MAT 25 Homework 5

Hardy Jones
999397426
Professor Bae
Fall 2013

1. 2.2.1

   (a) $\lim \frac{1}{6n^2+1} = 0$
   We need to show

$$\begin{aligned}\frac{1}{6n^2+1} &< \epsilon \\ \frac{1}{\epsilon} &< 6n^2+1 \\ \frac{1}{\epsilon} - 1 &< 6n^2 \\ \frac{1-\epsilon}{\epsilon} &< 6n^2 \\ \frac{1-\epsilon}{6\epsilon} &< n^2 \\ \sqrt{\frac{1-\epsilon}{6\epsilon}} &< n\end{aligned}$$

   Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \sqrt{\frac{1-\epsilon}{6\epsilon}}$.
   Let $n \geq N$. So, $n \geq N > \sqrt{\frac{1-\epsilon}{6\epsilon}} \implies \frac{1}{6n^2+1} < \epsilon$
   Thus $|a_n - 0| < \epsilon$.

   (b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$
   We need to show

$$\begin{aligned}\frac{3n+1}{2n+5} &< \epsilon \\ 3n+1 &< 2n\epsilon + 5\epsilon \\ 1 - 5\epsilon &< (2\epsilon - 3)n \\ \frac{1-5\epsilon}{2\epsilon - 3} &< n\end{aligned}$$

   Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \frac{1-5\epsilon}{2\epsilon-3}$.
   Let $n \geq N$. So, $n \geq N > \frac{1-5\epsilon}{2\epsilon-3} \implies \frac{3n+1}{2n+5} < \epsilon$.
   Thus $|a_n - \frac{3}{2}| < \epsilon$.

(c) $\lim \frac{2}{\sqrt{n+3}} = 0$

We need to show

$$\begin{aligned}\frac{2}{\sqrt{n+3}} &< \epsilon \\ \frac{2}{\epsilon} &< \sqrt{n+3} \\ \frac{4}{\epsilon^2} &< n+3 \\ \frac{4}{\epsilon^2} - 3 &< n\end{aligned}$$

Let $\epsilon > 0$. Choose $N \in \mathbb{N} | N > \frac{4}{\epsilon^2} - 3$.
Let $n \geq N$. So, $n \geq N > \frac{4}{\epsilon^2} - 3 \implies \frac{2}{\sqrt{n+3}} < \epsilon$.
Thus $|a_n - 0| < \epsilon$.

2. 2.2.5

(a) $a_n = \left\lfloor \frac{1}{n} \right\rfloor$

It is easy to see that after the first element in the sequence, all values are 0.
$\lim a_n = 0$

Proof. Let $\epsilon > 0$. Choose $N > 1$.
Let $n \geq N$. So, $n \geq N > 1 \implies \left\lfloor \frac{1}{n} \right\rfloor = 0 < \epsilon$.
Thus, $|a_n - 0| < \epsilon$. □

(b) $a_n = \left\lfloor \frac{10+n}{2n} \right\rfloor$

Again we see that after some elements all values are 0.
$\lim a_n = 0$

Proof. Let $\epsilon > 0$. Choose $N > 10$.
Let $n \geq N$. So, $n \geq N > 10 \implies \left\lfloor \frac{10+n}{2n} \right\rfloor = 0 < \epsilon$.
Thus, $|a_n - 0| < \epsilon$. □

3. 2.2.7

(a) A sequence (a_n) diverges to ∞ if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n| > \epsilon$
$\lim \sqrt{n} = \infty$

Proof. Let $\epsilon > 0$.
We want to show

$$\begin{aligned}\sqrt{n} &> \epsilon \\ n &> \epsilon^2\end{aligned}$$

Choose $N > \epsilon^2$.
Let $n \geq N$. So, $n \geq N > \epsilon^2 \implies \sqrt{n} > \epsilon$.
Thus, $|a_n| > \epsilon$. $\square$

(b) It states that this particular sequence does not diverge to ∞. The reason being, if you choose some $\epsilon > 0$, and any $N \in \mathbb{N}$, then $\exists n \geq N|$ either $n = 0$ or $n + 1 = 0$. So, $|a_n| \not\geq \epsilon, \forall n$.

