Virtual homological torsion in low dimensions

University of Haifa, Dec. 14 2025

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Overview

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- A 2-dimensional setting

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- What’s next?

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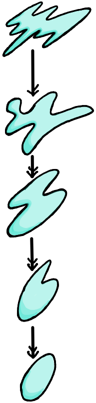
topological complexity of finite covers grows at a rate reflecting geometry



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Conjecture (Bergeron–Venkatesh)

Let M be a closed, hyperbolic 3-manifold and let

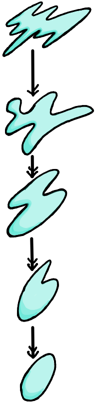
$$M = M_0 \twoheadleftarrow M_1 \twoheadleftarrow M_2 \twoheadleftarrow \cdots$$

be a cofinal tower of finite-sheeted normal covers of M, that is,

$$\bigcap_n \pi_1(M_n) = 1.$$

Then

$$\lim_{n\to\infty} \frac{\log(|\mathrm{Tor}(H_1(M_n;\mathbb{Z}))|)}{[\pi_1(M) : \pi_1(M_n)]} = \frac{\mathrm{vol}(M)}{6\pi}.$$



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- **Geometry $\leftrightsquigarrow$ Topology:** "Gauss-Bonnet on steroids" –

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  BV Conjecture: $\forall p$, $$\lim_{n\to\infty} \frac{\log(|\mathrm{Tor}_p(H_1(M_n;\mathbb{Z}))|)}{[\pi_1(M):\pi_1(M_n)]} = 0,$$

  so torsion must involve larger and larger sporadic primes.

Open problem

Is there a finitely presented residually finite group G, and a residual normal chain

$$G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$$

(i.e. $\bigcap_n G_n = \{1\}$), such that

$$\lim_{n\to\infty} \frac{\log(|\mathrm{Tor}(G_n^{\mathrm{ab}})|)}{[G : G_n]} > 0 \ ?$$

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- **Finite presentation:** (Kar–Kropholler–Nikolov '17) For every $f : \mathbb{N} \longrightarrow \mathbb{N}$ there is a fg G and an exhausting normal chain $G \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$ such that $|\mathrm{Tor}(G^{\mathrm{ab}})| > f([G : G_n])$.

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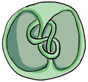
- **Trivial intersection:** (Silver–Williams '02) Cyclic covers of knot complements in $\mathbb{S}^3$.



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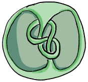

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- **Dimension:** (Avramidi–Okun–Schreve '21) Right-angled Artin groups where $\log|\mathrm{Tor}(H_2(-;\mathbb{Z}))|$ grows exponentially in the index.
- **Normality:** (Liu '19) Cofinal towers of finite-sheeted *non-normal* covers of closed, hyperbolic M^3 with exponential growth.

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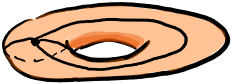

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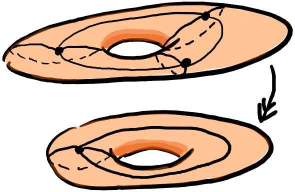

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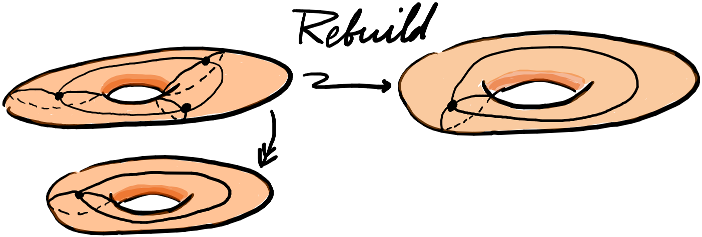

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Theorem (Sun '15)

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In graph manifolds:

- Solv or Seifert-fibred – restrictions on torsion.
- Non-trivial JSJ – any number can divide $|\mathrm{Tor}(H_1(\widehat{M};\mathbb{Z}))|$ (F–Hughes–Valiunas).

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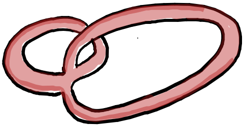

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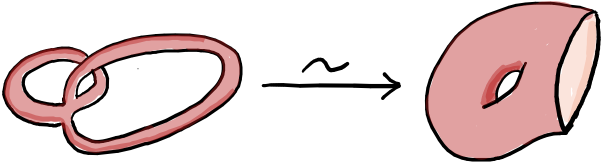

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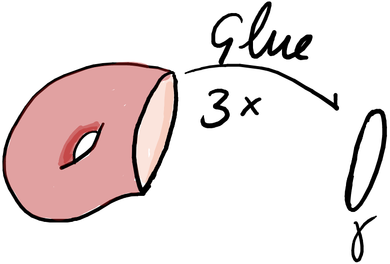

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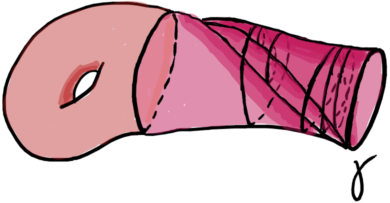

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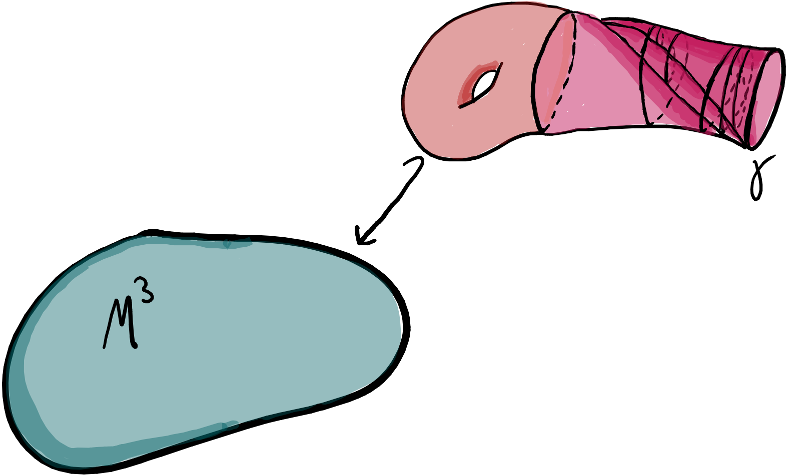

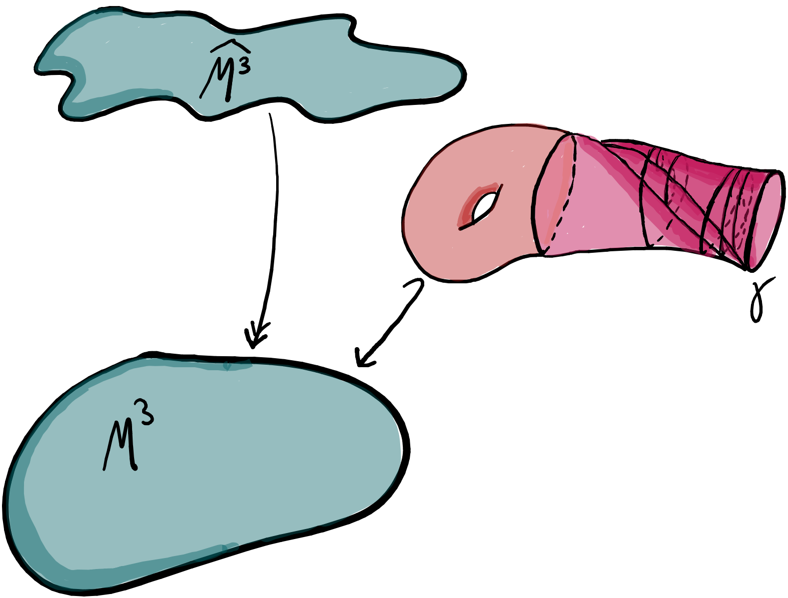
$\widehat{M}^3$
M^3
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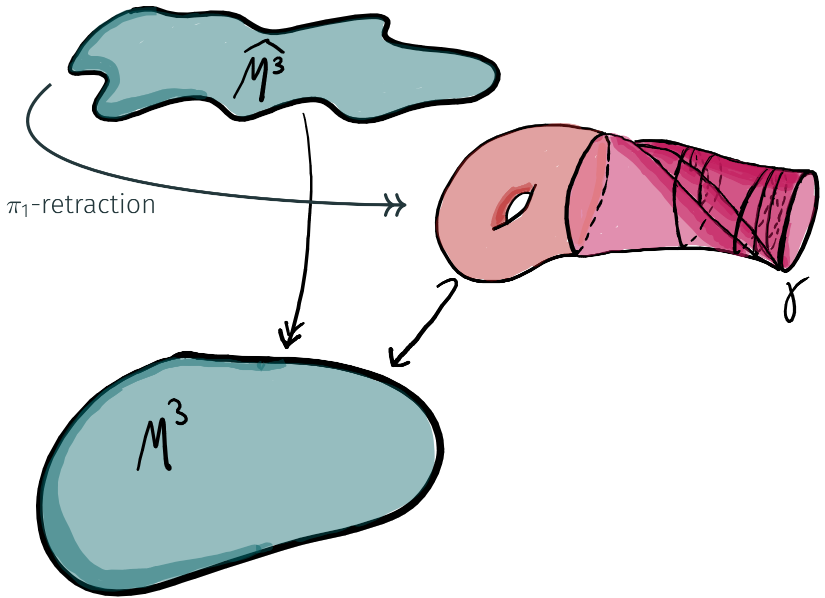

