

# Virtual homological torsion in low dimensions

HUJI Groups and Dynamics seminar, Dec. 11 2025

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University of Bonn

# Overview

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-  Detour: an application
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- What's next?

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**topological complexity of finite covers grows at a rate reflecting geometry**

**Conjecture (Bergeron–Venkatesh)**

Let  $M$  be a closed, hyperbolic 3-manifold and let

$$M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

be a cofinal tower of finite-sheeted normal covers of  $M$ , that is,

$$\bigcap_n \pi_1(M_n) = 1.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}(H_1(M_n; \mathbb{Z}))|)}{[\pi_1(M) : \pi_1(M_n)]} = \frac{\text{vol}(M)}{6\pi}.$$



# At the crossroads

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- **Geometry  $\rightsquigarrow$  Topology:** “Gauss-Bonnet on steroids” –

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BV Conjecture:  $\forall p,$  
$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}_p(H_1(M_n; \mathbb{Z}))|)}{[\pi_1(M): \pi_1(M_n)]} = 0,$$

so torsion must involve larger and larger sporadic primes.

# Beyond 3-manifolds

## Open problem

Is there a finitely presented residually finite group  $G$ , and a residual normal chain

$$G \triangleright G_1 \triangleright G_2 \triangleright \dots$$

(i.e.  $\bigcap_n G_n = \{1\}$ ), such that

$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}(G_n^{\text{ab}})|)}{[G : G_n]} > 0 ?$$

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- **Finite presentation:** (Kar–Kropholler–Nikolov '17) For every  $f : \mathbb{N} \longrightarrow \mathbb{N}$  there is a fg  $G$  and an exhausting normal chain  $G \triangleright G_1 \triangleright G_2 \triangleright \dots$  such that  $|\text{Tor}(G^{\text{ab}})| > f([G : G_n])$ .

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- **Dimension:** (Avramidi–Okun–Schreve '21) Right-angled Artin groups where  $\log |\text{Tor}(H_2(–; \mathbb{Z}))|$  grows exponentially in the index.
- **Normality:** (Liu '19) Cofinal towers of finite-sheeted *non-normal* covers of closed, hyperbolic  $M^3$  with exponential growth.

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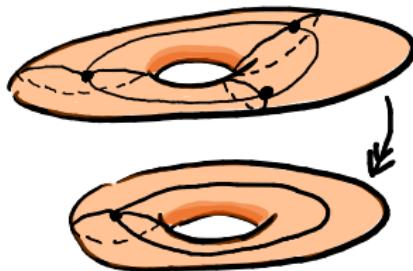


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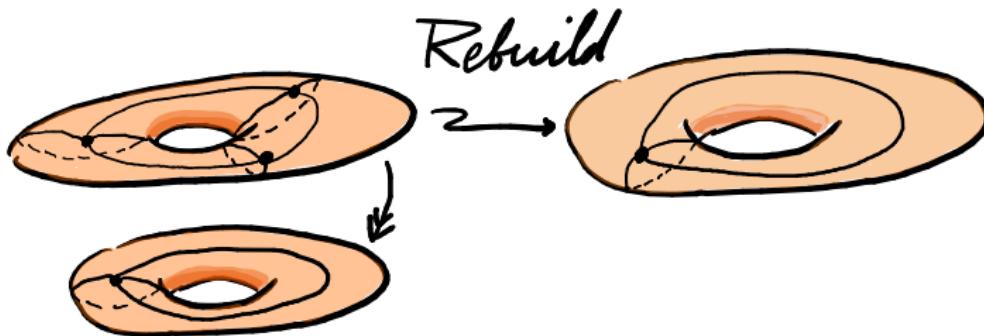


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### Theorem (Sun '15)

Let  $M$  be a closed hyperbolic 3-manifold and let  $A$  be a finite abelian group. There exists a finite-sheeted cover  $\hat{M} \rightarrow M$  such that

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In graph manifolds:

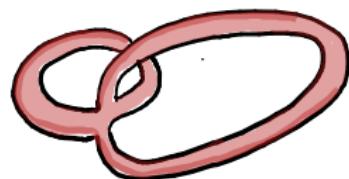
- Solv or Seifert-fibred – restrictions on torsion.
- Non-trivial JSJ – any number can divide  $|\text{Tor}(H_1(\widehat{M}; \mathbb{Z}))|$  (F–Hughes–Valiuunas).

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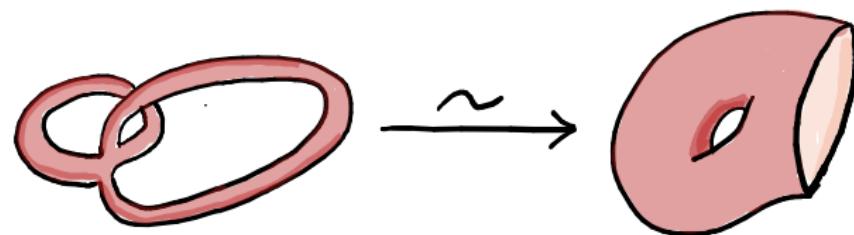
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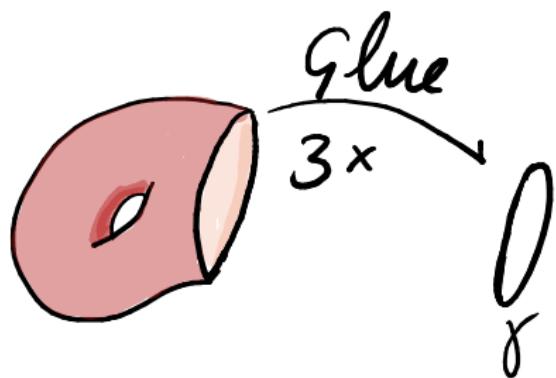
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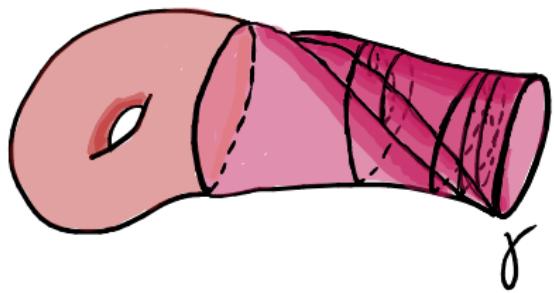
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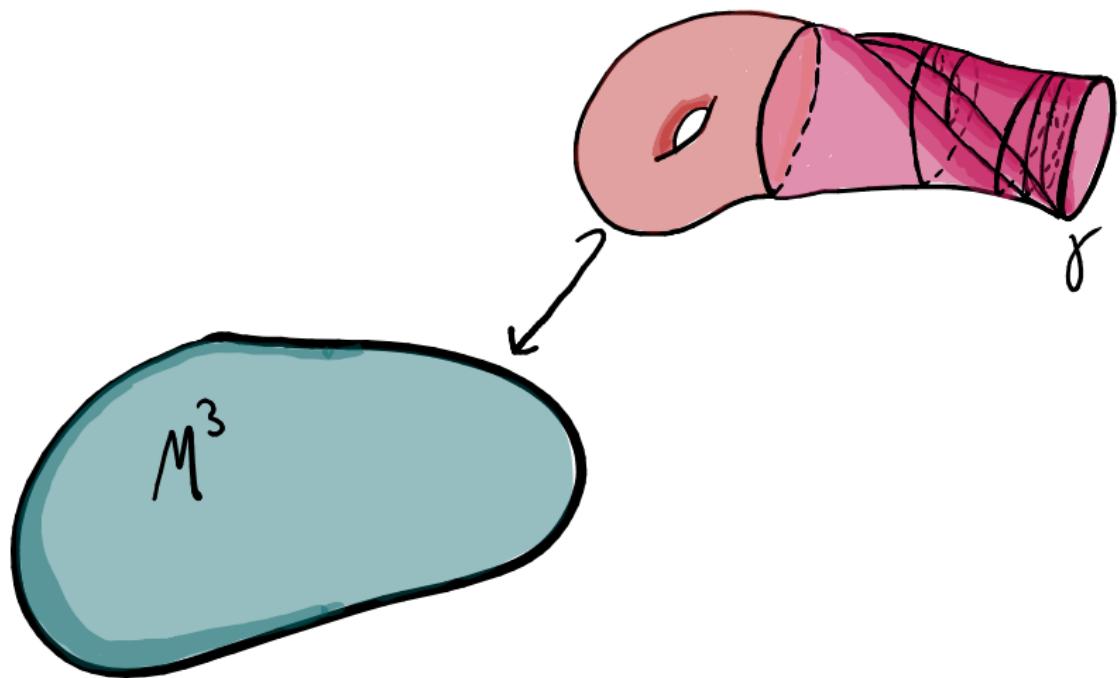
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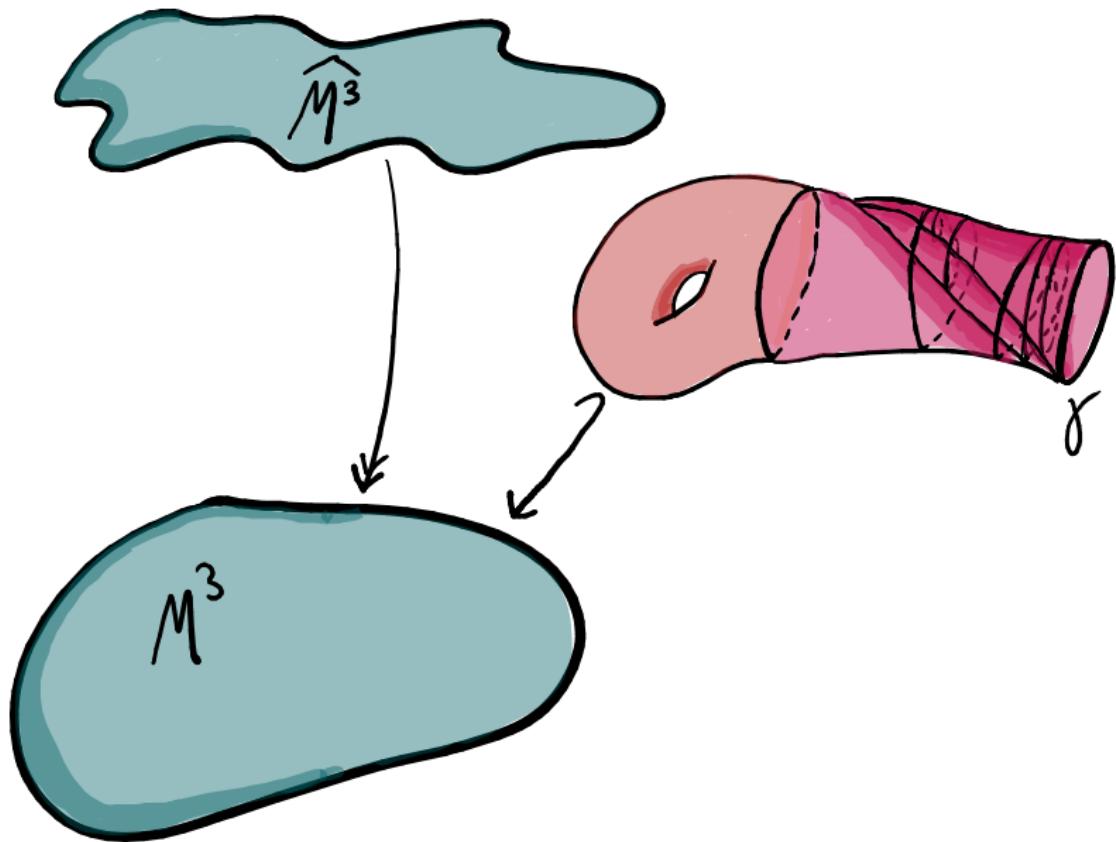
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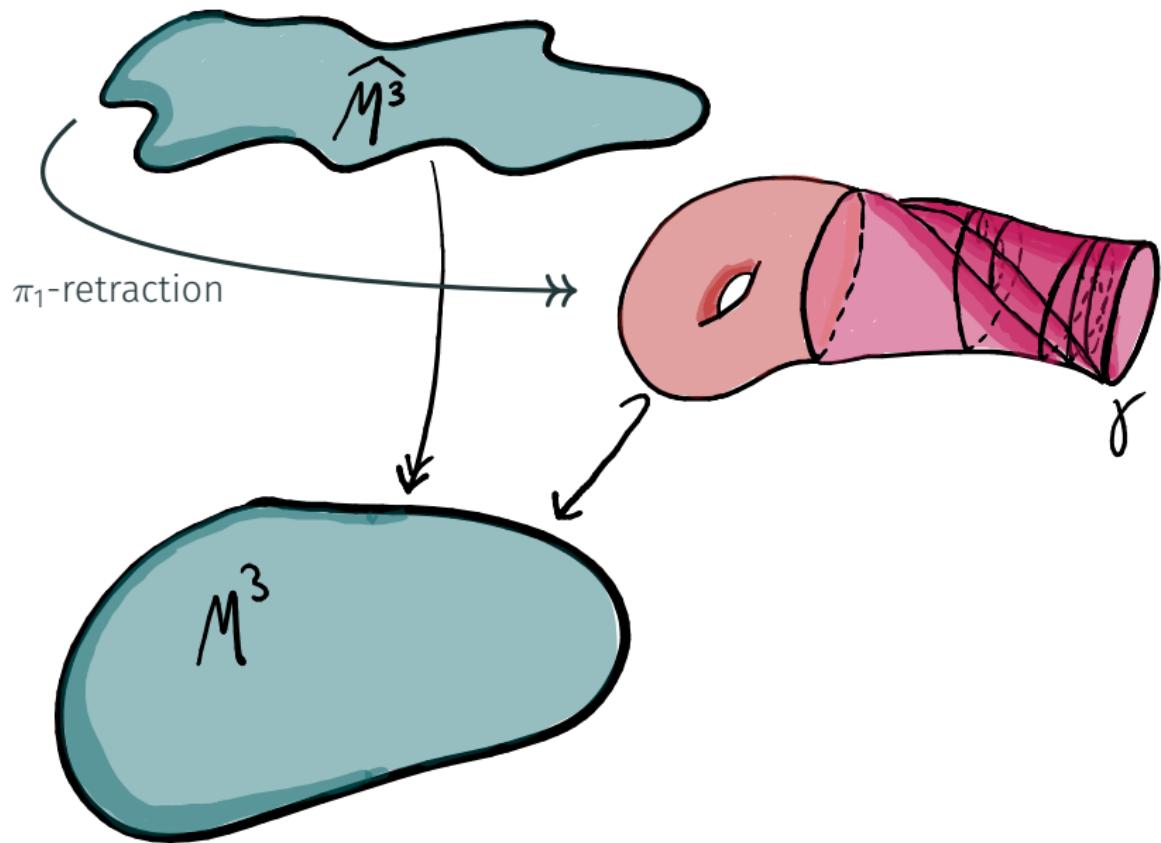
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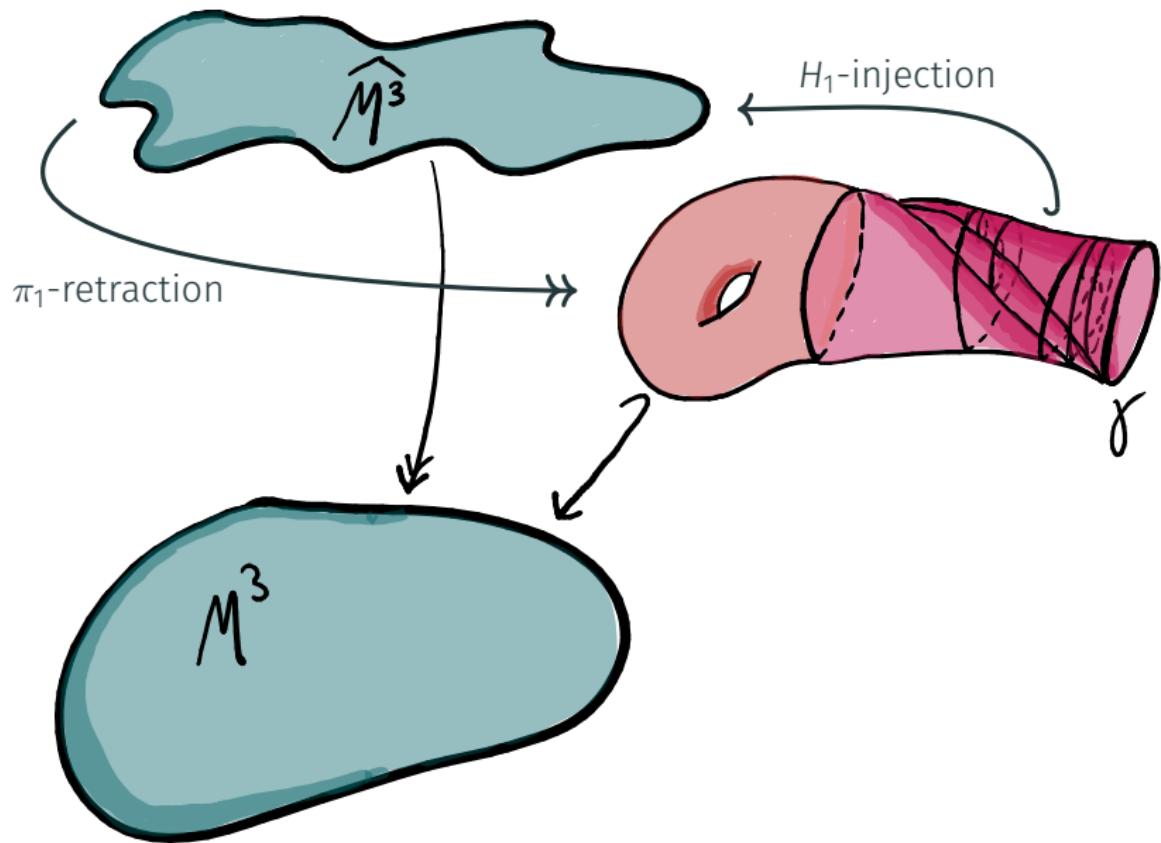
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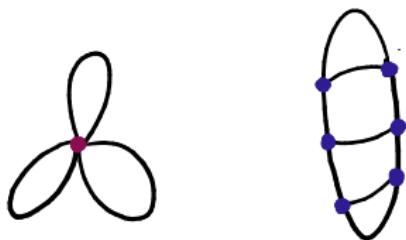
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(Fundamental groups of) graphs joined together by cylinders.

