MAT 150A Homework 3

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1. To show that f is an automorphism, we need to show that f is an isomorphism from $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$. To show that f is an isomorphism, we need to show that f is a homomorphism and bijective. To show that f is a homomorphism, we need to show that f is closed, and that it preserves the group operation.

$$GL_n(\mathbb{R}) := \{A_n | A \text{ is an } n \times n \text{ matrix}, |A| \neq 0\}$$

Proof. We begin by showing that f is a homomorphism.

- We first show closure.
  Choose any $A \in GL_n(\mathbb{R})$. $f(A) = (A^T)^{-1}$.
  The size of $f(A)$ has not changed. We know $|A^T| = |A|$ and $|A^{-1}| = |A|^{-1}$, so $|f(A)| = |(A^T)^{-1}| = |(A^T)|^{-1} = |A|^{-1} \neq 0$.
  So, f maps $GL_n(\mathbb{R}) \mapsto GL_n(\mathbb{R})$
  Thus, f is closed.
- Now we show that f preserves the group operation.
  Choose any $A, B \in GL_n(\mathbb{R})$.

  $$f(AB) = ((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = f(A)f(B)$$

  So, f preserves the group operation.
  Thus, $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is a homomorphism.

Now we need to show that f is bijective.

- We first show that $\ker f = \{I_n\}$
  So we want to find all $A \in GL_n(\mathbb{R})$ such that $f(A) = (A^T)^{-1} = I_n$.
  But we know that for any group the identity is its own inverse, so $A^T = I_n$.
  We also know that $I_n^T = I_n$, so $A = I_n$.
  And since we know that the identity is unique, we have that $\ker f = \{I_n\}$.
- Now we show that $\operatorname{im} f = GL_n(\mathbb{R})$
  It suffices to show that f has an inverse. Namely, $f^{-1}(A) = A^T$.

So, we have shown that $f : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is an isomorphism. And since f's domain is its co-domain, f is an automorphism. $\square$

2. We want to show that $\forall \varphi : G- > G'$ that are group homomorphisms, $\ker \varphi \leq G$ and $\operatorname{im} \varphi \leq G$.

   For both of these possible subgroups, it suffices to show two things:

   (a) The possible subgroup is non-empty.

   (b) For all a, b in the possible subgroup, ab^{-1} is also in the subgroup.

   - $\ker \varphi \leq G$

     Proof. We need to show the two conditions above.

     (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\ker \varphi$ is non-empty (as $e_G \in \ker \varphi$).

     (b) Choose $a, b \in \ker \varphi$.
     So we have, $\varphi(a) = e_{G'}$ and $\varphi(b) = e_{G'}$, and since φ is a homomorphism.

     $$\begin{aligned} \varphi(ab^{-1}) &= \varphi(a)\varphi(b^{-1}) \\ &= \varphi(a)\varphi(b)^{-1} \\ &= e_{G'}\varphi(b)^{-1} \\ &= e_{G'}e_{G'}^{-1} \\ &= e_{G'}e_{G'} \\ &= e_{G'} \end{aligned}$$

     Thus, $\ker \varphi \leq G$. $\square$

   - $\operatorname{im} \varphi \leq G$

     Proof. We need to show the two conditions above.

     (a) Since φ is a homomorphism, $\varphi(e_G) = e_{G'}$, so $\operatorname{im} \varphi$ is non-empty (as $e_G \in \operatorname{im} \varphi$).

     (b) Choose $a, b \in \operatorname{im} \varphi$.
     This means $\exists a', b' \in G$ s.t. $\varphi(a') = a, \varphi(b') = b$.
     Since φ is a homomorphism and $a'b'^{-1} \in G$.

     $$\begin{aligned} \varphi(a'b'^{-1}) &= \varphi(a')\varphi(b'^{-1}) \\ &= \varphi(a')\varphi(b')^{-1} \\ &= a\varphi(b)^{-1} \\ &= ab^{-1} \in \operatorname{im} \varphi \end{aligned}$$

     Thus, $\operatorname{im} \varphi \leq G$. $\square$

3. The subgroups of S_3 are:

$$\begin{aligned}
&\{id\} \\
&\{id, (1,2)\}, \{id, (1,3)\}, \{id, (2,3)\} \\
&\{id, (1,2,3)\}, \{id, (1,3,2)\} \\
&\{id, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\}
\end{aligned}$$

   The trivial subgroup and the group itself are normal.

4. Want to show $\varphi(x) = \varphi(y) \iff xy^{-1} \in \ker \varphi$

   Proof.
   - $(\Rightarrow)$
     Since φ is a homomorphism and $\varphi(x) = \varphi(y)$.

     $$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(x)^{-1} = \varphi(x)\varphi(x^{-1}) = \varphi(xx^{-1}) = \varphi(e_G) = e_{G'}$$

     By the definition of the kernel, $xy^{-1} \in \ker \varphi$.
   - $(\Leftarrow)$
     Since φ is a homomorphism and $xy^{-1} \in \ker \varphi$.

     $$\begin{aligned}
     \varphi(xy^{-1}) &= e_{G'} \\
     \varphi(x)\varphi(y^{-1}) &= e_{G'} \\
     \varphi(x)\varphi(y)^{-1} &= e_{G'} \\
     \varphi(x)\varphi(y)^{-1}\varphi(y) &= e_{G'}\varphi(y) \\
     \varphi(x)e_{G'} &= e_{G'}\varphi(y) \\
     \varphi(x) &= e_{G'}\varphi(y) \\
     \varphi(x) &= \varphi(y)
     \end{aligned}$$

     So, $\varphi(x) = \varphi(y)$

   Thus, we have shown both directions and $\varphi(x) = \varphi(y) \iff xy^{-1} \in \ker \varphi$. $\square$

5. (a) We need to show that gHg^{-1} is closed, has an identity and has inverses.