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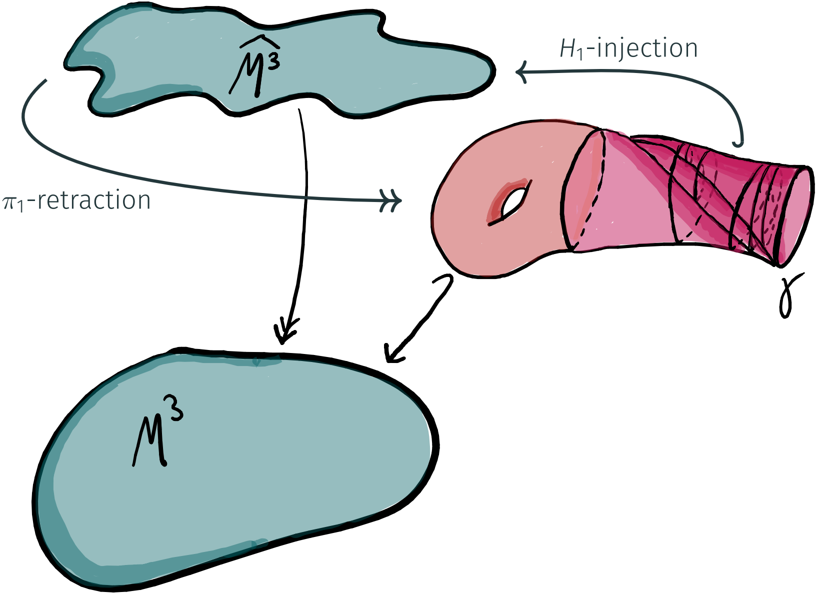

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(Fundamental groups of) **graphs joined together by cylinders.**

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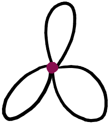

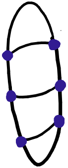

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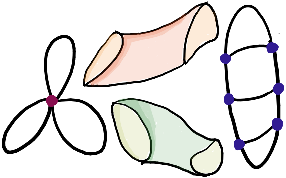

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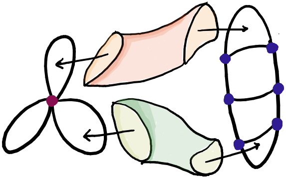

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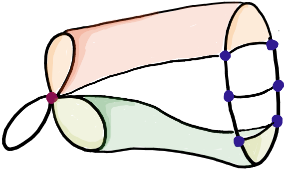

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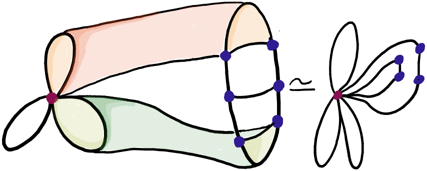

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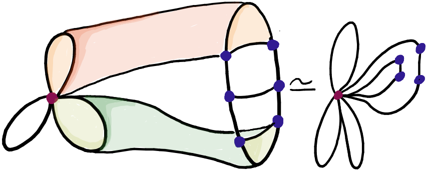

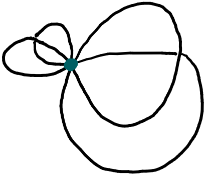

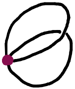

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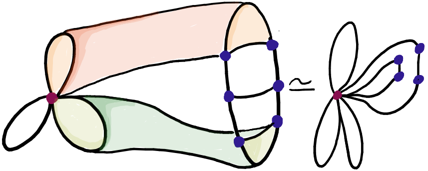

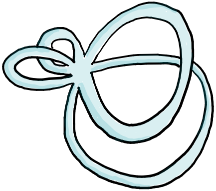

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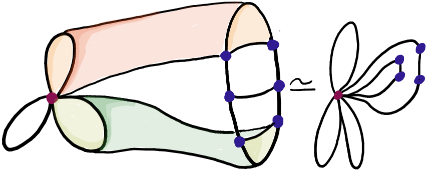

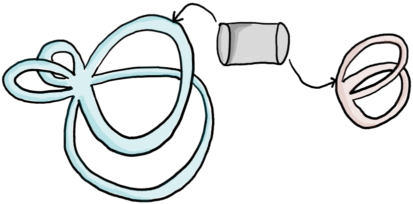

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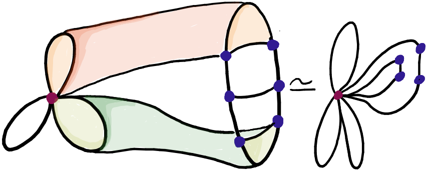

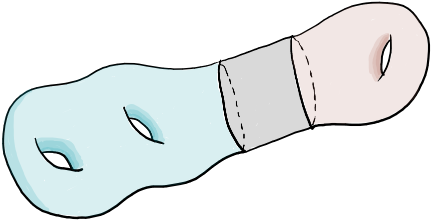

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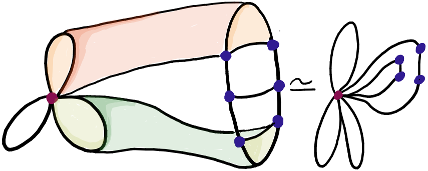

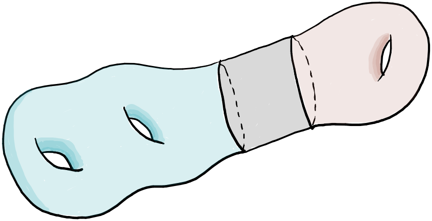

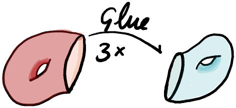

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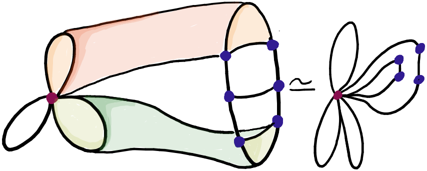

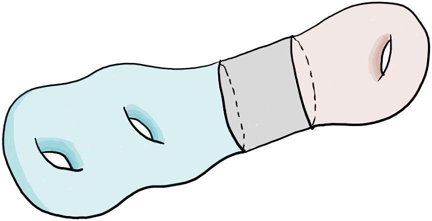

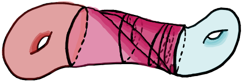

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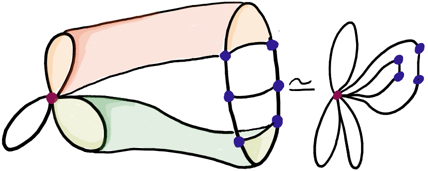

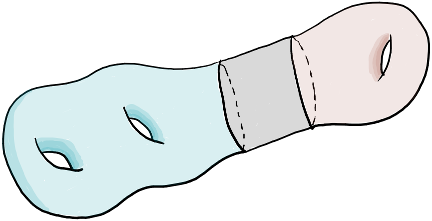

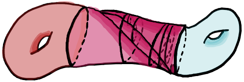

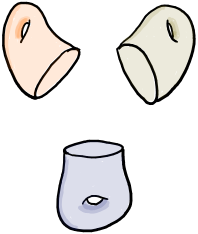

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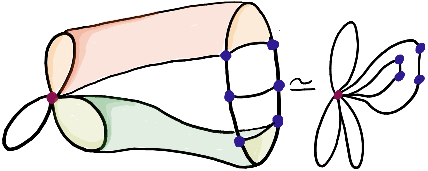

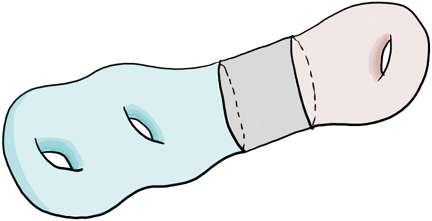

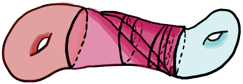

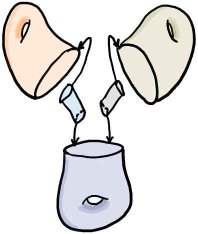

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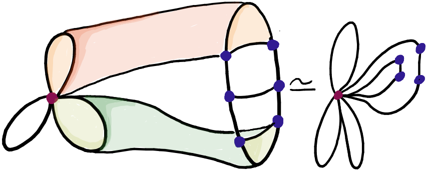
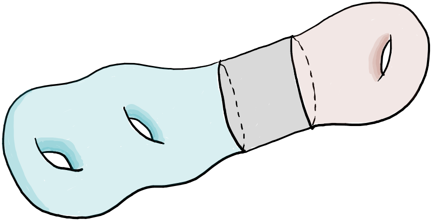
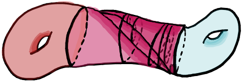
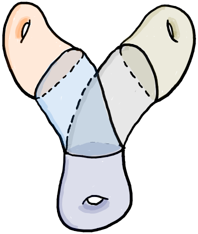


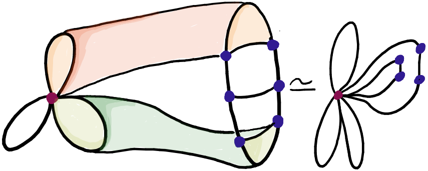


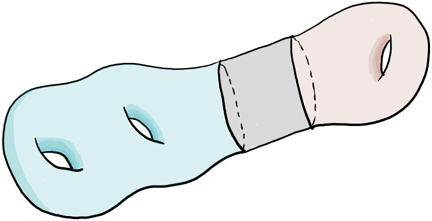


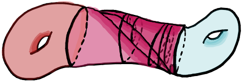


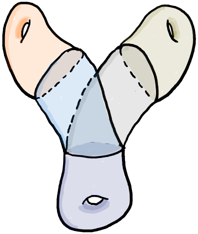
(Hyperbolic) Graphs of free groups with cyclic edges