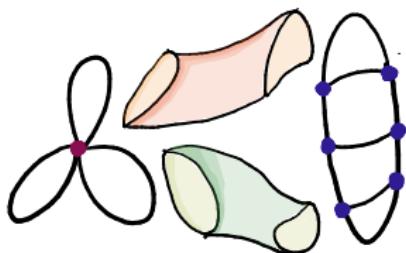
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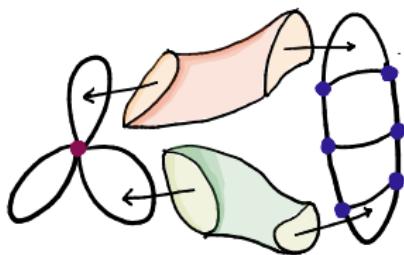
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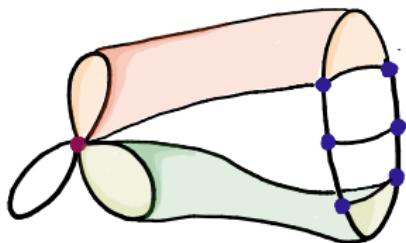
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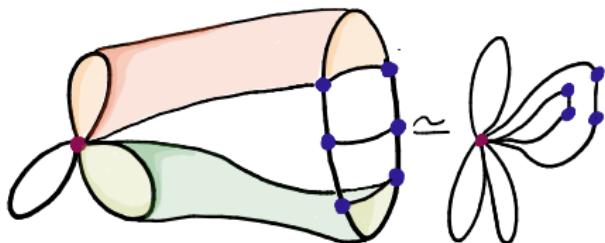
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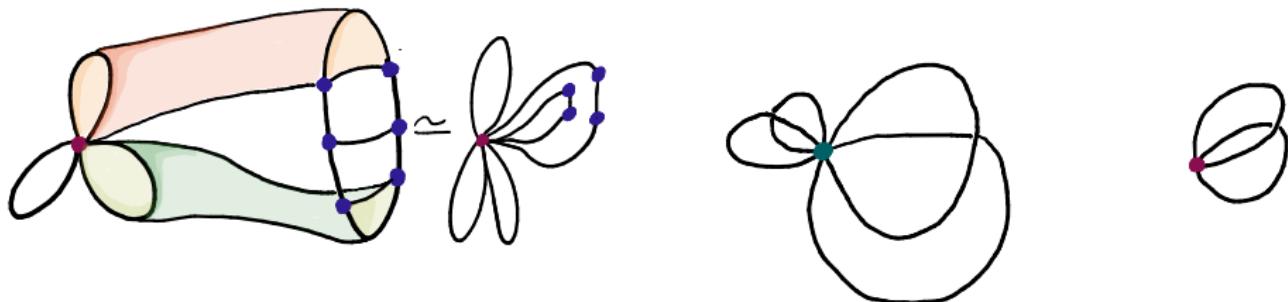
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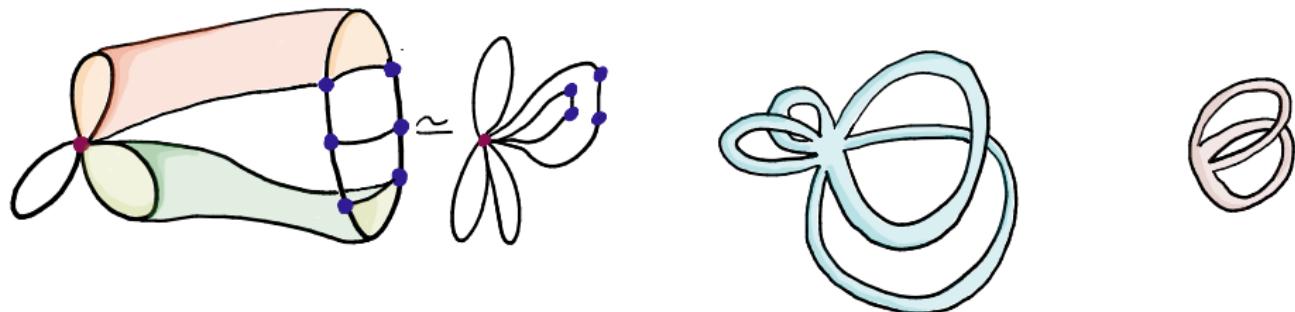
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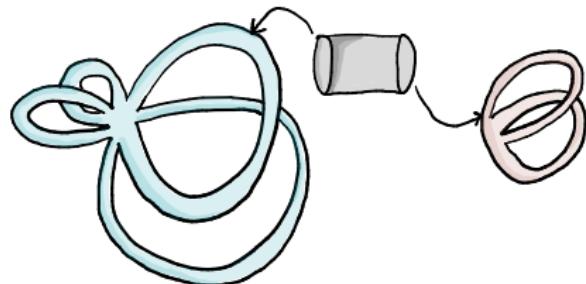
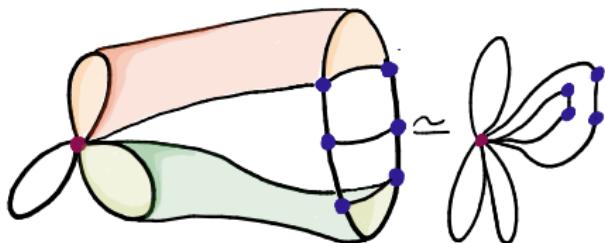
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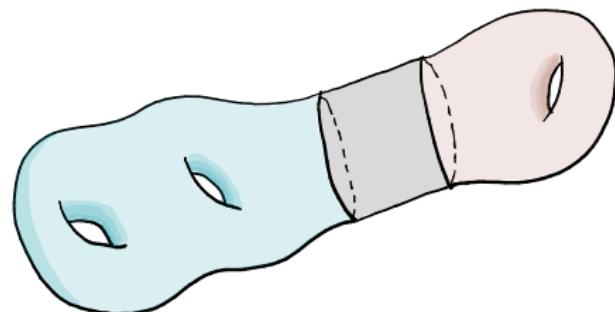
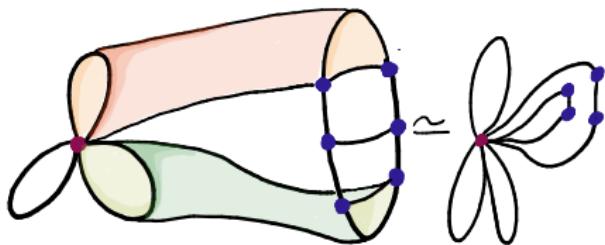
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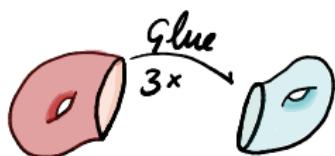
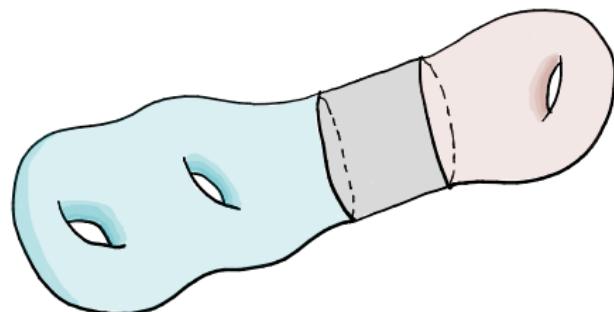
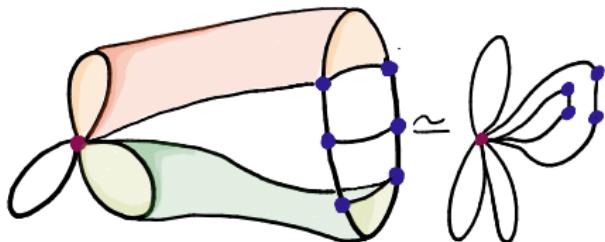
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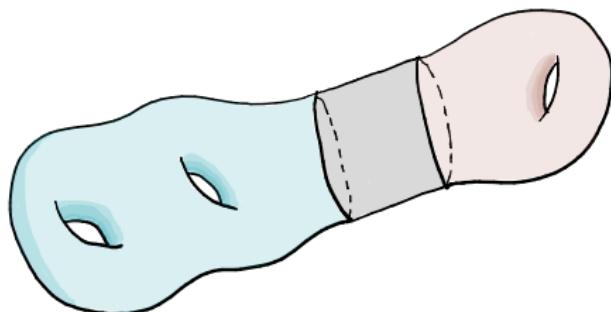
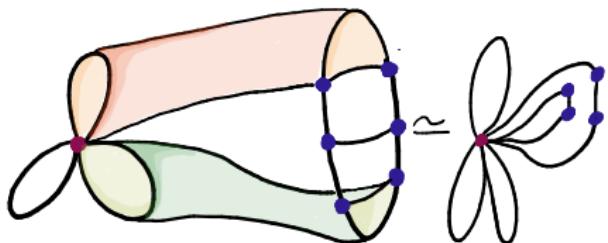
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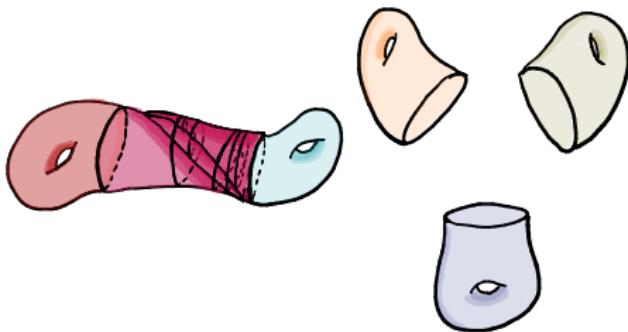
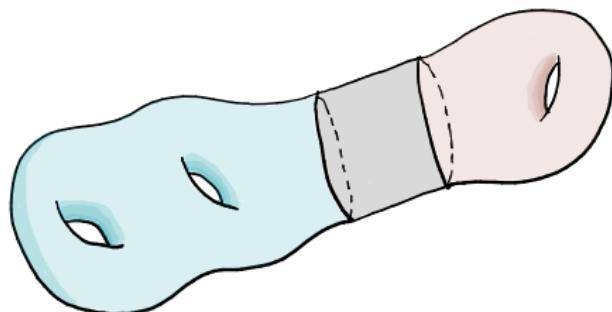
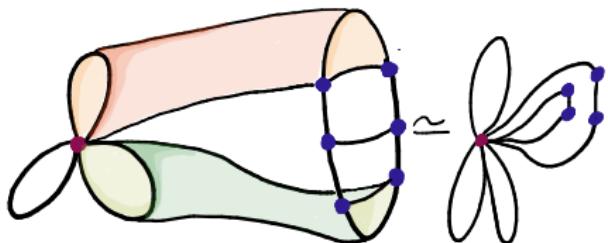
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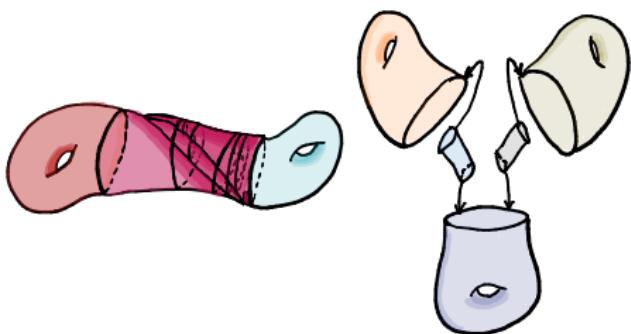
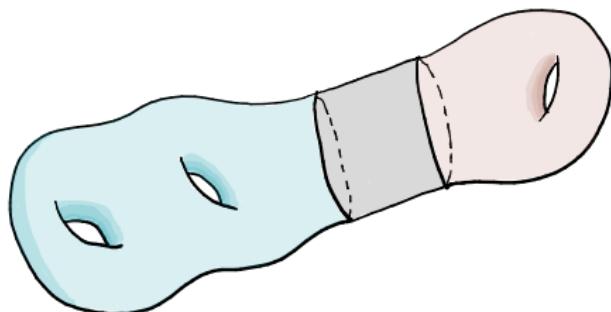
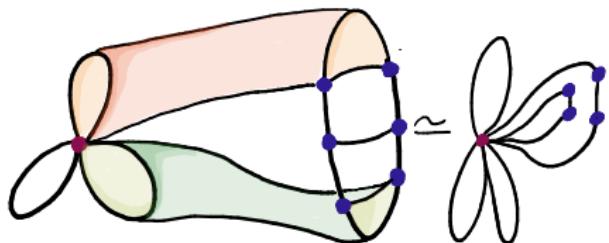
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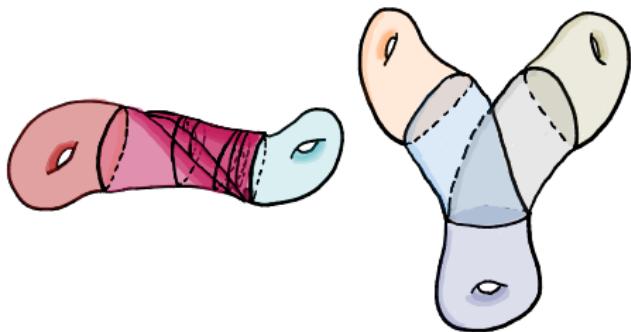
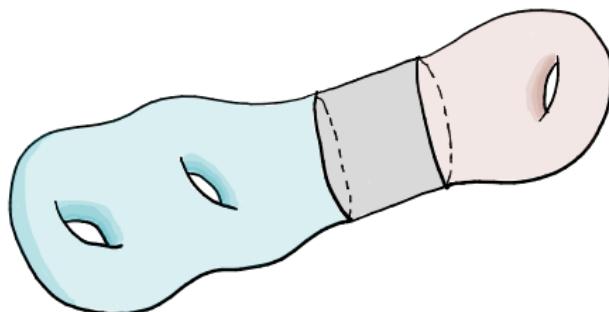
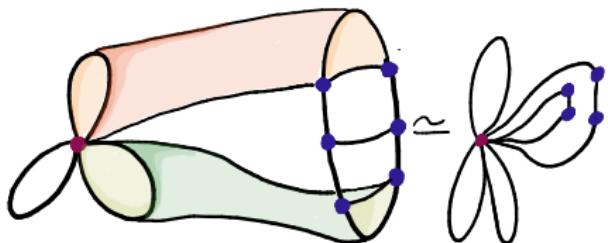
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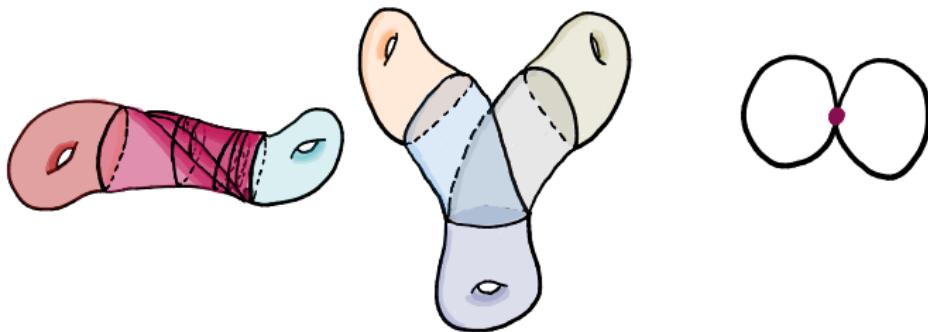
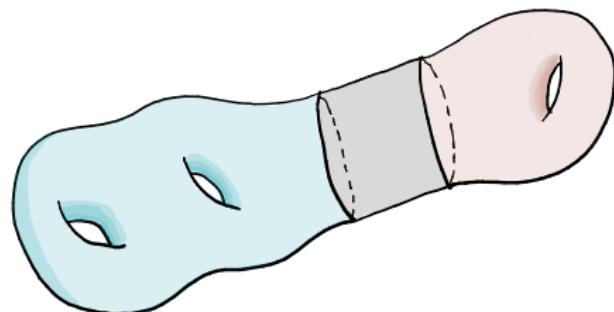
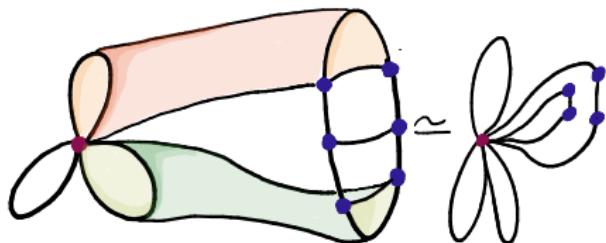
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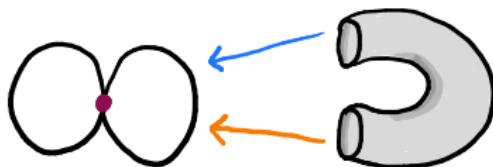
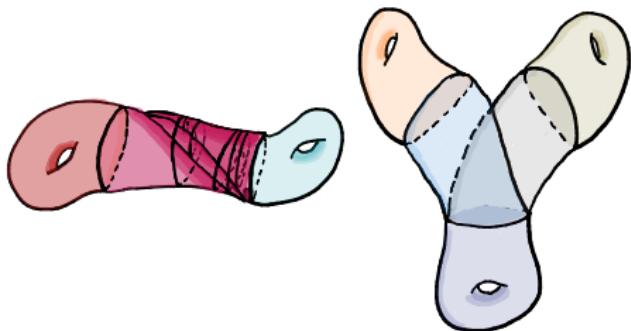
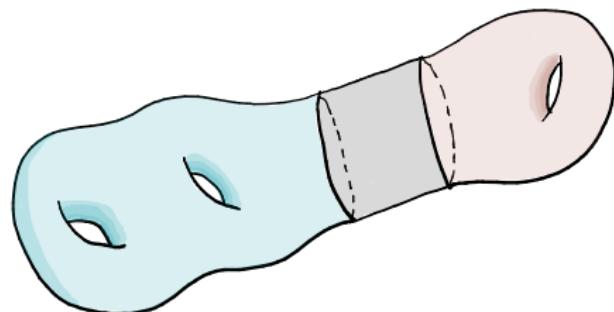
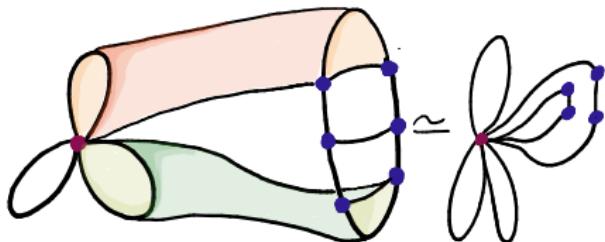
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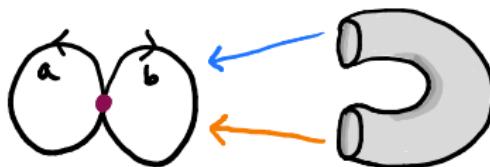
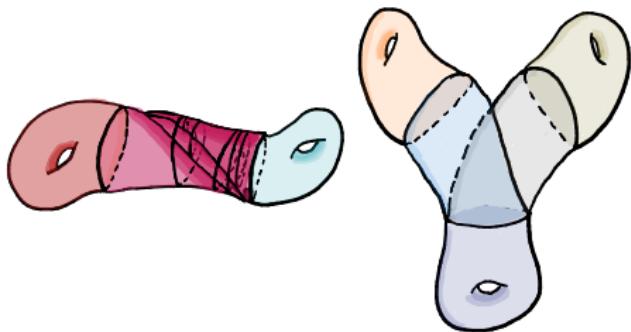
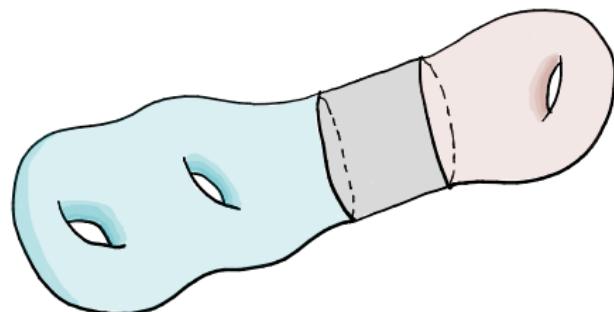
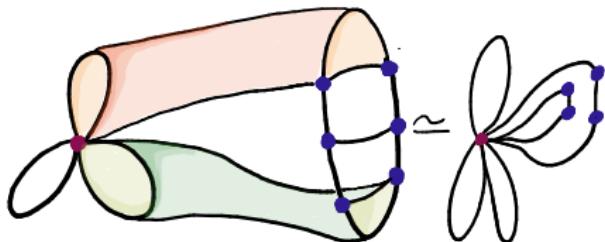
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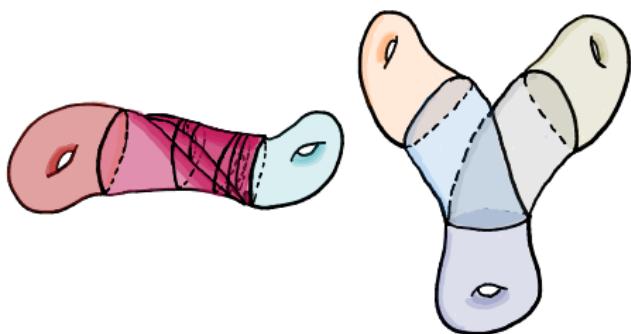
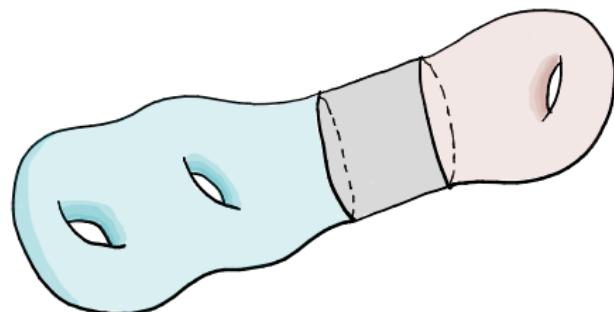
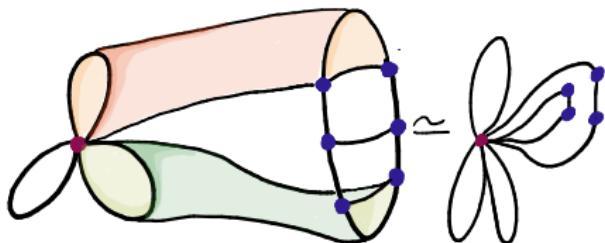
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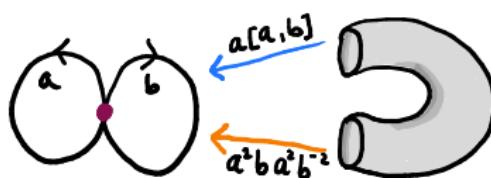
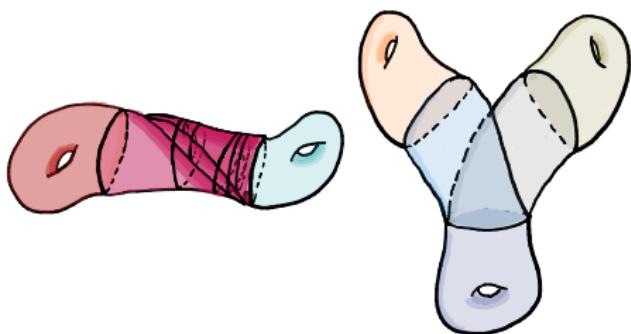
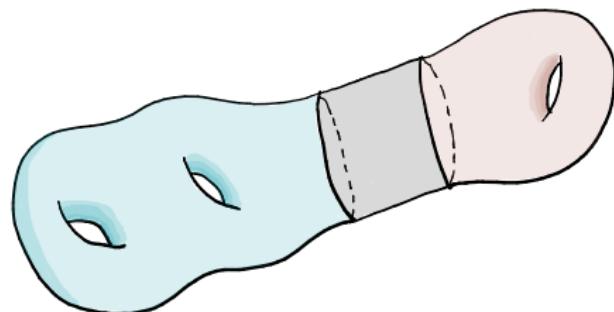
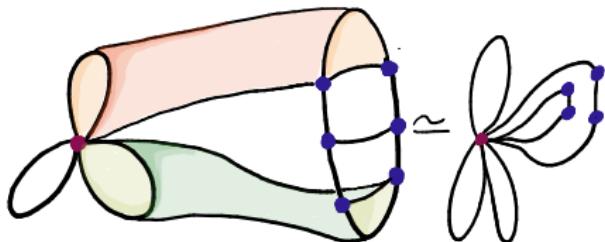
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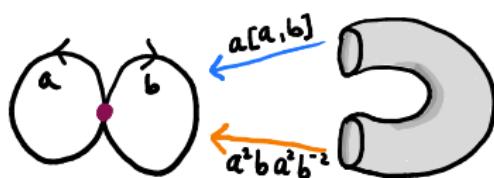
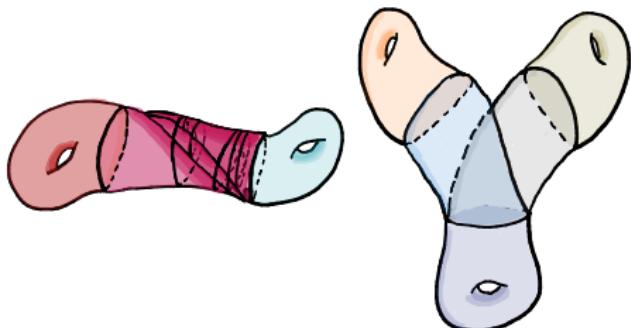
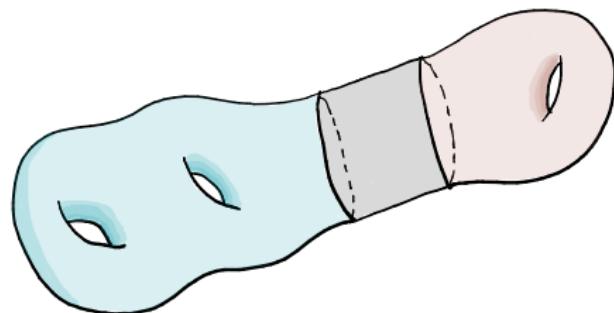
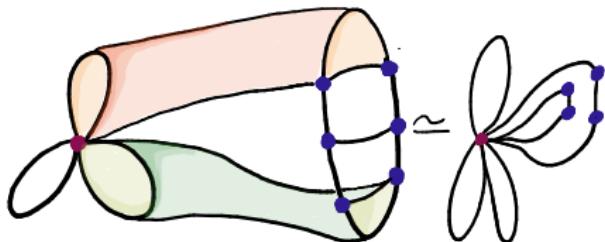
# (Hyperbolic) Graphs of free groups with cyclic edges