4. 2.3.4 Using the Algebraic Limit Theorem, if $\lim a_n = l_1$ and $\lim a_n = l_2$, then

$$\begin{aligned} \lim(a_n - a_n) &= 0 \\ l_1 - l_2 &= 0 \\ l_1 &= l_2 \end{aligned}$$

5. 2.3.7

(a) Since (a_n) is bounded, $\exists M > 0$ such that $|a_n| \leq M, \forall n \in \mathbb{N}$.
Also, since $\lim b_n = 0$ we can choose a special epsilon, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies |b_n| < \frac{\epsilon}{M}$
So, $|a_n||b_n| < M\frac{\epsilon}{M} \implies |a_n||b_n| < \epsilon$.
Which is equivalent to $|a_n b_n - 0| < \epsilon$.
By Definition 2.2.3, this sequence $(a_n b_n)$ goes to 0.
We were not allowed to use the Algebraic Limit Theorem because (a_n) is not necessarily convergent.

(b) The only thing we can conclude when $\lim b_n = b$ is that $(a_n b_n)$ is bounded by b times the bounds of (a_n).

6. 2.3.11 Since (x_n) converges to some number X we know $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n \geq N \implies |x_n| < \epsilon$

$$
\begin{aligned}
|y_n - X| &= \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - X\right| \\
&= \left|\frac{a_1 + a_2 + \cdots + a_n}{n} - \frac{nX}{n}\right| \\
&= \left|\frac{a_1 + a_2 + \cdots + a_n - nX}{n}\right| \\
&= \left|\frac{(a_1 - X) + (a_2 - X) + \cdots + (a_n - X)}{n}\right| \\
&= \left|\frac{a_1 - X}{n} + \frac{a_2 - X}{n} + \cdots + \frac{a_n - X}{n}\right| \\
&= \left|\frac{a_1 - X}{n}\right| + \left|\frac{a_2 - X}{n}\right| + \cdots + \left|\frac{a_n - X}{n}\right| \\
&= \frac{|a_1 - X|}{n} + \frac{|a_2 - X|}{n} + \cdots + \frac{|a_n - X|}{n} \\
&< \frac{\epsilon}{n} + \frac{\epsilon}{n} + \cdots + \frac{\epsilon}{n} \\
&< \epsilon
\end{aligned}
$$

So, $\lim y_n = X$.

An example of a sequence (x_n) that doesn't converge, but (y_n) does is $x_n = (-1)^n$

We can see this by looking at some terms

$$
\begin{aligned}
x_1 &= -1 \\
x_2 &= 1 \\
x_3 &= -1 \\
x_4 &= 1 \\
x_5 &= -1 \\
x_6 &= 1 \\
x_7 &= -1 \\
&\vdots
\end{aligned}
$$

Which clearly doesn't converge

$$
\begin{aligned}
y_1 &= \frac{-1}{1} &&= -1 \\
y_2 &= \frac{-1+1}{2} &&= \quad 0 \\
y_3 &= \frac{-1+1-1}{3} &&= -\frac{1}{3} \\
y_4 &= \frac{-1+1-1+1}{4} &&= \quad 0 \\
y_5 &= \frac{-1+1-1+1-1}{5} &&= -\frac{1}{5} \\
y_6 &= \frac{-1+1-1+1-1+1}{6} &&= \quad 0 \\
y_7 &= \frac{-1+1-1+1-1+1-1}{7} &&= -\frac{1}{7} \\
&\vdots
\end{aligned}
$$

It's easy to see that this sequence converges to 0.

(2)
Suppose (a_n) is a sequence such that $a_n \in \mathbb{Z}$. Suppose (a_n) converges to a.
Prove that $a \in \mathbb{Z}$.

Proof.
In order for (a_n) to converge to a the follow must hold: $|a_n - a| < \epsilon$ for any $\epsilon > 0$
Now, the only way this statement is true, is if $a_n - a = 0 \implies a_n = a$.
Since all $a_n \in \mathbb{Z}$ it follows that $a \in \mathbb{Z}$ $\square$