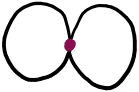
(Fundamental groups of) **graphs joined together by cylinders.**











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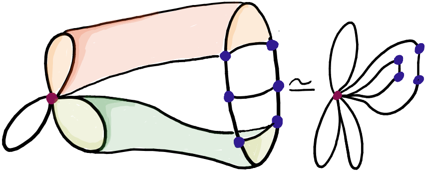

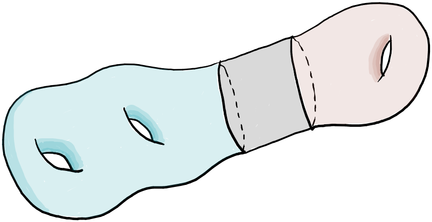

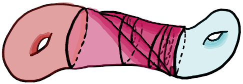

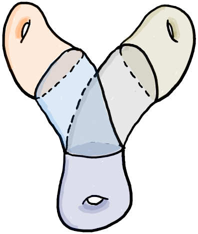

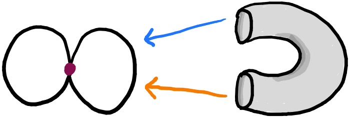

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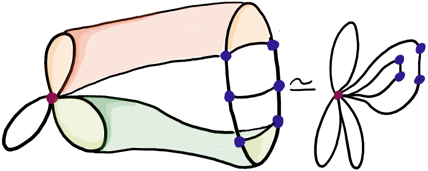

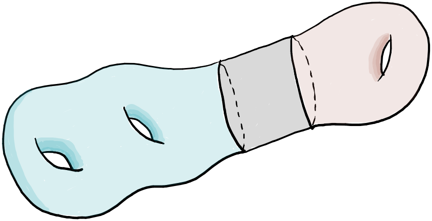

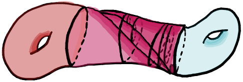

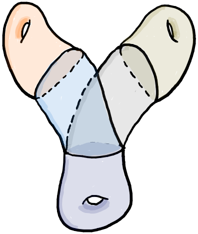

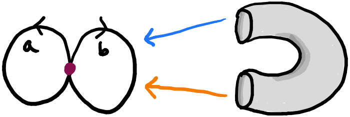

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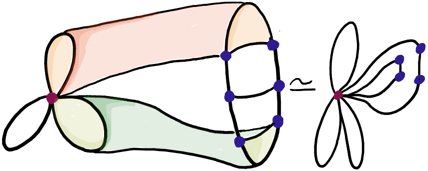

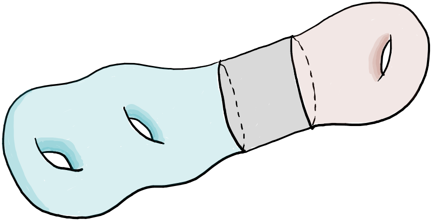

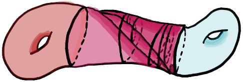

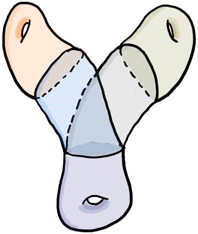

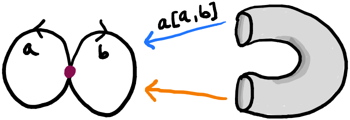

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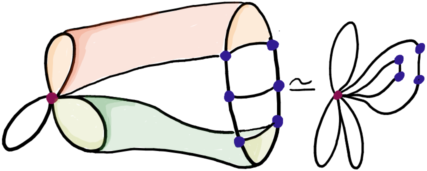

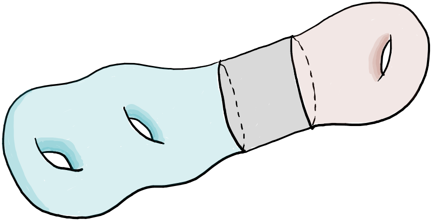

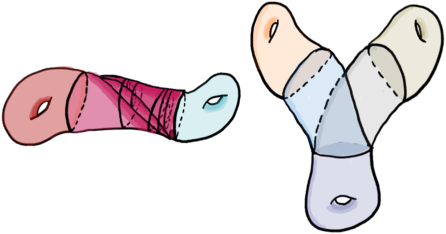

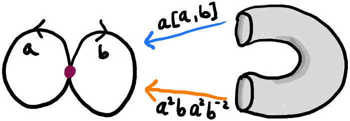

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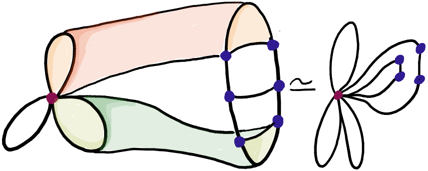
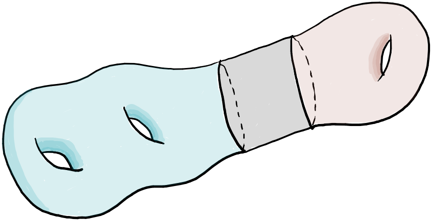
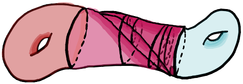
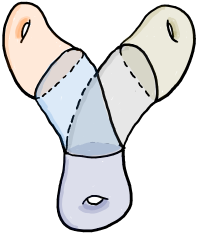
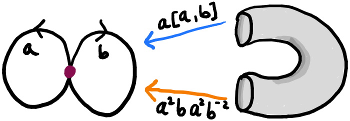
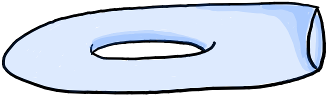






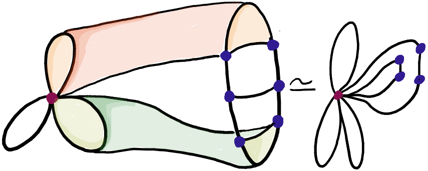


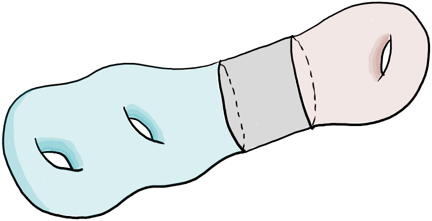


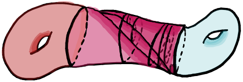


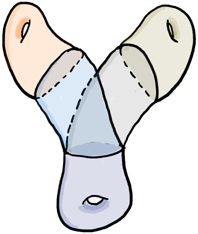
(Hyperbolic) Graphs of free groups with cyclic edges

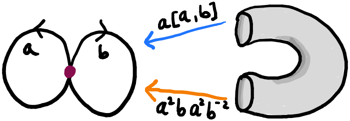
(Fundamental groups of) **graphs joined together by cylinders.**

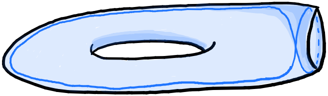












(Hyperbolic) Graphs of free groups with cyclic edges

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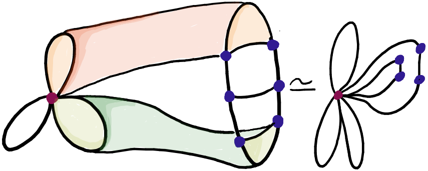

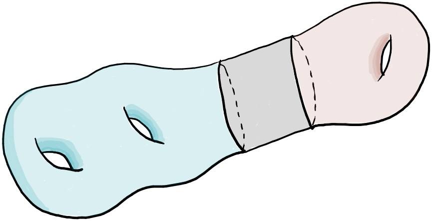

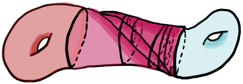

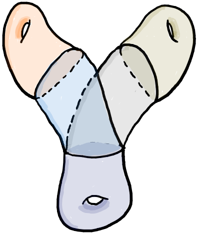

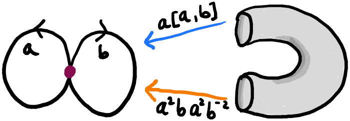

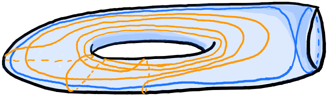

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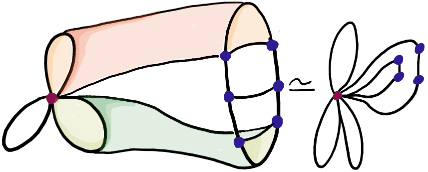

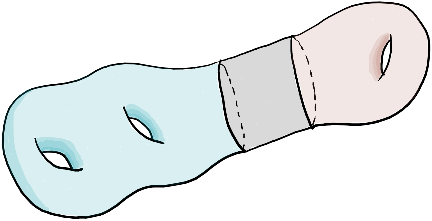

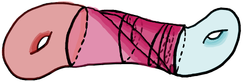

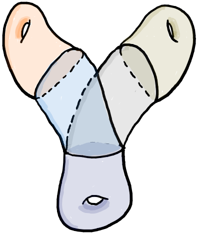

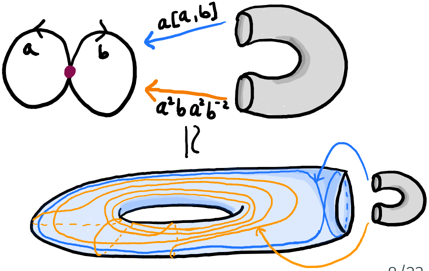

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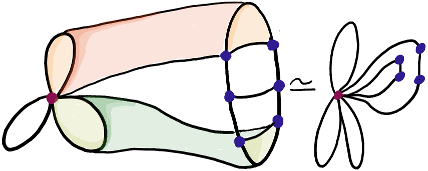

