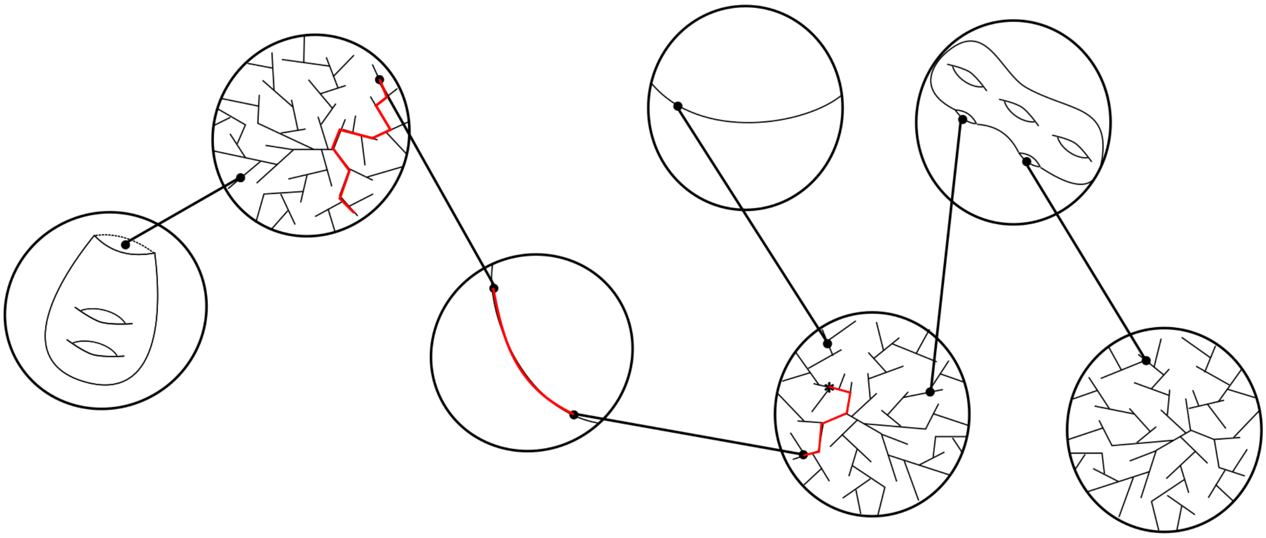
Groups, Geometry and Logic



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Introduction and overview

In this preliminary chapter we lay out foundational groundwork, giving a brief overview of fundamental concepts in the fields of first-order logic and model theory (with a focus on the study of groups from this perspective). We then survey important theorems from the past three decades concerning with the first-order theory of free groups.

Given a group G, our main goal is to understand the *theory of* G, which is (roughly speaking) the following set:

$$\mathrm{Th}(G) = \{\text{"things" that are true in } G\}.$$

To be able to talk about "things" in a precise manner, we will embrace the setting and conventions of *first-order logic*; not all "things" can be described in this setting, and as we will see different groups can have the same theory (whereas, under mild assumptions, others are completely determined by their theory). In the first part of the course, our discussion will focus on infinite groups, and in particular free groups. Our analysis of the theory of such groups will rely on topological and geometric tools.

Before making things precise, we give some assertions one can make about a group G in first-order logic. Such statements, are ones that involve:

1. **Quantifiers:** $\forall$ (for all) and $\exists$ (exists),
2. **Variables:** $x, y, z, \ldots$, which allow us to refer to group elements,
3. **Equality:** $=$ the equality sign,
4. **Logical connectors:** $\neg$ (not/negation), $\wedge$ (and/conjunction), $\vee$ (or/disjunction) and $\to$ (implies),

5. **Group theoretic symbols:** $\cdot$ (group multiplication), $-^{-1}$ (inverse operation) and 1 (trivial group element).

A few examples, that should give us better intuition as to one can or can not say about a group G using first-order statements, include:

Examples I.0.1. 1. The fact that a group G is trivial can be expressed by the sentence $\forall x \; x = 1$.

2. The fact that a group G is finite of size n can be expressed by the sentence

$$\varphi_n = \exists x_1 \cdots \exists x_n \Big(\bigwedge_{1 \le i < j \le n} x_i \neq x_j \Big) \wedge \Big(\forall y \bigvee_{1 \le i \le n} y = x_i \Big).$$

Note that this sentence depends on the constant n. Since quantifying over natural numbers is not allowed (and more generally, one can only quantify over *group elements*), there isn't a sentence that holds if and only if G is finite.

3. One can extend the idea above to construct a sentence that completely determines a finite group G: simply write down a finite multiplication table for G, and add $x_i \cdot x_j = x_k$ to the sentence above for every entry in the multiplication table.

4. One can encode the fact that G is abelian in a first-order sentence: $\forall x \forall y \; [x, y] = 1$; similarly, the negation of this sentence $\neg(\forall x \forall y \; [x, y] = 1$, which is equivalent to $\exists x \exists y \; [x, y] \neq 1$, asserts that G is not abelian.

5. Similar to the fact that being finite can not be encoded into a first-order sentence, there is no sentence which states that "G is torsion-free". However, given n, the sentence $\psi_n = \forall x \; x \neq 1 \to x^n \neq 1$ asserts that there are no elements of order n in G. We conclude that G is torsion-free if and only if the set of sentences $\{\psi_n : n \in \mathbb{N}\}$ is contained in the theory of G.

We seal the discussion with an exercise:

Exercise I. The point of this exercise is to show that free abelian groups are uniquely determined by their theory among finitely generated groups.

Let $G = \mathbb{Z}^k$ and let H be a finitely generated group such that $\mathrm{Th}(H) = \mathrm{Th}(G)$. From the examples above, we already know that H is abelian and torsion-free, and therefore $H \cong \mathbb{Z}^m$ for some m.

1. Suppose first that $k = 1$, that is $G = \mathbb{Z}$. Use the fact that every integer $z \in \mathbb{Z}$ is either *even* or *odd* to construct a sentence χ such that if $\chi \in \mathrm{Th}(H)$ then H must be isomorphic to $\mathbb{Z}$.

