MAT 125A HW 7

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6.5 2 (a) Choose $a_n = \frac{1}{n^2}$
(b) Choose $a_n = \frac{1}{n}$
(c)
(d) No, this is not possible. If the series converged absolutely at $x = 1$, then it the series would converge for all x_0 where $|x_0| \leq 1$. This includes $x_0 = -1$. So the series would have to converge absolutely at $x_0 = -1$ as well.

3 From Theorem 6.5.1, we get the set of points a power series converges to must be one of $\{0\}, \mathbb{R}$, or a bounded interval about 0 (really the first two are specific cases of the last one).

- If the set of convergence is $\{0\}$, then the series converges only at one point, which is less than 2 points.
- If the set of convergence is $\mathbb{R}$, then the series converges absolutely at every point.
- If the set of convergence is some interval $(-x, x), (-x, x], [-x, x)$, or $[-x, x]$, then the series converges absolutely at every point x_0 where $|x_0| < |x|$, and only at the end points is it possible to converge conditionally.

From these three cases, we see that at most two points can have conditional convergence.

4 (a)
(b)

9 *Proof.* First let's look at some derivatives.
Since we know

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n = g(x)$$

We can find the first few derivatives.

(a) 0^{th} derivative:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n = g(x)$$

(b) 1^{st} derivative:

$$f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=1}^{\infty} nb_n x^{n-1} = g'(x)$$

(c) 2^{nd} derivative:

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)b_n x^{n-2} = g''(x)$$

(d) 3^{rd} derivative:

$$f'''(x) = \sum_{n=3}^{\infty} n(n-1)(n-2)a_n x^{n-3} = \sum_{n=3}^{\infty} n(n-1)(n-2)b_n x^{n-3} = g'''(x)$$

Looks like there's a pattern.
The k-th derivative of f, g is

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) a_n x^{n-k} = \sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) b_n x^{n-k} = g^{(k)}(x)$$

We can prove this by induction:

- Base Case: $k = 0$

$$\begin{aligned}
f^{(0)}(x) = f(x) &= \sum_{n=0}^{\infty} a_n x_n \\
&= \sum_{n=0}^{\infty} (1) a_n x_n \\
&= \sum_{n=0}^{\infty} \left(\prod_{m=0}^{0-1} n - m \right) a_n x_n \\
&= \sum_{n=0}^{\infty} \left(\prod_{m=0}^{0-1} n - m \right) b_n x_n \\
&= \sum_{n=0}^{\infty} (1) b_n x_n \\
g^{(0)}(x) = g(x) &= \sum_{n=0}^{\infty} b_n x_n
\end{aligned}$$

- Inductive Case:
  Assume

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) a_n x^{n-k} = \sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) b_n x^{n-k} = g^{(k)}(x)$$

$$\begin{aligned}
f^{(k+1)}(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(f^{(k)}(x) \right) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(\sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) a_n x^{n-k} \right) \\
&= \sum_{n=k+1}^{\infty} (n-k) \left(\prod_{m=0}^{k-1} n - m \right) a_n x^{n-k-1} \\
&= \sum_{n=k+1}^{\infty} \left(\prod_{m=0}^{k} n - m \right) a_n x^{n-(k+1)} \\
&= \sum_{n=k+1}^{\infty} \left(\prod_{m=0}^{k} n - m \right) b_n x^{n-(k+1)} \\
&= \sum_{n=k+1}^{\infty} (n-k) \left(\prod_{m=0}^{k-1} n - m \right) b_n x^{n-k-1} \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(\sum_{n=k}^{\infty} \left(\prod_{m=0}^{k-1} n - m \right) b_n x^{n-k} \right) \\
g^{(k+1)}(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(g^{(k)}(x) \right)
\end{aligned}$$

By induction we have proved our conjecture.
Now we can move on to the actual proof.
Since we know

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n = g(x)$$

for all $x \in (-R, R)$, choose $x = 0$.
Then we have:

$$
\begin{aligned}
f(0) &= \sum_{n=0}^{\infty} a_n 0^n \\
&= a_0 0^0 + a_1 0^1 + a_2 0^2 + \dots \\
&= a_0 \\
&= b_0 \\
&= b_0 0^0 + b_1 0^1 + b_2 0^2 + \dots \\
g(0) &= \sum_{n=0}^{\infty} b_n 0^n
\end{aligned}
$$

So $a_0 = b_0$.
Now, assuming $a_k = b_k$:

$$
\begin{aligned}
f^{(k+1)}(0) &= \sum_{n=k+1}^{\infty} \left(\prod_{m=0}^{k} n - m \right) a_n 0^{n-(k+1)} \\
&= \left(\prod_{m=0}^{k} (k+1) - m \right) a_{k+1} 0^0 + \left(\prod_{m=0}^{k} (k+2) - m \right) a_{k+2} 0^1 + \dots \\
&= \left(\prod_{m=0}^{k} (k+1) - m \right) a_{k+1} \\
&= \left(\prod_{m=0}^{k} (k+1) - m \right) b_{k+1} \\
&= \left(\prod_{m=0}^{k} (k+1) - m \right) b_{k+1} 0^0 + \left(\prod_{m=0}^{k} (k+2) - m \right) b_{k+2} 0^1 + \dots \\
g^{(k+1)}(0) &= \sum_{n=k+1}^{\infty} \left(\prod_{m=0}^{k} n - m \right) b_n 0^{n-(k+1)}
\end{aligned}
$$

So we have shown by induction that $a_n = b_n, \forall n \in \{0, 1, 2, \dots\}$. $\square$

6.6 1 At the point $x = 1$, the series is:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

and this series converges by the alternating series test.

