STA 032 Homework 3

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§ 2.4 5 (1)

x	$f(x)$
1	0.7
2	0.15
3	0.1
4	0.03
5	0.02

(2)

$$P(X \leq 2) = P(X = 1) + P(X = 2) = 0.7 + 0.15 = 0.85$$

(3)

$$P(X > 3) = P(X = 4) + P(X = 5) = 0.03 + 0.02 = 0.05$$

(4)

$$\begin{aligned} \mu_X &= x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + x_4 f(x_4) + x_5 f(x_5) \\ &= 1(0.7) + 2(0.15) + 3(0.1) + 4(0.03) + 5(0.02) \\ &= 0.7 + 0.3 + 0.3 + 0.12 + 0.1 \\ &= 1.52 \end{aligned}$$

(5)

$$\begin{aligned} \sigma_X &= \sqrt{x_1^2 f(x_1) + x_2^2 f(x_2) + x_3^2 f(x_3) + x_4^2 f(x_4) + x_5^2 f(x_5) - \mu_X^2} \\ &= \sqrt{1^2(0.7) + 2^2(0.15) + 3^2(0.1) + 4^2(0.03) + 5^2(0.02) - 1.52^2} \\ &= \sqrt{1(0.7) + 4(0.15) + 9(0.1) + 16(0.03) + 25(0.02) - 2.3104} \\ &= \sqrt{0.8696} \\ &\approx 0.93 \end{aligned}$$

8 (1) We want to find

$$P(X \leq 2) = F(2) = 0.83$$

(2) We want to find

$$P(X > 3) = F(4) - F(3) = 1.00 - 0.95 = 0.05$$

(3) We want to find

$$P(X = 1) = F(1) - F(0) = 0.72 - 0.41 = 0.31$$

(4) We want to find

$$P(X = 0) = F(0) = 0.41$$

(5) If we look at the probability of each number of error, we can answer this.

$$\begin{aligned}
&P(X = 0) = 0.41\\
&P(X = 1) = 0.31\\
&P(X = 2) = F(2) - F(1) = 0.83 - 0.72 = 0.11\\
&P(X = 3) = F(3) - F(2) = 0.95 - 0.83 = 0.12\\
&P(X = 4) = 0.05
\end{aligned}$$

Since $P(X = 0)$ has the largest probability, it is most probable that 0 errors will be detected.

15 (1)

$$\begin{aligned}
\mu_t &= \int_{-\infty}^{\infty} t f(t)\ \mathrm{d}t\\
&= 0.1 \int_{0}^{\infty} t e^{-0.1t}\ \mathrm{d}t
\end{aligned}$$

Using the tabular method for integration by parts

u	$\mathrm{d}v$
t	$e^{-0.1t}$
1	$-10e^{-0.1t}$
0	$100e^{-0.1t}$

So we have

$$\begin{aligned}
\mu_t &= 0.1\left(t(-10e^{-0.1t}) - 1(100e^{-0.1t})\right)\Big|_0^\infty\\
&= 0.1\left(-(10t + 100)(e^{-0.1t})\right)\Big|_0^\infty\\
&= -\frac{t + 10}{e^{0.1t}}\Big|_0^\infty\\
&= 0 - (-10)\\
&= 10
\end{aligned}$$

(2)

$$
\begin{aligned}
\sigma_t &= \sqrt{\int_0^\infty t^2 f(t)\ \mathrm{d}t - \mu_t^2} \\
&= \sqrt{0.1 \int_0^\infty t^2 e^{-0.1t}\ \mathrm{d}t - 10^2} \\
&= \sqrt{0.1 \int_0^\infty t^2 e^{-0.1t}\ \mathrm{d}t - 100}
\end{aligned}
$$

Using the tabular method for integration by parts

u	$\mathrm{d}v$
t^2	$e^{-0.1t}$
$2t$	$-10e^{-0.1t}$
2	$100e^{-0.1t}$
0	$-1000e^{-0.1t}$

So we have

$$
\begin{aligned}
\sigma_t &= \sqrt{0.1\left(t^2(-10e^{-0.1t}) - 2t(100e^{-0.1t}) + 2(-1000e^{-0.1t})\right)\Big|_0^\infty - 100} \\
&= \sqrt{0.1\left((-10t^2 - 200t - 2000)e^{-0.1t}\right)\Big|_0^\infty - 100} \\
&= \sqrt{\frac{-t^2 - 20t - 200}{e^{0.1t}}\bigg|_0^\infty - 100} \\
&= \sqrt{(0 - (-200)) - 100} \\
&= \sqrt{100} \\
&= 10
\end{aligned}
$$

(3)

$$
\begin{aligned}
F(x) &= \int_0^x f(t)\ \mathrm{d}t \\
&= 0.1 \int_0^x e^{-0.1t}\ \mathrm{d}t \\
&= -e^{-0.1t}\Big|_0^x \\
&= -\frac{1}{e^{0.1t}}\bigg|_0^x \\
&= -\frac{1}{e^{0.1x}} - (-1) \\
&= 1 - \frac{1}{e^{0.1x}}
\end{aligned}
$$

(4) We want to find $P(X < 12) = F(11)$, since we are dealing with discrete random variables.

$$F(11) = 1 - \frac{1}{e^{0.1(11)}} \approx 0.6671$$

So the probability that the lifetime will be less than 12 months is approximately 66.71%.

