MAT 108 HW 2

Hardy Jones
999397426
Professor Bandyopadhyay
Spring 2015

§1.4 5 (e) With integers x and y, we want to show:
If x and y are odd, then $x + y$ is even.

Proof. Suppose x and y are odd.
Then there exist some integers p, q such that $x = 2p + 1$ and $y = 2q + 1$.
Then we have $x + y = (2p + 1) + (2q + 1) = 2p + 2q + 2 = 2(p + q + 1)$.
Since $p + q + 1$ is an integer, we can rename as $r = p + q + 1$.
So $x + y = 2r$, and is even.
Thus if x and y are odd, then $x + y$ is even □

(f) With integers x and y, we want to show:
If x and y are odd, then $3x - 5y$ is even.

Proof. Suppose x and y are odd.
Then there exist some integers p, q such that $x = 2p + 1$ and $y = 2q + 1$.
Then we have $3x-5y = 3(2p+1)-5(2q+1) = 6p+3-10q-5 = 6p-10q-2 = 2(3p - 5q - 1)$.
Since $3p - 5q - 1$ is an integer, we can rename as $r = 3p - 5q - 1$.
So $3x - 5y = 2r$, and is even.
Thus if x and y are odd, then $3x - 5y$ is even □

(i) With integers x and y, we want to show:
If x is even and y is odd, then xy is even.

Proof. Suppose x is even and y is odd.
Then there exist some integers p, q such that $x = 2p$ and $y = 2q + 1$.
Then we have $xy = (2p)(2q + 1) = 4pq + 2p = 2(2pq + p)$.
Since $2pq + p$ is an integer, we can rename as $r = 2pq + p$.
So $xy = 2r$, and is even.
Thus if x is even and y is odd, then xy is even □

6 (d) With real numbers a, b we want to prove $|a + b| \leq |a| + |b|$.

Proof. We prove this by cases.
For cases 2 and 3, we choose without loss of generality $a \geq 0, b < 0$. The exact same argument holds for $a < 0, b \geq 0$.

Case 1 $a \geq 0, b \geq 0$
Since $a \geq 0, b \geq 0, a + b \geq 0$.
So $|a + b| = a + b$.
Also, $|a| = a$ and $|b| = b$.
So $|a + b| = a + b = |a| + |b|$.
Thus, $|a + b| \leq |a| + |b|$.

Case 2 $a \geq 0, b < 0, a+b \geq 0$

Since $a+b \geq 0, |a+b| = a+b$.

Also, $|a| = a$ and $|b| = -b$.

Since $b < 0 \implies 2b < 0 \implies b < -b \implies a+b < a+(-b)$,

we have $|a+b| = a+b < a+(-b) = |a|+|b|$.

Thus, $|a+b| \leq |a|+|b|$.

Case 3 $a \geq 0, b < 0, a+b < 0$

Since $a+b < 0, |a+b| = -(a+b) = -a-b$.

Also, $|a| = a$ and $|b| = -b$.

Since $0 \leq a \implies 0 \leq 2a \implies -a \leq a \implies -a-b \leq a-b$,

we have $|a+b| = -a-b \leq a-b = a+(-b) = |a|+|b|$.

Thus, $|a+b| \leq |a|+|b|$.

Case 4 $a < 0, b < 0$

Since $a < 0, b < 0, a+b < 0$.

So $|a+b| = -(a+b) = -a-b$.

Also, $|a| = -a$ and $|b| = -b$.

So $|a+b| = -a-b = -a+(-b) = |a|+|b|$.

Thus, $|a+b| \leq |a|+|b|$.

Since these are all the possible cases, we have proven by exhaustion that $|a+b| \leq |a|+|b|$. □

(e) With real numbers a, b we want to prove if $|a| \leq b$, then $-b \leq a \leq b$.

Proof. We prove this by cases.

Case 1 $a \geq 0$

Assume $|a| \leq b$.

Since $a \geq 0$, we have $|a| = a$.

Now, we know $|a| = a \leq b$.

Also $0 \leq |a| = a \leq b \implies 0 \leq b$. And since both a and b are non-negative, $a+b$ is also non-negative.

So we have $0 \leq a+b \implies -b \leq a$.

Then we have both $-b \leq a$ and $a \leq b$ or $-b \leq a \leq b$.

Thus if $|a| \leq b$, then $-b \leq a \leq b$.

Case 2 $a < 0$

Assume $|a| \leq b$.

Since $a < 0$, we have $|a| = -a$.

Then, $|a| = -a \leq b \implies -a-b \leq 0 \implies -b \leq a$.

Now, by the definition of absolute value, $|a| \leq b \implies 0 \leq b$.

And $a < 0 \implies a < 0 \leq b \implies a < b \implies a \leq b$.

Then we have both $-b \leq a$ and $a \leq b$ or $-b \leq a \leq b$.

Thus if $|a| \leq b$, then $-b \leq a \leq b$.

Since these are the only possible cases, we have proved by exhaustion that: If $|a| \leq b$, then $-b \leq a \leq b$. □

(f) With real numbers a, b we want to prove if $-b \leq a \leq b$, then $|a| \leq b$.

