MAT 67 Homework 2

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1. Let V be a vector space over the field $\mathbb{F}$

   Given $a \in \mathbb{F}, \vec{v} \in V, a\vec{v} = \vec{0}$. Prove either $a = 0$ or $\vec{v} = \vec{0}$.

   Proof. Assume $a \neq 0$. Then $a^{-1} \in \mathbb{F}$, and $a^{-1}a = 1$ by definition.

   $$\vec{v} = 1\vec{v} = (a^{-1}a)\vec{v} = a^{-1}(a\vec{v})$$

   But we are given $a\vec{v} = \vec{0}$, so

   $$a^{-1}(a\vec{v}) = a^{-1}\vec{0} = \vec{0}$$

   Thus, $\vec{v} = \vec{0}$.

   By contraposition, we have: if $\vec{v} \neq \vec{0}$, then $a = 0$.

   Thus we have $a = 0$ or $\vec{v} = \vec{0}$. □

2. Show that the set $V = \{(x_1, x_2.x_3) \in \mathbb{F} : x_1 + 2x_2 + 2x_3 = 0\}$ is a vector space.

   We need to show that the 10 axioms hold.

   Given $\vec{u}, \vec{v}, \vec{w} \in V; a, b \in \mathbb{F}$ where $\vec{u} = (u_1, u_2, u_3).\vec{v} = (v_1, v_2, v_3).\vec{w} = (w_1, w_2, w_3)$ and the operators:

   $$\oplus : V \times V \to V$$
   $$\vec{u} \oplus \vec{v} = (u_1, u_2, u_3) \oplus (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$
   $$* : \mathbb{F} \times V \to V$$
   $$a * \vec{u} = a * (u_1, u_2, u_3) = (a \cdot u_1, a \cdot u_2, a \cdot u_3)$$

(a) Closure over vector addition.

$$\begin{aligned}\vec{u} \oplus \vec{v} &= (u_1, u_2, u_3) \oplus (v_1, v_2, v_3)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3)\end{aligned}$$

Now we need to show that our equation is still valid.

$$\begin{aligned}(u_1 + v_1, u_2 + v_2, u_3 + v_3) &= (u_1 + v_1) + 2(u_2 + v_2) + 2(u_3 + v_3)\\ &= u_1 + v_1 + 2u_2 + 2v_2 + 2u_3 + 2v_3)\\ &= (u_1 + 2u_2 + 2u_3) + (v_1 + 2v_2 + 2v_3)\\ &= 0 + 0\\ &= 0\end{aligned}$$

So, $\vec{u} \oplus \vec{v} \in V$. Thus, V is closed over vector addition.

(b) Associativity over vector addition.

$$\begin{aligned}\vec{u} \oplus (\vec{v} \oplus \vec{w}) &= (u_1, u_2, u_3) \oplus ((v_1, v_2, v_3) \oplus (w_1, w_2, w_3))\\ &= (u_1, u_2, u_3) \oplus (v_1 + w_1, v_2 + w_2, v_3 + w_3)\\ &= (u_1 + (v_1 + w_1), u_2 + (v_2 + w_2), u_3 + (v_3 + w_3))\\ &= ((u_1 + v_1) + w_1, (u_2 + v_2) + w_2, (u_3 + v_3) + w_3)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \oplus (w_1, w_2, w_3)\\ &= ((u_1, u_2, u_3) \oplus (v_1, v_2, v_3)) \oplus (w_1, w_2, w_3)\\ &= (\vec{u} \oplus \vec{v}) \oplus \vec{w}\end{aligned}$$

Thus, association holds over vector addition.

(c) Commutativity over vector addition.

$$\begin{aligned}\vec{u} \oplus \vec{v} &= (u_1, u_2, u_3) \oplus (v_1, v_2, v_3)\\ &= (u_1 + v_1, u_2 + v_2, u_3 + v_3)\\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3)\\ &= (v_1, v_2, v_3) \oplus (u_1, u_2, u_3)\\ &= \vec{v} \oplus \vec{u}\end{aligned}$$

Thus, commutation holds over vector addition.