24 (1) We want to solve the following equation for c:

$$\begin{aligned}
\int_1^\infty \frac{c}{x^3}\ \mathrm{d}x &= 1 \\
\int_1^\infty cx^{-3}\ \mathrm{d}x &= \\
-\frac{cx^{-2}}{2}\Big|_1^\infty &= \\
-\frac{c}{2x^2}\Big|_1^\infty &= \\
0 - (-\frac{c}{2}) &= \\
\frac{c}{2} &= \\
c &= 2
\end{aligned}$$

So $f(x)$ is a valid probability density function when $c = 2$.

(2)

$$\begin{aligned}
\int_1^\infty xf(x)\ \mathrm{d}x &= \int_1^\infty \frac{2x}{x^3}\ \mathrm{d}x \\
&= \int_1^\infty 2x^{-2}\ \mathrm{d}x \\
&= -2x^{-1}\Big|_1^\infty \\
&= -\frac{2}{x}\Big|_1^\infty \\
&= 0 - (-2) \\
&= 2
\end{aligned}$$

(3)

$$\begin{aligned} F(x) &= \int_1^x f(t)\ \mathrm{d}t \\ &= \int_1^x \frac{2}{t^3}\ \mathrm{d}t \\ &= -\frac{1}{t^2}\bigg|_1^x \\ &= -\frac{1}{x^2} - (-1) \\ &= 1 - \frac{1}{x^2} \end{aligned}$$

(4) The median is the 50^{th} percentile, so we can find:

$$\begin{aligned} P(X = x_{50}) &= \frac{1}{2} \\ F(x_{50}) &= \\ 1 - \frac{1}{x_{50}} &= \\ \frac{1}{2} &= \frac{1}{x_{50}} \\ x_{50} &= 2 \end{aligned}$$

So the median particle size is 2 µm.

(5) We can plug in this value in our cumulative density function.

$$F(10) = 1 - \frac{1}{10^2} = 1 - \frac{1}{100} = \frac{99}{100}$$

So 99% of the contaminating particles are smaller than 10 µm.

(6) We can plug in this value in our cumulative density function.

$$F(2.5) = 1 - \frac{1}{2.5^2} = 1 - \frac{1}{6.25} = 1 - \frac{4}{25} = \frac{84}{100}$$

So 84% of the contaminating particles are smaller than 2.5 µm.

(7) The proportion $PM_{2.5}$ of PM_{10} is

$$\frac{\frac{84}{100}}{\frac{99}{100}} = \frac{84}{99} = 0.\overline{84}$$

or 84.85%.

§ 2.5 8 We call the original fill volume mean $\mu_x = 20.01$, and the standard deviation $\sigma_x = 0.02$.

(1)

$$\mu_y = 24(\mu_x) = 24(20.01) = 480.24$$

So the mean of the total volume of the case is 480.24 oz.

(2)

$$\sigma_y = 24(\sigma_x) = 24(0.02) = 0.48$$

So the standard deviation of the total volume of the case is 0.48 oz.

(3)

$$\mu_z = \frac{\mu_y}{24} = \frac{480.24}{24} = 20.01$$

So the mean of the average volume per bottle of the case is 20.01 oz.

(4)

$$\sigma_z = \frac{\sigma_x}{\sqrt{24}} = \frac{0.02}{\sqrt{24}} \approx 0.004082$$

So the standard deviation of the average volume per bottle of the case is 0.0041 oz.

(5) We want to solve the following equation:

$$\begin{aligned}
\frac{0.02}{\sqrt{x}} &= 0.0025 \\
\frac{0.02}{0.0025} &= \sqrt{x} \\
\frac{0.02}{0.0025}^{2} &= x \\
64 &= x
\end{aligned}$$

So 64 bottles are needed in each case for the standard deviation of the average volume per bottle in each case is 0.0025 oz.

10 (1)

$$\mu_{daily} = 2.60(1500) + 2.75(500) + 2.90(300) = 3900 + 1375 + 870 = 6145$$

So the mean daily revenue is \$6145.00.

(2)

$$\sigma_{daily} = \sqrt{180^2 + 90^2 + 40^2} = \sqrt{32400 + 8100 + 1600} = 205.1828$$

So the standard deviation of the daily revenue is \$205.18.