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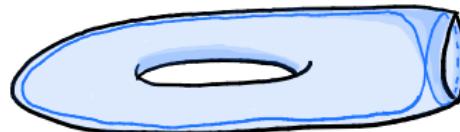
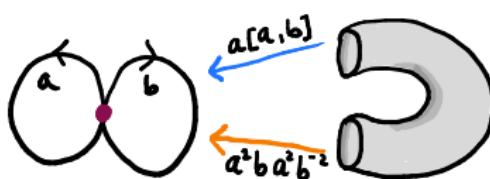
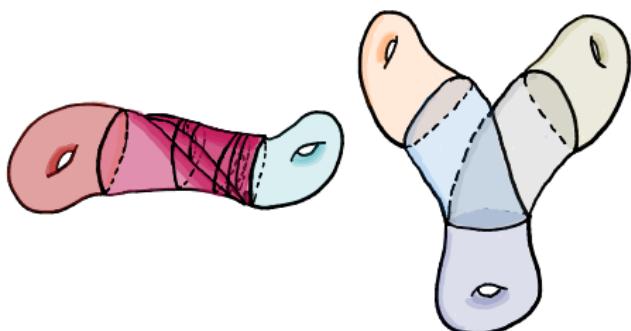
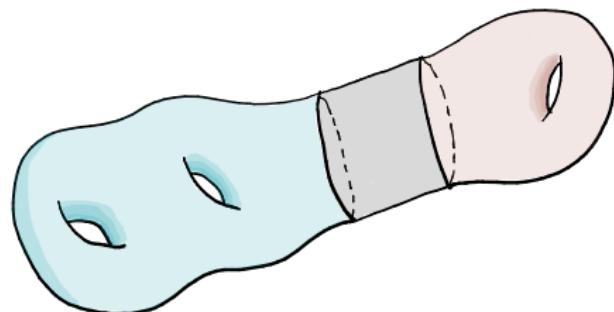
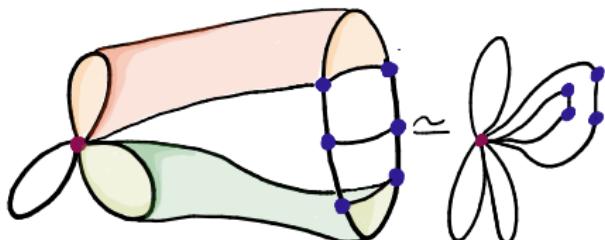
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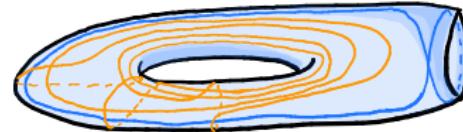
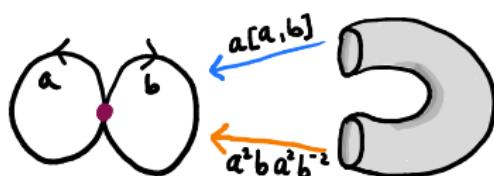
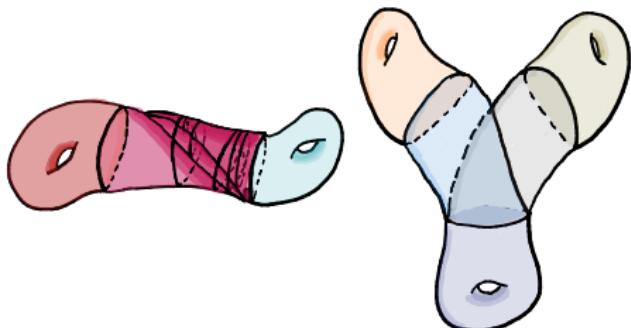
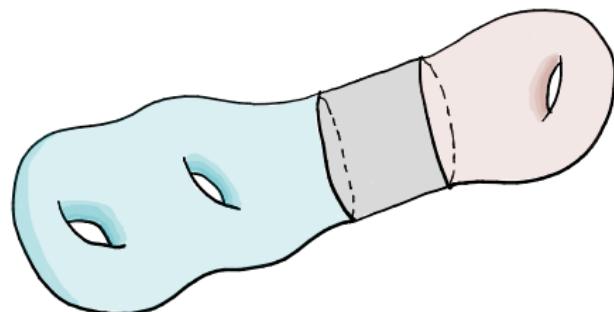
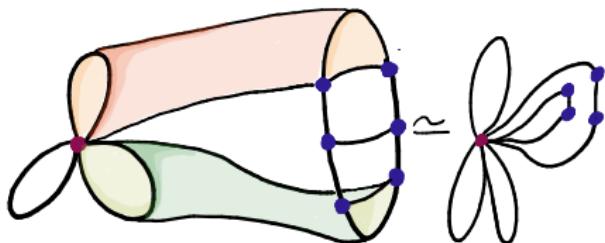
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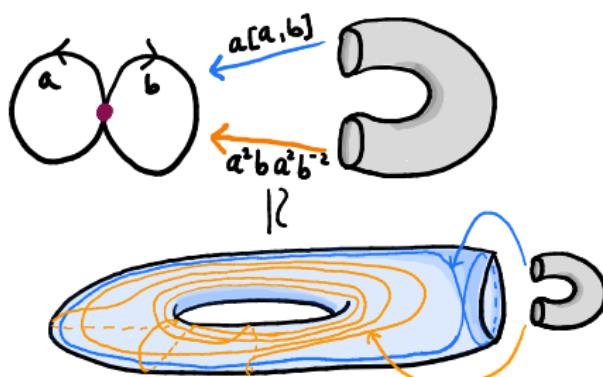
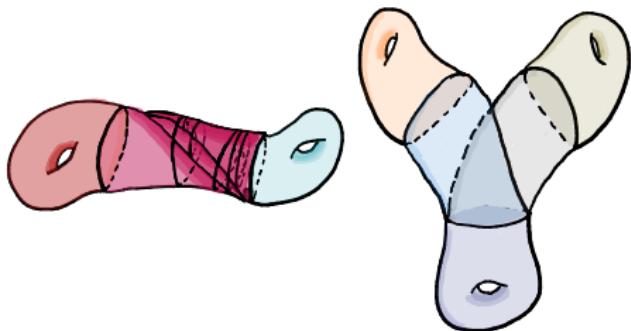
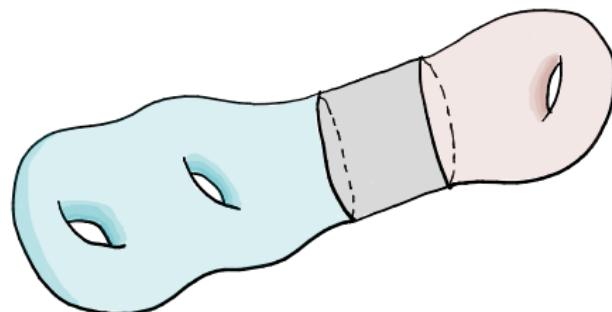
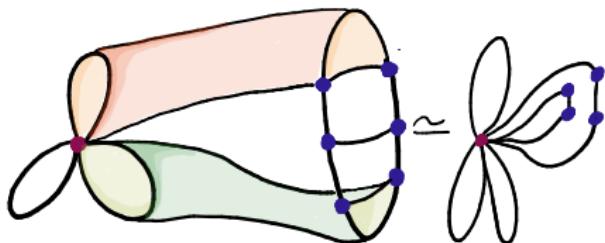
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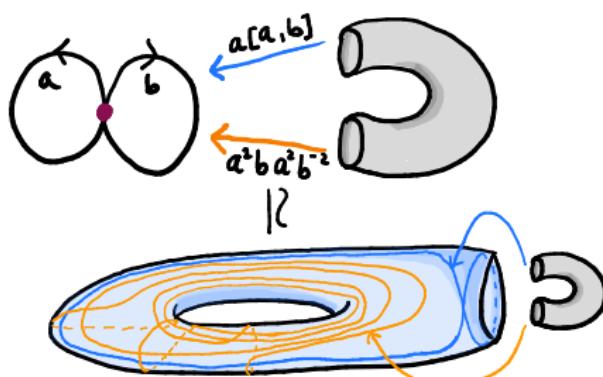
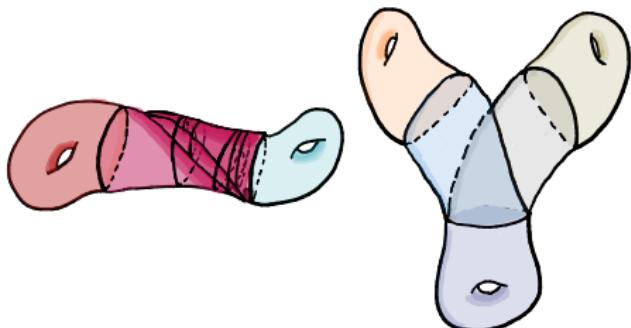
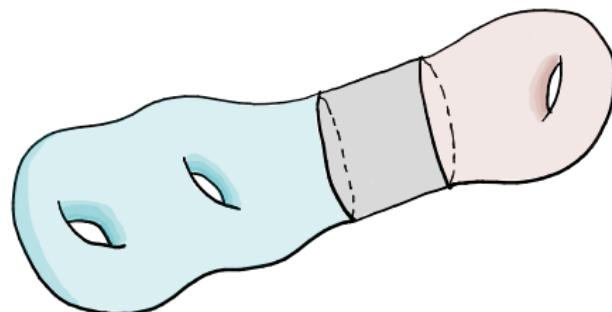
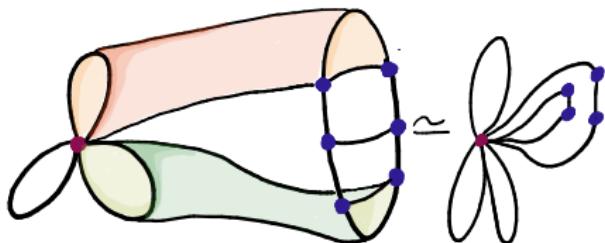
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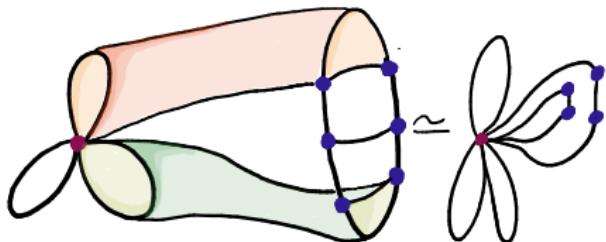
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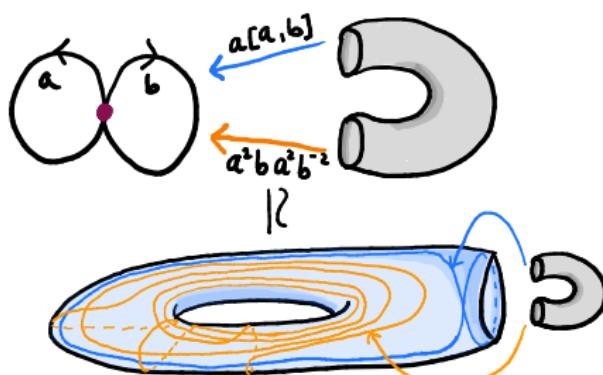
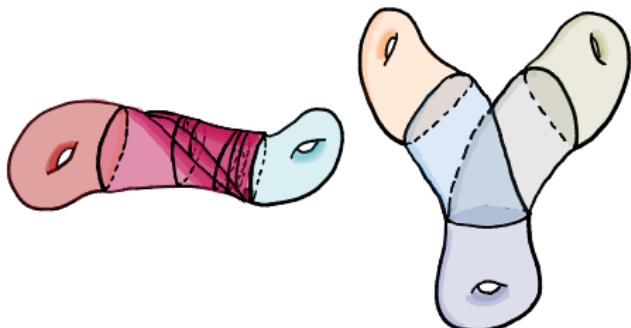
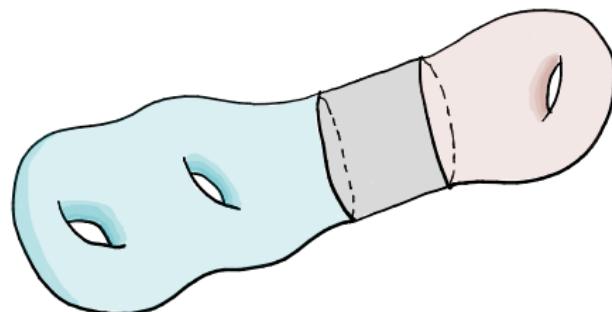