(d) Existence of identity vector. Let $\vec{0} = (0, 0, 0) \in V$

$$\begin{aligned}\vec{u} \oplus \vec{0} &= (u_1, u_2, u_3) \oplus (0, 0, 0)\\ &= (u_1 + 0, u_2 + 0, u_3 + 0)\\ &= (u_1, u_2, u_3)\\ &= \vec{u}\end{aligned}$$

Thus, $\vec{0} = (0, 0, 0)$ exists as the identity vector.

(e) Existence of vector inverse.

$$\begin{aligned}
\vec{u} \oplus (-\vec{u}) &= (u_1, u_2, u_3) \oplus (-u_1, -u_2, -u_3) \\
&= (u_1 + (-u_1), u_2 + (-u_2), u_3 + (-u_3)) \\
&= (u_1 - u_1, u_2 - u_2, u_3 - u_3) \\
&= (0, 0, 0) \\
&= \vec{0}
\end{aligned}$$

Thus, there exists an inverse for all vectors.

(f) Closure over scalar multiplication.

$$\begin{aligned}
a * \vec{u} &= a * (u_1, u_2, u_3) \\
&= (a \cdot u_1, a \cdot u_2, a \cdot u_3)
\end{aligned}$$

Now we need to ensure that our equation holds.

$$\begin{aligned}
(a \cdot u_1, a \cdot u_2, a \cdot u_3) &= a \cdot u_1 + 2(a \cdot u_2) + 2(a \cdot u_3) \\
&= a(u_1 + 2u_2 + 2u_3) \\
&= a \cdot 0 \\
&= 0
\end{aligned}$$

So, $a * \vec{u} \in V$. Thus, V is closed over scalar multiplication.

(g) Existence of scalar multiplication identity.

$$\begin{aligned}
1 * \vec{u} &= 1 * (u_1, u_2, u_3) \\
&= (1 \cdot u_1, 1 \cdot u_2, 1 \cdot u_3) \\
&= (u_1, u_2, u_3) \\
&= \vec{u}
\end{aligned}$$

Thus, 1 exists as the scalar multiplication identity.

(h) Associativity over scalar multiplication.

$$\begin{aligned}
a * (b * \vec{u}) &= a * (b * (u_1, u_2, u_3)) \\
&= a * (b \cdot u_1, b \cdot u_2, b \cdot u_3) \\
&= (a(b \cdot u_1), a(b \cdot u_2), a(b \cdot u_3)) \\
&= ((ab) \cdot u_1, (ab) \cdot u_2, (ab) \cdot u_3) \\
&= (ab) * (u_1, u_2, u_3) \\
&= (ab) * \vec{u}
\end{aligned}$$

Thus, associativity holds over scalar multiplication.

(i) Distributivity of scalar multiplication over vector addition.

$$\begin{aligned}
a * (\vec{u} \oplus \vec{v}) &= a * ((u_1, u_2, u_3) \oplus (v_1, v_2, v_3)) \\
&= a * (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\
&= (a(u_1 + v_1), a(u_2 + v_2), a(u_3 + v_3)) \\
&= (a \cdot u_1 + a \cdot v_1, a \cdot u_2 + a \cdot v_2, a \cdot u_3 + a \cdot v_3) \\
&= (a \cdot u_1, a \cdot u_2, a \cdot u_3) \oplus (a \cdot v_1, a \cdot v_2, a \cdot v_3) \\
&= a * (u_1, u_2, u_3) \oplus a * (v_1, v_2, v_3) \\
&= a * \vec{u} \oplus a * \vec{v}
\end{aligned}$$

Thus, distributivity of scalar multiplication over vector addition holds.

(j) Distributivity of scalar addition over scalar multiplication.

$$\begin{aligned}
(a + b) * \vec{u} &= (a + b) * (u_1, u_2, u_3) \\
&= ((a + b)u_1, (a + b)u_2, (a + b)u_3) \\
&= (a \cdot u_1 + b \cdot u_1, a \cdot u_2 + b \cdot u_2, a \cdot u_3 + b \cdot u_3) \\
&= (a \cdot u_1, a \cdot u_2, a \cdot u_3) \oplus (b \cdot u_1, b \cdot u_2, b \cdot u_3) \\
&= a * (u_1, u_2, u_3) \oplus b * (u_1, u_2, u_3) \\
&= a * \vec{u} \oplus b * \vec{u}
\end{aligned}$$

Thus, distributivity of scalar addition over scalar multiplication holds.

So we conclude, since all 10 of the axioms hold, that V is a vector space.