MAT 67 Homework 6

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We are asked to determine if there is a

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

such that $T\alpha_1 = \beta_1$, $T\alpha_2 = \beta_2$, and $T\alpha_3 = \beta_3$.

Proof. We can use theorem 6.1.3 to help in our proof. So we need to show two things:

1. $(\alpha_1, \alpha_2, \alpha_3)$ form a basis for $\mathbb{R}^2$
2. $(\beta_1, \beta_2, \beta_3)$ is a list of vectors in $\mathbb{R}^2$

1. In order to show that $(\alpha_1, \alpha_2, \alpha_3)$ is a basis for $\mathbb{R}^2$, we need to show that $(\alpha_1, \alpha_2, \alpha_3)$ is linearly independent and $\mathbb{R}^2 = \text{span}(\alpha_1, \alpha_2, \alpha_3)$.

   By inspection, we can see that $(\alpha_1, \alpha_2, \alpha_3)$ is linearly dependent, since $\alpha_1 = -\alpha_2 - \alpha_3$, so we're better off using the basis reduction theorem.

   Let's start by showing that $\text{span}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{R}^2$.

   We simply need to show that

$$\forall v \in \mathbb{R}^2; x, y, a_1, a_2, a_3 \in \mathbb{R}; v = (x, y) = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$\begin{aligned} v = (x, y) &= a_1(1, -1) + a_2(2, -1) + a_3(-3, 2) \\ &= (a_1 + 2a_2 - 3a_3, -a_1 - a_2 + 2a_3) \end{aligned}$$

   This is equivalent to solving this system of equations.

$$\begin{aligned} a_1 + 2a_2 - 3a_3 &= x \\ -a_1 - a_2 + 2a_3 &= y \end{aligned}$$

   This reduces to:

$$\begin{aligned} a_1 + 0a_2 - a_3 &= x \\ 0a_1 + a_2 - a_3 &= y \end{aligned}$$

So, $v = (a_1\alpha_1 - a_3\alpha_3, a_2\alpha_2 - a_3\alpha_3)$

So we can represent any vector in $\mathbb{R}^{\not\models}$ as a linear combination of $(\alpha_1, \alpha_2, \alpha_3)$. Thus, $\text{span}(\alpha_1, \alpha_2, \alpha_3) = \mathbb{R}^2$

Now, we start our recursive reduction.

$\alpha_1 \neq 0$, so we leave it alone, and our list is still $(\alpha_1, \alpha_2, \alpha_3)$

$\alpha_2 \neq a_1\alpha_1$, so leave it in our list, which is still $(\alpha_1, \alpha_2, \alpha_3)$

$\alpha_3 = -\alpha_1 - \alpha_2$, so remove it and now our list is (α_1, α_2)

So, we're done, we now have a linearly independent list of vectors which span $\mathbb{R}^2$.

Thus, we have our basis for $\mathbb{R}^2$, namely (α_1, α_2)

2. It is trivial to show that $(\beta_1, \beta_2, \beta_3)$ is a list of vectors in $\mathbb{R}^2$

From both of these, we can finally use theorem 6.1.3 to show that there exists some

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

such that $T\alpha_1 = \beta_1$, $T\alpha_2 = \beta_2$, and $T\alpha_3 = \beta_3$. $\square$