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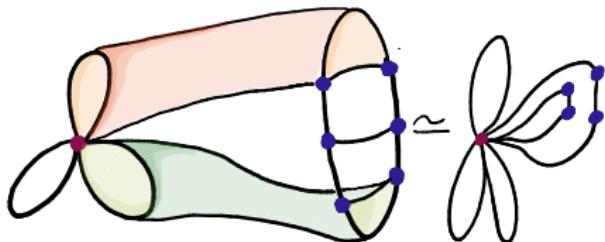


Free and surface groups  
(and their free products)

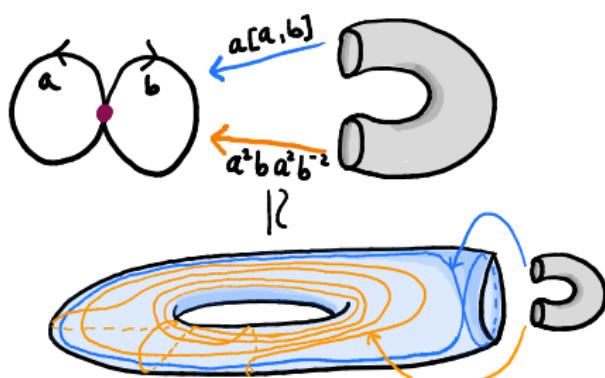
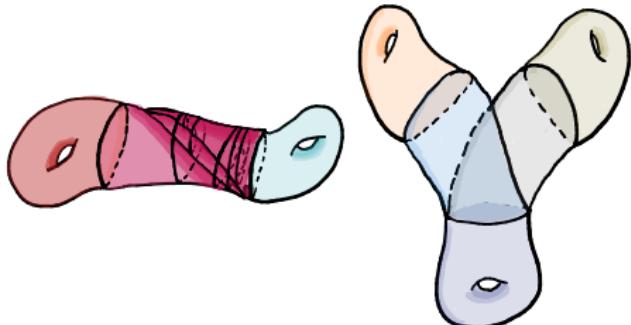
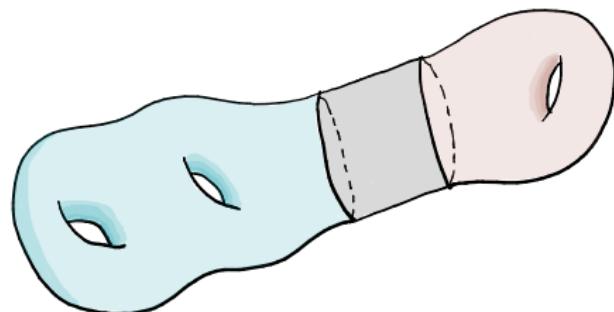


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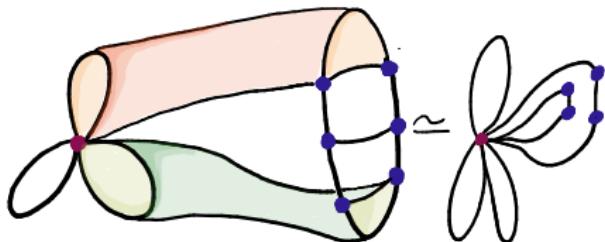


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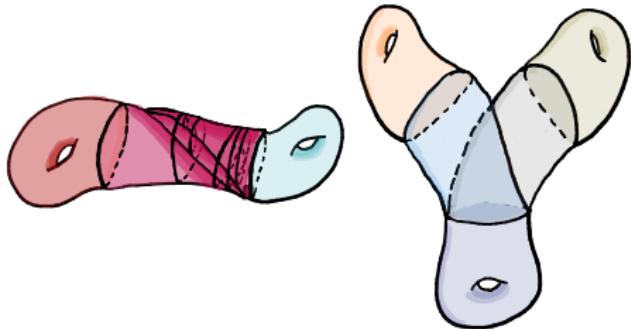
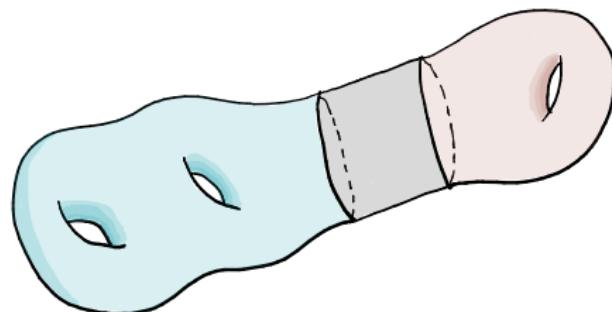


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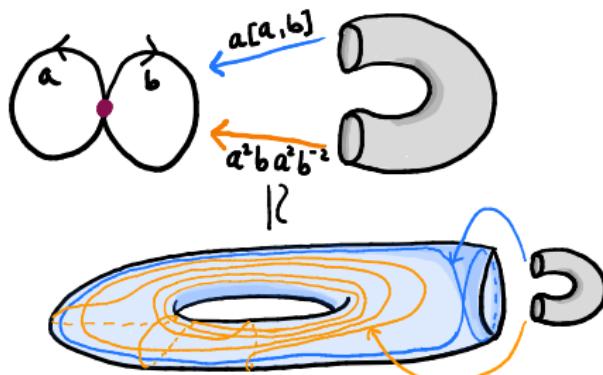
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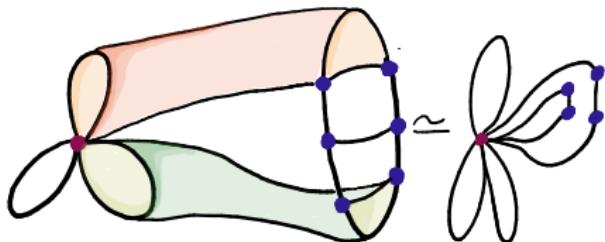


“More than a surface”

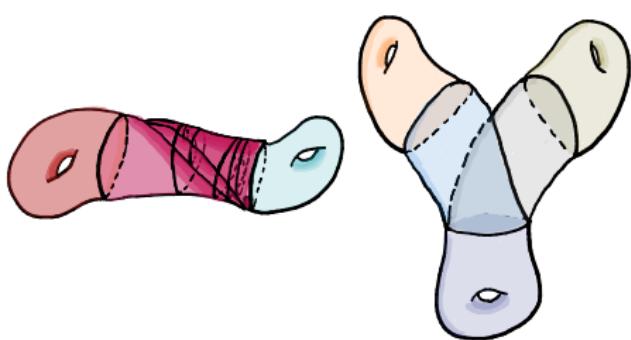
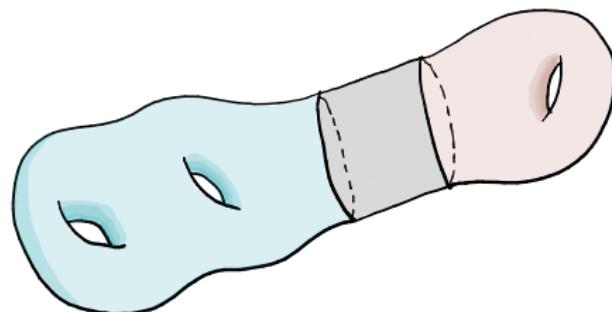


# (Hyperbolic) Graphs of free groups with cyclic edges

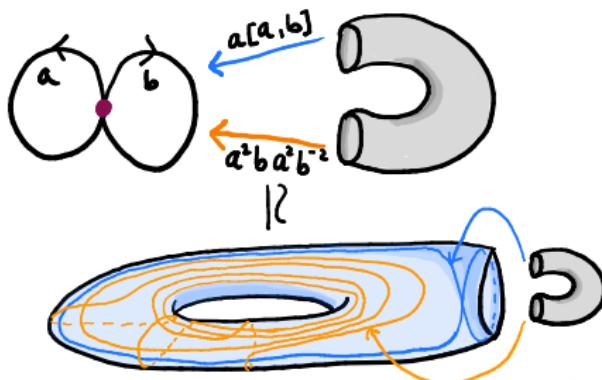
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Free and surface groups  
(and their free products)



“More than a surface”



“Has a rigid vertex”

## Homological torsion: Existence

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## Theorem (Ascari-F '25)

Let  $G$  be a hyperbolic graph of free groups with  $\mathbb{Z}$  edges, which is not a free product of free and surface groups. Then  $G$  has abundant virtual homological torsion, that is,

for every finite abelian group  $A$ , there is a finite-index subgroup  $G_0 \leq G$  such that

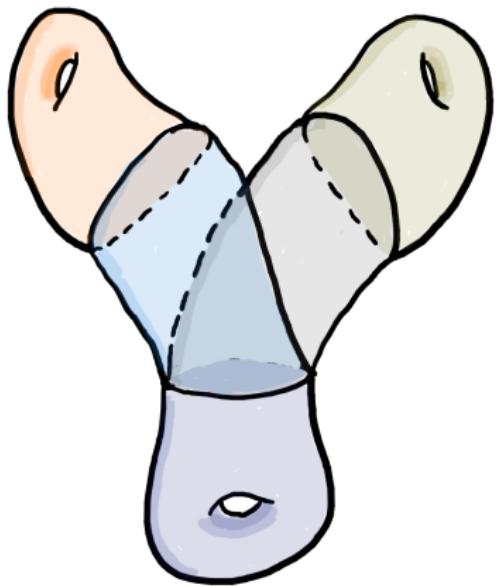
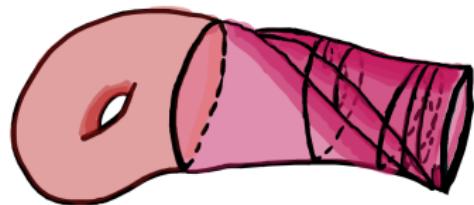
$$G_0^{\text{ab}} = A \oplus \cdots.$$