By Abel's Theorem, the series converges uniformly on $[0, 1]$.

Since we assume $\arctan(x)$ is continuous on $[0, 1]$, we must necessarily have $\arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$.

After trying and failing miserably to find a nice identity, I plugged $\arctan(1)$ into a calculator and found $\frac{\pi}{4}$.

2 Following the example (in reverse), we want

$$\begin{aligned} \tfrac{\mathrm{d}\ln(1+x)}{\mathrm{d}x} &= \frac{1}{1+x} \\ \ln(1+x) &= \int_0^x \frac{1}{1+t}\,\mathrm{d}t \end{aligned}$$

So we need to substitute $-t$ for t:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 - \dots$$

So if we integrate this, we get

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

So this expression if valid for all $x \in (-1, \infty)$. Though it only converges for $x \in (-1, 1]$.

5 *Proof.* To prove that $S_N(x)$ converges uniformly to $\sin(x)$ on $[-2, 2]$, we need to show:

$\forall \epsilon > 0, \exists M \in \mathbb{N}$, such that $\forall m \geq M, x \in [-2, 2], \left|S_m(x) - \sin(x)\right| < \epsilon$.

It suffices to show that $E_N(x) \to 0$ as $N \to \infty$, so what we really want is:

$\forall \epsilon > 0, \exists M \in \mathbb{N}$, such that $\forall m \geq M, x \in [-2, 2], \left|E_m(x)\right| < \epsilon$.

From Lagrange's Remainder Theorem, and recalling that derivatives of $\sin, \cos$ cycle between each other, we have:

$$\begin{aligned} \left|E_N(x)\right| &= \left|\frac{\sin^{(N+1)}(c)}{(N+1)!}x^{N+1}\right| \\ &\leq \left|\frac{1}{(N+1)!}x^{N+1}\right| && \text{since } \left|\sin(x)\right|, \left|\cos(x)\right| \leq 1, \forall x \\ &= \frac{1}{(N+1)!}\left|x^{N+1}\right| \\ &\leq \frac{1}{(N+1)!}\left|2^{N+1}\right| && \text{since } x \in [-2, 2] \\ &\leq \frac{1}{(N+1)!}2^{N+1} && \text{since } x \in [-2, 2] \end{aligned}$$

So for any $\epsilon > 0$, choose $M \in \mathbb{N}$ such that $\frac{1}{M+1} 2^{M+1} < \epsilon$.
Then we have, for any $m \geq M$ and for all $x \in [-2, 2]$,
$\left| E_m(x) \right| \leq \frac{1}{M+1} 2^{M+1} < \epsilon$.
So $S_N(x)$ converges uniformly to $\sin(x)$ on $[-2, 2]$. $\square$
We can generalize this proof to any interval $[-R, R]$ by substituting R for 2.

Proof. For any $\epsilon > 0$,
choose $M \in \mathbb{N}$ such that $\frac{1}{M+1} R^{M+1} < \epsilon$.
Then we have, for any $m \geq M$ and for all $x \in [-R, R]$,
$\left| E_m(x) \right| \leq \frac{1}{M+1} R^{M+1} < \epsilon$.
So $S_N(x)$ converges uniformly to $\sin(x)$ on $[-R, R]$. $\square$

10 Let's rewrite this first.

$$g(x) = e^{-x^{-2}}$$

Now we have:

$$g'(x) = e^{-x^{-2}} \left(2x^{-3}\right)$$

$$\begin{aligned} g''(x) &= e^{-x^{-2}} \left(2x^{-3}\right) \left(2x^{-3}\right) + e^{-x^{-2}} \left(-6x^{-4}\right) \\ &= e^{-x^{-2}} \left(4x^{-6} - 6x^{-4}\right) \end{aligned}$$

$$\begin{aligned} g'''(x) &= e^{-x^{-2}} \left(2x^{-3}\right) \left(4x^{-6} - 6x^{-4}\right) + e^{-x^{-2}} \left(-24x^{-7} + 24x^{-5}\right) \\ &= e^{-x^{-2}} \left(8x^{-9} - 36x^{-7} + 24x^{-5}\right) \end{aligned}$$

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$$\begin{aligned} g''(0) &= \lim_{x \to 0} \frac{g'(x)}{x} \\ &= \lim_{x \to 0} \frac{e^{-x^{-2}} \left(2x^{-3}\right)}{x} \\ &= \lim_{x \to 0} e^{-x^{-2}} \left(2x^{-4}\right) \end{aligned}$$