MAT 150A Homework 8

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1. Choose a subgroup $H \leq G$.

   Then $\exists \varepsilon > 0$ s.t.:

   (a) $\forall t_a \in H, \|a\| \geq \varepsilon$ as $t_a \in G$

   (b) $\forall \rho_\theta \in H, |\theta| \geq \varepsilon$ as $\rho_\theta \in G$

   Since our choice of H was arbitrary, any subgroup $H \leq G$ is discrete.

2. If a discrete group G is rotations about the origin, let $\rho_\theta \in G$ be the smallest rotation.

   Choose $\rho_\psi \in G$, then $\exists \varphi \in \mathbb{R}, n \in \mathbb{N}$ such that $\psi = n\theta + \varphi$ and $0 \leq \varphi < \theta$.

   Then

$$
\begin{aligned}
\rho_\psi &= \rho_{n\theta+\varphi} \\
&= \rho_{n\theta}\rho_\varphi \\
&= \rho_\theta^n \rho_\varphi \\
\rho_\theta^{-n}\rho_\psi &= (\rho_\theta^{-n}\rho_\theta^n)\rho_\varphi \\
\rho_\theta^{-n}\rho_\psi &= \rho_\varphi
\end{aligned}
$$

   Since the LHS is in G, the RHS is also in G. But, as we chose θ as the smallest rotation angle, this implies $\varphi = 0$.

$$
\begin{aligned}
\rho_\theta^{-n}\rho_\psi &= \rho_0 \\
(\rho_\theta^n\rho_\theta^{-n})\rho_\psi &= \rho_\theta^n\rho_0 \\
\rho_\psi &= \rho_\theta^n \\
\rho_\psi &= \rho_{n\theta}
\end{aligned}
$$

   So we have that $\psi = n\theta$ for some $n \in \mathbb{N}$. Since our choice of ρ_ψ was arbitrary, this shows that every rotation in G is generated by the smallest rotation. In other words, a discrete group G of rotations about the origin is a cyclic group generated by the smallest rotation $\rho_\theta \in G$.

3. Choose two elements of G as $a = t_a\rho_\theta, b = t_b\rho_\psi$.

$$\begin{aligned}
aba^{-1}b^{-1} &= (t_a\rho_\theta)(t_b\rho_\psi)(t_a\rho_\theta)^{-1}(t_b\rho_\psi)^{-1} \\
&= (t_a\rho_\theta)(t_b\rho_\psi)(\rho_\theta^{-1}t_a^{-1})(\rho_\psi^{-1}t_b^{-1}) \\
&= (t_a\rho_\theta)(t_b\rho_\psi)(\rho_{-\theta}t_{-a})(\rho_{-\psi}t_{-b}) \\
&= (t_a\rho_\theta)t_b(\rho_\psi\rho_{-\theta})t_{-a}(\rho_{-\psi}t_{-b}) \\
&= t_a(\rho_\theta t_b)\rho_{\psi-\theta}t_{-a}(\rho_{-\psi}t_{-b}) \\
&= t_a(t_{b'}\rho_\theta)\rho_{\psi-\theta}t_{-a}(\rho_{-\psi}t_{-b}) \\
&= t_{a+b'}(\rho_\theta\rho_{\psi-\theta})t_{-a}(\rho_{-\psi}t_{-b}) \\
&= t_{a+b'}(\rho_{\theta+\psi-\theta}t_{-a})(\rho_{-\psi}t_{-b}) \\
&= t_{a+b'}(\rho_\psi t_{-a})(\rho_{-\psi}t_{-b}) \\
&= (t_{a+b'}t_{-a'})(\rho_\psi\rho_{-\psi})t_{-b} \\
&= t_{a+b'-a'}(\rho_\psi\rho_{-\psi})t_{-b} \\
&= t_{a+b'-a'}\rho_{\psi-\psi}t_{-b} \\
&= t_{a+b'-a'}t_{-b} \\
&= t_{a+b'-a'-b}
\end{aligned}$$

So G contains a translation.

4. Since $\mathcal{O}_2$ does not contain translations, we only need concern ourselves with showing:

$$\forall \text{ discrete subgroups } G \le \mathcal{O}_2, \exists\varepsilon > 0 \text{ s.t. } \forall\rho_\theta \in G, |\theta| \ge \varepsilon$$

Choose a discrete subgroup $G \le \mathcal{O}_2$.

Suppose G is infinite. That is, there are an infinite number of angles.

Then, for any $\varepsilon > 0$ we can choose an $n \in \mathbb{N}$ and construct n divisions from 0 to 2π each of size $\frac{2\pi\varepsilon}{n}$.

Since G is infinite, there must be at least two non-zero angles θ, ψ such that $|\theta - \psi| < \frac{2\pi}{n} < \varepsilon \le |\theta + \psi``|$.

This means that $\rho_\theta\rho_\psi \in G$.

Which also means that $\rho_\theta\rho_\psi^{-1} = \rho_\theta\rho_{-\psi} = \rho_{\theta-\psi} \in G$.

But $\theta - \psi < \varepsilon$, which would make this not a discrete subgroup by construction.

So our assumption was incorrect and G must not be infinite.

Thus any discrete subgroup $G \le \mathcal{O}_2$ is finite.

5. G acts transitively on G-set S if, $\forall s_1, s_2 \in S, s_2$ is the orbit of s_1 in S.

That is, $O_{s_1} = \{s_2 \in S | s_2 = gs_1 \text{ for some } g \in G\}$

6. We want to show that G acts faithfully on $S \iff \forall g, g' \in G, s \in S, (g \neq g' \implies gs \neq g's)$

   - $(\Longrightarrow)$
     Choose $e, g \in G$ where $e \neq g$.

     $$\begin{aligned} es &\neq gs \\ s &\neq gs \end{aligned}$$

     Since G acts faithfully on S.
     So, we have shown for any $g, g' \in G, s \in S, (g \neq g' \implies gs \neq g's)$
   - $(\Longleftarrow)$
     Choose $e, g \in G$ where $e \neq g$ and any $s \in S$.
     We know that $gs \neq es$ so $gs \neq s$.
     This is the contrapositive of G acting faithfully on G-set S.
     So we have that G acts faithfully on G-set S.

   From these two, we have shown both directions.
   Thus, G acts faithfully on $S \iff \forall g, g' \in G, s \in S, (g \neq g' \implies gs \neq g's)$