## Homological torsion: By the book

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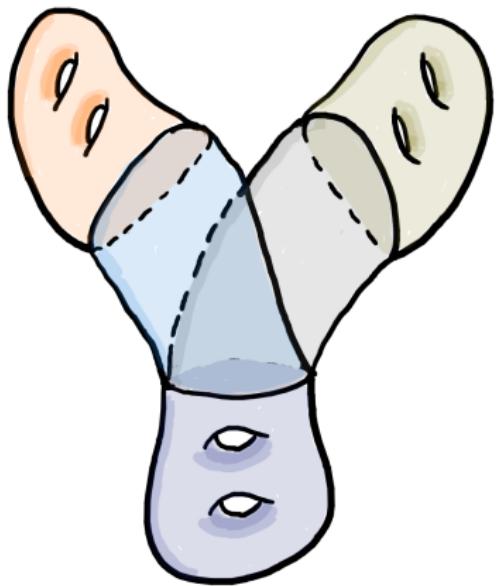
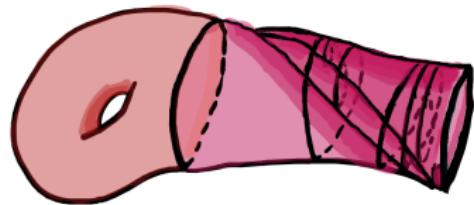
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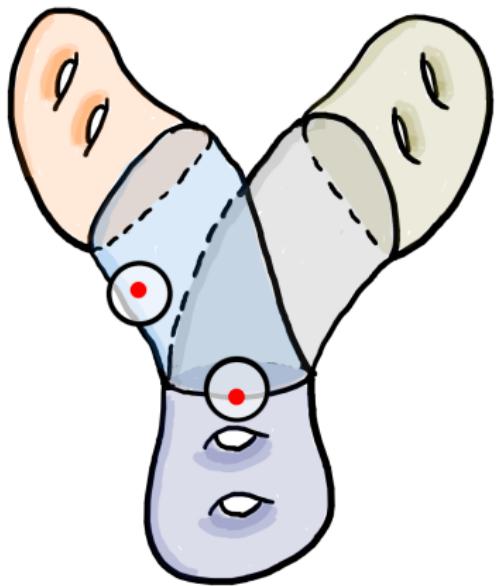
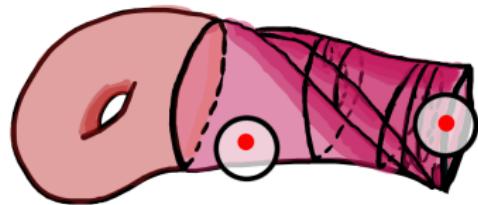
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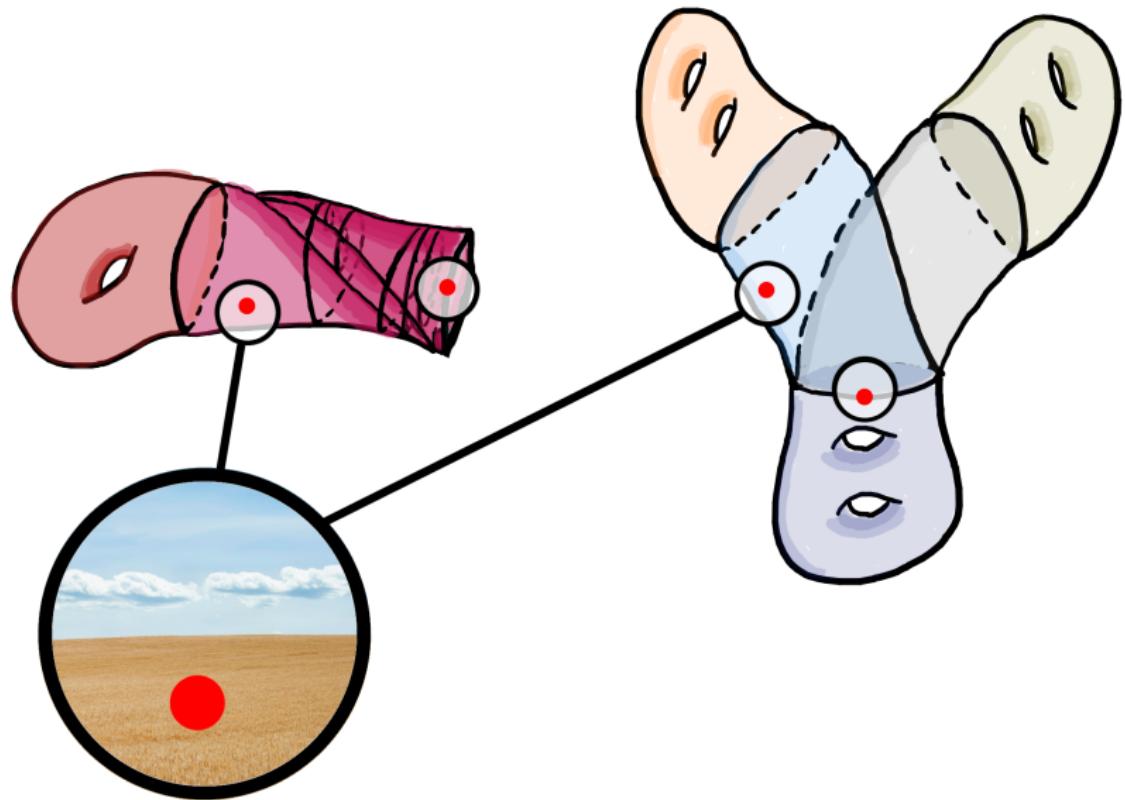


## Homological torsion: By the book

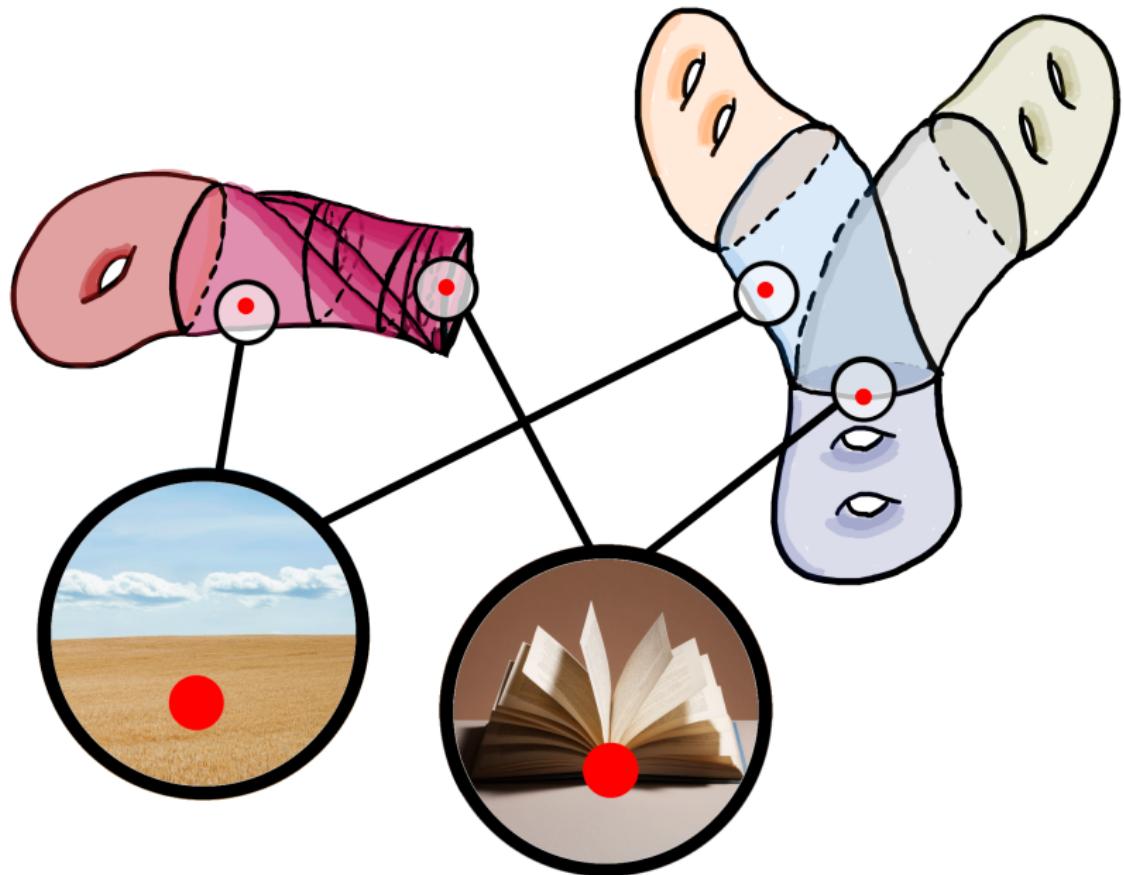
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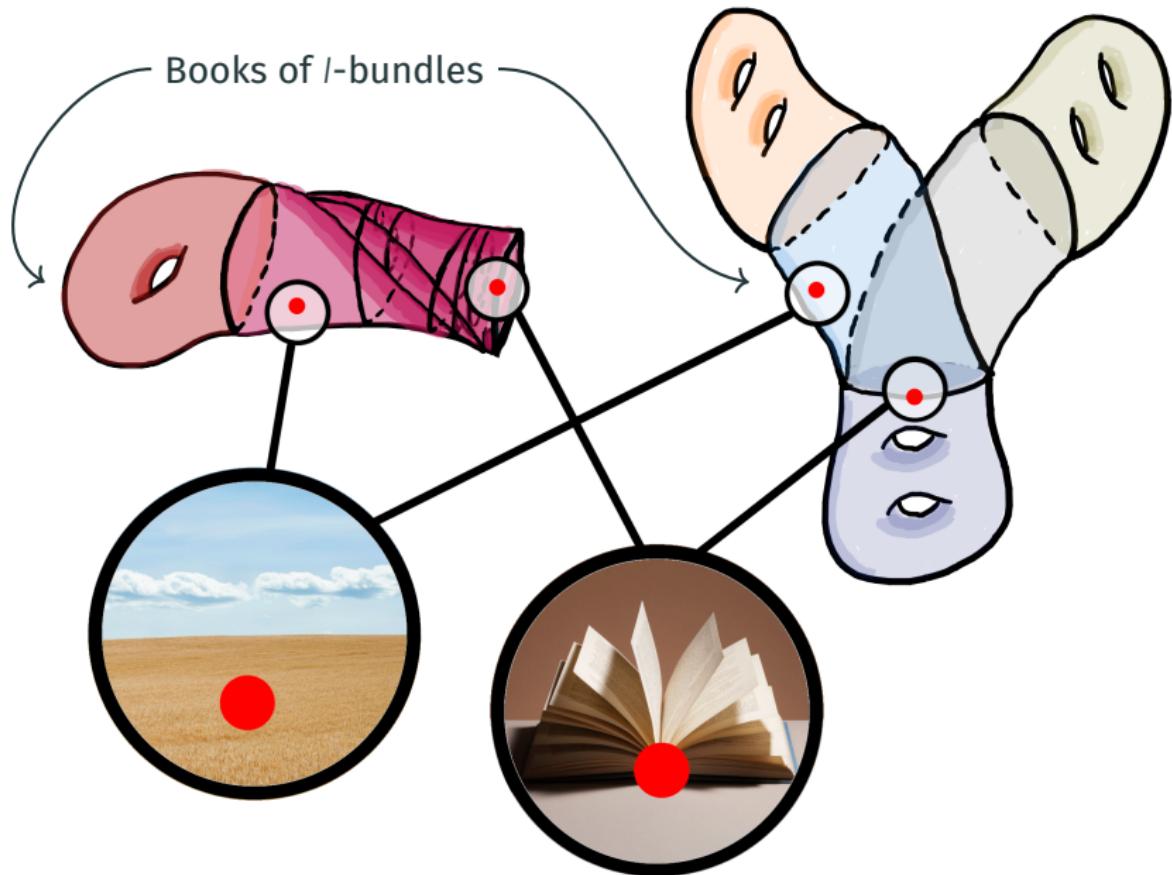
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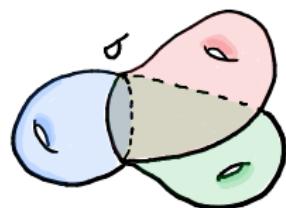


# Homological torsion: Systems of surfaces

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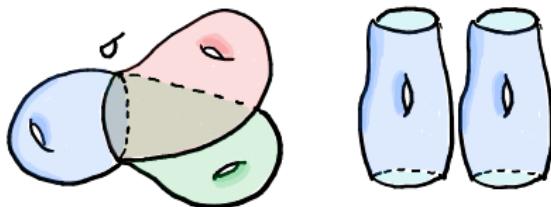
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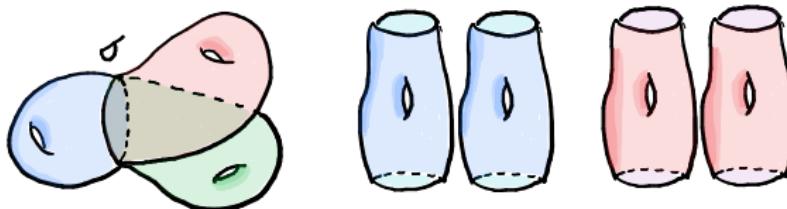
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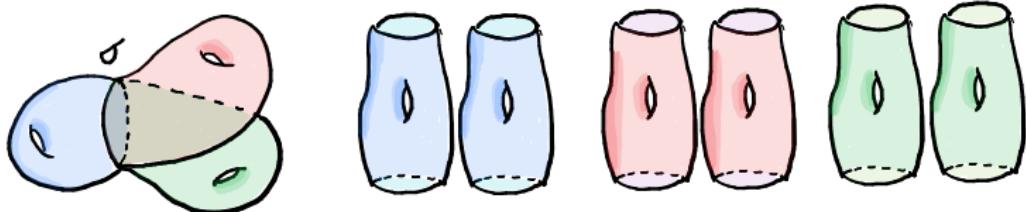
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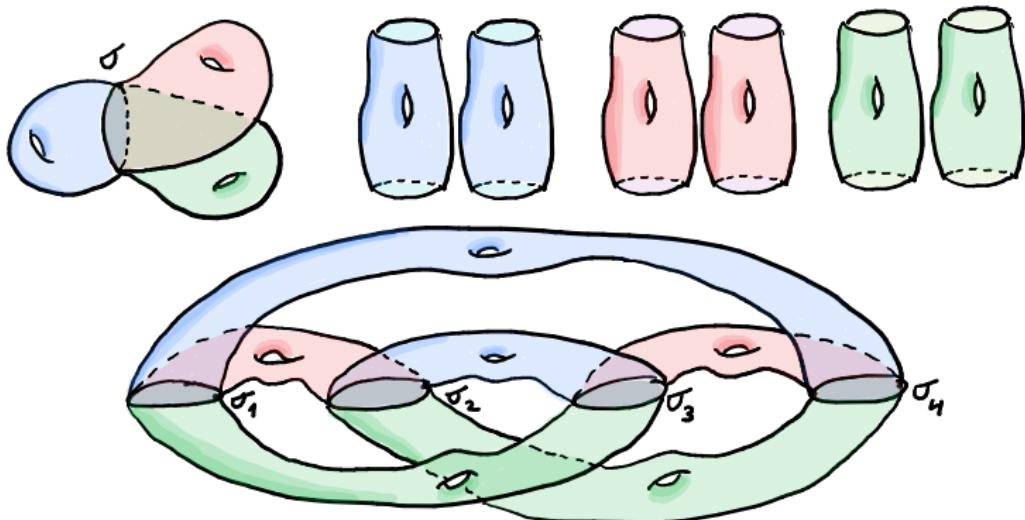