2. Generalize the idea above and construct a similar sentence when $k > 1$.

I.1 Basic concepts in first-order logic

To make statements, one needs a language. In the context of first-order logic, a *language* $\mathcal{L}$ is a set of symbols, which come in one of three types:

1. *constants*, usually denoted by $c_1, c_2, \ldots$,

2. *predicates, or relations*, usually denoted by $P_1, P_2, \ldots$, and each of these will interpreted later as a multivariable function with target in $\{\mathtt{True}, \mathtt{False}\}$,

3. *functions*, usually denoted by $f_1, f_2, \ldots$, which will be interpreted as multivariable functions that create new elements from given elements.

Remark I.1.1. Some authors refer to "=" as a 2-variable predicate, while others use the convention that $=$ is a logical symbol (such as $\wedge$ or $\neg$). We will stick to the latter.

Given a language, one can form *structures*:

Definition I.1.2. Given a language $\mathcal{L}$, an $\mathcal{L}$*-structure* M consists of:

1. A set called a *universe*, or a *domain*. The universe of M is usually denoted by $|M|$.

2. An interpretation of the constant symbols: for each constant symbol $c \in \mathcal{L}$, we specify an element $x \in |M|$ (usually denoted by c^M).

3. An interpretation of the predicate symbols: for each n-ary predicate symbol $P \in \mathcal{L}$, we associate a function $P^M : |M|^n \to \{\mathtt{True}, \mathtt{False}\}$.

4. An interpretation of the function symbols: for each n-ary function symbol $f \in \mathcal{L}$, we associate a function $f^M : |M|^n \to |M|$.

Example I.1.3. Let $\mathcal{L} = \mathcal{L}^{\mathrm{gp}} = \{\cdot, ^{-1}, 1\}$ be the language of groups, which consists of a constant symbol 1, and two function symbols: a 2-ary function symbol $\cdot$ and a 1-ary function symbol $^{-1}$. Then every group G is an $\mathcal{L}$-structure, where 1 is interpreted as the identity element, and $\cdot$ and $^{-1}$ are interpreted as the group multiplication and inverse operations.

Statements in a language (usually) only make sense if they are phrased adhering to certain syntactic rules. We continue by defining appropriate syntax, which will enable us to make statements in a language $\mathcal{L}$ about its structures. The first-order syntax coincides with what one would intuitively expect, and the definitions (as well as classical proofs in the study of first-order logic), are made by induction:

Definition I.1.4. A *term* t in a language $\mathcal{L}$ is either

1. a variable,

2. a constant symbol, or

3. $f(t_1, \dots, t_n)$ where $\in \mathcal{L}$ is an n-ary function symbol, and $t_1, \dots, t_n$ are terms.

Example I.1.5. In the language of groups, the commutator of two variables is a term (involves applying $^{-1}$ twice and $\cdot$ thrice), and more generally any nested commutator of variables, that is an expression of the form $[x_1, [x_2, \cdots, [x_{n-1}, x_n] \cdots]$ is a term.

Definition I.1.6. A *formula* φ in a language $\mathcal{L}$ is either

1. *atomic*, that is φ has one of the following two forms

   (a) $t_1 = t_2$ where t_1 and t_2 are $\mathcal{L}$-terms,

   (b) $P(t_1, \dots, t_n)$ where $P \in \mathcal{L}$ is an n-ary predicate symbol, and $t_1, \dots, t_n$ are terms.

2. obtained from formulas using logical symbols, e.g. $\neg\varphi$ or $\varphi \wedge \psi$ where φ and ψ are formulas, or

3. obtained from a formula φ by *bounding* a variable by a quantifier, that is $\forall x \varphi$ and $\exists x \varphi$ are both formulas.

We remark that, in order to make things clear, we will *always* wrap formulas in brackets brackets when constructing new formulas from old ones. This way, for example, if φ and ψ are formulas, the formula "φ implies ψ" will be written as $(\varphi) \to (\psi)$. A quick inspection of different formulas reveals that variables can come in two flavours: they either appear *free*, or they are *bound* by a quantifier. We make this statement precies:

Definition I.1.7. Let φ be a formula. Looking at φ as a string $a_1 a_2 \cdots a_n$. Given a variable x, look at a specific occurence of x in φ, that is choose some i with $a_i = x$. We say that this occurrence of x is *bound* by a quantifier if for some $j < i-1$, $a_j \in \{\exists, \forall\}$, $a_{j+1} = x$ and the *subformula* starting at a_j (that is, the minimal string starting at a_j which is itself a formula, or in other words it has a balanced number of parentheses), has the form $a_j x(a_{j+3} \dots a_i \dots)$. Otherwise, we say that the occurrence of x is *free* in φ.

Remark I.1.8. Note that the same variable can occur both boundedly and freely in a single formula, for example in the formula $\varphi = (\forall x(x \cdot x = 1)) \vee x \cdot x \cdot x = 1$ the first occurence of x is bounded, whereas the second one is free.

Armed with these definitions, we can now determine when a formula is *satisfied* in a model:

Definition I.1.9. Let φ be a formula in $\mathcal{L}$ and let M be an $\mathcal{L}$-structure. We say that M satisfies φ, and denote it by $M \vDash \varphi$, if the following holds:

Let σ be an *assignment* of values for the variables in M, that is σ is a function from the collection of variables into $|M|$. Using the inductive definition of terms, σ can be easily extended to a map from the collection of all $\mathcal{L}$-terms into $|M|$. The assignment σ gives rise to a valuation v_σ, which is a map from the collection of all $\mathcal{L}$-formulas into $\{\texttt{True}, \texttt{False}\}$, defined inductively as follows:

1. if φ is atomic of the form $t_1 = t_2$, $v_\sigma(\varphi) = \texttt{True}$ if and only if $\sigma(t_1) = \sigma(t_2)$ in $|M|$,

2. if φ is atomic of the form $P(t_1, \dots, t_n)$ then $v_\sigma(\varphi) = P(\sigma(t_1), \dots, \sigma(t_n)) \in \{\texttt{True}, \texttt{False}\}$,

3. if φ was obtained from formulas using logical symbols, we deduce the value of $v_\sigma(\varphi)$ using the truth tables for the logical connectors, e.g. if $\varphi = \psi \to \chi$ then $v_\sigma(\varphi) = \texttt{True}$ if and only if either $v_\sigma(\psi) = \texttt{False}$ or $v_\sigma(\psi) = v_\sigma(\chi) = \texttt{True}$,

4. if $\varphi = \exists x \psi$ or $\varphi = \forall x \psi$, consider the collection of assignments $\sigma_{x,m}$ ($m \in |M|$), each defined by
$$\sigma_{x,m}(y) = \begin{cases} \sigma(y) & y \neq x \\ m & y = x \end{cases}.$$
Note that we already know the values of the different $v_{\sigma_{x,m}}(\psi)$ by induction; we say that $v_\sigma(\exists x \varphi) = \texttt{True}$ if and only if $v_{\sigma_{x,m}}(\psi) = \texttt{True}$ for some $m \in |M|$, and that $v_\sigma(\forall x \varphi) = \texttt{True}$ if and only if $v_{\sigma_{x,m}}(\psi) = \texttt{True}$ for all $m \in |M|$.

