0 \\ l_1 - l_2 &= 0 \\ l_1 &= l_2 \end{aligned}$$

5. 2.3.7

(a) Since (a_n) is bounded, $\exists M > 0$ such that $|a_n| \leq M, \forall n \in \mathbb{N}$.
Also, since $\lim b_n = 0$ we can choose a special epsilon, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies |b_n| < \frac{\epsilon}{M}$
So, $|a_n||b_n| < M\frac{\epsilon}{M} \implies |a_n||b_n| < \epsilon$.
Which is equivalent to $|a_n b_n - 0| < \epsilon$.
By Definition 2.2.3, this sequence $(a_n b_n)$ goes to 0.
We were not allowed to use the Algebraic Limit Theorem because (a_n) is not necessarily convergent.

(b) The only thing we can conclude when $\lim b_n = b$ is that $(a_n b_n)$ is bounded by $b(a_n)$.