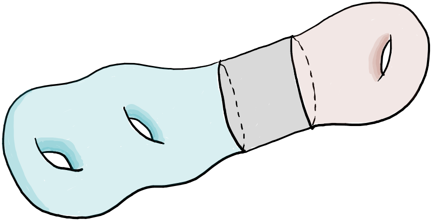

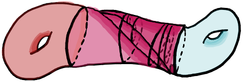

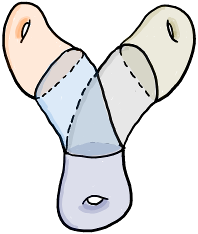

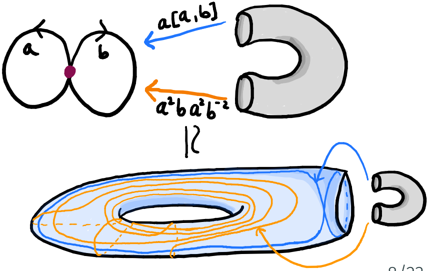

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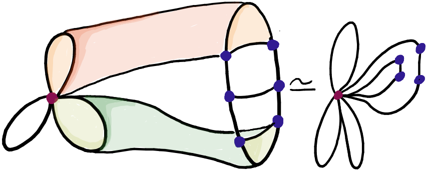
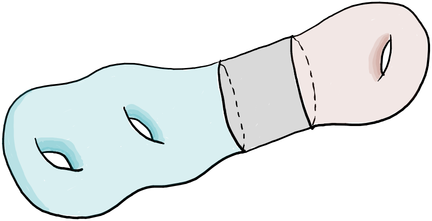
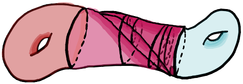
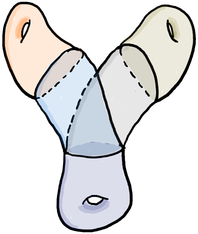
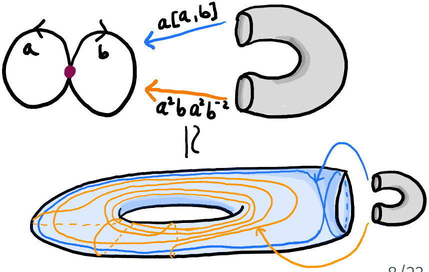


Free and surface groups
(and their free products)









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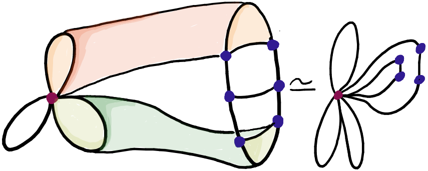

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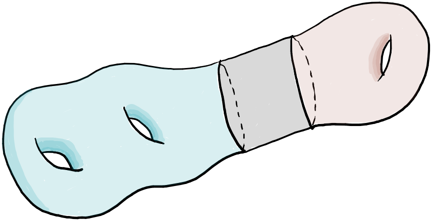

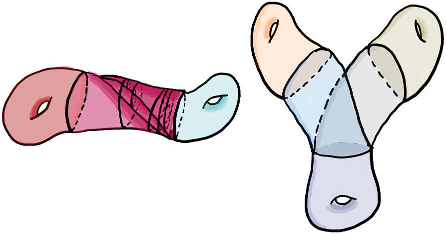

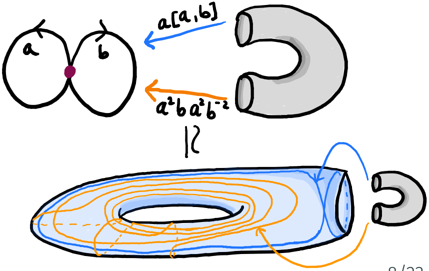

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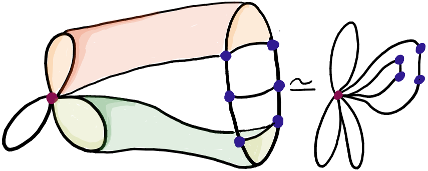

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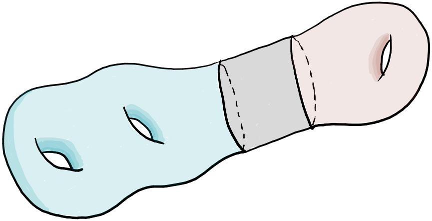
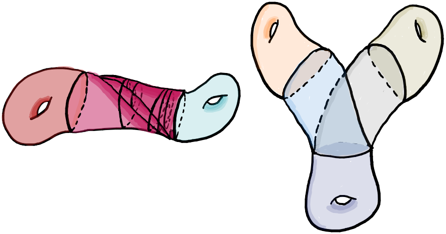
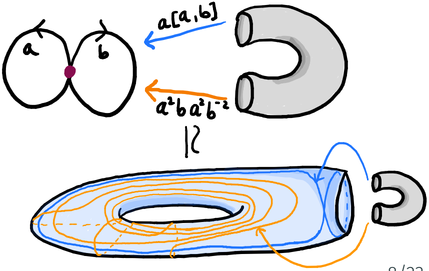




"More than a surface"



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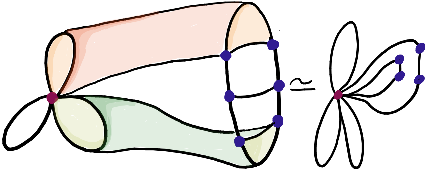

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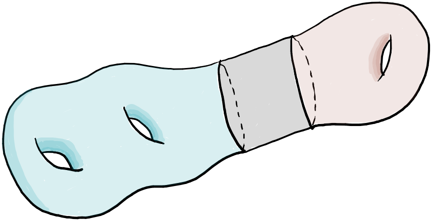

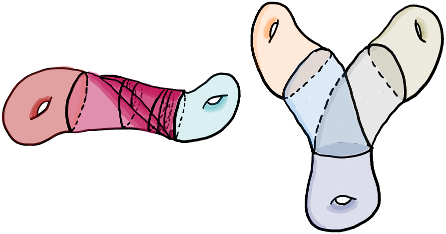

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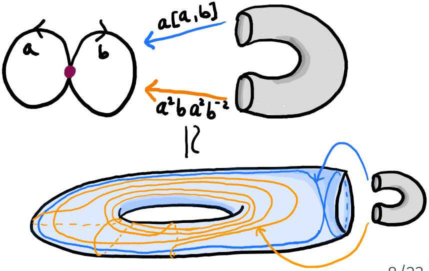


"Has a *rigid* vertex"

Homological torsion: Existence

Homological torsion: Existence

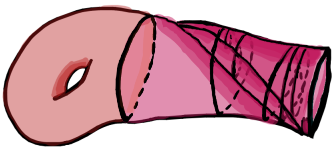
Theorem (Ascari-F '25)

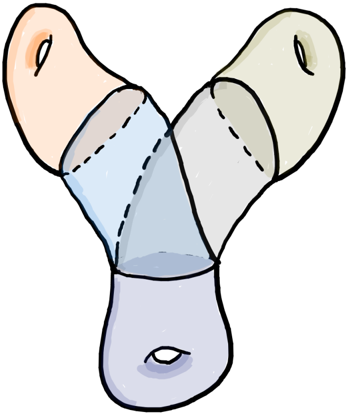
Let G be a hyperbolic graph of free groups with $\mathbb{Z}$ edges, which is not a free product of free and surface groups. Then G has abundant virtual homological torsion, that is,

for every finite abelian group A, there is a finite-index subgroup $G_0 \leq G$ such that

$$G_0^{\mathrm{ab}} = A \oplus \cdots .$$

Homological torsion: By the book





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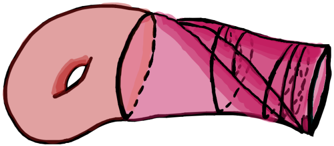

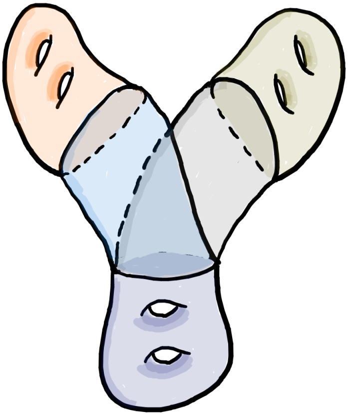

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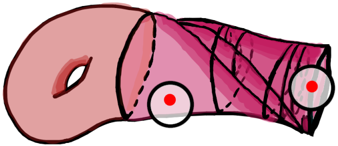

