MAT 125A HW 2

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Exercise 4.3.6 (a) *Proof.* This proof was partially inspired by John Hunter's lecture notes.

We have Dirichlet's function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

This is a function from $\mathbb{R} \to \mathbb{R}$.

Choose 0 as a limit point in $\mathbb{R}$.

Now we construct a sequence from $x_n = \frac{e}{n}$. So $(x_n) \to 0$, by The Algebraic Limit Theorem.

Now, $f(x_n) \to 0$, but $f(0) = 1$.

So, by Corollary 4.3.3, Dirichlet's function is not continuous at 0.

We can extend this to any $c \in \mathbb{Q}$ by constructing a new sequence $y_n = x_n + c$.

Following similar arguments, it can be shown that Dirichlet's function is not continuous at any point in $\mathbb{Q}$.

A similar argument holds for showing that Dirichlet's function is not continuous on $\mathbb{I}$. We choose some sequence of rationals such that for any limit point c, $(z_n) \to c$. Then we have that $f(z_n) \to 1$, but $f(c) = 0$. So Dirichlet's function is not continuous on $\mathbb{I}$ either.

Thus, Dirichlet's function is not continuous on $\mathbb{R}$. □

(b) *Proof.* We have

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

.

Choose some rational number c in lowest terms $\frac{m}{n}$.

Now construct any sequence (x_n) from $\mathbb{I}$ such that, $(x_n) \to c$.

So, $f(x_n) \to 0$, but $f(c) = \frac{1}{n}$.

Now, since $f : \mathbb{R} \to \mathbb{R}$, c is a limit point in $\mathbb{R}$, $(x_n) \to c$, but $f(x_n) \neq f(c)$, we conclude that Thomae's function is not continuous at any rational point. □

(c) *Proof.* We have

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

.

Choose some $c \in \mathbb{I}$. Then we know that $f(c) = 0$, since $c \notin \mathbb{Q}$.

Now given, some $\epsilon > 0$, choose $\delta = \epsilon$.

So for $x \in \mathbb{I}$ where $|x - c| < \delta$, we have $|f(x) - f(c)| = |0 - 0| = 0 < \epsilon$.

Thus, by Theorem 4.3.2, f is continuous on $\mathbb{I}$. □

Exercise 4.3.7 *Proof.* Let c be a limit point in K. Then there exists some sequence (x_n) in K such that $\lim_{x \to c} x_n = c$

Since h is continuous on $\mathbb{R}$, $\lim_{x \to c} h(x_n) = h(c)$.

But since all x_n are in K, all $h(x_n) = 0$.

So $\lim_{x \to c} h(x_n) = h(c) = 0$.

Since $h(c) = 0, c \in K$, so K contains its limit points.

Thus K is a closed set. □

Exercise 4.3.8 (a) *Proof.* Let the continuous function be f.

Let c be an arbitrary point in $\mathbb{I}$.

Then there exists some sequence (x_n) in $\mathbb{Q}$ such that $\lim_{x \to c} x_n = c$. Since f is continuous, we know that $\lim_{x \to c} f(x_n) = f(c)$. And since all $x_n \in \mathbb{Q}$, all $f(x_n) = 0$.

So $\lim_{x \to c} f(x_n) = f(c) = 0$.

Since our choice of c was arbitrary, we have that for all $c \in \mathbb{I}, f(c) = 0$.

So f is 0 on all of $\mathbb{I}$ and $\mathbb{Q}$.

Thus f is 0 on all of $\mathbb{R}$. □

(b) No, the two functions do not have to be the same since there is not restriction that the functions be continuous.

Let

$$f(x) = 1, g(x) = \begin{cases} 1 & \text{if } \in \mathbb{Q} \\ 0 & \text{if } \notin \mathbb{Q} \end{cases}$$

Then these two functions are not the same, yet both equal each other when $x \in \mathbb{Q}$.

Exercise 4.3.9 (a) Since the given information looks quite similar to the definition of continuity, we should try to manipulate it a bit.

If we had $|f(x) - f(y)| \leq c|x - y| < \epsilon$ for any $\epsilon > 0$, we'd be all set.