Proof. We prove this by cases.

Case 1 $a \geq 0$

Assume $-b \leq a \leq b$.
Since $a \geq 0$, we have $|a| = a$.
We have $-b \leq a \leq b \implies a \leq b$.
Then, $a = |a| \leq b$.
Thus, if $-b \leq a \leq b$, then $|a| \leq b$.

Case 2 $a < 0$

Assume $-b \leq a \leq b$.
Since $a < 0$, we have $|a| = -a$.
We have $-b \leq a \leq b \implies -b \leq a \implies -a - b \leq 0 \implies -a \leq b$.
Then $-a = |a| \leq b$.
Thus, if $-b \leq a \leq b$, then $|a| \leq b$.

Since these are the only possible cases, we have proved by exhaustion that:
If $-b \leq a \leq b$, then $|a| \leq b$. $\square$

9 (b) With integers a, b, c we work backward to prove:
if a divides b and a divides $b + c$, then a divides $3c$.
If a divides $3c$, then there exists some integer p such that $3c = ap \implies 3b + 3c = 3b + ap \implies 3(b + c) = 3b + ap$.
Now, if a divides b, then there exists some integer q such that $b = aq$. And also, if a divides $b + c$, then there exists some integer r such that $b + c = ar$
So $3(b+c) = 3aq+ap \implies 3ar = 3aq+ap \implies 3r = 3q+p \implies 3r-3q = p \implies 3(r - q) = p$.
Looks like we can construct a proof if we can assume $p = 3(r - q)$.

Proof. Assume a divides b and a divides $b + c$.
These mean there exist integers q, r such that $b = aq, b + c = ar$.
Now construct another integer $p = 3(r - q)$.
Then,

$$\begin{aligned} 3(r - q) &= p \\ 3a(r - q) &= ap \\ 3(ar - aq) &= ap \\ 3(b + c - b) &= ap \\ 3c &= ap \end{aligned}$$

So, we have that a divides $3c$.
Thus, if a divides b and a divides $b + c$, then a divides $3c$. $\square$

(c)

(d) With the real number x we work backward to prove:
if $x^3 + 2x^2 < 0$, then $2x + 5 < 11$.
We find $2x + 5 < 11 \implies 2x < 6 \implies x < 3$.
If we work a bit forward from the antecedent we see $x^3+2x^2 < 0 \implies x+2 < 0 \implies x < -2$.
Now, we should have enough to construct a proof.

Proof. Assume $x^3 + 2x^2 < 0$.
Then we have $x^3 + 2x^2 < 0 \implies x + 2 < 0 \implies x < -2$.
Now, if x is less than -2, then x is also less than 3.
So we have $x < 3 \implies 2x < 6 \implies 2x + 5 < 11$.
Thus, if $x^3 + 2x^2 < 0$, then $2x + 5 < 11$. $\square$

11 (b) The claim is solid.
The proof has the correct idea, however, it is incorrect.
When constructing the factors of c, a new integer should be chosen as otherwise, b and c are the same integer.
While this is also a true claim, it does not prove what the original claim suggests.
One way to fix the proof is as follows.

Proof. Suppose a divides b and a divides c. Then for some integer $q, b = aq$, and for some integer $\mathbf{r}, c = ar$.
Then $b + c = aq + ar = a(q + r)$. Since $q + r$ is also an integer, we rename $q + r = s$. So $b + c = as$, and a divides $b + c$. $\square$

So on the scale of **A**, **C**, **F**, this proof gets a grade of **C**.

(e) The claim is solid.
The proof is also correct.
So on the scale of **A**, **C**, **F**, this proof gets a grade of **A**.

§1.5 3 (c) We want to show by contraposition:
if x^2 is not divisible by 4, then x is odd.
The contrapositive of this statement is:
If x is even, then x^2 is divisible by 4.

Proof. Assume x is even.
Then there exists some integer p such that $x = 2p$.
So $x^2 = (2p)^2 = 4p^2$.
Since p is an integer, p^2 is also an integer. So we can replace it with $p^2 = q$.
Then $x^2 = 4q$, meaning that 4 divides x^2, or equivalently x^2 is divisible by 4.
Therefore, x^2 is divisible by 4.
Thus, if x is even, then x^2 is divisible by 4.
Therefore, if x^2 is not divisible by 4, then x is odd. $\square$

(d) We want to show by contraposition:
if xy is even, then either x or y is even.
The contrapositive of this statement is:
if x and y are both odd, then xy is odd.

Proof. Assume both x and y odd.
Then there exists some integers p, q such that $x = 2p + 1, y = 2q + 1$.
So $xy = (2p + 1)(2q + 1) = 4pq + 2p + 2q + 1 = 2(2pq + p + q) + 1$.
Since $2pq + p + q$ is an integer, we can replace it with $2pq + p + q = r$.
Then $xy = 2r + 1$.
Therefore, xy is odd.
Thus, if x and y are both odd, then xy is odd.

Therefore, if xy is even, then either x or y is even. □

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6 (a)

(b)

7 (c)

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