MAT 25 Homework 2

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Fall 2013

1. 1.2.3 (c)
   Show that $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

   Proof. We need to show two things: $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$

   (a) By definition of the compliment,

   $$(A \cup B)^c = \{x : x \notin (A \cup B)\}$$

   This means, given some x in the set: $x \notin A$ and $x \notin B$.
   Or using the definition of complement: $x \in A^c$ and $x \in B^c$.
   From the definition of intersection we get: $x \in A^c \cap B^c$
   Since x was an arbitrary choice, this result holds for all x in the set.
   Or more succinctly, $\forall x \in (A \cup B)^c, x \in A^c \cap B^c$.
   By the definition of inclusion, we can say:

   $$(A \cup B)^c \subseteq A^c \cap B^c$$

   (b)
   $$A^c \cap B^c = \{x : x \in A^c \text{and} x \in B^c\}$$

   This means, given some x in the set, we can say: $x \notin A$ and $x \notin B$.
   If x is not in either A or B, then it cannot be in the union of those two sets. That is: $x \notin A \cup B$
   Using the definition of complement we can say: $x \in (A \cup B)^c$.
   Again, since x was arbitrary, the result holds for all elements of the set.
   Or more succinctly, $\forall x \in A^c \cap B^c, x \in (A \cup B)^c$.
   By the definition of inclusion, we can say:

   $$A^c \cap B^c \subseteq (A \cup B)^c$$

   From a and b we have both sides of inclusion, so by the definition of set equality:

   $$(A \cup B)^c = A^c \cap B^c$$

   □

2. 1.2.7

Given $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$ let $f^{-1}(B) = \{x \in D : f(x) \in B\}$

(a) Let $f(x) = x^2$. if $A = [0, 4]$ and $B = [-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$.
Does $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$?
Does $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$?
$f^{-1}(A) = \{x \in [-2, 2]\}$ $f^{-1}(B) = \{x \in [-1, 1]\}$

$$\begin{aligned}
f^{-1}(A \cap B) &= f^{-1}([0,4] \cap [-1,1]) \\
&= f^{-1}([0,4] \cap [-1,1]) \\
&= f^{-1}([0,1]) \\
&= \{x \in [-1,1]\} \\
&= \{x \in [-2,2]\} \cap \{x \in [-1,1]\} \\
&= f^{-1}(A) \cap f^{-1}(B)
\end{aligned}$$

So this part is true.

$$\begin{aligned}
f^{-1}(A \cup B) &= f^{-1}([0,4] \cup [-1,1]) \\
&= f^{-1}([0,4] \cup [-1,1]) \\
&= f^{-1}([-1,4]) \\
&= \{x \in [-2,2]\} \\
&= \{x \in [-2,2]\} \cup \{x \in [-1,1]\} \\
&= f^{-1}(A) \cup f^{-1}(B)
\end{aligned}$$

So this part is true.