MAT 168 Calculation 1

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2.1 We first start by rewriting as a dictionary:

$$\begin{aligned}
\zeta &= \quad + 6x_1 + 8x_2 + 5x_3 + 9x_4 \\
x_5 &= 5 - 2x_1 - \ \ x_2 - \ \ x_3 - 3x_4 \\
x_6 &= 3 - \ \ x_1 - 3x_2 - \ \ x_3 - 2x_4
\end{aligned}$$

Then we can begin by entering with the largest variable, x_4.

We look at the constraints and see:

- $x_5 \implies x_4 \leq \frac{5}{3}$
- $x_6 \implies x_4 \leq \frac{3}{2}$

The more restrictive constraint is that $x_4 \leq \frac{3}{2}$, so set $x_4 = \frac{3}{2}$.

So we can let x_4 enter and x_6 leave.

$$x_4 = \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_6$$

This gives a new value for x_5. $x_5 = \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_6$

This gives a new value for ζ. $\zeta = \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}x_6$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= \frac{27}{2} + \frac{3}{2}x_1 - \frac{11}{2}x_2 + \frac{1}{2}x_3 - \frac{9}{2}x_6 \\
x_5 &= \frac{1}{2} - \frac{1}{2}x_1 + \frac{7}{2}\ x_2 + \frac{1}{2}x_3 + \frac{3}{2}x_6 \\
x_4 &= \frac{3}{2} - \frac{1}{2}x_1 - \frac{3}{2}\ x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_6
\end{aligned}$$

Now, we can continue optimizing since x_1 has a positive coefficient.

We look at the constraints and see:

- $x_5 \implies x_1 \leq 1$
- $x_4 \implies x_1 \leq 3$

The more restrictive constraint is that $x_1 \leq 1$, so set $x_1 = 1$.

So we can let x_1 enter and x_5 leave.

$$x_1 = 1 + 7x_2 + x_3 + 3x_6 - 2x_5$$

This gives a new value for x_4. $x_4 = 1 - 5x_2 - x_3 - 2x_6 + x_5$

This gives a new value for ζ. $\zeta = 15 + 5x_2 + 2x_3 - 3x_5$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= 15 + 5x_2 + 2x_3 \qquad\quad - 3x_5 \\
x_1 &= \;\;1 + 7x_2 + \;\; x_3 + 3x_6 - 2x_5 \\
x_4 &= \;\;1 - 5x_2 - \;\; x_3 - 2x_6 + \;\; x_5
\end{aligned}$$

Now, we can continue optimizing since x_2 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_2 \geq 0$
- $x_4 \implies x_2 \leq \frac{1}{5}$

The more restrictive constraint is that $x_2 \leq \frac{1}{5}$, so set $x_2 = \frac{1}{5}$.

So we can let x_2 enter and x_4 leave.

$$x_2 = \frac{1}{5} - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}x_6 + \frac{1}{5}x_5$$

This gives a new value for x_1. $x_1 = \frac{12}{5} - \frac{7}{5}x_4 - \frac{2}{5}x_3 + \frac{1}{5}x_6 - \frac{3}{5}x_5$

This gives a new value for ζ. $\zeta = 16 - x_4 + x_3 - 2x_6 - 2x_5$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= \;16 - \;\; x_4 + \;\; x_3 - \; 2x_6 - 2\,x_5 \\
x_1 &= \frac{12}{5} - \frac{7}{5}x_4 - \frac{2}{5}x_3 + \frac{1}{5}x_6 - \frac{3}{5}x_5 \\
x_2 &= \;\;\frac{1}{5} - \frac{1}{5}x_4 - \frac{1}{5}x_3 - \frac{2}{5}x_6 + \frac{1}{5}x_5
\end{aligned}$$

Now, we can continue optimizing since x_3 has a positive coefficient.

We look at the constraints and see:

- $x_1 \implies x_3 \leq 6$
- $x_2 \implies x_3 \leq 1$

The more restrictive constraint is that $x_3 \leq 1$, so set $x_3 = 1$.