Finally, we say that $M \vDash \varphi$ if $v_\sigma(\varphi) = \texttt{True}$ for every assignment σ.

The definition above is somewhat cumbersome, but it goes hand-in-hand with one's intuition. This way, for example, the formula $x = x$ is satisfied in any structure (of any language), and the formula $\xi = x \neq 1 \to x \cdot x = 1$ is satisfied in reflection groups, while in other groups it can be valued as `True` or `False` depending on the assignment (and therefore, generally, $G \nvDash \xi$ (meaning that ξ is not satisfied in G). We next draw attention to the fact that a valuation does not depend on the value that the assignment assigns to bounded variables. In particular, if all of the occurrences of variables in a formula φ are bounded, the truth value of $v_\sigma(\varphi)$ does not depend on the valuation (or the assignment). This motivates the following definitions:

Definition I.1.10. A formula φ is called a *closed formula* or a *sentence* if every occurrence of a variable in φ is bounded. A *Theory* is a set of sentences.

Definition I.1.11. The *theory* of an $\mathcal{L}$-structure M is

$$\mathrm{Th}(M) = \{\varphi |\, \varphi \text{ is a sentence and } M \vDash \varphi\}.$$

If for two $\mathcal{L}$-structures M and N $\mathrm{Th}(M) = \mathrm{Th}(N)$, we say that M and N are *elementarily equivalent* and denote it by $M \equiv N$.

Before continuing with our (shallow) overview of first-order logic and model theory, we are finally in a position to state Sela's incredible result, which will be the focal point of the first part of the course:

Theorem I.1.12 ([16, *et seq.*], *cf.* [8])**.** *Let* $2 \le m, n$*, then*

$$\mathrm{Th}(F_m) = \mathrm{Th}(F_n).$$

Remark I.1.13. We make the following obvious remark: the assumption that F_m and F_n are free of rank at least 2 is crucial: the group $F_1 \cong \mathbb{Z}$ is abelian, so its first-order theory can not coincide with that of a non-abelian group. In fact, we have seen in an exercise that any finitely generated group G with $\mathrm{Th}(G) = \mathrm{Th}(\mathbb{Z})$ has to be isomorphic to $\mathbb{Z}$. Lastly, we would like to draw the reader's attention to the fact that the theorem above holds also when one of the groups in question is countable, but not finitely generated.

We continue amassing a few definitions and theorems. First-order theory deals with *provability*: which axioms (that is, sentences) allow one to prove certain theorems, and similar questions. More formally, there is a notion of a *formal system* which consists of a language and a *deductive system*. A deductive system is a set of rules that allow one to deduce new sentences from existing ones (for example, if we assume both φ and ψ, we can deduce that $\varphi \wedge \psi$ is a theorem). Given a theory T, the fact that one can prove φ from the axioms in T is denoted by $T \vdash \varphi$. We will not dive into defining deductive systems and first-order logic proofs. Luckily for us, there are two important theorems that allow model theorists to "forget" about deductive systems, and simply look at how a sentence behaves models of a given theory instead:

Theorem I.1.14 ($\Rightarrow$ Soundness Theorem $\Leftarrow$ Gödel's Completeness Theorem)**.** *Let $\mathcal{L}$ be a language, T be a theory in $\mathcal{L}$ and φ an $\mathcal{L}$-sentence. Then $T \vdash \varphi$ if and only if for every $\mathcal{L}$-structure M such that $M \vDash T$, also $M \vDash \varphi$.*

Remark I.1.15. This theorem encourages model theorists to abuse notation, and write $T \vDash \varphi$ instead of $T \vdash \varphi$.

We mention another fundamental theorem in first-order logic:

Definition I.1.16. An $\mathcal{L}$-theory T is called *consistent* if it does not prove a contradiction. In other words, T is consistent if and only if $T \nvdash \forall x(x \neq x)$. A theory is *satisfiable* if it has a model, that is there is an $\mathcal{L}$-structure M such that $M \vDash T$.

Remark I.1.17. A variation on Gödel's completeness theorem, commonly referred to as the "Model Existence Theorem", shows that every consistent theory has a model. Since every set of formulas satisfied in a model has to be consistent, this theorem shows that consistency and satisfiability are in fact equivalent.

Theorem I.1.18 (Compactness Theorem)**.** *Let T be a theory. If T is* finitely satisfiable*, meaning that every finite $T_0 \subset T$ has a model, then T is consistent, and therefore has a model.*

The compactness theorem is a standard technique for proving that theories are consistent. We give a simple example that illustrates how one can use the compactness in the context of group theory:

Example I.1.19. Suppose that G is a torsion group, and suppose that every group H which is a model of $\mathrm{Th}(G)$ is a torsion group. We will prove that G is of *bounded exponent*, meaning that there exists $N \in \mathbb{N}$ such that $g^N = 1$ for every $g \in G$.

Let $\mathcal{L}' = \mathcal{L} \cup \{c\}$, where $\mathcal{L}$ is the language of groups and c is a constant symbol. Assume for a contradiction that G is not of bounded exponent. Let $T = \mathrm{Th}(G) \cup \{c^n \neq 1 | n \in \mathbb{N}\}$, and let T_0 be a finite subset of T. Let m be the largest such that $c^m \neq 1$

appears in T_0, and since G is not of bounded exponent there is $g \in G$ such that $g, g^2, \dots, g^m \neq 1$. Interpreting c as g in G, we obtain that G is an $\mathcal{L}'$-structure and that $G \vDash T_0$. It follows that T is finitely satisfiable, and therefore T has a model H. H is also a model of $\mathrm{Th}(G)$ and is therefore a torsion group, but $c^{|H|}$ has to be of infinite order in H, a contradiction.