## Homological torsion: Systems of surfaces

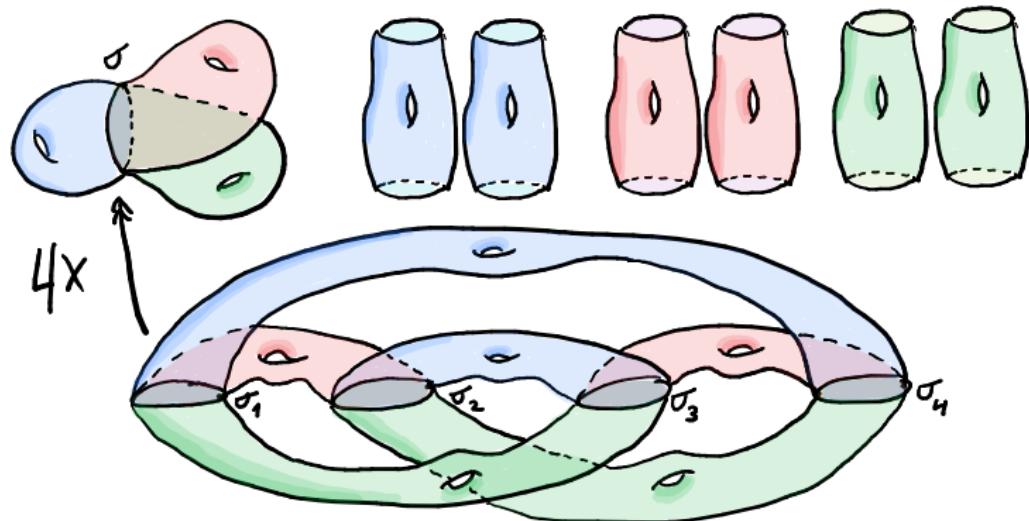
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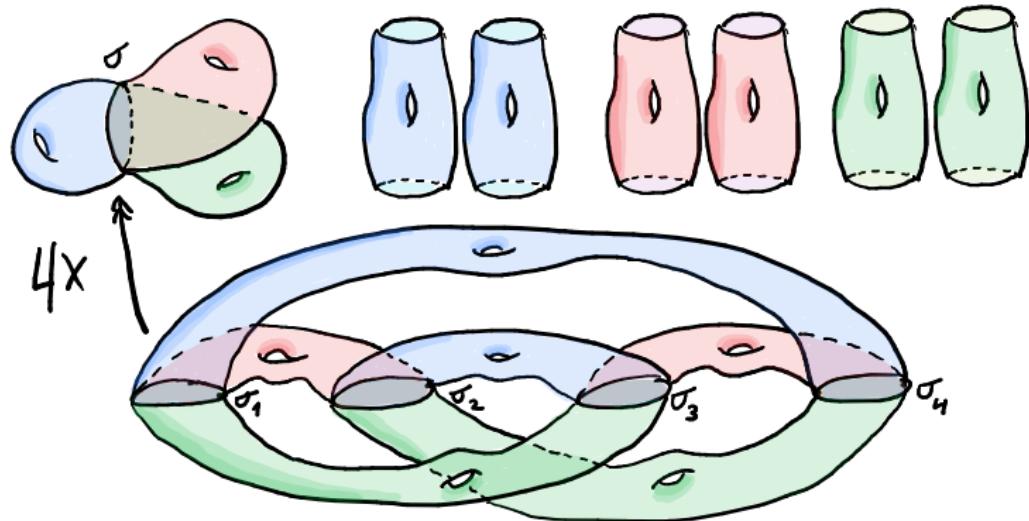
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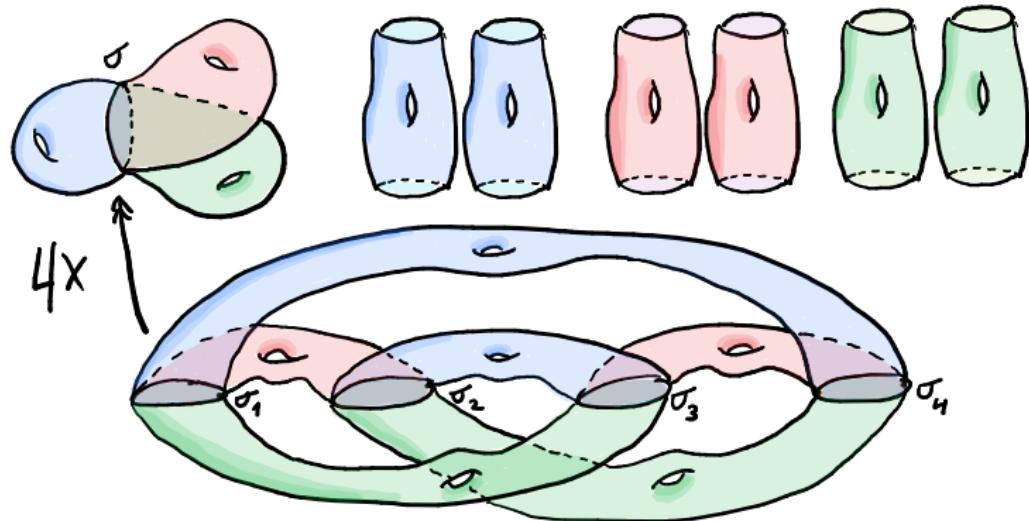


## Homological torsion: Systems of surfaces



In first homology:

## Homological torsion: Systems of surfaces



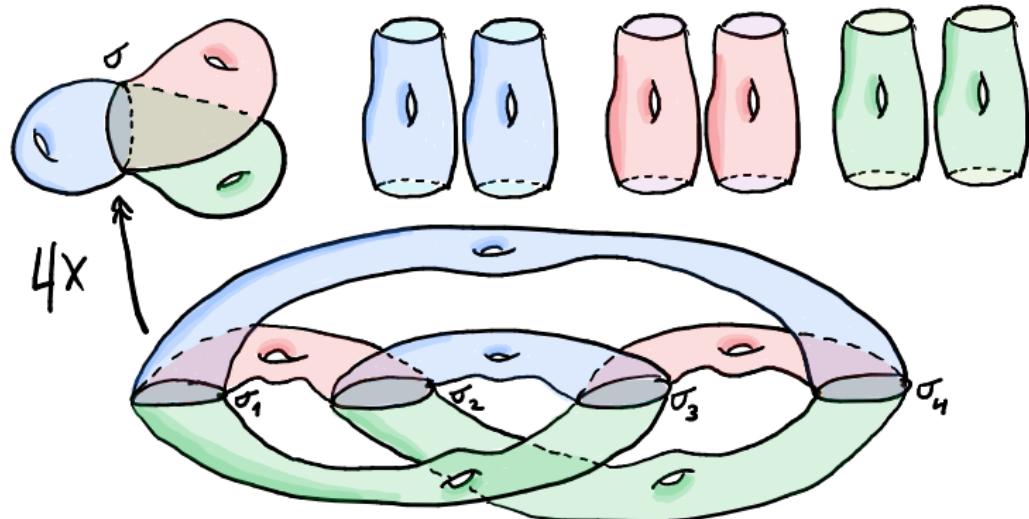
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$$\Rightarrow$$

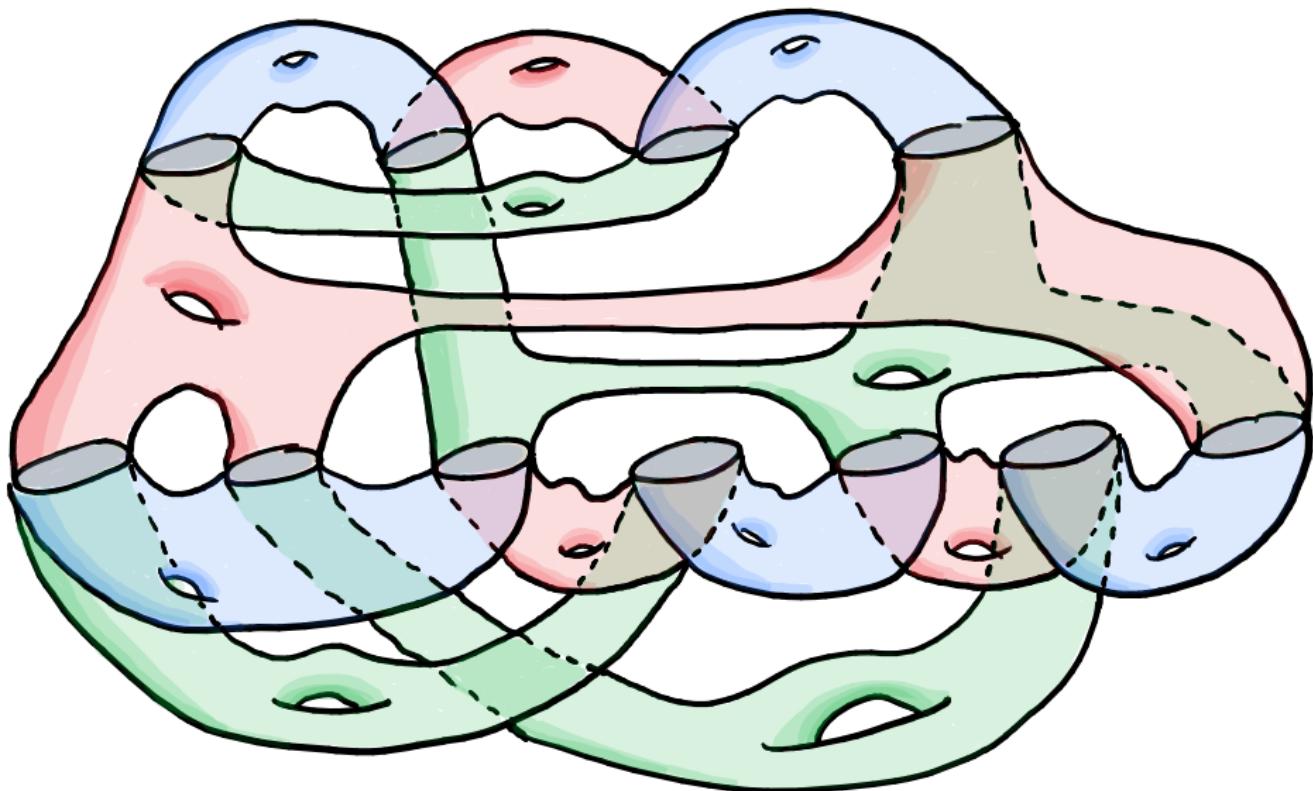
$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$$

$$2 \cdot \sigma_1 = 0$$

## Homological torsion: Two and three

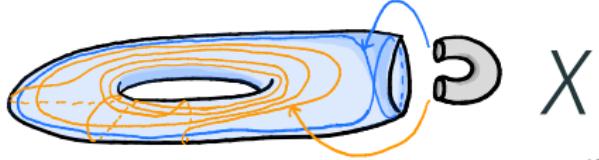
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## Homological torsion: Two and three



Enter a rigid vertex

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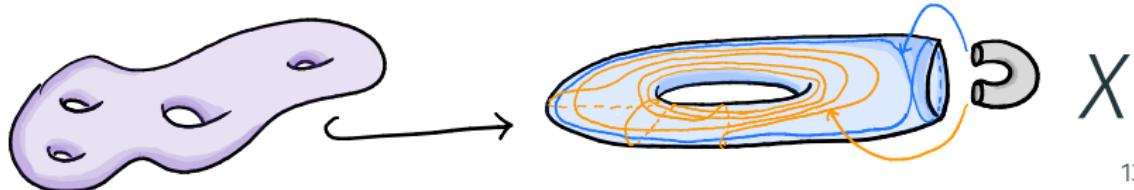


13/23

# Enter a rigid vertex

Theorem (Wilton '18)

If  $G$  is not free then  $G$  has a surface subgroup.

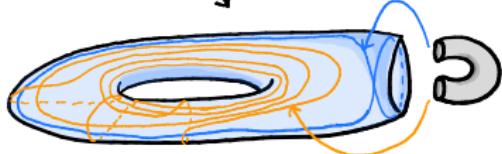
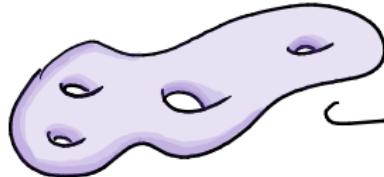
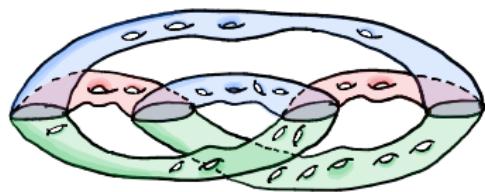


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B



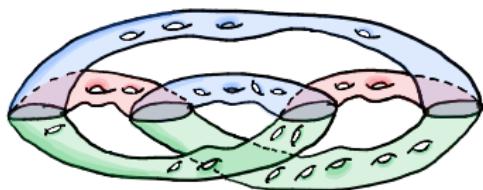
X

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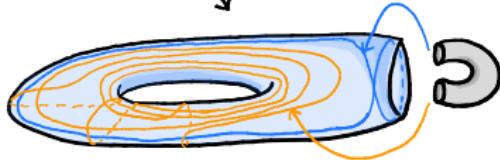
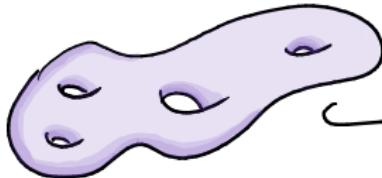
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B



Just a map (not  $\pi_1$ -injective)



X

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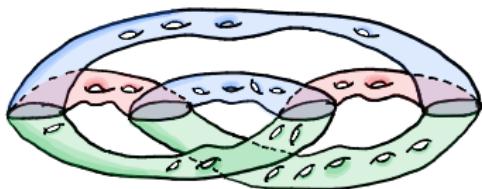
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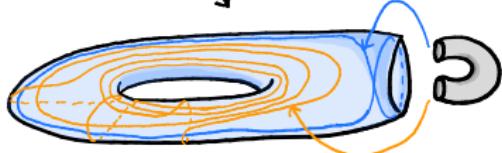
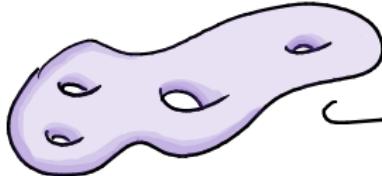
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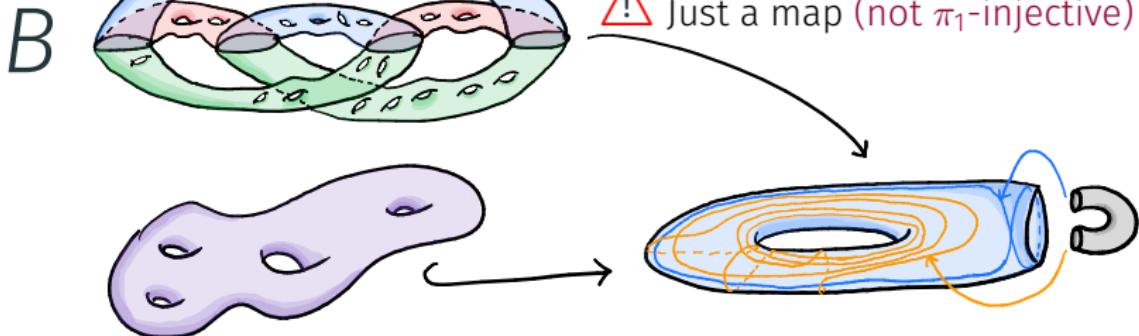
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$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n} \in \mathbb{Q}.$$



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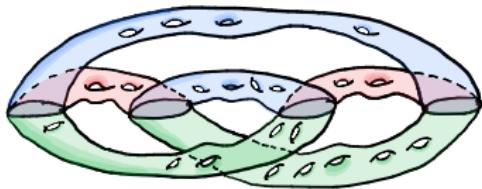
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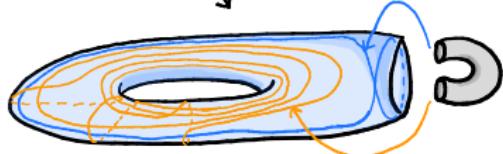
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$\rightsquigarrow$  has a topological interpretation in terms of surfaces

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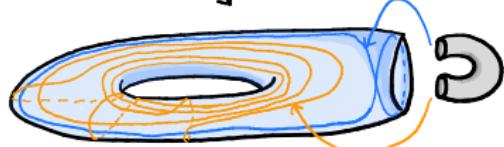
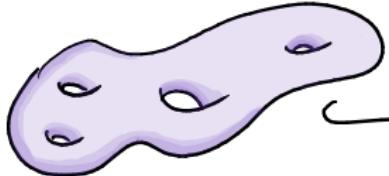
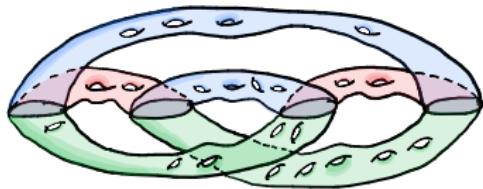
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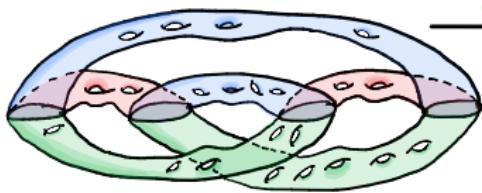
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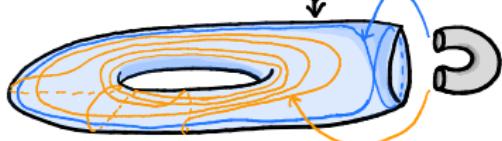
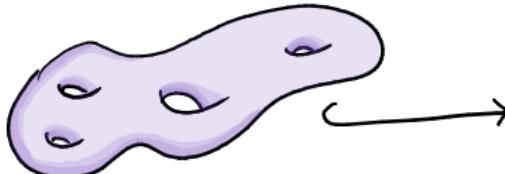
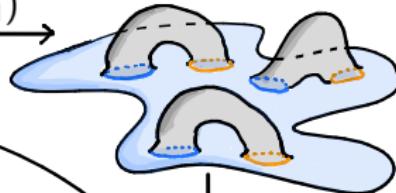
$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n} \in \mathbb{Q}.$$

↷ has a topological interpretation in terms of surfaces

B



$$\xrightarrow{\text{Tor}(H_1) \text{ injective}}$$



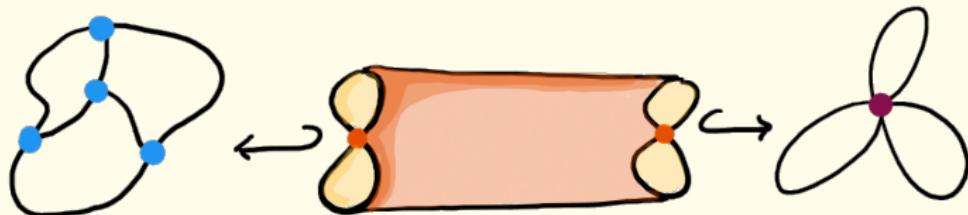
X̂

X

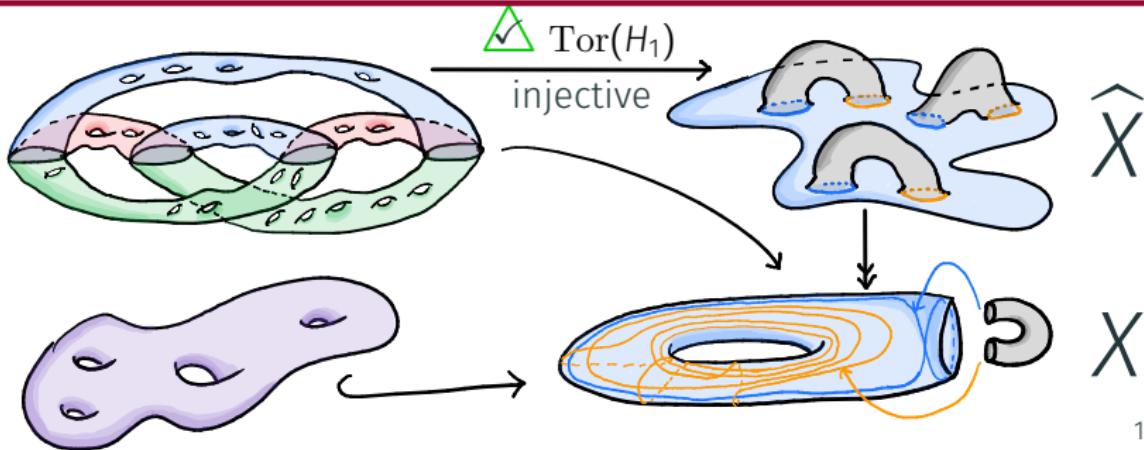
# Enter a rigid vertex

Applicable to other settings?

e.g. graphs of graphs.



