MAT 67 Homework 8

Hardy Jones
999397426
Professor Bandyopadhyay
Fall 2013

1. Let V and W be vector spaces over $\mathbb{F}$ and suppose that $T \in \mathcal{L}(V, W)$ is injective.

   Given a linearly independent list $(v_1, \ldots, v_n)$ of vectors in V, prove that the list $(T(v_1), \ldots, T(v_n))$ is linearly independent in W.

   Proof. Suppose $a_1, a_2, \ldots, a_n \in \mathbb{F}$ and $a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0$.

   Now since T is a linear map,

   $$0 = a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) \qquad = T(a_1v_1 + a_2v_2 + \cdots + a_nv_n)$$

   And since T is injective we have that there is one vector, namely 0, in its kernel by proposition 6.2.6.

   That is:

   $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$$

   Now since $(v_1, \ldots, v_n)$ is linearly independent, $a_1 = a_2 = \cdots = a_n = 0$.

   From this, we see that $a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) = 0$ must be linearly independent. $\square$

2. Let V and W be vector spaces over $\mathbb{F}$ and suppose that $T \in \mathcal{L}(V, W)$ is surjective.

   Given a spanning list $(v_1, \ldots, v_n)$ for V, prove that $\text{span}(T(v_1), \ldots, T(v_n)) = W$.

   Proof. Suppose $\exists w \in W$, then since T is surjective, we have a $v \in V$ such that $T(v) = w$.

   Since $(v_1, \ldots, v_n)$ spans V, we can make a linear combination for v

   $$a_1v_1 + a_2v_2 + \cdots + a_nv_n = v, \text{ where } a_1, a_2, \ldots, a_n \in \mathbb{F}$$

   Now, since T is a linear map,

   $$\begin{aligned} w &= T(v) \\ &= T(a_1v_1 + a_2v_2 + \cdots + a_nv_n) \\ &= T(a_1v_1) + T(a_2v_2) + \cdots + T(a_nv_n) \\ &= a_1T(v_1) + a_2T(v_2) + \cdots + a_nT(v_n) \end{aligned}$$

   Since our choice for w was arbitrary, we can find all $w \in W$ this way.

   So we have that $\forall w \in W, w \in$ span $(T(V))$.

   Or put another way, $\text{span}(T(v_1), \ldots, T(v_n)) = W$ $\square$

3. Let V be a finite-dimensional vector space over $\mathbb{F}$ with $S, T \in \mathcal{L}(V, V)$.

   Prove that $T \circ S$ is invertible if and only if both S and T are invertible.

   Proof. We need to show both ways.

   (a) Suppose $T \circ S$ is invertible.
   We only need to check that $null(T) = \{0\}$ and $null(S) = \{0\}$
   In order for $T \circ S$ to be invertible we must have that $T(0) = 0$. So there must exist some $v \in V$ such that $S(v) = 0$.
   If $v \neq 0$, then $(T \circ S)(v) = T(S(v)) = T(0) = 0$ but that would imply that $null(T \circ S) \neq \{0\}$, which contradicts the fact that $T \circ S$ is invertible.
   So $v = 0$ and $S(0) = 0 \implies T(0) = 0$.
   These two imply that $null(S) = \{0\}$ and $null(T) = \{0\}$.
   Thus S and T are invertible.

   (b) Suppose T, S are invertible. It suffices to check that $\text{null}(T \circ S) = \{0\}$
   For some $v \in V$,
   If $v \in null(T \circ S)$, then $S(v) \in null(T)$.
   So, $S(v) = 0$.
   This means that $v \in null(S)$ and so $v = 0$.
   So, $null(T \circ S) = \{0\}$.
   Thus $T \circ S$ is invertible.

   Thus, from (a) and (b) we have that $T \circ S$ is invertible if and only if both S and T are invertible. $\square$