Exercise II. Prove (using the compactness theorem) that there isn't a finite list of $\mathcal{L}$-axioms $A = \{\varphi_1, \dots, \varphi_n\}$ which axiomatize the class of infinite groups (that is, for a group G, $G \vDash A$ if and only if G is infinite).

Notation I.1.20. Let $\varphi = \varphi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula with free variables $x_1, \dots, x_n$. Let M be an $\mathcal{L}$-structure and let $a_1, \dots, a_n \in |M|$. We denote by $\varphi(a_1, \dots, a_n)$ the *formal expression* one obtains by replacing each occurrence of x_i in φ by a_i, and refer to the a_i as parameters. In this case, we can assign a truth value for $\varphi(a_1, \dots, a_n)$ in M, and we say that $M \vDash \varphi(a_1, \dots, a_n)$ if for some (equivalently, every, since $x_1, \dots, x_n$ are the only free variables in φ) assignment σ with $\sigma(x_i) = a_i$, $v_\sigma(\varphi) = \mathtt{True}$.

Armed with this notation, we can define the notion of *elementary embeddings.* We remark that an *isomorphism* of $\mathcal{L}$-structures $f : M \to N$ is a bijection $|M| \to |N|$, such that the interpretations of all constant, function and relation symbols in M agree with those in N (applied to the image of the relevant arguments under f). Similarly, we say that M is a *substructure* or *submodel* of M if $|N| \subset |M|$, the values of all constant symbols agree between N and M, and so are the values of all function and relation symbols (when they are applied to elements from $|N|$).

Definition I.1.21. We say that a map f between two L-structures $f : N \to M$ is an *elementary embedding* if:

1. f is injective,
2. $f(N)$, with the suitable interpretations, is a substructure of M (and we will therefore assume, without loss of generality, that N is a substructure of M),
3. for any formula $\varphi(x_1, \dots, x_n)$ and every $a_1, \dots, a_n \in |N|$, $N \vDash \varphi(a_1, \dots, a_n) \iff M \vDash \varphi(a_1, \dots, a_n)$.

In this case, we will often abuse notation and say that N is an *elementary submodel* (or *substructure*) of M and denote it by $N \preceq M$.

Remark I.1.22. If $N \preceq M$ then clearly $N \equiv M$. Moreover, consider the language $\mathcal{L}_N$, obtained from L by adding a constant symbol c_n for every $n \in N$. Then, interpreting each c_n as n, we can view N and M as $\mathcal{L}_N$ structures. Now if $N \preceq M$ as $\mathcal{L}$-structures, we have that $N \equiv M$ as $\mathcal{L}_N$ structures.

We can now state a stronger form of Sela's remarkable theorem:

Theorem I.1.23 ([16, *et seq.*]]). *Let $2 \leq m < n \leq \omega$ and let $f : F_m \to F_n$ be the natural embedding (that is, $F_n = f(F_m) * F_{n-m}$). Then f is an elementary embedding.*

A converse of this theorem, was proven by Perín:

Theorem I.1.24 ([13]). *Let F be a finitely-generated, non-abelian free group and let H be a subgroup of H. If $H \preceq F$ then H is a free factor of F.*

We will sketch a proof of this result, different to the standard one appearing in the literature, later on; the proof that we will sketch is a corollary of a relative version of the following theorem.

Definition I.1.25. Let M be an $\mathcal{L}$-structure, and let $\overline{a} = (a_1, \ldots, a_n) \in |M|^n$ be an n-tuple of elements of $|M|$. The *type* of $\overline{a}$ is

$$\mathrm{tp}^M(\overline{a}) = \{\varphi(x_1, \ldots, x_n) | M \vDash \varphi(\overline{a})\}.$$

More generally, an $\mathcal{L}$-*type* of arity n, $p(x_1, \ldots, x_n)$, is a maximal set of consistent formulas with n free variables (by maximal, we mean that for every $\varphi(x_1, \ldots, x_n)$, either $\varphi(\overline{x}) \in p(\overline{x})$ or $\neg\varphi(\overline{x}) \in p(\overline{x})$; note that the type of a tuple coming from a structure is always maximal).

Remark I.1.26. When it is clear from the context, we will omit the superscript M when referring to types of tuples of elements from M. One can also define types over sets of parameters A, by allowing parameters from A to appear in the formulas in the type. The type of a tuple $\overline{a}$ over a set of parameters A is usually denoted by $\mathrm{tp}(\overline{a}/A)$.

Free groups are commonly defined by their *universal property*: the free group F_n maps onto any n-generated group G by mapping the basis of F_n to the n generators of G. A similar-spirited phenomenon occurs also from a model-theoretic perspective:

Theorem I.1.27 ([14]). *Let F be a non-abelian free group and let $\overline{a}$ and $\overline{b}$ be two n-tuples of elements from F. If $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$ then there is an automorphism $f : F \to F$ such that $f(a_i) = b_i$.*

Moreover, if F is finitely generated, A is a subgroup of F and $\mathrm{tp}(\overline{a}/A) = \mathrm{tp}(\overline{b}/A)$ then there is an automorphism $f : F \to F$ which maps $\overline{a}$ to $\overline{b}$, and which restricts to the identity on A.

In the next section, we will prove this theorem for F_2; this was originally proven by Nies [11], but the proof that we give is different to Nies' original proof, and is one of a more topological flavour (it also resembles the proof of Theorem I.1.27 more closely).

I.2 Homogeneity in F_2

In this section, we will prove a "model-theoretic" universal property for the free group of rank 2; that is, we will show that if two tuples of elements in F_2 are indistinguishable from a first-order logic perspective, then there is an automorphism of F_2 taking one to the other. More formally, we will prove the following theorem:

Theorem I.2.1 ([11]). *Let $\overline{a} = (a_1, \ldots, a_n)$ and $\overline{b} = (b_1, \ldots, b_n)$ be two tuples of variables in F_2. If $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$ then there is an automorphism $f : F_2 \to F_2$ with $f(a_i) = b_i$.*

We remark that the converse to this statement is easy to prove:

Exercise III. Let f be an automorphism of a non-abelian, finitely generated free group F, and let $\overline{a}$ be a tuple of elements from F. Show that $\mathrm{tp}(\overline{a}) = \mathrm{tp}(f(\overline{a}))$.

