

ECS 120 Problem Set 1

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Spring 2014

Problem 1 We can gain some insight by looking at the first few n -digit palindromes.

n	elements	$ n $
1	1,2,3,4,5,6,7,8,9	9
2	11,22,33,44,55,66,77,88,99	9
3	101,202,303,404,505,606,707,808,909	90
	111,212,313,414,515,616,717,818,919	
	\vdots	
	191,292,393,494,595,696,797,898,999	
4	1001,2002,3003,4004,5005,6006,7007,8008,9009	90
	1111,2112,3113,4114,5115,6116,7117,8118,9119	
	\vdots	
	1991,2992,3993,4994,5995,6996,7997,8998,9999	

If we continue in this way we see that the number of palindromes increases by 10 for every 2 digits added.

We can see a recurrence relation here:

$$D_1 = 9$$

$$D_2 = 9$$

$$D_n = D_{n-2} \cdot 10, \text{ for } n > 2$$

From this we can manipulate it to a closed form:

$$D_n = 9 \cdot 10^{\lfloor \frac{n-1}{2} \rfloor}$$

We can prove this by induction.

Proof. Base Cases:

$$n = 1$$

$$D_1 = 9 \cdot 10^{\lfloor \frac{1-1}{2} \rfloor} = 9 \cdot 10^0 = 9$$

$$n = 2$$

$$D_2 = 9 \cdot 10^{\lfloor \frac{2-1}{2} \rfloor} = 9 \cdot 10^0 = 9$$

Inductive Hypothesis:

$$D_n = 9 \cdot 10^{\lfloor \frac{n-1}{2} \rfloor}$$

Inductive Case:

$$\begin{aligned} D_{n+1} &= D_{n-1} \cdot 10 \\ &= 9 \cdot 10^{\lfloor \frac{(n-1)-1}{2} \rfloor} \cdot 10 \\ &= 9 \cdot 10^{\lfloor \frac{n-2}{2} \rfloor} \cdot 10 \\ &= 9 \cdot 10^{\lfloor \frac{n}{2} - \frac{2}{2} \rfloor} \cdot 10 \\ &= 9 \cdot 10^{\lfloor \frac{n}{2} - 1 \rfloor} \cdot 10 \\ &= 9 \cdot 10^{\lfloor \frac{n}{2} - 1 + 1 \rfloor} \\ &= 9 \cdot 10^{\lfloor \frac{n}{2} \rfloor} \end{aligned}$$

Thus, we have proved that our formula is correct.

□

Now, we can calculate $D_{20} = 9 \cdot 10^{\lfloor \frac{20-1}{2} \rfloor} = 9 \cdot 10^9 = 9000000000$

Problem 2 Let's begin by enumerating some of the first few strings in lexographic order.

w_1	ϵ
w_2	0
w_3	1
w_4	00
w_5	01
w_6	10
w_7	11
w_8	000

Interestingly, this looks like the binary representation of each n in w_n without the leading digit. In the case of $n = 1$, removing the leading (only) digit leaves the empty string.

So we have a mapping: $n \in \mathbb{N} \rightarrow w_n$.

This means that $w_{1234567}$ can be easily computed. We just find the binary representation for 1234567_{10} and remove the first digit.

$$1234567_{10} = 100101101011010000111_2$$

Removing the first digit, we end up with: 00101101011010000111

Problem 3 (a) We can take some intuition from linear algebra for this. If we view each of the strings as a vector, concatenation as vector addition, and the number of characters in a string as the magnitude of the vector, then we can see that there are only

two linearly independent vectors in L . For simplicity, we take a and b to be these two independent vectors.

The question then becomes, how many different ways can we arrange these two vectors so that the resulting vector has magnitude 10?

Let's enumerate the first few lengths.

length	elements
0	ϵ
1	a, b
2	aa, ab, ba, bb
3	$aaa, aab, aba, abb, baa, bab, bba, bbb$

So it looks like the number of elements is 2^l , where l is the length of the string.

This means that there are $2^{10} = 1024$ strings of length 10 from this language.

(b) We can start by enumerating some lengths

length	elements
0	ϵ
1	a
2	aa, bb
3	aaa, abb, bba
4	$aaaa, aabb, abba, bbaa, bbbb$
5	$aaaaa, aaabb, aabba, abbaa, abbbb, bbaaa, bbabb, bbbba$

Here it looks like the number of elements is $fib(l)$, where l is the length of the string and $fib(l) = fib(l-1) + fib(l-2)$.

This means there are $fib(10) = 89$ strings of length 10 from this language.

Problem 4 We can use the division formula here: $s = qN + r$,

where $s \in \mathcal{S}$, $N = 314159265359$, $q, r \in \mathbb{Z}$, $q, r \geq 0$.

Since \mathcal{S} is infinite, we have more elements in \mathcal{S} than there are congruence classes modulo N . So, by the pigeon hole principle, at least two elements in \mathcal{S} must have the same remainder when divided by N .

Using the division formula:

$$s_1 = q_1N + r \text{ and } s_2 = q_2N + r$$

If we manipulate and substitute for r :

$$\begin{aligned} s_1 &= q_1N + (s_2 - q_2N) \\ s_1 - s_2 &= q_1N - q_2N \\ &= (q_1 - q_2)N \end{aligned}$$

Thus, since $q_1 - q_2$ is an integer, $s_1 - s_2$ is a multiple of N .

More succinctly, the difference of two elements of \mathcal{S} is a multiple of N .