MAT 125A HW 3

Hardy Jones
999397426
Professor Slivken
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4.4 4 *Proof.* Assume f is continuous on $[a, b]$ and for all $x \in [a, b], f(x) > 0$.

So we have $[a, b]$ is compact, then we know that f is compact on $[a, b]$, since f is continuous on $[a, b]$.

By the extreme value theorem, f has a minimum and maximum value. So there is some $x_0 \in [a, b]$ such that for all $x \in [a, b], f(x_0) \leq f(x)$.

Now, since we know for all $x \in [a, b], f(x) > 0$,

we have $f(x_0) \leq f(x) \implies \frac{1}{f(x)} \leq \frac{1}{f(x_0)}$.

Then for all $x \in [a, b]$, $\frac{1}{f(x)}$ is bounded by $\frac{1}{f(x_0)}$. □

6 (a) Let $f(x) = \frac{1}{x}$. This function is continuous on $(0, 1)$.
If we take the sequence $(x_n) = \frac{1}{n}$, then (x_n) is Cauchy.
But $f(x_n) = \frac{1}{\frac{1}{n}} = n$, and this sequence is unbounded. So it is not Cauchy.

(b)

(c)

(d) Let $f(x) = -\left(x - \frac{1}{2}\right)^2$.
Then f has a maximum at the root of the polynomial, but no minimum value.

8 (a) We want to show if there exists some $b > 0$ such that f is uniformly continuous on $[b, \infty)$, then f is uniformly continuous on $[0, \infty)$.

Proof. Given $f : [0, \infty) \to \mathbb{R}$ is continuous at every point in $[0, \infty)$.
Assume that f is not uniformly continuous on $[0, \infty)$, then we want to show that for any $b > 0$ f is not uniformly continuous on $[b, \infty)$.
We can rephrase the first part as:
There exists some $\epsilon_0 > 0$ such that for all $\delta_0 > 0$, $|x - y| < \delta_0$ and $|f(x) - f(y)| \geq \epsilon_0$.
And the second part as for any $b > 0$, there exists some $\epsilon_1 > 0$ such that for all $\delta_1 > 0$, $|x - y| < \delta_1$ and $|f(x) - f(y)| \geq \epsilon_1$.
If we choose $\epsilon_0 = \epsilon_1$, then for any $\delta_1 > 0$ there exists some $0 < \delta_0 < \delta_1$, where $|x - y| < \delta_0$ and $|f(x) - f(y)| \geq \epsilon_0$.
Then we have that f is not uniformly continuous , as we desired.
Thus, by contraposition:
Given $f : [0, \infty) \to \mathbb{R}$ is continuous at every point in $[0, \infty)$, if there exists some $b > 0$ such that f is uniformly continuous on $[b, \infty)$, then f is uniformly continuous on $[0, \infty)$. □

(b) *Proof.* From part (a), we need an interval $[b, \infty)$ with f continuous at every point.
If we choose $b = 1$, then for any $\epsilon > 0$ we can choose $\delta = \epsilon$.
We also know that $\sqrt{x} + \sqrt{y} \geq 1 + 1 = 2$
Now,

$$\begin{aligned}
|x - y| &< \epsilon \\
|\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| &< \epsilon \\
|\sqrt{x} - \sqrt{y}| &< \frac{\epsilon}{|\sqrt{x} + \sqrt{y}|} \\
|\sqrt{x} - \sqrt{y}| &\leq \frac{\epsilon}{2} \\
|\sqrt{x} - \sqrt{y}| &< \epsilon
\end{aligned}$$

So f is uniformly continuous on $[1, \infty)$.
From part (a), we conclude that f is uniformly continuous on $[0, \infty)$.
Thus, $f(x) = \sqrt{x}$ uniformly continuous on $[0, \infty)$. □

9 (a) *Proof.* Assume $f : A \to \mathbb{R}$ is Lipschitz.
Then there exists some $M > 0$ such that for all $x, y \in A$,