Theorem I.2.1 was originally proven by Nies in 2003, but the proof that we give is a topological one, reminiscent of the general proof of Perín and Sklinos. We begin with the (rather easy) task of constructing a *homomorphism* $f : F_2 \to F_2$ with $f(a_i) = b_i$.

Lemma I.2.2. *There is a homomorphism $f : F_2 \to F_2$ mapping $\overline{a}$ to $\overline{b}$.*

Proof. Let g and h be a free basis for F_2, and write each a_i as a word in the generators g and h; we fix the notation $a_i = w_{a_i}(g, h)$. Therefore, for each a_i, $F_2 \models \exists x \exists y\ a_i = w_{a_i}(x, y)$ and

$$F_2 \models \varphi_{\overline{a}}(\overline{a}) = \exists x \exists y \bigwedge_{i=1}^{n} a_i = w_{a_i}(x, y).$$

It follows that $F_2 \models \varphi_{\overline{a}}(\overline{b})$, which implies that there are $g', h' \in F_2$ such that for every i, $b_i = w_{a_i}(g', h')$. This yields a homomorphism $f_{\overline{a}} : F_2 \to F_2$ with $f(a_i) = b_i$. □

Reversing the roles of $\overline{a}$ and $\overline{b}$, we also obtain $f_{\overline{b}} : F_2 \to F_2$ with $f(b_i) = a_i$. Now set $f = f_b \circ f_a$. Our next goal, is to ensure that f is an automorphism. To do so, we will show that finitely generated, non-abelian free groups satisfy two properties defined by Heinz Hopf:

Definition I.2.3. A group G is called

1. *Hopfian*, if it is not isomorphic to any of its proper quotients; in other words, every epimorphism $G \to G$ is an automorphism.

2. *co-Hopfian*, if it is not isomorphic to any of its proper subgroups; in other words, every monomorphism $G \to G$ is an automorphism.

Remark I.2.4. A cautious reader would have probably noticed that free groups are *not* co-Hopfian; in fact, every two non-commuting elements in F_2 generate a copy of F_2. However, free groups satisfy a *relative* version of this property.

Theorem I.2.5. *Let F be a finitely generated, non-abelian free group, and let H be a subgroup of F which is not contained in a proper free factor of F. Then every monomorphism $f : F \to F$ whose restriction to H is the identity is an automorphism.*

We remind that a subgroup H of a group G is a *free factor* if G admits a free product decomposition of the form $G = H * H'$. This means that G has a presentation of the form $\langle S_H, S_{H'} | R\rangle$ such that S_H and $S_{H'}$ generate H and H' respectively, and each relation in R is written with generators coming from *exactly one* of X_H or $X_{H'}$. Free products admit a nice topological description:

Exercise IV. Let X and Y be two path connected topological spaces, and let $X \vee Y$ be their wedge sum. Prove that $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$.

We will not prove Theorem I.2.5 today, but it is worth mentioning that it follows rather easily from the *Shortening Argument* Theorem II.3.25. However, we will prove that free groups are Hopfian.

Definition I.2.6. A group G is called *residually finite* if for every $1 \neq g \in G$ there is a finite quotient $q : G \to Q$ such that $q(g) \neq 1$.

Lemma I.2.7. *Let G be a finitely generated, residually finite group. Then G is Hopfian.*

Proof. Let $f : G \to G$ be an epimorphism, and suppose for a contradiction that there is $1 \neq g \in G$ in $\ker f$. Since G is residually finite, there is a finite quotient $q : G \to Q$ such that $q(g) \neq 1$. Consider the collection of quotients $q_i = q \circ f^i : G \to Q$; we will show that $q_i \neq q_j$ for $i \neq j$, contradicting the fact that there are only finitely many homomorphisms $G \to Q$.

Let $i < j$, and note that f^i is surjective so there is $h \in G$ such that $f^i(h) = g$. It follows that $q_i(h) = \alpha(f^i(h)) \neq 1$. On the other hand,

$$q_j(h) = \alpha(f \circ \cdots \circ f \circ f^i(h)) = \alpha(f \circ \cdots \circ f(g)) = \alpha(1) = 1.$$

□

Corollary I.2.8. *Finitely generated, non-abelian free groups are Hopfian.*

Proof. By Lemma I.2.7, it is enough to prove that such an F is residually finite. We will give a simple topological proof. Suppose that F is generated by n elements, and realize F as the fundamental group of a rose graph R with n petals. Recall that finite-sheeted covering spaces of R stand in one-to-one correspondence with finite-index subgroups of F. It is therefore enough to construct for every $g \in F$ a finite-sheeted covering R_g of R such that $g \notin \pi_1(R_g)$: indeed, $[F : \pi_1(R_g)] = n < \infty$, and one readily sees that $N = \bigcap_{[F:H]\le n} H$ (note that, since F is finitely generated, there are only finitely many groups participating in the intersection) is a finite-index normal subgroup of F which doesn't contain g. The map $F \to F/N$ is the desired finite quotient.

To construct R_g, we note that a graph Γ covers R if and only if every vertex admits an in-going and out-going edge for each petal of R; moreover, $g \notin \pi_1(\Gamma)$ if and only if the lift of g to Γ does not close a loop. So to construct R_g, we fix a basepoint x and draw a loop that traverses the word g^2. Since every edge in that loop leaves one vertex and enters another, one can add edges between the vertices of the loop to obtain a covering R_g of R. Since no new vertices were added to the loop, R_g has finitely many vertices and it is a finite-sheeted cover of R. □

A drawing will be added later

We will also use the following auxiliary lemma in the proof of Theorem I.2.1:

Lemma I.2.9. *There is a formula $\psi_2(x, y)$ such that for g, h in a non-abelian free group F, $F \vDash \psi_2(g, h)$ if and only if $\langle g, h\rangle$ is a free group of rank 2.*

Exercise V. Prove that one can't form such formulas $\psi_k(x_1, \dots, x_k)$ for $k > 2$. *Hint:* what can be said about the sentence $\exists x_1 \exists x_2 \forall x_3 \cdots \forall x_k \psi_2(x_1, x_2) \wedge \neg\psi_k(x_1, \dots, x_k)$?