Proof. Let y be a limit point of $\mathbb{R}$.
For any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{c}$.
Then we have $0 < |x-y| < \delta = \frac{\epsilon}{c} \implies |f(x) - f(y)| \leq c|x-y|$.
Since $|x-y| < \frac{\epsilon}{c} \implies |f(x) - f(y)| \leq c|x-y| < c\frac{\epsilon}{c} = \epsilon$.
So f is continuous at y. But our choice of y was arbitrary, so f is continuous on all of $\mathbb{R}$. $\square$

(b)

(c)

(d)

Exercise 4.3.10 (a)

- Choose $x = y = 0$.
  Then:

$$\begin{aligned} f(0+0) &= f(0) + f(0) \\ f(0) &= f(0) + f(0) \\ 0 &= f(0) \end{aligned}$$

  Thus, $f(0) = 0$.
- Since $f(0) = 0$.
  Then:

$$\begin{aligned} f(0) &= 0 \\ f(x-x) &= 0, \text{for any } x \in \mathbb{R} \\ f(x+(-x)) &= 0 \\ f(x) + f(-x) &= 0 \\ f(-x) &= -f(x) \end{aligned}$$

  Thus, $f(-x) = -f(x)$ for any $x \in \mathbb{R}$.

(b) *Proof.* Assume f is continuous at 0.
Choose some $c \in \mathbb{R}$ as a limit point.
Then there exists a sequence $(x_n) \to c$. We also know that $(c) \to c$.
So we can compute $(x_n - c) \to c - c = 0$.

Since f is continuous at 0, we have

$$\begin{aligned}
\lim_{x\to 0} f(x_n - c) &= f(0) \\
\lim_{x\to 0} f(x_n + (-c)) &= 0 \\
\lim_{x\to 0} f(x_n) + f(-c) &= \\
\lim_{x\to 0} f(x_n) - f(c) &= \\
\lim_{x\to 0} f(x_n) - \lim_{x\to 0} f(c) &= \\
\lim_{x\to 0} f(x_n) &= \lim_{x\to 0} f(c)
\end{aligned}$$

So f is continuous at c.

Since our choice of c was arbitrary, the result holds for all $c \in \mathbb{R}$.

Thus, if f is continuous at 0 then f is continuous at all points in $\mathbb{R}$. □

(c)
- *Proof.* We can show this by induction.
  - Base $f(1) = k = k(1)$, so the base case is true.
  - Inductive Assume $f(n) = kn$. Then $f(n+1) = f(n) + f(1) = kn + k = k(n+1)$. So the inductive case in true.

  Thus we have shown by induction that $f(n) = kn, \forall n \in \mathbb{N}$ □
- We need to look at three case:
  - $n > 0$
    This is exactly the case proven above.
  - $n = 0$
    We are given by definition $f(0) = 0 = k(0)$
  - $n < 0$
    In this case we can let $-m = n \implies m = -n$ and $m > 0$.
    Then we have $f(-m) = -f(m) = -k(m) = -k(-n) = k(n)$

  From these three we have shown that $f(n) = kn, \forall n \in \mathbb{Z}$
- Given some $f\left(\frac{p}{q}\right)$, we can rewrite this as

$$\begin{aligned}
\underbrace{f\left(\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}\right)}_{p \text{ times}} &= \underbrace{f\left(\frac{1}{q}\right) + f\left(\frac{1}{q}\right) + \cdots + f\left(\frac{1}{q}\right)}_{p \text{ times}} \\
&= pf\left(\frac{1}{q}\right)
\end{aligned}$$

So we can rewrite

$$
\begin{aligned}
f(1) &= f\underbrace{\left(\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}\right)}_{n \text{ times}} \\
&= \underbrace{f\left(\frac{1}{n}\right)+f\left(\frac{1}{n}\right)+\cdots+f\left(\frac{1}{n}\right)}_{n \text{ times}} \\
&= nf\left(\frac{1}{n}\right) && = k
\end{aligned}
$$

So, $k = nf\left(\frac{1}{n}\right) \implies f\left(\frac{1}{n}\right) = \frac{k}{n}$
Then, $f\left(\frac{p}{q}\right) = pf\left(\frac{1}{q}\right) = p\frac{k}{q} = k\frac{p}{q}$.
Thus $f(n) = k(n), \forall n \in \mathbb{Q}$

(d) *Proof.* Choose $x \in \mathbb{I}$ as a limit point.
Then there exists some $(x_n) \to x$ with all $x_n \in \mathbb{Q}$.
Since f is continuous on $\mathbb{R}$, f is continuous at x.
So, $\lim_{n\to\infty} f(x_n) = f(x)$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} kx_n = kx$
Then $f(x) = kx$.
Thus we conclude any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin. □