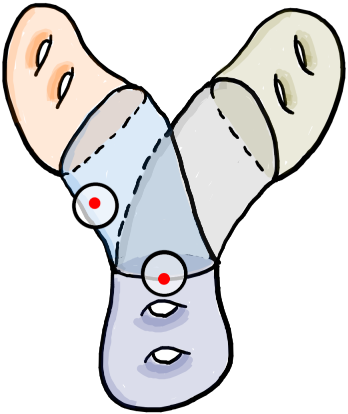

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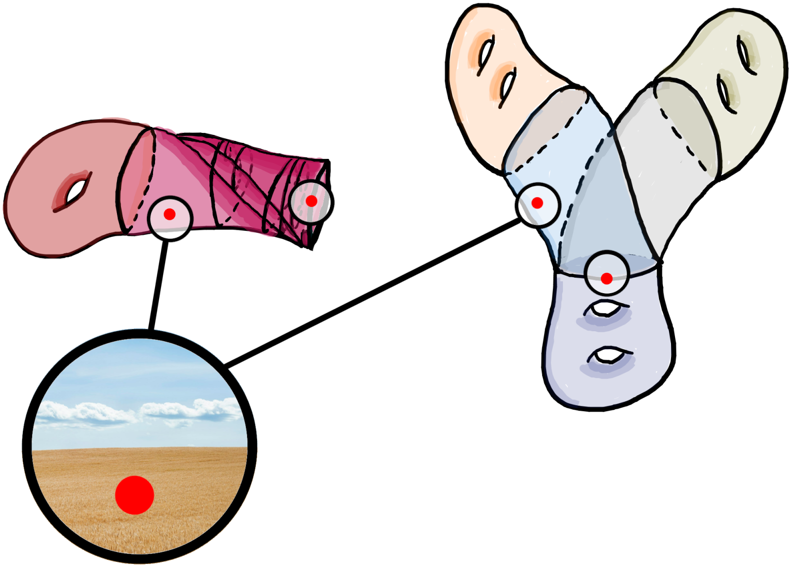

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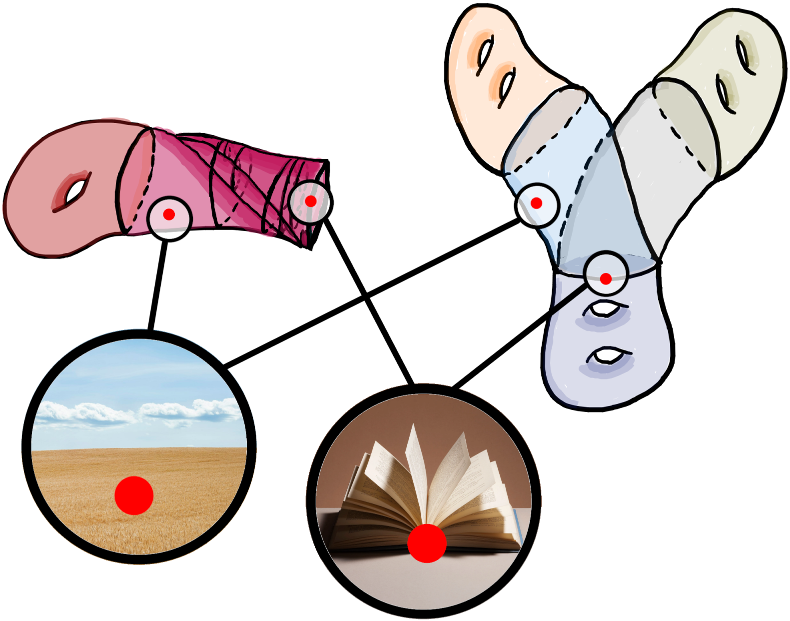

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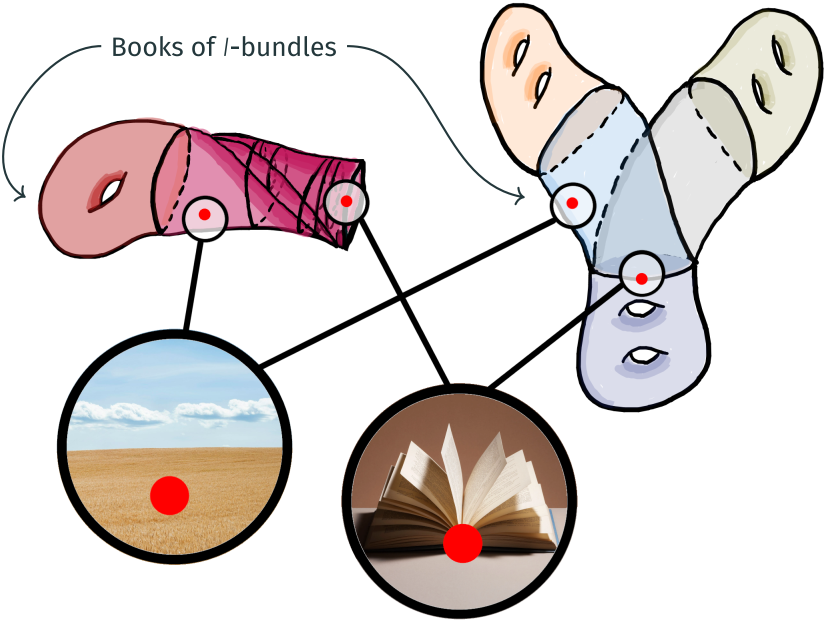

Homological torsion: Systems of surfaces

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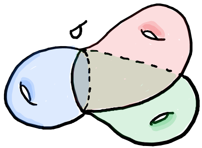

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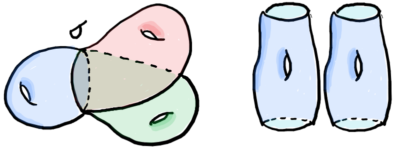

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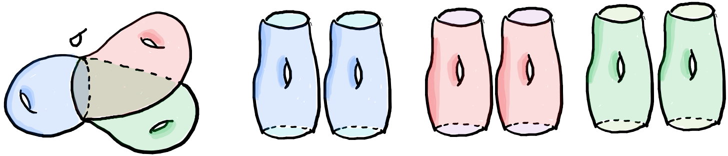

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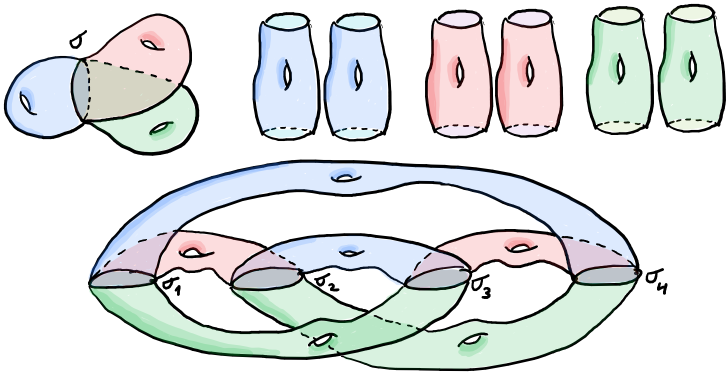

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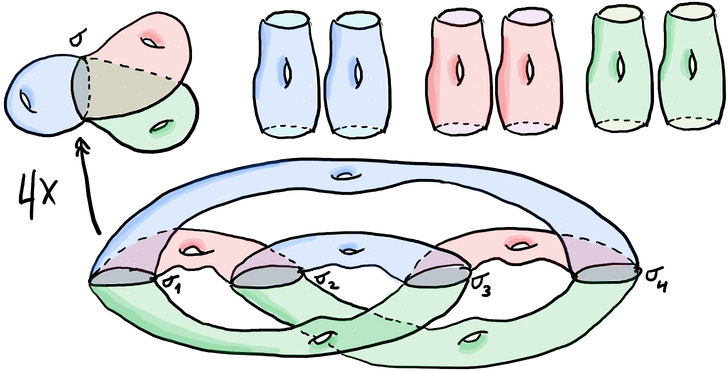

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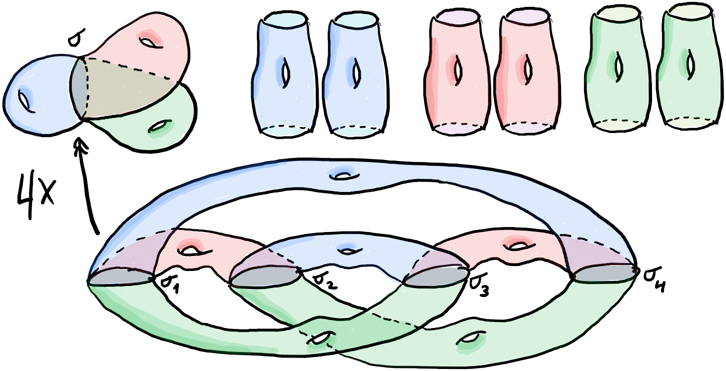

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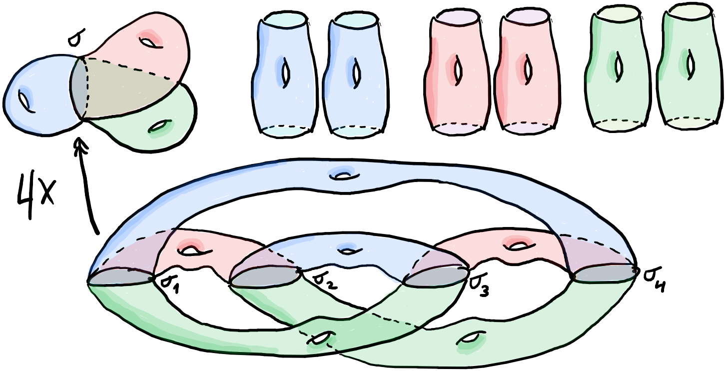

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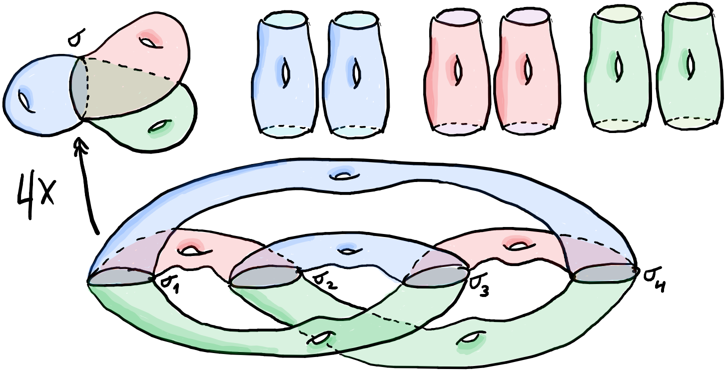

