MAT 125A HW 5

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Spring 2015

6.2 1 Given: $f_n(x) = \frac{nx}{1+nx^2}$

(a)

$$\begin{aligned}
\lim_{n\to\infty} \frac{nx}{1+nx^2} &= \lim_{n\to\infty} \frac{\frac{1}{x}\left(nx^2\right)}{1+nx^2} \\
&= \lim_{n\to\infty} \frac{\frac{1}{x}\left(1-1+nx^2\right)}{1+nx^2} \\
&= \lim_{n\to\infty} \frac{\frac{1}{x}\left(1+nx^2\right)-\frac{1}{x}}{1+nx^2} \\
&= \lim_{n\to\infty} \left[\frac{\frac{1}{x}\left(1+nx^2\right)}{1+nx^2} - \frac{\frac{1}{x}}{1+nx^2}\right] \\
&= \lim_{n\to\infty} \left[\frac{1}{x} - \frac{\frac{1}{x}}{1+nx^2}\right] \\
&= \lim_{n\to\infty} \frac{1}{x} - \lim_{n\to\infty} \frac{\frac{1}{x}}{1+nx^2} \\
&= \lim_{n\to\infty} \frac{1}{x} - 0 \\
&= \lim_{n\to\infty} \frac{1}{x} \\
&= \frac{1}{x}
\end{aligned}$$

So the point-wise limit of (f_n) is $\frac{1}{x}$

(b) On $(0, \infty)$ we check:

$$\begin{aligned}
|f_n(x) - f(x)| &= \left|\frac{nx}{1+nx^2} - \frac{1}{x}\right| \\
&= \left|\frac{nx^2 - (1+nx^2)}{x+nx^3}\right| \\
&= \left|\frac{-1}{x+nx^3}\right| \\
&= \frac{1}{x+nx^3}
\end{aligned}$$

And we need to find some $N \in \mathbb{N}$ for any ϵ, such that, if $n \geq N$, then $\frac{1}{x+nx^3} < \epsilon$
We can rewrite a bit:

$$\begin{aligned}
\frac{1}{x+nx^3} &< \epsilon \\
\frac{1}{\epsilon} &< x + nx^3 \\
\frac{1}{\epsilon} - x &< nx^3 \\
\frac{1}{\epsilon x^3} - \frac{1}{x^2} &< n \\
\frac{1}{\epsilon x^3} - \frac{1}{x^2} &< N \leq n
\end{aligned}$$

But when $x \to 0$, $\frac{1}{\epsilon x^3} - \frac{1}{x^2} \to \infty$.
So there's no N we can choose for all ϵ on $(0, \infty)$.
Thus, the convergence is not uniform on $(0, \infty)$.

(c) On $(0, 1)$ we run into the same issues as previous.
So the convergence is not uniform on $(0, 1)$.

(d) On $(1, \infty)$ we check:
The computation is the same up to $\frac{1}{\epsilon x^3} - \frac{1}{x^2} < N \leq n$.
For all x in $(1, \infty)$, $\frac{1}{\epsilon x^3} - \frac{1}{x^2} < \frac{1}{\epsilon} - 1$
So we can choose $N > \frac{1}{\epsilon} - 1$.
Then we have $|f_n(x) - f(x)| < \epsilon$.
So the convergence is uniform on $(1, \infty)$.

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$$\begin{aligned}
\lim_{n\to\infty} \frac{nx + \sin(nx)}{2n} &= \lim_{n\to\infty} \left(\frac{nx}{2n} + \frac{\sin(nx)}{2n} \right) \\
&= \lim_{n\to\infty} \left(\frac{x}{2} + \frac{\sin(nx)}{2n} \right) \\
&= \lim_{n\to\infty} \frac{x}{2} + \lim_{n\to\infty} \frac{\sin(nx)}{2n} \\
&= \frac{x}{2} + 0 \\
&= \frac{x}{2}
\end{aligned}$$

So the pointwise limit of (g_n) is $\frac{x}{2}$

- On $[-10, 10]$ we check:

$$
\begin{aligned}
|g_n(x) - g(x)| &= \left|\frac{x}{2} - \frac{nx + \sin(nx)}{2n}\right| \\
&= \left|\frac{nx - nx + \sin(nx)}{2n}\right| \\
&= \left|\frac{\sin(nx)}{2n}\right|
\end{aligned}
$$

Since $-1 \leq \sin(nx) \leq 1$, we have:

$$
\begin{aligned}
|g_n(x) - g(x)| &= \left|\frac{\sin(nx)}{2n}\right| \\
&\leq \left|\frac{1}{2n}\right|
\end{aligned}
$$

And we want $\frac{1}{2n} < \epsilon \implies \frac{1}{2\epsilon} < n$.
So choose $N > \frac{1}{2\epsilon}$, then we have $|g_n(x) - g(x)| < \epsilon$.
So the convergence is uniform on $[-10, 10]$.