B



## Profinite rigidity 101

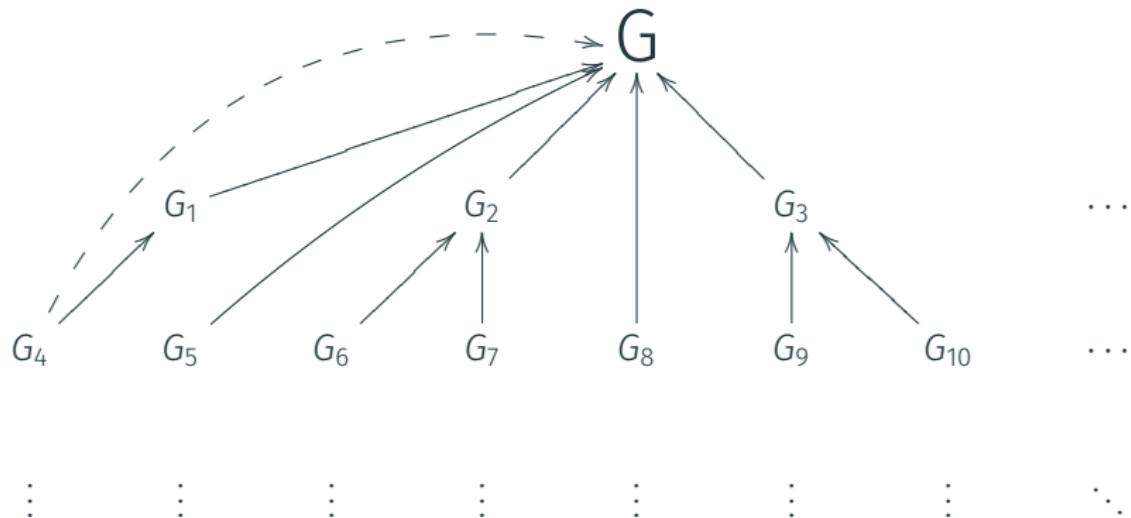
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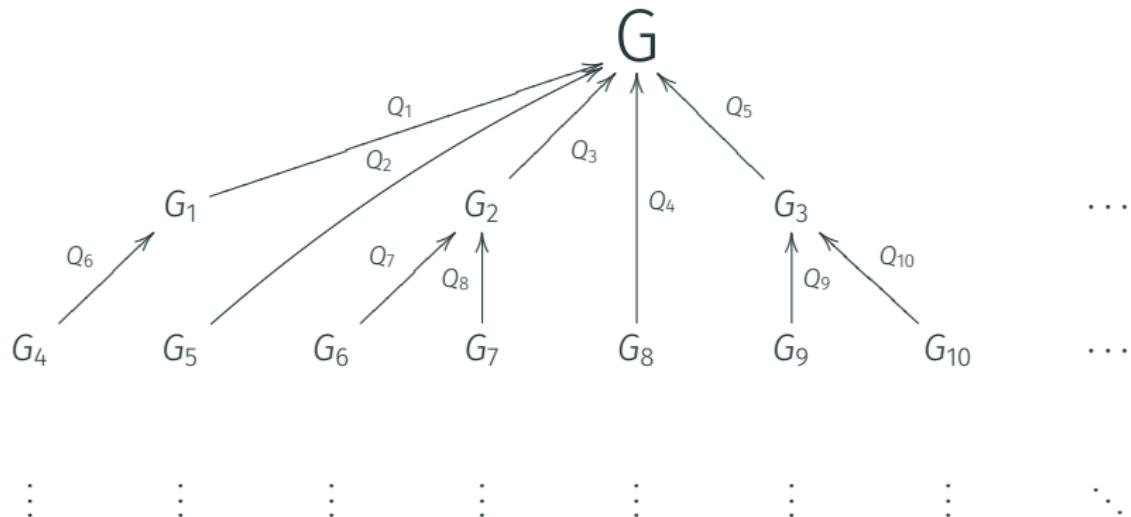
## Lattice/poset of finite-index normal subgroups of $G$



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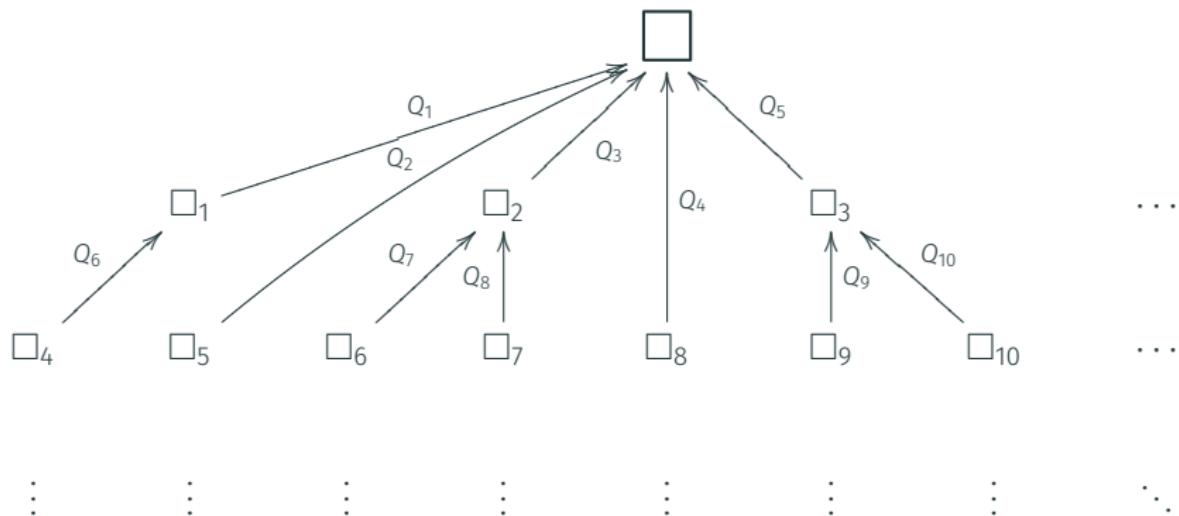
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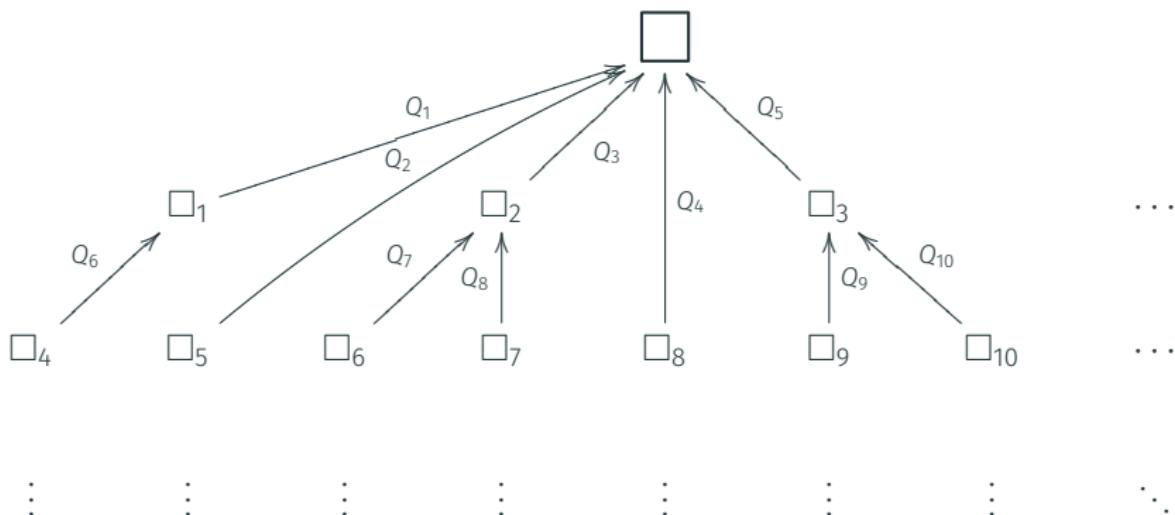
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## Question

Which fg residually finite groups  $H$  can fill the empty  $\square$ 's?

# Profinite rigidity 101

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## The basics

- Full lattice structure encoded in the set of finite quotients of  $G$  (“Dixon’s theorem”, DFPR ’82).

# Profinite rigidity 101

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- The lattice determines the abelianization of each  $\square$ :

If  $G$  and  $H$  have the same lattice, then

$$G_n^{\text{ab}} \cong H_n^{\text{ab}}.$$

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## Question (Remeslennikov)

Are finitely generated (non-abelian) free groups profinitely rigid?

# Profinite rigidity 201

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## Theorem (Ascari-F '25)

*Free products of free and surface groups are profinitely rigid amongst hyperbolic graphs of free groups with  $\mathbb{Z}$  edges.*

## Growth: non-normal towers

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### Theorem (Ascari-F '25)

*There exists a tower of finite-index non-normal subgroups*

$$G \geq G_1 \geq G_2 \geq \cdots, \quad \text{with} \quad \bigcap_n G_n = \{1\},$$

*such that simultaneously for every prime  $p$ ,*

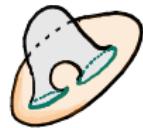
$$\lim_{n \rightarrow \infty} \frac{\log(|\mathrm{Tor}_p(G_n^{\mathrm{ab}}))|)}{[G : G_n]} > 0.$$

## Fractal-like towers

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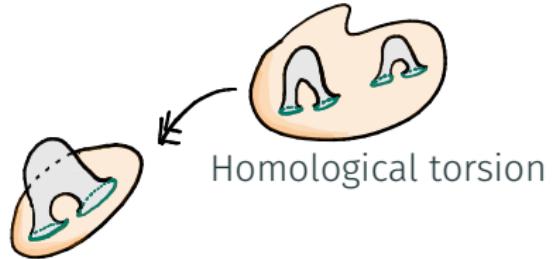
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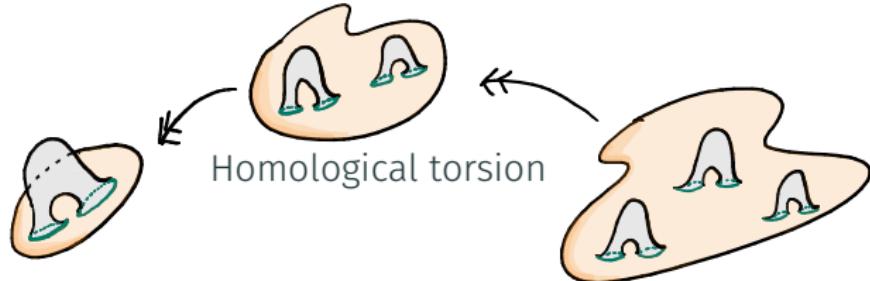
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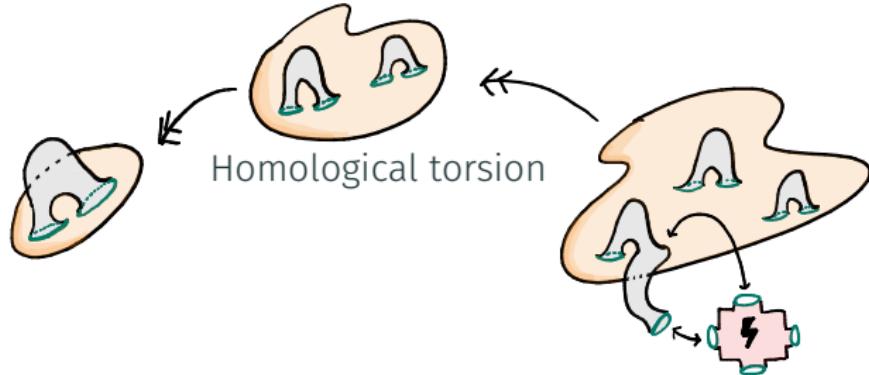
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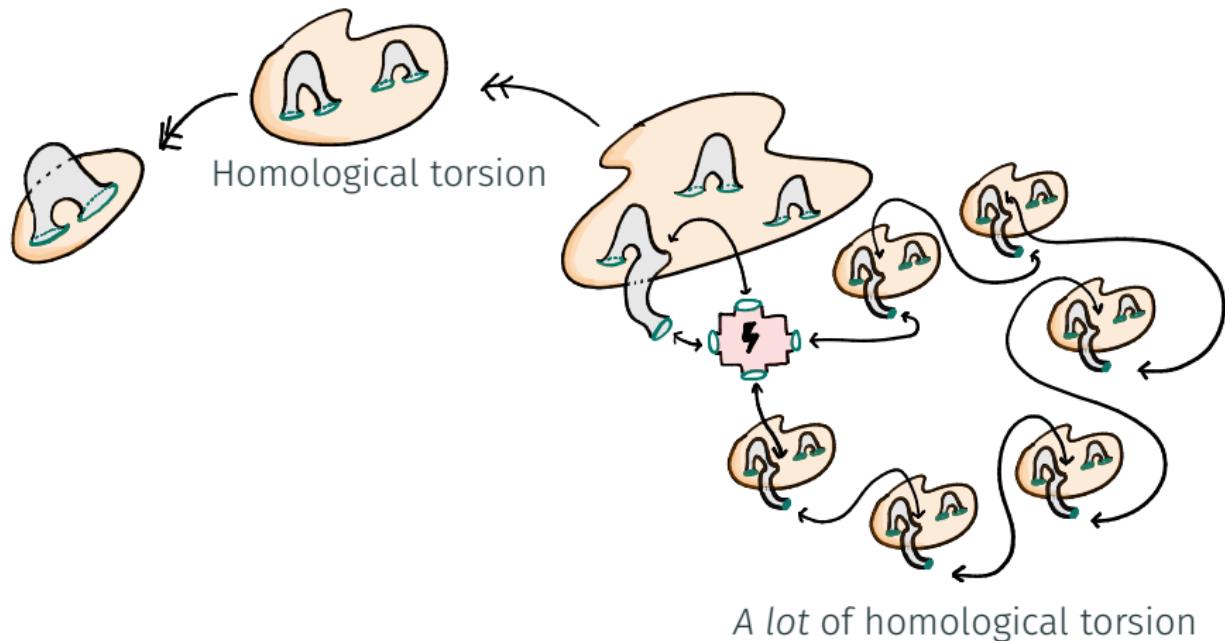