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$$\implies \quad \begin{gathered} \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 \\ 2 \cdot \sigma_1 = 0 \end{gathered}$$

Homological torsion: Two and three

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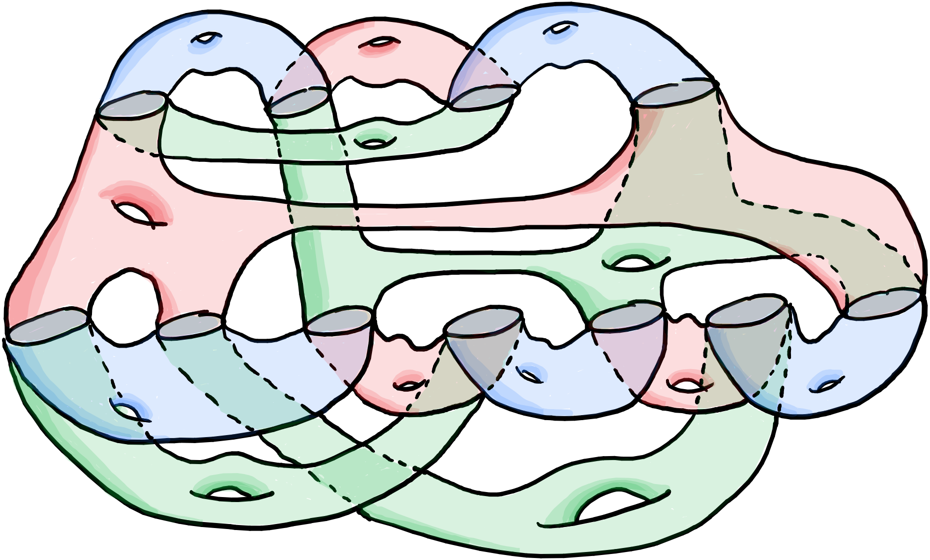
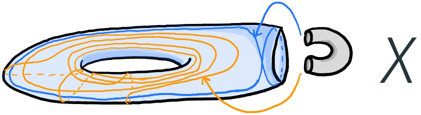


Enter a rigid vertex



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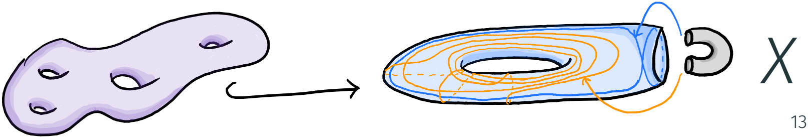
If G is not free then G has a surface subgroup.



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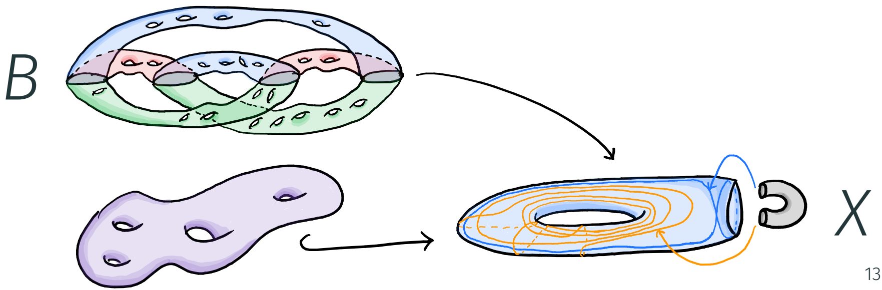

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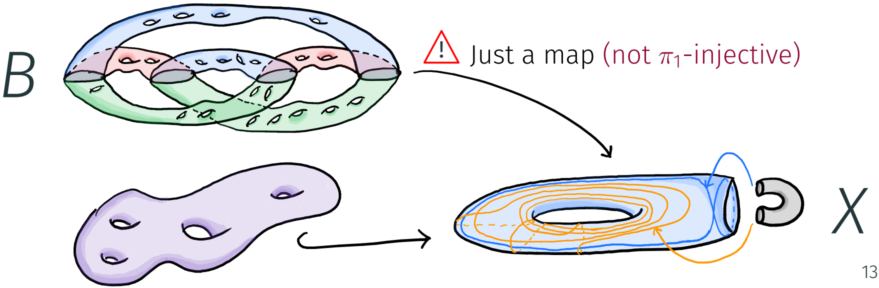

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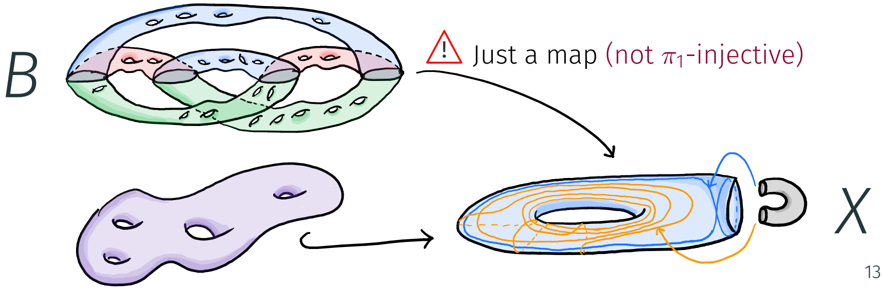

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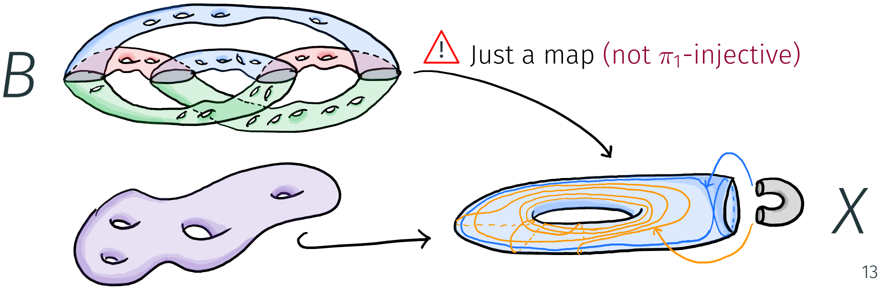

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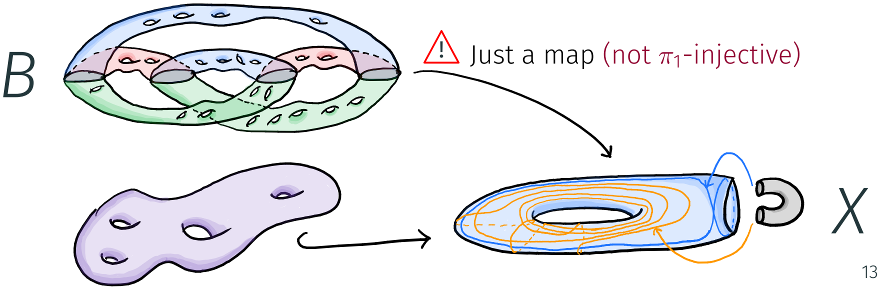

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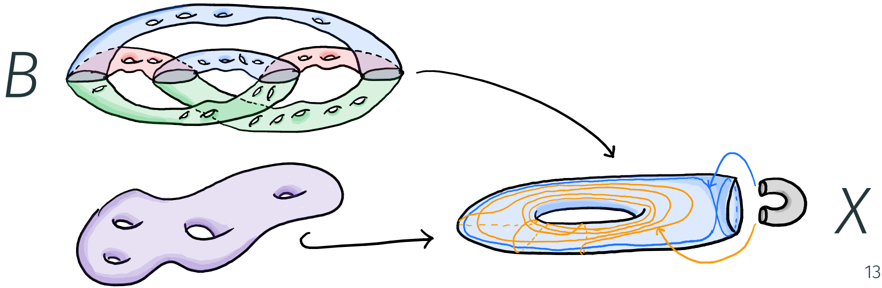

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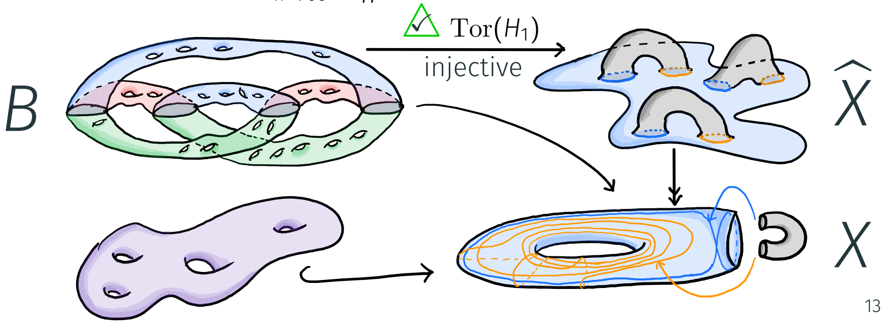
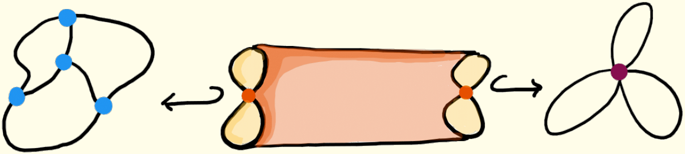
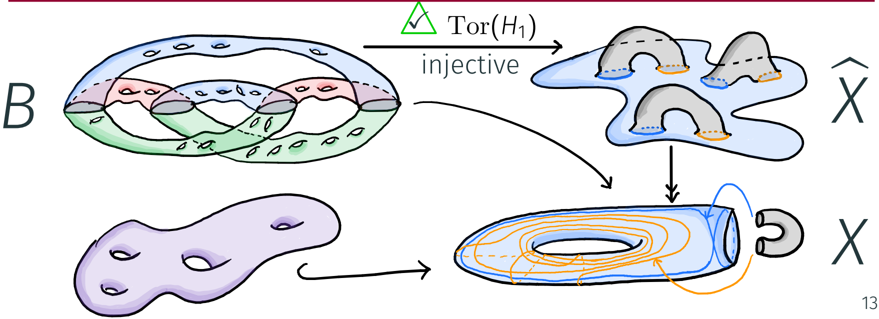


Enter a rigid vertex

Applicable to other settings?

e.g. graphs of graphs.