Proof of Lemma I.2.9. The formula ψ_2 is incredibly simple: we set $\psi_2(x, y) = [x, y] \neq 1$. Suppose now that $F \vDash \psi_2(g, h)$. It follows that $\langle g, h\rangle$ is non-abelian, so it can't be a free group of rank 1 (i.e. $\mathbb{Z}$). Since every subgroup of a free group is free (this statement is known as the Nielsen-Schreier theorem, and it follows from the fact that every covering space of a graph is a graph, and fundamental groups of graphs are always free), and since a free group with a basis of size $k > 2$ can't be generated by 2 generators, we deduce that $\langle g, h\rangle = F_2$. □

We are finally ready to prove that F_2 is homogeneous:

Proof of Theorem I.2.1. Recall that we have two tuples in F_2, $\overline{a}$ and $\overline{b}$, with $\mathrm{tp}(\overline{a}) = \mathrm{tp}(\overline{b})$. Let g, h be a basis of F_2, and note that

$$F_2 \vDash \chi_{\overline{a}}(\overline{a}) = \exists x \exists y \Big(\Big(\bigvee_{i=1}^{n} a_i = w_{a_i}(x,y) \Big) \wedge ([x,y] \neq 1) \Big).$$

Now $F_2 \vDash \chi_{\overline{a}}(\overline{b})$ so there are g', h' such that $w_{a_i}(g', h') = b_i$ and $\langle g', h' \rangle \cong F_2$. By mapping g and h to g' and h' respectively, we get a homomorphism $f_a : F_2 \to F_2$ which maps $\overline{a}$ to $\overline{b}$. Moreover, the image of f_a is isomorphic to F_2, and since F_2 is Hopfian by Corollary I.2.8, we deduce that f_a is injective. Similarly, we obtain an injective homomorphism $f_b : F_2 \to F_2$ which maps $\overline{b}$ to $\overline{a}$. Their composition $f = f_b \circ f_a$ is a monomorphism of F_2 which fixes $\overline{a}$.

If $H = \langle \overline{a} \rangle$ is not contained in a proper free factor of F_2, then the relative co-Hopf property Theorem I.2.5 implies that f is an automorphism; this implies that f_a is an isomorphism. Otherwise, we have that $\overline{a}$ is contained in a proper free factor of F_2.

Suppose now that $\overline{a}$ is contained a proper free factor K; such a free factor of F_2 must be a free group of rank 1 (that is, $\mathbb{Z}$), so write $K = \langle g \rangle$. In this situation, every a_i is a power of g. We first claim that $g \in \mathrm{Im}(f_b)$. Write $a_1 = g^{m_1}$ so $F_2 \vDash \chi_1(\overline{a}) = \exists x a_1 = x^{m_1}$; it follows that $F_2 \vDash \chi_1(\overline{b})$ and there is some $g' \in F_2$ such that $b_1 = g'^{m_1}$. Since $f_b(b_1) = a_1$, we must have that $f_b(g') = g$ (recall that roots of elements in a free group are unique).

To finish the proof, it suffices to show that $\langle g' \rangle$ is also a free factor: if this is the case, the restriction $f_b|_{\langle g' \rangle} : \langle g' \rangle \to \langle g \rangle$ which maps each b_i to a_i can be extended to an automorphism of F_2. Since f_b is injective, it is an isomorphism $F_2 \to \mathrm{Im}(f_b)$. It is therefore enough to prove that $K \leq \mathrm{Im}(f_b)$ is a free factor of $\mathrm{Im}(f_b)$. In the spirit of Corollary I.2.8, we will give a topological proof [1].

Recall that in Exercise IV we saw that if a space Z is the wedge of two spaces X and Y, then $\pi_1(X)$ and $\pi_1(Y)$ are free factors of $\pi_1(Z)$. Since $\langle g \rangle$ is a free factor of F_2, we can realize F_2 as the fundamental group of a two-petaled rose R in which one of the petals corresponds to g. There is a (possibly infinite-sheeted) cover R' of R which corresponds to the subgroup $\mathrm{Im}(f_b)$; denote its basepoint by v. Since $g \in \mathrm{Im}(f_b)$, there is a single-edged loop, which is a lift of g to R', based at v. It

[1] In fact, the following more general claim holds: whenever G is a finitely generated group, K is a free factor of G and $H \leq G$, $K \cap H$ is a free factor of H. This is a simple consequence of the Kurosh subgroup theorem. In more detail, there is an action of G on a simplicial tree T with trivial edge stabilizers, and such that K fixes a vertex $v \in T$. Any $H \leq G$ also acts on T with trivial edge stabilizers, and $K \cap H$ fixes v, so $K \cap H$ is a free factor of H.

follows that R' is the wedge of this loop and another graph Γ. Therefore

$$\mathrm{Im}(f_b) = \pi_1(R') = \langle g \rangle * \pi_1(\Gamma)$$

$\square$

A drawing will be added later

II

Algebraic Geometry over Groups

We dive into understanding the first-order theory and start with the simplest type of formulas: those without quantifier and without inequalities, or in other words collections of systems of equations over free groups. Such formulas correspond to homomorphisms from a finitely generated group to a free group, and these can be encoded into a tree-like diagram called the *Makanin-Razborov* diagram. The vertices of such a diagram are *limit groups*, which will be introduced and studied in this chapter.

Our quest towards understanding the first-order theory of a finitely generated group G begins with understanding the behaviour of the simplest type of formulas: *positive* atomic formulas, or in other words atomic formulas that do not involve inequalities. Such formulas are always equivalent to one of the form $w(x_1, \ldots, x_n) = 1$ (where w is a word in a free group of rank n).

Definition II.0.1. A (group theoretic) *equation* in variables $\overline{x} = (x_1, \ldots, x_n)$ is simply an element $w \in F(\overline{x})$. We will sometimes refer to the equation w as $w(\overline{x})$ or $w(\overline{x}) = 1$, depending on the context. An *inequality* (or an *inequation*) is an expression of the form $w(\overline{x}) \neq 1$ where $w \in F(\overline{x})$.