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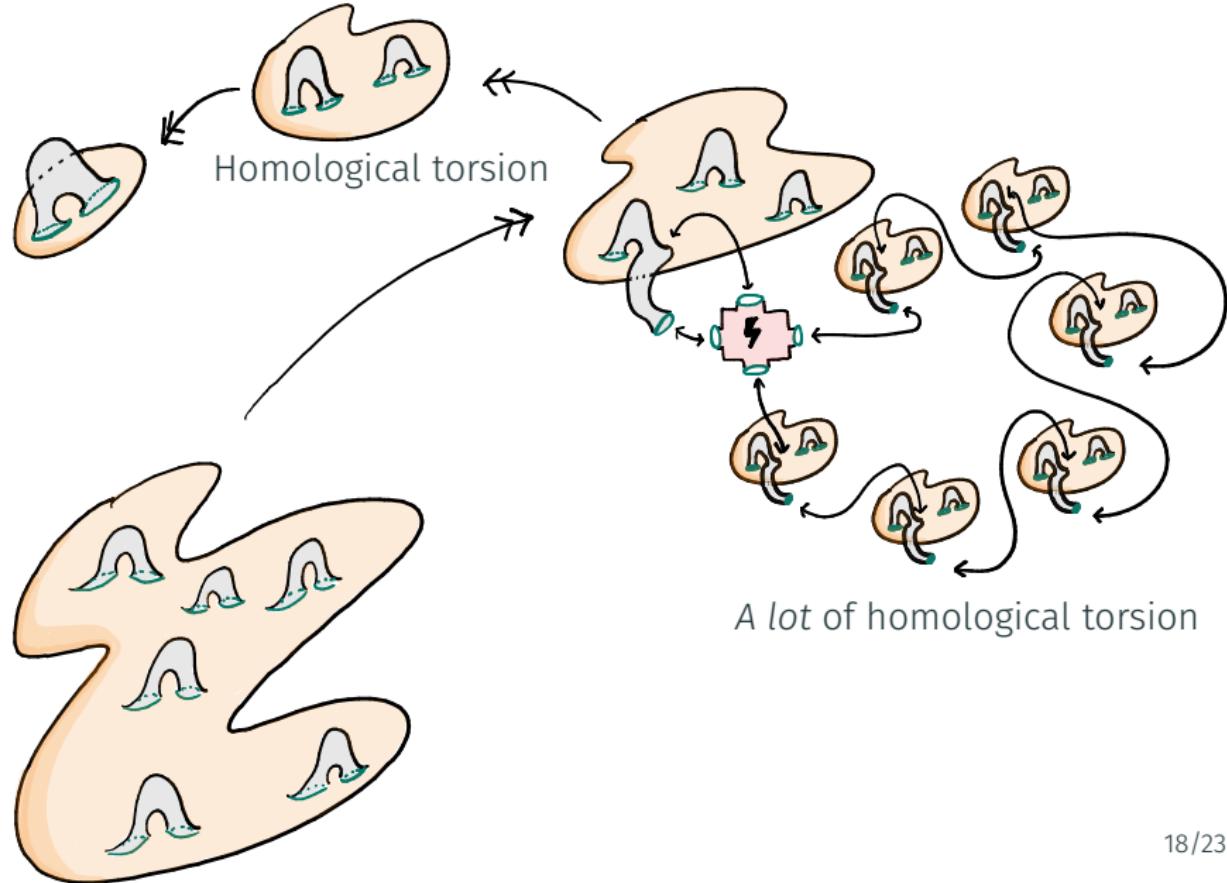
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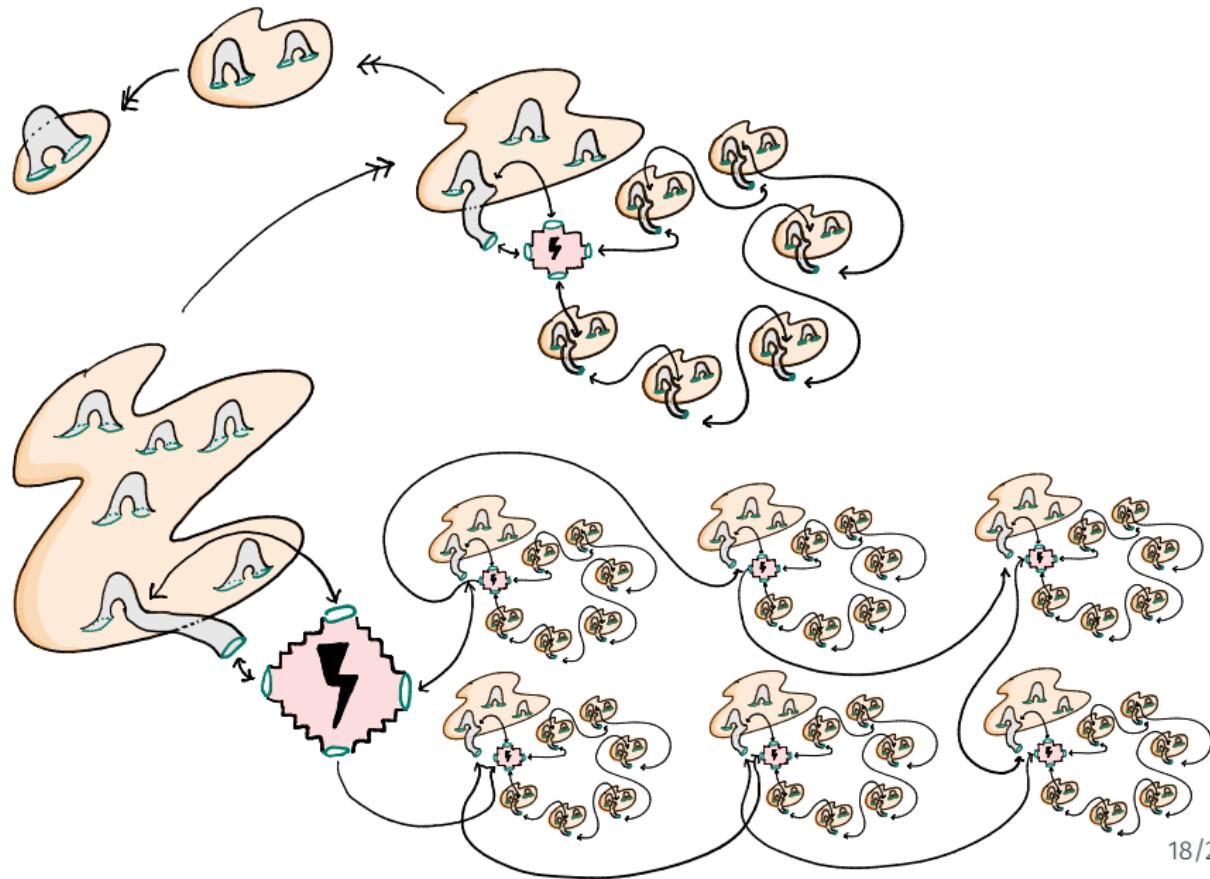
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## Growth: normal towers

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### Theorem (F '25)

*In books of  $\mathbb{Z}$ -bundles, homological torsion grows subexponentially along any exhausting tower of finite-index normal subgroups.*

## Growth: normal towers

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No infinite amenable normal subgroups

**Theorem (F '25)**



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- **Tension:** Books of  $I$ -bundles can have abundant virtual homological torsion.
- Bergeron-Venkatesh: such behaviour expected in dimension  $\geq 4$ .

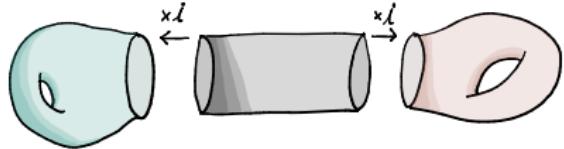
## A computation: unwrapping cylinders

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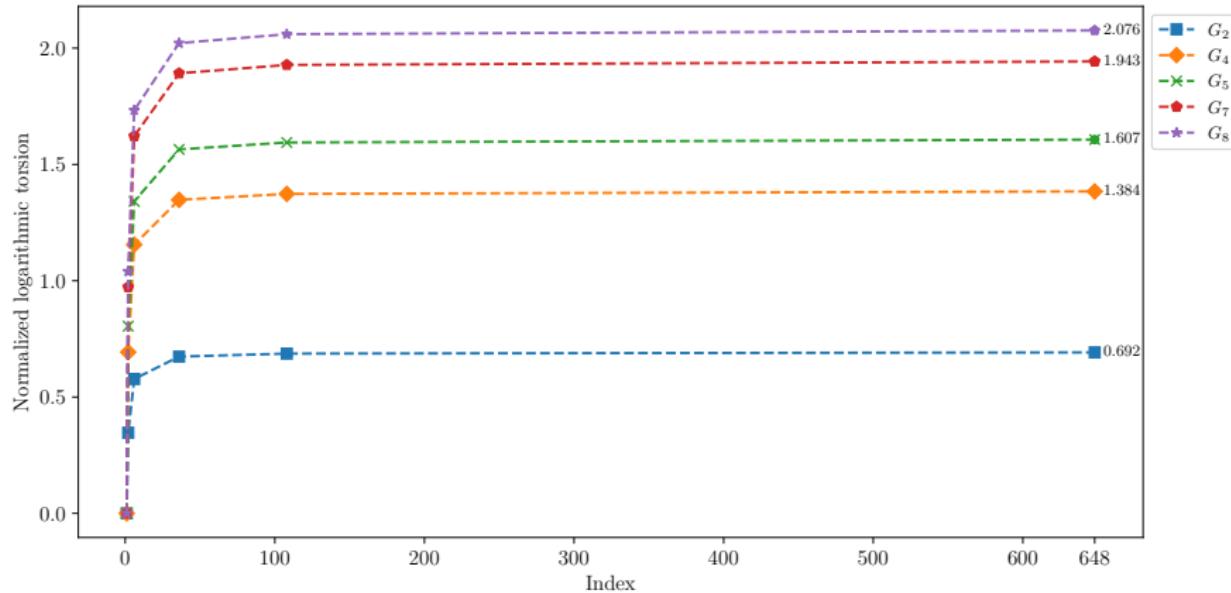
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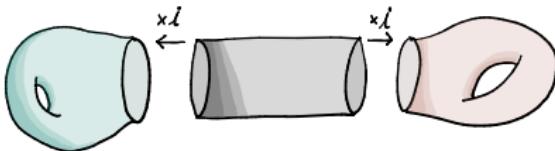
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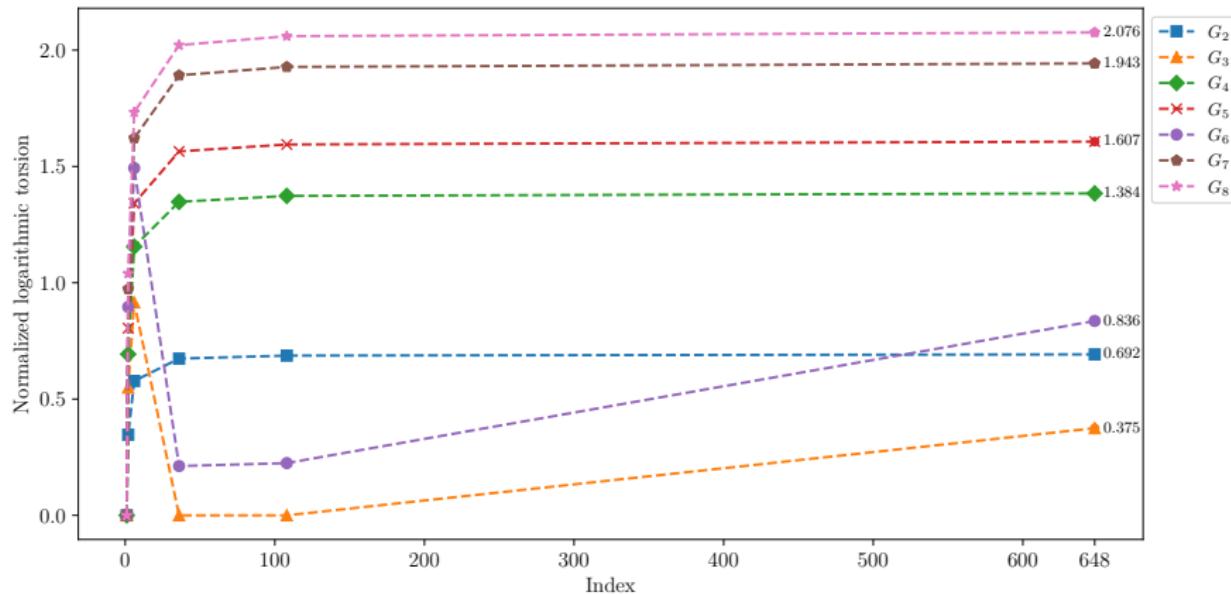
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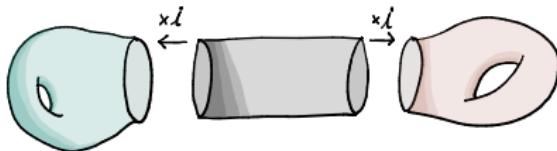
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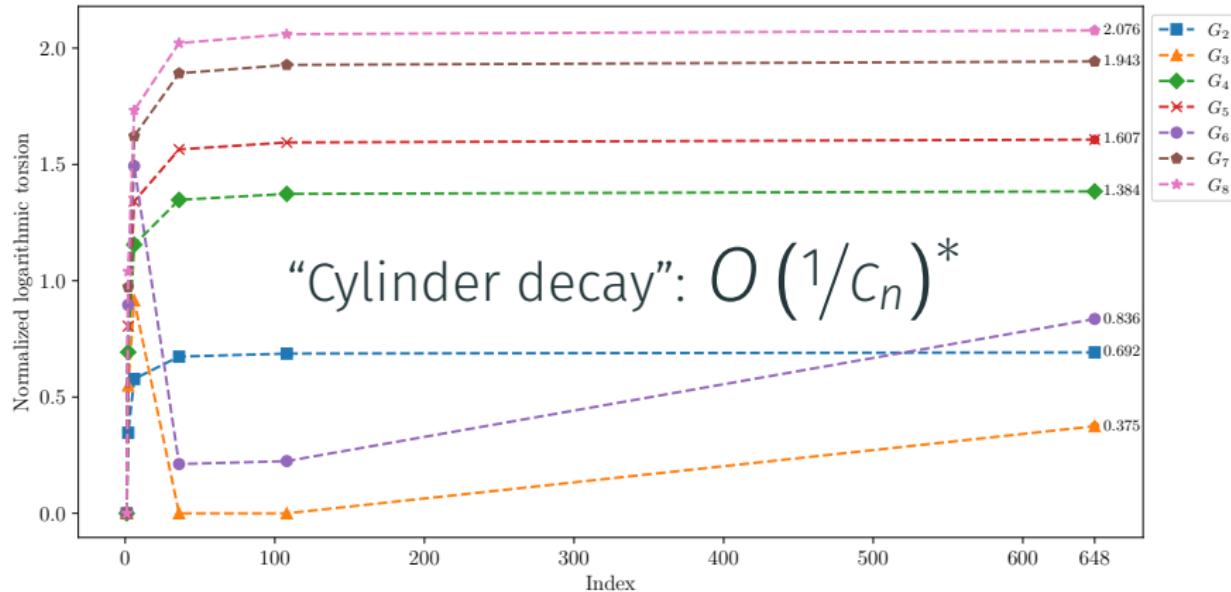
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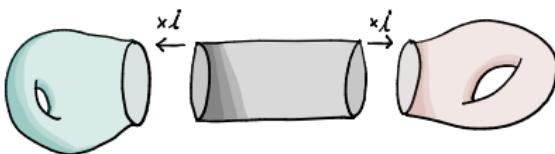


# A computation: unwrapping cylinders



\*  $C_n$  measures the wrapping degree of the cylinder in the  $n^{\text{th}}$  cover.

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## Enter a rigid vertex: revisited

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-   **Restrictions lifted:** Torsion no longer confined to cylinders.

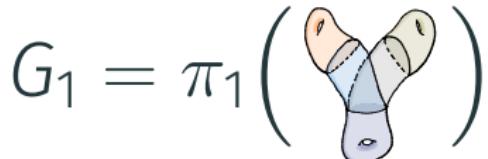
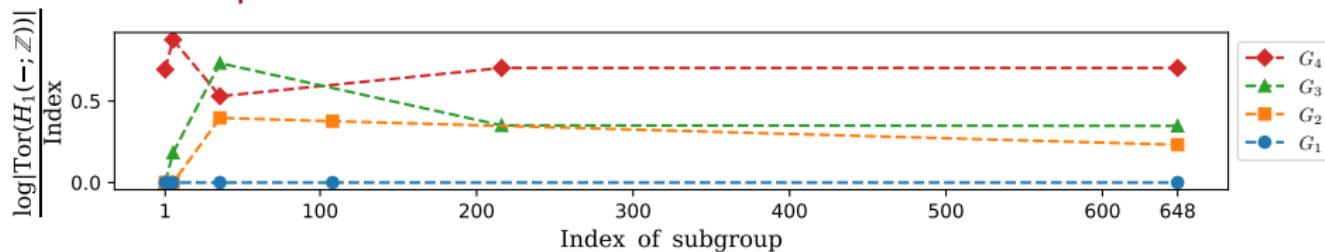
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- **Computations:**



$G_{i+1}$  obtained from  $G_i$  by adding a single letter to its presentation

# The landscape

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$\text{Tor}(H_1)$	(Hyp) 3-manifolds	Surface groups	Books of $I$ -bundles	Rigid vertex
Abundance	(Sun '15)		(Ascari-F '25)	(Ascari-F '25)
Non-normal exponential growth	(Liu '19)		(Ascari-F '25)	(Ascari-F '25)
Normal growth rate	(Exp.)		Subexp. (F '25)	

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Normal growth rate	(Exp.)		Subexp. (F '25)	Subexp. (F, recently)
"Cylinder decay" rate			$O(1/c_n)$	$O(\log(c_n)/c_n)$

## What's next?

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$\rightsquigarrow$  cross the threshold?



Thank you for listening!