   Proof.
   - Choose $x = ghg^{-1}, y = gh'g^{-1} \in ghg^{-1}$.

     $$xy = (ghg^{-1})(gh'g^{-1}) = gh(g^{-1}g)h'g^{-1} = gheh'g^{-1} = ghh'g^{-1}$$

     Now, since $hh' \in H$, we have that $ghh'g^{-1} = xy \in H$.
     Thus, gHg^{-1} is closed.

- Choose $e \in G$ as the identity.

$$ghg^{-1}e = ghg^{-1} = eghg^{-1}$$

  Thus, gHg^{-1} has an identity.
- $\forall x = ghg^{-1} ing Hg^{-1}$, choose $x^{-1} = gh^{-1}g^{-1}$

$$(ghg^{-1})(gh^{-1}g^{-1}) = gh(g^{-1}g)h^{-1}g^{-1} = gheh^{-1}g^{-1} = g(hh^{-1})g^{-1} = geg^{-1} = gg^{-1} = e$$

  Thus, gHg^{-1} has inverses.

From the three results shown, $gHg^{-1} \leq G$ is a subgroup. □

(b) We want to prove:
$H \triangleleft G$ is normal $\iff \forall g \in G, gHg^{-1} = H$

Proof. We need to show both sides of this equivalence.

(⇒) Choose $h \in H$, we know that $\forall g \in G, gH = Hg$, since $H \triangleleft G$ is normal.
So, choose $g \in G$, we have:

$$\begin{aligned} gh &= hg \\ ghg^{-1} &= hgg^{-1} \\ ghg^{-1} &= he \\ ghg^{-1} &= h \end{aligned}$$

Since our choices for g, h were arbitrary, we have that this result holds for all $g \in G, h \in H$.
Thus we have that $gHg^{-1} = H$.

(⇐) Choose $g \in G$, we know $\forall g \in G, gHg^{-1} = H$.
So, choose $h \in H$, we have:

$$\begin{aligned} ghg^{-1} &= h \\ ghg^{-1}g &= hg \\ ghg^{-1}g &= hg \\ gh(g^{-1}g) &= hg \\ ghe &= hg \\ gh &= hg \end{aligned}$$

Since our choices for g, h were arbitrary, we have that this result holds for all $g \in G, h \in H$.
Thus we have that $gH = Hg$, in other words, $H \triangleleft G$ is normal.

□

6. We need to show two things:
   - The center of a group is a subgroup.
     For this we need to show three things.
     - The subgroup is closed.
     - The subgroup has an identity.
     - the subgroup has inverses.
   - The center of a group is normal.

   Proof.
   - We show that $Z(G) \leq G$ is a subgroup.
     - **Closure**
       Choose $z, z' \in Z(G)$.
       Since $Z(G)$ is the center of G, we know that $z, z' \in G$.
       So, we know that $zz' = z'z$.
       Thus, $Z(G)$ is closed.
     - **Identity**
       Choose $e \in G$.
       Since $eg = g = ge \forall g \in G, e \in Z(G)$.
       Thus, $Z(G)$ has an identity.
     - **Inverse**
       Choose $g \in G$.
       Since G is a group, $\forall g \in G, \exists g^{-1} \in G$ s.t. $g^{-1}g = e = gg^{-1}$.
       So, $\forall g \in G, g^{-1} \in Z(G)$.
       Thus, $Z(G)$ has inverses.

     From these results, we see that $Z(G) \leq G$ is a subgroup.
   - We show that $Z(G) \triangleleft G$ is normal. Equivalently, $\forall g \in G, gZ(G) = Z(G)g$.
     By definition of $Z(G) = \{z \in G | \forall g \in G, zg = gz\}$.
     This is exactly the definition of the normal.
     Thus, $Z(G) \triangleleft G$ is normal by construction.

   From these two results, we have shown that the center of a group is a normal subgroup. □

7. We need to consider four cases here.
   (a) $|G|$ is infinite, $|H|$ is infinite.
       In this case, since the two sets are infinite, they have infinite order.
       Thus, $\forall x \in G, x \neq e_G, |\varphi(x)| = |x|$.
   (b) $|G|$ is infinite, $|H|$ is finite.
       The two sets cannot have the same order, as $\forall x \in G, x \neq e_G, |x| = \infty$,
       While $\forall x \in H, |x| \leq |H|$
   (c) $|G|$ is finite, $|H|$ is infinite.
       A similar argument holds here as before.

(d) $|G|$ is finite, $|H|$ is finite.

- Since G is finite, $\forall x \in G$ of order $n \in \mathbb{Z}^+$, we have $x^n = e_G$.
  So, we have:

$$\begin{aligned} \varphi(x^n) &= \varphi(x)^n \\ \varphi(e_G) &= \varphi(x)^n \\ e_H &= \varphi(x)^n \end{aligned}$$

  Thus $\forall x \in G$ of order $n \in \mathbb{Z}^+, |\varphi(x)| = |x|$
- Since $\varphi : G \to H$ is an isomorphism, every element in G maps to exactly one element in H and vice versa.
  We know that $|x| = |\varphi(x)|$, so H has exactly the same number of elements of order n for each $n \in \mathbb{Z}^+$. Since our G and H are arbitrary groups, the result holds for all isomorphic groups.
- No, for instance, choose the homomorphism

$$h : (\{-1, 0, 1\}, +) \to (\{0\}, +)$$

$$n \mapsto 0$$

  The domain has two elements of order 2 and one element of order 1, and the co-domain has just one element of order 1.
  This cannot be an isomorphism, and the map is not bijective.