MAT 167 Homework 3

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Find the dimension and basis for the four fundamental subspaces for

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1. C(A)

   dimension: 2

   basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

   C(U)

   dimension: 2

   basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$

2. N(A)

   dimension: 2

   basis:9 $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   N(U)

   dimension: 2

   basis: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3. C(A^T)

   dimension: 2

   basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

   C(U^T)

   dimension: 2

   basis: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

4. N(A^T)

   dimension: 1

   basis: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

   N(U^T)

   dimension: 1

   basis: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

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Why is there no matrix whose row space and nullspace both contain (1,1,1)?

The only vector in common between the row space and null space is the zero vector, since these two spaces are orthogonal by definition.

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Find the rank of A and write the matrix as uv^T

1.
$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 6 \end{bmatrix}$$

   rank: 1

$$A = uv^T = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 3 \end{bmatrix}$$

2.

$$A = \begin{bmatrix} 2 & -2 \\ 6 & -6 \end{bmatrix}$$

rank: 1

$$A = uv^T = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \end{bmatrix}$$

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Find a left-inverse and/or a right-inverse (when they exist) for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The rank of this matrix is 2, which is the same as the number of rows, and it is a rectangular matrix, so it only has a right inverse, though there many inverses.

We can construct the "best" right inverse by $A^T(AA^T)^{-1}$

$$\begin{aligned} A^{-1} = A^T(AA^T)^{-1} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \right)^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned}$$

In this case, we have only one inverse.

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The rank of this matrix is 2, which is the same as the number of columns, and it is a rectangular matrix, so it only has a left inverse which is unique.

We can use the fact that $M^T = A$ to use $(A^{-1})^T$ as the left inverse.

$$M^{-1} = (A^{-1})^T = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$T = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

The rank of this matrix is 2, which is the same as the number of rows and columns, so it has both a left and a right inverse which are the same.

This we can use the closed form to calculate $T^{-1} = \frac{1}{\det(T)} \text{adj}(T)$

$$T^{-1} = \frac{1}{a^2} \begin{bmatrix} a & -b \\ 0 & a \end{bmatrix}$$

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Construct a matrix with the required property, or explain why you can't.

1. Column space contains $\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right]$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2. Column space has basis $\left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$, nullspace has basis $\left[\begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}\right]$.

   Not possible since the rank would be 1, and the nullity 1, but it has 3 rows, and $3 - 1 \neq 1$

3. Dimension of nullspace = 1 + dimension of left nullspace.

$$\begin{bmatrix} 1 & 0 \end{bmatrix}$$

4. Left nullspace contains $\left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, row space contains $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$.

$$\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$$

5. Row space = column space, nullspace $\neq$ left nullspace.

   Not possible since row space = column space implies a square matrix, and for square matrices, nullspace = left nullspace.

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A combination of the rows of A has produced the zero row.

What combination is it?

The combination is spelled out in the right hand side.

1 row 3 - 2 row 2 + 1 row 1 = the zero row.

Which vectors are in the nullspace of A^T and which are in the nullspace of A?

The same vectors are in both spaces: scalar multiples of $\left[\begin{smallmatrix} 1 \\ -2 \\ 1 \end{smallmatrix}\right]$

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1

For the 3-node triangular graph in the figure following, write the 3 by 3 incidence matrix A.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Find a solution to $Ax = 0$ and describe all other vectors in the nullspace of A.
First we reduce to row echelon form.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So one solution to this is: $x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

And all vectors in the nullspace of A are of the form: $\begin{bmatrix} c \\ c \\ c \end{bmatrix}$ for some constant c.

Find a soluion to $A^T y = 0$ and describe all other vectors in the left nullspace of A.
First reduce the transpose to row echelon form.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So one solution is: $y = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

And all vectors in the left nullspace of A are of the form: $\begin{bmatrix} c \\ c \\ -c \end{bmatrix}$ for some constant c.

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For the same 3 by 3 matrix, show directly from the columns that every vector b in the column space will satisfy $b_1 + b_2 - b_3 = 0$.

$$b_1 + b_2 - b_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Derive the same thing from the three rows—the equations in the system $Ax = b$.

We can go look at the nullspace. for the derivation of this.

$$b_1 + b_2 - b_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}^T + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}^T - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T$$

What does that mean about potential differences around a loop?
This means the sum of potenial differences around the loop must be zero.

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Show directly from the rows that every vector f in the row space will satisfy $f_1+f_2+f_3 = 0$.
We can show this by adding the elements in each row.

$$\begin{bmatrix} 1-1+0 \\ 0+1-1 \\ 1+0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Derive the same thing from the three equations $A^T y = f$.
We can say for $y = \left[\begin{smallmatrix} y_1 \\ y_2 \\ y_3 \end{smallmatrix}\right]$.
So, $f_1 = y_1 + y_3$, $f_2 = -y_1 + y_2$, $f_3 = -y_2 - y_3$.
Thus $f_1 + f_2 + f_3 = y_1 + y_3 - y_1 + y_2 - y_2 - y_3 = 0$.
What does that mean when the f's are current into the nodes?
This means, there is no current flow into the loop from outside the loop.

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Compute the 3 by 3 matrix $A^T A$, and show that it is symmetric but singular—what vectors are in its nullspace?

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

We can see by inspection that $A^T A$ is symmetric.
We can see it is singular by taking the determinate
$det(A^T A) = 2(4-1) + 1(-2-1) - 1(1+2) = 6 - 3 - 3 = 0$
Since the determinate is 0, the $A^T A$ is singular.
Removing the last column of A (and the last row of A^T) leaves the 2 by 2 matrix in the upper left corner; show that it is *not* singular.

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

We can take the determinate of this and see it is not singular.
$det(\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]) = 1 \neq 0$

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Put the diagonal matrix C with entries c_1, c_2, c_3 in the middle and compute A^TCA.

$$A^TCA = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2c_1 & -1c_1 & -1c_1 \\ -1c_2 & 2c_2 & -1c_2 \\ -1c_3 & -1c_3 & 2c_3 \end{bmatrix}$$

Show again that the 2 by 2 matrix in the upper left corner is invertible.

$$\begin{vmatrix} 2c_1 & -1c_1 \\ -1c_2 & 2c_2 \end{vmatrix} = 4c_1c_2 + c_1c_2 = 5c_1c_2$$

So this matrix is invertible so long as $c_1 \neq 0$ and $c_2 \neq 0$.