Profinite rigidity 101

Let G be a finitely generated *residually finite* group (every $g \in G$ survives in some finite quotient $q : G \twoheadrightarrow Q$).

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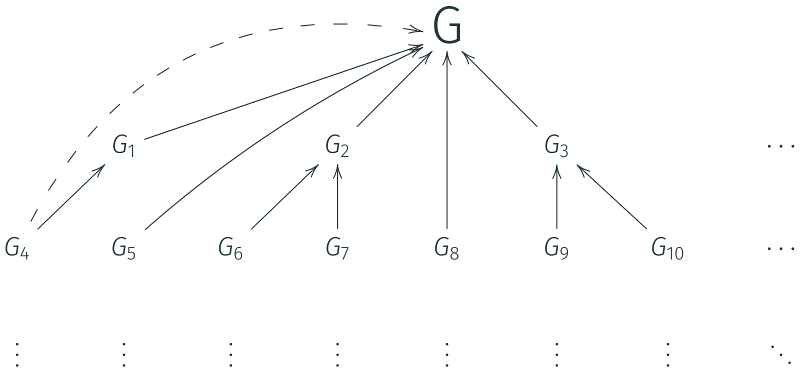

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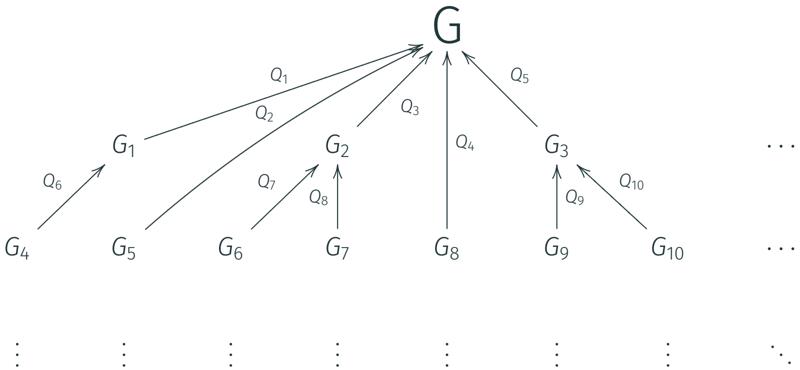

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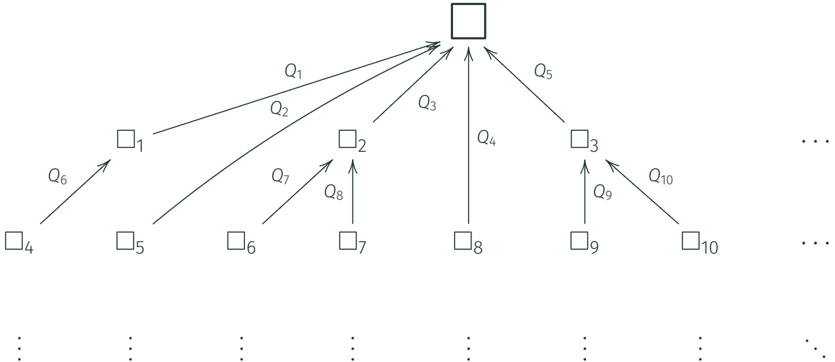

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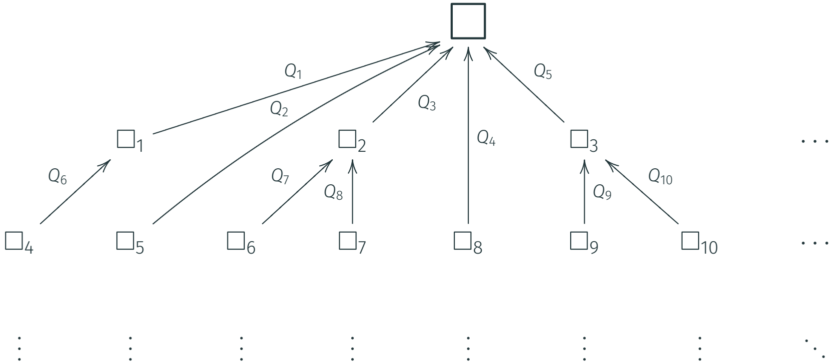

Question

Which fg residually finite groups H can fill the empty $\square$'s?

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Question (Remeslennikov)

Are finitely generated (non-abelian) free groups profinitely rigid?

The largest natural class of groups where Remeslennikov's question has an answer:

Profinite rigidity 201

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Theorem (Ascari-F '25)

Free products of free and surface groups are profinitely rigid amongst hyperbolic graphs of free groups with $\mathbb{Z}$ edges.

Growth: non-normal towers

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Theorem (Ascari-F '25)

There exists a tower of finite-index non-normal subgroups

$$G \geq G_1 \geq G_2 \geq \cdots, \qquad \textit{with} \quad \bigcap_n G_n = \{1\},$$

such that simultaneously for every prime p,

$$\lim_{n\to\infty} \frac{\log(|\mathrm{Tor}_p(G_n^{\mathrm{ab}}))|)}{[G : G_n]} > 0.$$

Fractal-like towers

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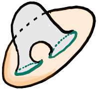

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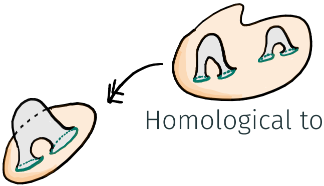

Homological torsion

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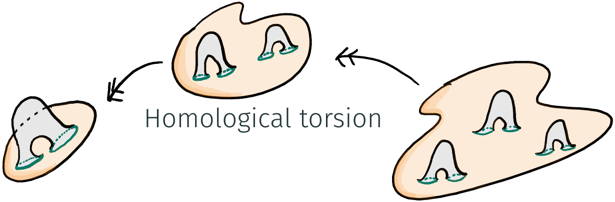

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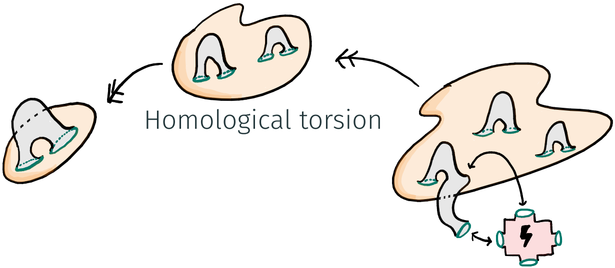

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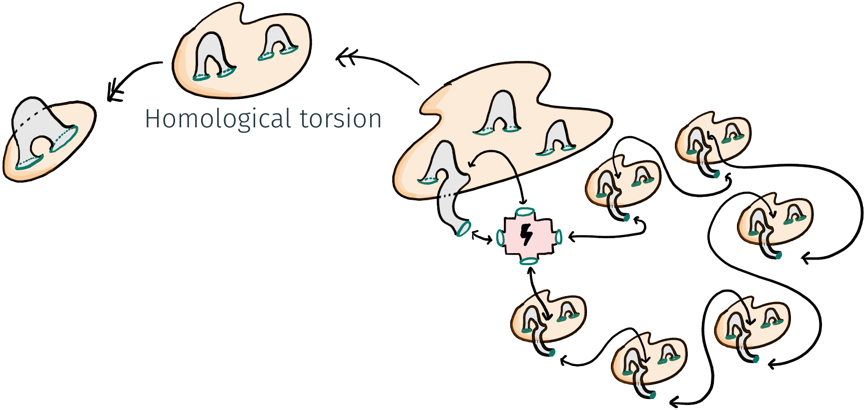

A lot of homological torsion

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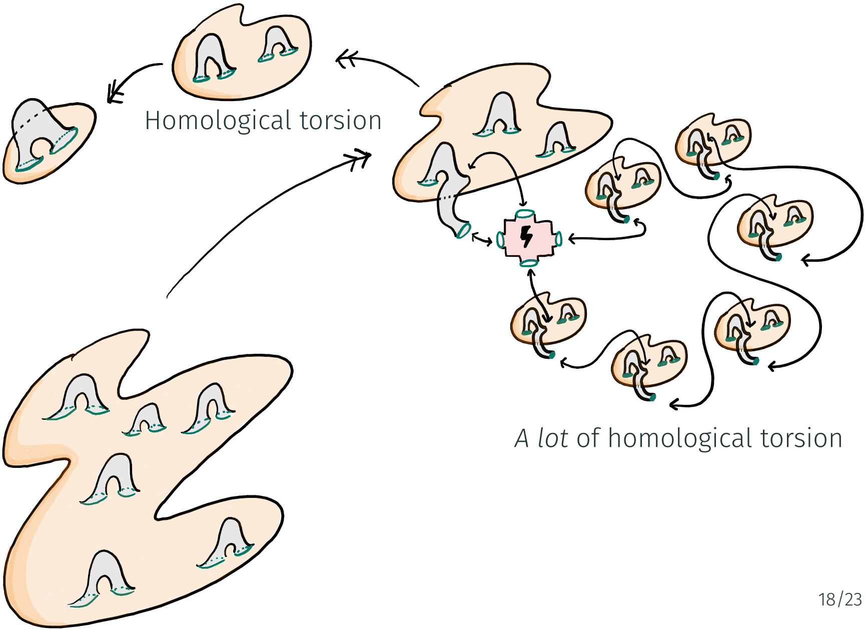