So we can let x_3 enter and x_2 leave.

$$x_3 = 1 - x_4 - 5x_2 - 2x_6 + x_5$$

This gives a new value for x_1. $x_1 = 2 - x_4 + 2x_2 + x_6 - x_5$

This gives a new value for ζ. $\zeta = 17 - 2x_4 - 5x_2 - 4x_6 - x_5$

So we have a new dictionary:

$$\begin{aligned} \zeta &= 17 - 2x_4 - 5x_2 - 4x_6 - x_5 \\ x_1 &= 2 - x_4 + 2x_2 + x_6 - x_5 \\ x_3 &= 1 - x_4 - 5x_2 - 2x_6 + x_5 \end{aligned}$$

Since we have no more optimizable variables (all variable coefficients of ζ are non-positive), we can no longer maximize ζ.

Then we have an optimal solution with $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0$, and value 17.

2.2 We first start by rewriting as a dictionary:

$$\begin{aligned} \zeta &= + 2x_1 + x_2 \\ x_3 &= 4 - 2x_1 - x_2 \\ x_4 &= 3 - 2x_1 - 3x_2 \\ x_5 &= 5 - 4x_1 - x_2 \\ x_6 &= 1 - x_1 - 5x_2 \end{aligned}$$

Then we can begin by entering with the largest variable, x_1.

We look at the constraints and see:

- $x_3 \implies x_1 \leq 2$
- $x_4 \implies x_1 \leq \frac{3}{2}$
- $x_5 \implies x_1 \leq \frac{5}{4}$
- $x_6 \implies x_1 \leq 1$

The more restrictive constraint is that $x_1 \leq 1$, so set $x_1 = 1$.

So we can let x_1 enter and x_6 leave.

$$x_1 = 1 + 5x_2 - x_6$$

This gives a new value for x_3. $x_3 = 2 - 9x_2 + 2x_6$

This gives a new value for x_4. $x_4 = 1 + 7x_2 + 2x_6$

This gives a new value for x_5. $x_5 = 1 + 19x_2 + 4x_6$

This gives a new value for ζ. $\zeta = 2 - 9x_2 - 2x_6$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= 2 - 9x_2 - 2x_6 \\
x_3 &= 2 - 9x_2 + 2x_6 \\
x_4 &= 1 + 7x_2 + 2x_6 \\
x_5 &= 1 + 19x_2 + 4x_6 \\
x_1 &= 1 + 5x_2 - x_6
\end{aligned}$$

Since we have no more optimizable variables (all variable coefficients of ζ are non-positive), we can no longer maximize ζ.

Then we have an optimal solution with $x_1 = 1, x_2 = 0$, and value 2.

2.3 Since the right hand side has some non-negative values, we must first create an auxiliary problem that takes care of this.

We want

$$\begin{array}{rr}
\text{maximize} & -x_0 \\
\text{subject to} & -x_1 - x_2 - x_3 - x_0 \leq -2 \\
 & 2\,x_1 - x_2 + x_3 - x_0 \leq 1 \\
 & x_i \geq 0
\end{array}$$

Now we can attempt to optimize this program.

We first start by rewriting as a dictionary:

$$\begin{aligned}
\zeta &= 0 - x_0 \\
x_4 &= -2 + x_1 + x_2 + x_3 + x_0 \\
x_5 &= 1 - 2x_1 + x_2 - x_3 + x_0
\end{aligned}$$

But this is infeasible. We need to pivot on x_0.

We look at the constraints and see x_4 is the most infeasible constraint.

We choose $x_0 = 2$ and pivot.

So we can let x_0 enter and x_4 leave.

This gives a new value for x_0. $x_0 = 2 - x_1 - x_2 - x_3 + x_4$

This gives a new value for x_5. $x_5 = 3 - 3x_1 - 2x_3 + x_4$

This gives a new value for ζ. $\zeta = -2 + x_1 + x_2 + x_3 - x_4$

$$\begin{aligned}
\zeta &= -2 + x_1 + x_2 + x_3 - x_4 \\
x_0 &= 2 - x_1 - x_2 - x_3 + x_4 \\
x_5 &= 3 - 3x_1 - 2x_3 + x_4
\end{aligned}$$

Now, this dictionary is feasible, so we can start to optimize it. We choose x_1 to pivot on.