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

.
For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$.
Then, when $|x - y| < \delta$,

$$\begin{aligned}
\left| \frac{f(x) - f(y)}{x - y} \right| &\leq M \\
\frac{|f(x) - f(y)|}{|x - y|} &\leq M \\
|f(x) - f(y)| &\leq |x - y| \, M \\
|f(x) - f(y)| &< \frac{\epsilon}{M} M \\
|f(x) - f(y)| &< \epsilon
\end{aligned}$$

So, f is uniformly continuous .
Thus, if $f : A \to \mathbb{R}$ is Lipschitz, then f is uniformly continuous . □

(b) No, not all uniformly continuous functions are Lipschitz.
Let $f(x) = \sqrt{x}$ on $[0, \infty)$, then we know this is uniformly continuous by a problem above.
But if $y = 0$ then

$$\left|\frac{\sqrt{x}-\sqrt{0}}{x-0}\right| = \left|\frac{\sqrt{x}}{x}\right| = \frac{\sqrt{x}}{x}$$

and this grows unbounded when $x \to 0$.
So f is not Lipschitz.

13 (a) *Proof.* Assume $f : A \to \mathbb{R}$ is uniformly continuous , and $(x_n) \subseteq A$ is Cauchy. Then we have, for all $\epsilon > 0$ there exists some $\delta > 0$ such that, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$.
And also, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that, for all $m, n \geq N, |x_m - x_n| < \epsilon$.
So, given some $\epsilon > 0$, choose $\delta = \epsilon$.
Since (x_n) is Cauchy, there exists $N_0 \in \mathbb{N}$ such that, for all $m, n \geq N, |x_m - x_n| < \delta$.
And since we know f is uniformly continuous , we have $|f(x_m) - f(x_n)| < \epsilon$.
So $f(x_n)$ is also Cauchy. □

(b) *Proof.* We need to prove both directions.

- $(\Longrightarrow)$
- $(\Longleftarrow)$
  Since g is continuous and $[a, b]$ is compact, g is uniformly continuous on $[a, b]$.
  And Since $(a, b) \subseteq [a, b]$, g is uniformly continuous on (a, b).

□

4.5 1 *Proof.* The closed interval $[a, b] = E$ is connected.
Given Theorem 4.5.2, $f(E)$ is also connected.
If we have some L between $f(a)$ and $f(b)$, then $L \in f(E)$. And if $L \in f(E)$ there must be some $c \in E$ such that $f(c) = L$.
This is the Intermediate Value Theorem. □

2 (a) False.
Let $f(x) = \frac{1}{x}$.
Then on the bounded open interval $(0, 1)$, the range of f is $(0, \infty)$, which is unbounded.

(b) False.
Let $f(x) = x^2$.
Then on the bounded open interval $(-1, 1)$, the range of f is $[0, 1)$, which is a not an open set.

(c) True.
Let f be continuous on a bounded closed interval A. Then this interval is compact, and so $f(A)$ is also compact.
And since $f(A)$ is compact, $f(A)$ is an interval.

7 *Proof.* Assume f is continuous on $[0, 1]$. We can construct a function $g(x) = x$ which is also continuous with the same domain and range.

g is continuous if we take $\epsilon = \delta$ for any $\epsilon > 0$, then $|x - y| < \delta = \epsilon$.

Then we can construct another continuous function, $h(x) = f(x) - g(x)$.

Now $h(x)$ has domain $[0, 1]$ and range $[-1, 1]$.

Where at $x = 0$, $h(0) = f(0) - g(0) = f(0)$ so $h(0) \in [0, 1]$. And at $x = 1$, $h(1) = f(1) - g(1) = f(1) - 1$ so $h(1) \in [-1, 0]$.

So by the Intermediate Value Theorem, there must exist some $c \in [0, 1]$ such that $h(c) = 0 = f(c) - g(c) \implies f(c) = g(c) \implies f(c) = c$. $\square$