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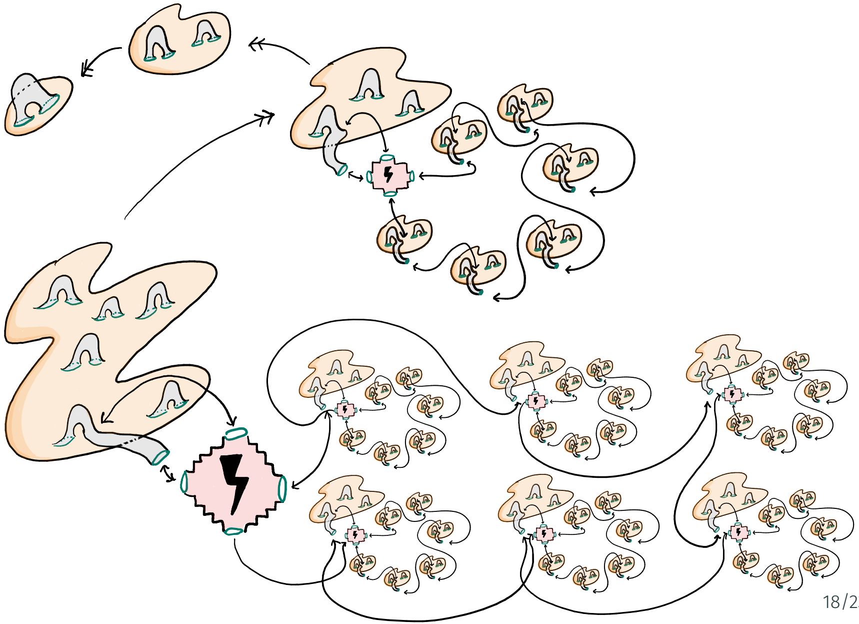

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- Bergeron-Venkatesh: such behaviour expected in dimension ≥ 4.

A computation: unwrapping cylinders

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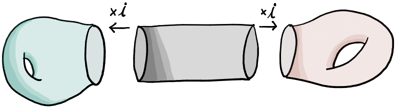
$$G_i = \langle a, b \rangle_{[a,b]^i = [c,d]^i} \langle c, d \rangle$$



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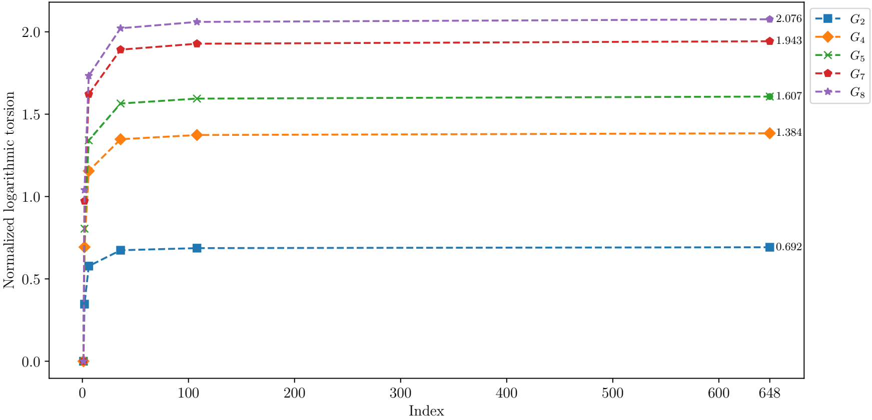

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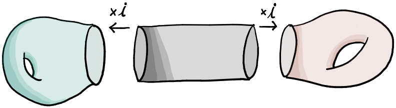

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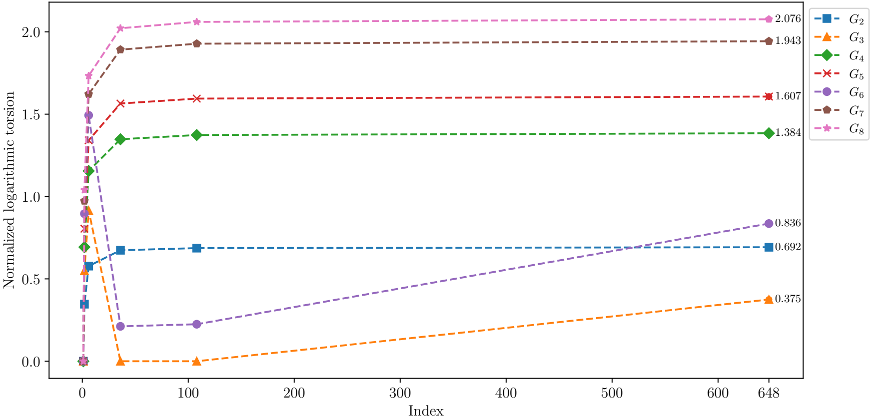

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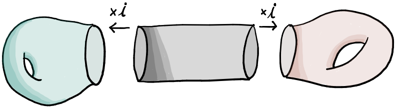

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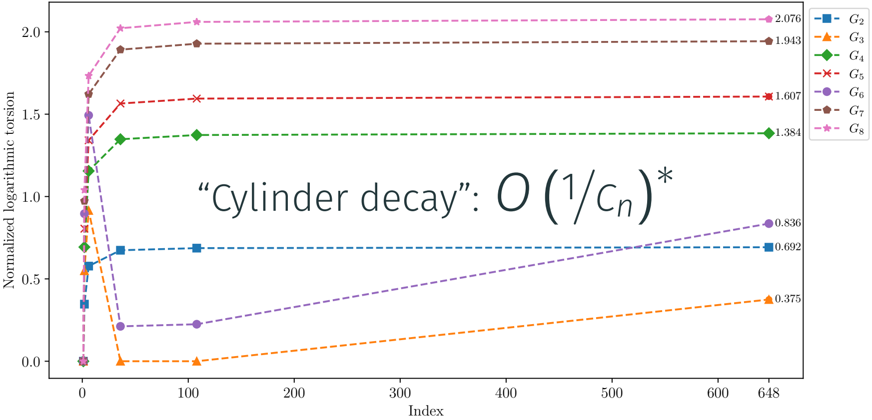

"Cylinder decay": $O\left({}^{1}\!/{C_n}\right)^*$

* C_n measures the wrapping degree of the cylinder in the n^{th} cover.

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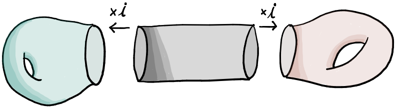

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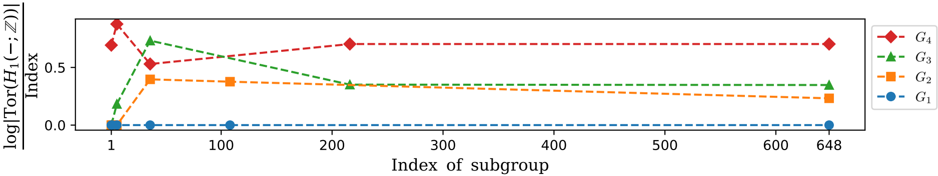
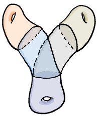
- **Computations:**



$G_1 = \pi_1\Bigg($  $\Bigg)$

G_{i+1} obtained from G_i by adding a single letter to its presentation

The landscape

The landscape

$\mathrm{Tor}(H_1)$	(Hyp) 3-manifolds	Surface groups	Books of I-bundles	Rigid vertex
Abundance	✓ (Sun '15)	X	✓ (Ascari-F '25)	✓ (Ascari-F '25)
Non-normal exponential growth	✓ (Liu '19)	X	✓ (Ascari-F '25)	✓ (Ascari-F '25)
Normal growth rate	? (Exp.)	X	✓ Subexp. (F '25)	?

The landscape

$\mathrm{Tor}(H_1)$	(Hyp) 3-manifolds	Surface groups	Books of I-bundles	Rigid vertex
Abundance	✓ (Sun ’15)	✗	✓ (Ascari-F ’25)	✓ (Ascari-F ’25)
Non-normal exponential growth	✓ (Liu ’19)	✗	✓ (Ascari-F ’25)	✓ (Ascari-F ’25)
Normal growth rate	? (Exp.)	✗	✓ Subexp. (F ’25)	✓ Subexp. (F, recently)

The landscape

$\mathrm{Tor}(H_1)$	(Hyp) 3-manifolds	Surface groups	Books of I-bundles	Rigid vertex
Abundance	✓ (Sun ’15)	X	✓ (Ascari-F ’25)	✓ (Ascari-F ’25)
Non-normal exponential growth	✓ (Liu ’19)	X	✓ (Ascari-F ’25)	✓ (Ascari-F ’25)
Normal growth rate	? (Exp.)	X	✓ Subexp. (F ’25)	✓ Subexp. (F, recently)
“Cylinder decay” rate			$O\left({}^{1}/{C_n}\right)$	$O\left({}^{\log(C_n)}/{C_n}\right)$

What’s next?

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- "Cylinder decay" rate:

  matching lower bounds $\rightsquigarrow$ applications to profinite rigidity

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- **"Cylinder decay" rate:**

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- **Graphs of graphs:**

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  $\rightsquigarrow$ cross the threshold? 

Thank you for listening!