We look at the constraints and see:

- $x_0 \implies x_1 \leq 2$
- $x_5 \implies x_1 \leq 1$

The more restrictive constraint is that $x_1 \leq 1$, so set $x_1 = 1$.

So we can let x_1 enter and x_5 leave.

This gives a new value for x_1. $x_1 = 1 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5$

This gives a new value for x_0. $x_0 = 1 - x_2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5$

This gives a new value for ζ. $\zeta = -1 + x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 - \frac{1}{3}x_5$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= -1 + x_2 + \frac{1}{3}x_3 - \frac{2}{3}x_4 - \frac{1}{3}x_5 \\
x_0 &= 1 - x_2 - \frac{1}{3}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 \\
x_1 &= 1 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5
\end{aligned}$$

Now, we can continue optimizing since x_2 has a positive coefficient.

We look at the constraints and see:

- $x_0 \implies x_2 \leq 1$

Since there's only one constraint, we set $x_2 = 1$

So we can let x_2 enter and x_0 leave.

This gives a new value for x_2. $x_2 = 1 - \frac{1}{3}x_3 - x_0 + \frac{2}{3}x_4 + \frac{1}{3}x_5$

This gives a new value for x_1. $x_1 = 1 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5$

This gives a new value for ζ. $\zeta = -x_0$

So we have a new dictionary:

$$
\begin{aligned}
\zeta &= \qquad\qquad\quad - x_0 \\
x_2 &= 1 - \frac{1}{3}x_3 - x_0 + \frac{2}{3}x_4 + \frac{1}{3}x_5 \\
x_1 &= 1 - \frac{2}{3}x_3 \qquad\quad + \frac{1}{3}x_4 - \frac{1}{3}x_5
\end{aligned}
$$

Since there are no more positive variable coefficients in the objective function, we have an optimal solution for this auxiliary program.

Then we should be able to convert back to the original problem and find an optimal solution.

We need to get rid of x_0, which is non-basic, so $x_0 = 0$. Then we bring back the original objective function $\zeta = 2x_1 - 6x_2 = -4 + \frac{2}{3}x_3 - \frac{10}{3}x_4 - \frac{8}{3}x_5$

So we have the following dictionary to try and optimize.

$$
\begin{aligned}
\zeta &= -4 + \frac{2}{3}x_3 - \frac{10}{3}x_4 - \frac{8}{3}x_5 \\
x_2 &= \quad 1 - \frac{1}{3}x_3 + \frac{2}{3}x_4 + \frac{1}{3}x_5 \\
x_1 &= \quad 1 - \frac{2}{3}x_3 + \frac{1}{3}x_4 - \frac{1}{3}x_5
\end{aligned}
$$

Now, we can continue optimizing since x_3 has a positive coefficient.

We look at the constraints and see:

- $x_2 \implies x_3 \leq 3$
- $x_1 \implies x_3 \leq \frac{3}{2}$

The more restrictive constraint is that $x_3 \leq \frac{3}{2}$, so set $x_3 = \frac{3}{2}$.

So we can let x_3 enter and x_1 leave.

This gives a new value for x_3. $x_3 = \frac{3}{2} - \frac{3}{2}x_1 + \frac{1}{2}x_4 - \frac{1}{2}x_5$

This gives a new value for x_2. $x_2 = \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_4 + \frac{1}{2}x_5$

This gives a new value for ζ. $\zeta = -3 - x_1 - 3x_4 - 3x_5$

So we have a new dictionary:

$$\begin{aligned}
\zeta &= -3 - x_1 - 3x_4 - 3\,x_5 \\
x_2 &= \frac{1}{2} + \frac{1}{2}x_1 + \frac{1}{2}x_4 + \frac{1}{2}x_5 \\
x_3 &= \frac{3}{2} - \frac{3}{2}x_1 + \frac{1}{2}x_4 - \frac{1}{2}x_5
\end{aligned}$$

Since we have no more optimizable variables (all variable coefficients of ζ are non-positive), we can no longer maximize ζ.

Then we have an optimal solution with $x_1 = 0, x_2 = \frac{1}{2}, x_3 = \frac{3}{2}$, and value -3.