- Since x does not affect our decision of N, the same reasoning can be used to choose $N > \frac{1}{2\epsilon}$ on all of $\mathbb{R}$.
  So the convergence is uniform on $\mathbb{R}$.

3 (a) We have to consider three cases:

- $0 \leq x < 1$
- $x = 1$

$$
\lim_{n\to\infty} \frac{1}{1 + 1^n} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}
$$

- $1 < x$

$$
\lim_{n\to\infty} \frac{x}{1 + x^n} = 0
$$

(b) Since there is a jump discontinuity on $h_n(x)$ when $x = 1$, and when $x > 1$, $h_n(x)$ is not continuous. And from Theorem 6.2.6, h_n is not continuous.
Since h_n is not continuous, it cannot have convergence on $[0, \infty)$

(c) Choose $(1, \infty)$.
Then we check:

$$
\begin{aligned}
|h_n(x) - h(x)| &= \left| \frac{x}{1+x^n} - 0 \right| \\
&= \left| \frac{x}{1+x^n} \right| \\
&= \frac{x}{1+x^n} \\
&< 1
\end{aligned}
$$

Then we can choose $N = 1$.
And for any ϵ

4 We take some derivatives to find maxima and minima.

$$
\begin{aligned}
f_n'(x) &= \frac{(1+nx^2)\cdot 1 - x(2nx)}{(1+nx^2)^2} \\
&= \frac{1+nx^2-2nx^2}{(1+nx^2)^2} \\
&= \frac{1-nx^2}{(1+nx^2)^2}
\end{aligned}
$$

$$
\begin{aligned}
f_n''(x) &= \frac{(1+2nx^2+n^2x^4)(-2nx) - (4nx+4n^2x^3)(1-nx^2)}{(1+nx^2)^4} \\
&= \frac{-6nx - 4n^2x^3 + 2n^3x^5}{(1+nx^2)^4}
\end{aligned}
$$

Setting the first derivative to zero, we can find possible maxima and minima.

$$
\begin{aligned}
\frac{1-nx^2}{(1+nx^2)^2} &= 0 \\
1 - nx^2 &= 0 \\
1 &= nx^2 \\
\frac{1}{n} &= x^2 \\
\pm\frac{1}{\sqrt{n}} &= x
\end{aligned}
$$

Now, we can plus into the second derivative.

$$
\begin{aligned}
f_n''\left(\frac{1}{\sqrt{n}}\right) &= \frac{-6n\left(\frac{1}{\sqrt{n}}\right) - 4n^2\left(\frac{1}{\sqrt{n}}\right)^3 + 2n^3\left(\frac{1}{\sqrt{n}}\right)^5}{\left(1+n\left(\frac{1}{\sqrt{n}}\right)^2\right)^4} \\
&= \frac{-6\sqrt{n} - 4n^{2-\frac{3}{2}} + 2n^{3-\frac{5}{2}}}{(1+1)^4} \\
&= \frac{-6\sqrt{n} - 4\sqrt{n} + 2\sqrt{n}}{16} \\
&= \frac{-8\sqrt{n}}{16} \\
&= \frac{-\sqrt{n}}{2} \\
&< 0
\end{aligned}
$$

$$
\begin{aligned}
f_n''\left(\frac{-1}{\sqrt{n}}\right) &= \frac{-6n\left(\frac{-1}{\sqrt{n}}\right) - 4n^2\left(\frac{-1}{\sqrt{n}}\right)^3 + 2n^3\left(\frac{-1}{\sqrt{n}}\right)^5}{\left(1+n\left(\frac{-1}{\sqrt{n}}\right)^2\right)^4} \\
&= \frac{6\sqrt{n} + 4n^{2-\frac{3}{2}} - 2n^{3-\frac{5}{2}}}{(1+1)^4} \\
&= \frac{6\sqrt{n} + 4\sqrt{n} - 2\sqrt{n}}{16} \\
&= \frac{8\sqrt{n}}{16} \\
&= \frac{\sqrt{n}}{2} \\
&> 0
\end{aligned}
$$

So the maximum occurs at $\frac{-1}{\sqrt{n}}$ and the minimum occurs at $\frac{1}{\sqrt{n}}$