Virtual homological torsion in graphs of free groups with cyclic edge groups

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The **Exponential Torsion Growth Conjecture** predicts that torsion in first homology grows exponentially in finite covers of 3-manifolds (with empty or toroidal boundary).

This remains open, but there is positive evidence: Every finite abelian group appears in the first homology of a finite cover of any closed, hyperbolic 3-manifold [3, 2].

We prove an analogue for **hyperbolic graphs of free groups with cyclic edge groups**, an important testbed for ideas in geometric group theory.

Theorem ([1, Theorem A])
Let G be a hyperbolic graph of free groups with cyclic edge groups that is not a free product of free and surface groups. Then every finite abelian group appears as a direct summand in the abelianization of a finite-index subgroup of G.

Model spaces for homological torsion

We use **branched surfaces**—compact surfaces with boundary glued along boundary components—to produce homological torsion.

Each surface piece in a branched surface B contributes a boundary equation in homology; together, these form a system of **∂-equations**.

For every system of linear equations Ψ in a free abelian group, there exists a finite-sheeted cover B' of B whose associated system of ∂-equations is equivalent to Ψ.

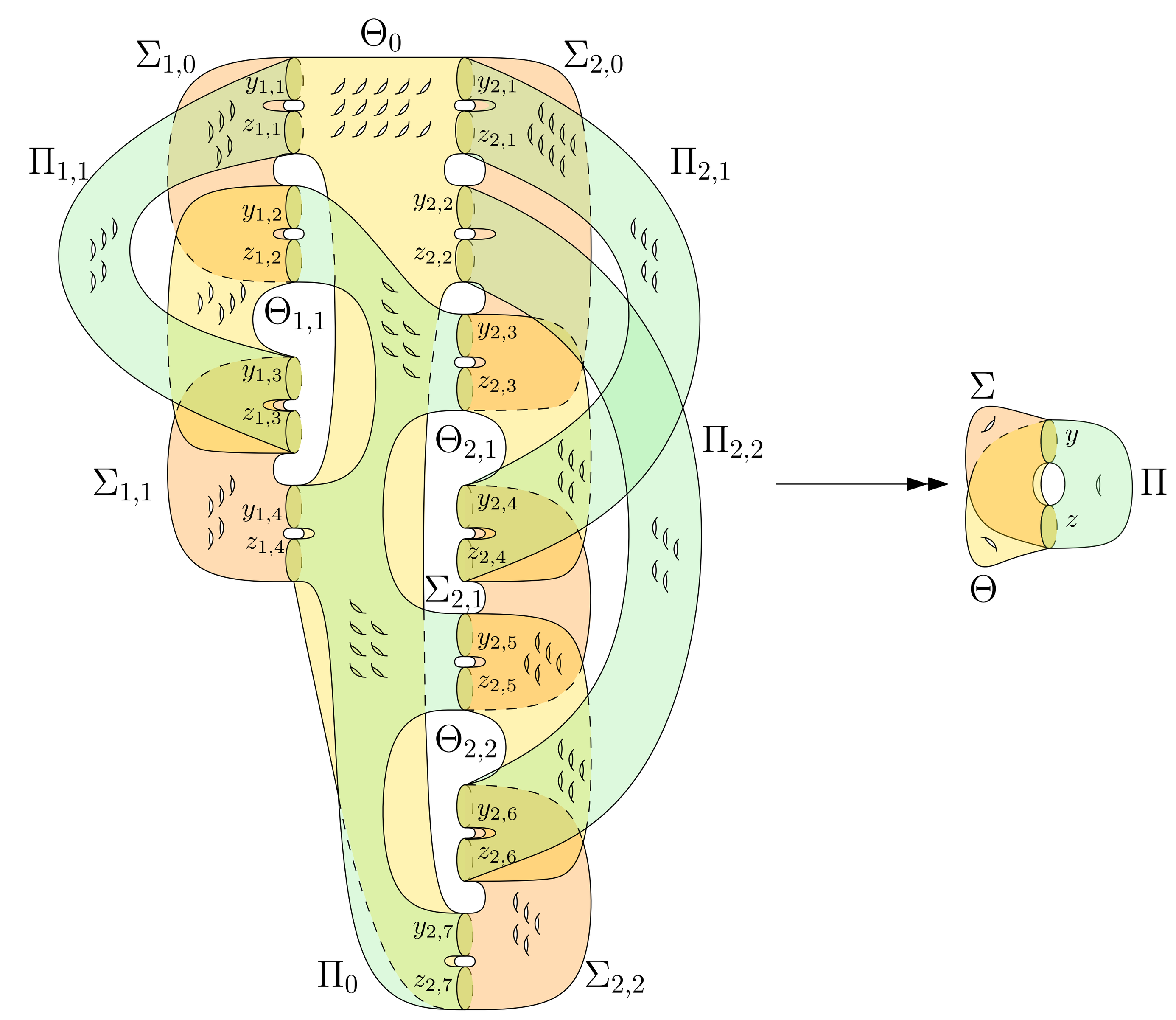

Figure: A finite-sheeted cover of a branched surface with a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ direct factor in its abelianization

Artificial branching

Wilton [4] showed that surfaces immerse into non-free hyperbolic graphs of free groups, using non-freeness as a lower bound on the complexity of vertex links.