An *equation over a group G* in variables $\overline{x}$ is an element $w \in F(\overline{x}) * G$. Recall that such w is an alternating product of elements from $F(\overline{x})$ and G. As before, we will often refer to w as $w(\overline{x}, \overline{a})$ where $\overline{a}$ is a tuple of elements from G. An *inequality over a group G* is an expression of the form $w(\overline{x}, \overline{a}) \neq 1$.

Note that we can always refer to an equation $w(\overline{x})$ as an equation over a group G. In this case we say that the equation is *without parameters.*

Notation II.0.2. Given a tuple of variables $\overline{x} = (x_1, \dots, x_n)$, $w \in F(\overline{x})$ and $\overline{g} = (g_1, \dots, g_n) \in G^n$, we write $w(\overline{g})$ to denote the element of G obtained from w by replacing each $x_i^{\pm 1}$ with the corresponding $g_i^{\pm 1}$. For $w \in F(\overline{x}) * G$, $w(\overline{g}, \overline{a})$ is defined similarly.

As with "standard" equations, group-theoretic equations can also have *solutions*:

Definition II.0.3. A *solution* to an equation $w(\overline{x}, \overline{a}) = 1$ over a group G consists of a tuple $\overline{g} = (g_1, \dots, g_n)$ of elements from G, of the same arity as $\overline{x}$, such that $w(\overline{g}, \overline{a}) = 1$ holds in G.

Examples II.0.4.

1. The set of solutions to the equation $[x, y] \in F(x, y)$ over any group G is set of all pairs (g_1, g_2) such that g_1 and g_2 commute. In a free group, such a pair of commuting elements must satisfy one of the following:
   - $g_1 = 1$ or $g_2 = 1$,
   - g_1 is a power of g_2, or
   - g_2 is a power of g_1.
2. Now fix a group G and let $a \in G$. Consider the equation $w = [x, a] \in F(x) * G$ over G. Then the set of solutions of w is the centraliser of a in G.

Our next observation, is that there is a one-to-one correspondence between the set of solutions to a system of equations $\Sigma(\overline{x}, \overline{a}) = \{w_i(\overline{x}, \overline{a})\}_{i \in I} \subset F(\overline{x}) * G$ over a group G and the set of homomorphisms from the group

$$G_\Sigma = \langle \overline{x}, \overline{a} \mid R(\overline{a}) \cup \Sigma(\overline{x}, \overline{a}) \rangle$$

to G; here, $R(\overline{a})$ is a set of relations for which $\langle \overline{a} \mid R(\overline{a}) \rangle$ is a presentation of the subgroup of G generated by $\overline{a}$.

If $\overline{g}$ is a solution to $\Sigma(\overline{x}, \overline{a}) = 1$, there exists a homomorphism $\varphi : G_\Sigma \to G$ mapping $\overline{x}$ to $\overline{g}$ and $\overline{a}$ to $\overline{a}$; on the other hand, given such a homomorphism φ, the tuple $\varphi(\overline{x})$ is a solution to $\Sigma(\overline{x}, \overline{a}) = 1$ over G. Therefore, in order to understand the behaviour of atomic formulas, and more generally positive formulas without quantifiers, over G, we need to understand the different spaces $\mathrm{Hom}(G_\Sigma, G)$. This amounts to understanding the space of all homomorphisms from finitely generated groups to F.

A schematic illustrating this correspondence will be added later

Examples II.0.5. 1. $\mathrm{Hom}(F_n, F) = F^n$.

2. $\mathrm{Hom}(\mathbb{Z}^n, F)$ can be described as follows: let $f : \mathbb{Z}^n \to F$ and denote by $p : \mathbb{Z}^n \to \mathbb{Z}$ the projection to the first coordinate. Then there is an automorphism $\alpha \in \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$ such that $f \circ \alpha$ factors through p. Since homomorphisms from $\mathbb{Z}$ to F correspond to choosing an element of F, $\mathrm{Hom}(\mathbb{Z}^n, F)$ can be parametrized by $\mathrm{GL}_n(\mathbb{Z}) \times F$ (but different points in $\mathrm{GL}_n(\mathbb{Z}) \times F$ can yield the same point in $\mathrm{Hom}(\mathbb{Z}^n, F)$.

3. A similar phenomenon occurs in $\mathrm{Hom}(\pi_1(\Sigma), F)$, where Σ is an orientable surface of genus n, and F is of rank n. Let $f : \pi_1(\Sigma) \to F$, and denote by $r : \pi_1(\Sigma) \to F$ the map induced from "filling" the surfae, turning it into a solid handlebody. Then there is $\alpha \in \mathrm{Aut}(\pi_1(\Sigma))$ such that $f \circ \alpha$ factors through r.

II.1 From equations over free groups to limit groups

The contents of this section are based on Champatier's and Guirardel's introductory paper [4]. Some of the proofs appearing here are omitted in [4]; however, proofs of more advanced theorems appear in Champatier's and Guirardel's paper and we recommend an enthusiastic reader to complement this section with [4].

In what follows, we will restrict our attention to the case where $G = F$ is a non-abelian free group. With slight modifications, the constructions, assertions and observations described in this chapter apply to any finitely generated group G. We invite the reader to think and make these modifications wherever they apply.

Consider the collection of all homomorphisms from finitely generated groups to F. In order to understand this collection, we first turn it into a topological space by equipping it with a topology; however, this will not be enough for our purposes. It is much more convenient to work with compact topological spaces. In order to compactify this space, we will add to it "points at infinity", or *limit* points. These will be homomorphisms from finitely generated groups to groups which are not free, namely *limit groups*.

We begin by taking a topological approach towards limit groups; later on we will give an equivalent algebraic definition of limit groups, as well as a residual one. The algebraic definition will later reveal its full geometri power to us: it is through this construction that one gets an action of a limit group on an interesting metric space.

II.1.1 Marked groups

As part of his study of groups of polynomial growth, Gromov came up with an idea as to how to equip a set of groups with a topology [6]. This idea led Grigorchuk to define a topological *space of marked groups* in his work on groups of intermediate growth from 1985 [5]. This topological space gives a natural setting in which to define limit groups.

