MAT 150A Homework 2

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1. We need to show that $a(bc) = (ab)c$

   Proof.

$$\begin{aligned} a(bc) &= a(b) \\ &= ab \\ &= a \end{aligned} \tag{left}$$

$$\begin{aligned} (ab)c &= (ab) \\ &= ab \\ &= a \end{aligned} \tag{right}$$

   Since left = right, we have shown that the operation is associative. □

   This law is an identity for sets with exactly one element.

   Proof. Assume that a set with more than one element had this law.
   Choose $a \in S$ with e as the identity.
   Then we want that $ae = a = ea$.
   But we see that $ea = e \neq a$.
   It can be shown that if $e = a$ then the identity law holds. As $ee = e = ee$. □

2. We need to show

   - $\star$ is closed

     Proof. Choose $a, b \in G^O$. $a \star b = ba$ and we know that $ba \in G$, so since the set is the same between G and G^O, we also know $ba \in G^O$.
     Thus, $\star$ is closed. □

   - $\forall a, b, c \in G^O, a \star (b \star c) = (a \star b) \star c$

Proof. Choose $a, b, c \in G^O$.

$$\begin{aligned} a \star (b \star c) &= a \star (cb) \\ &= (cb)a \qquad \text{(left)} \end{aligned}$$

$$\begin{aligned} (a \star b) \star c &= c(a \star b) \\ &= c(ba) \qquad \text{(right)} \end{aligned}$$

Since we know the underlying group G, we know that it is associative. So left = right since G is associative.
Thus, we have shown that the associativity law holds. □

- $\exists e \in G^O$ s.t. $\forall a \in G^O, a \star e = a = e \star a$

  Proof. Choose $a \in G^O$.
  $a \star e = ea = a$ and $e \star a = ae = a$.
  Thus, we have shown that the identity law holds. □

- $\forall a \in G^O, \exists a^{-1} \in G^O$ s.t. $a \star a^{-1} = e = a^{-1} \star a$

  Proof. Choose $a \in G^O$.
  $a \star a^{-1} = a^{-1}a = e$ and $a^{-1} \star a = aa^{-1} = e$.
  Thus, we have shown that the inverse law holds. □

Since we have shown all four properties of a group, we conclude G^O is a group.

3. Let's name our matrix.

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^4 = A^3A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A^5 = A^4A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^6 = A^5A = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since we have generated the identity, we have generated all possible elements of this cyclic group.