The assumption that G is not a free product of free and surface groups gives a sharper lower bound, allowing us to construct *artificial branching*—pieces that mimic the branching behaviour in a branched surface.

This gives maps from branched surfaces to G that inject torsion in abelianization.

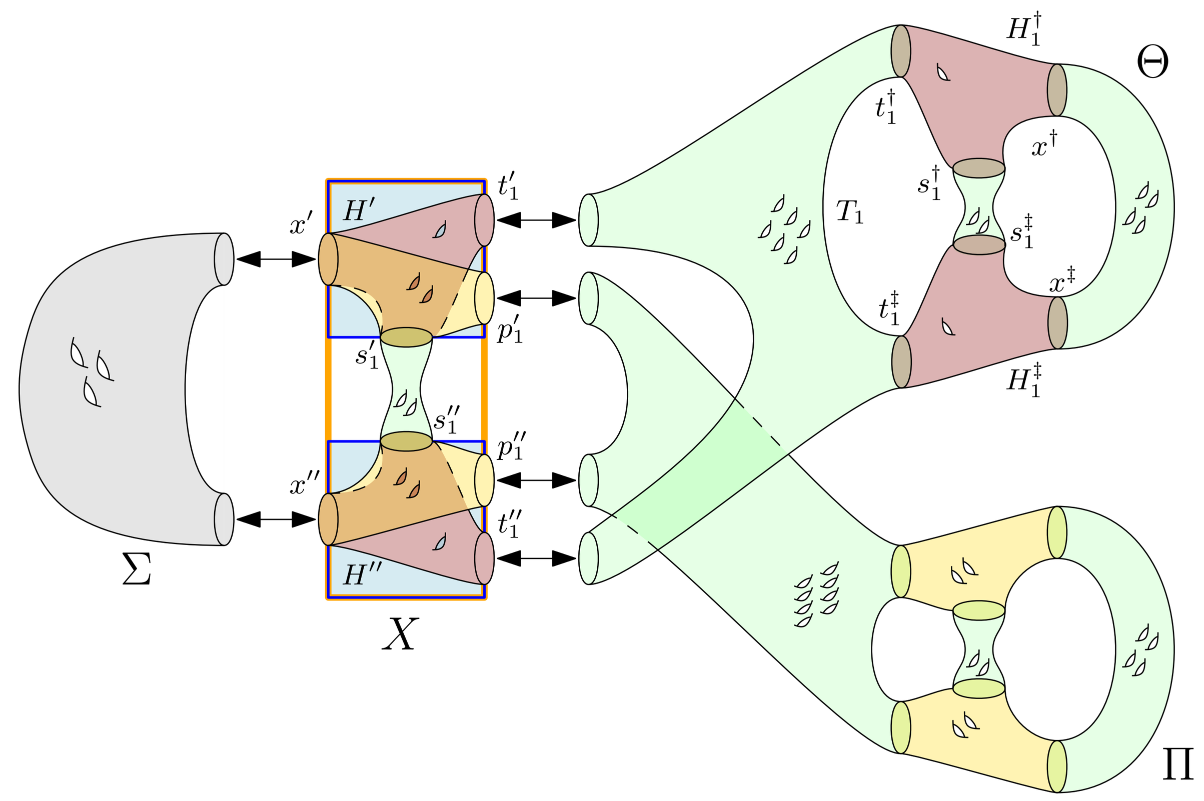

Figure: A precover of a graphs of free groups with cyclic edge groups exhibiting artificial branching

Profinite rigidity of words

A word $w \in F_k$ is **profinitely rigid** if its profinite automorphic orbit intersects F_k exactly in its usual automorphic orbit, i.e.,

$$\mathrm{Aut}(\widehat{F}_k).w \cap F = \mathrm{Aut}(F_k).w.$$

Very few non-power words (3 or 4, depending on k) are known to be profinitely rigid in F_k: primitive words, commutators of basis elements and (orientable and non-orientable) surface words.

Corollary ([1, Corollary D])
The following words are profinitely rigid in F_k:
- $[x_1, x_2], \ldots, [x_{2n-1}, x_{2n}]$ for $2n < k$,
- $x_1^2 \cdots x_n^2$ for $n < k$.

[1] Dario Ascari and Jonathan Fruchter. "Virtual homological torsion in graphs of free groups with cyclic edge groups". In: (2025). arXiv: 2505.20960.
[2] Michelle Chu and Daniel Groves. "Prescribed virtual homological torsion of 3-manifolds". In: *Journal of the Institute of Mathematics of Jussieu* 22.6 (2022), pp. 2931–2941.
[3] Hongbin Sun. "Virtual homological torsion of closed hyperbolic 3-manifolds". In: *Journal of Differential Geometry* 100.3 (2015), pp. 547–583.
[4] Henry Wilton. "Essential surfaces in graph pairs". In: *Journal of the American Mathematical Society* 31.4 (2018), pp. 893–919. ISSN: 1088-6834. DOI: 10.1090/jams/901. URL: http://dx.doi.org/10.1090/jams/901.

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