Given a finitely generated group G with a generating tuple $S = (g_1, \dots, g_n)$, every homomorphism f from G to a free group F yields a possible generating tuple for the free group $\mathrm{Im}(f)$: $f(S) = \{f(g_1), \dots, f(g_n)\}$ (note that $f(S)$ does not have to be a basis for $\mathrm{Im}(f)$).

Definition II.1.1. A *marked group* is a pair of the form (G, S) where G is a group and S is a finite tuple whose elements generate G. Given $n \in \mathbb{N}$, denote by $\mathcal{G}_n$ the set of all marked groups (G, S) with $|S| = n$.

Remark II.1.2. We emphasize that S in Definition II.1.1 above is a tuple, that is S is ordered and repetitions are allowed. In addition, if $G, (g_1, \dots, g_n)$ and $G', (g'_1, \dots, g'_n)$ are two marked groups, and the map $g_i \mapsto g'_i$ extends to an isomorphism $G \to G'$, we consider $G, (g_1, \dots, g_n)$ and $G', (g'_1, \dots, g'_n)$ as the same point in $\mathcal{G}_n$.

The next step is to define a topology on $\mathcal{G}_n$; we do so by defining a metric on $\mathcal{G}_n$. Note that a priori this metric is a pseudometric, but identifying points as in Remark II.1.2 makes it a metric. Given $(G, S), (G', S') \in \mathcal{G}_n$, set $v((G, S), (G', S'))$ to be the maximal $N \in \mathbb{N} \cup \{\infty\}$ such that $w(S) = 1$ in G if and only if $w(S') = 1$ in G' for every word $w \in F_n$ of length at most N (if (G, S) and (G', S') are isomorphic as marked groups, and they represent the same point in $\mathcal{G}_n$, set $v((G, S), (G', S')) = \infty$).

Definition II.1.3. The metric $d_n : \mathcal{G}_n \times \mathcal{G}_n \to \mathbb{R}_{\geq 0}$ is given by

$$d_n((G, S), (G', S')) = e^{-v((G,S),(G',S'))}.$$

We remind the reader that the *Cayley graph* of a group G with resepct to a (finite) generating set S is a labelled, oriented graph whose vertices are the elements of G, and $g, h \in G$ are connected by an edge with a label s if and only if $h = gs$ (or $g = hs$, in which case the edge is oreitned from h to g). We denote the Cayley graph of G with respect to S by $X(G, S)$. The ball of radius N around 1 in $X(G, S)$ (which is isomorphic to any N-ball in $X(G, S)$) will be denoted by $B_N(X(G, S))$.

Remark II.1.4. Note that $v((G,S),(G',S'))$ above can be replaced with the following quantity: the maximal $N \in \mathbb{N}$ for which the Cayley graphs $X(G,S)$ and $X(G',S')$ are isomorphic. This N is smaller than $v((G,S),(G',S'))$, but the topologies induced by these metric agree. We will use both definitions interchangeably (and sometimes use $v((G,S),(G',S'))$ to refer to N above), and choose the one which makes our proofs easier to follow.

Finally, we can give an alternative definition of limit groups:

Definition II.1.5. A group G is a *limit group* if there is $n \in \mathbb{N}$ and a generating tuple S of G of size n such that (G,S) is the limit of a sequence (G_i,S_i) in $\mathcal{G}_n$ and every G_i is a free group.

Examples II.1.6. We give a few examples that illustrate convergence in the space of marked groups:

1. The marked group $(\mathbb{Z},1)$ is the limit of the sequence $(\mathbb{Z}/n\mathbb{Z},1)_{n\in\mathbb{N}}$ in $\mathcal{G}_1$. Indeed, the balls of radius $n/100$ in $X(\mathbb{Z},1)$ and $X(\mathbb{Z}/n\mathbb{Z},1)$ are isomorphic (for n large enough).

2. $(\mathbb{Z}^2,((1,0),(0,1))$ is the limit of the sequence $(\mathbb{Z},(1,n))_{n\in\mathbb{N}}$ in $\mathcal{G}_2$. This is best explained with a drawing: A figure illustrating this will be added later. This implies that $\mathbb{Z}^2$ (and in fact, any finitely generated free abelian group) is a limit group.

Exercise VI. 1. Prove that every finitely generated, residually finite group is the limit of finite groups in the space of makred groups.

2. Prove that if $(G,S) \in \mathcal{G}_n$ is a finitely presented marked group, then there is a neighborhood $N(G,S)$ in $\mathcal{G}_n$ such that every $(G',S') \in N(G,S)$ is a quotient of (G,S) (under the map which sends S to S').

Convergence in the space of marked groups is not always easy to understand; we will later give alternative descriptions of limit groups which will help us to give more examples.

We next turn to explain how limit groups compactify the space of free groups within $\mathcal{G}_n$; the explanation is rather elementary, and it will be mostly left as an exercise.

Note that

$$\overline{\{(G,S)\in\mathcal{G}_n | G \text{ is free}\}} = \{(G,S)\in\mathcal{G}_n | G \text{ is free}\} \cup \{(G,S)\in\mathcal{G}_n | G \text{ is a limit group}\}$$

and it is therefore enough to establish that $\mathcal{G}_n$ is a compact space. To do so, we identify $\mathcal{G}_n$ with a subset of 2^{F_n}. By Tychonoff's Theorem, 2^{F_n} is compact, so it suffices to prove the following two claims:

Claim II.1.7. *The topology that $\mathcal{G}$ inherits from 2^{F_n} coincides with the topology induced by the metric d_n.*

Claim II.1.8. *$\mathcal{G}$ is a closed subset of 2^{F_n}.*

Endow F_n with a basis $\{x_1, \ldots, x_n\}$ and identify $\mathcal{G}$ with a subset of 2^{F_n} in the following manner: every $(G, S = (s_1, \ldots, s_n)) \in \mathcal{G}_n$ can be seen as a quotient of F_n through the map $q_{(G,S)} : F_n \to G$ which sends x_i to s_i. Note that (G, S) and (G', S') are isomorphic as marked groups if and only if the maps $q_{(G,S)} : F_n \to G$ and $q_{(G',S')} : F_n \to G'$ have the same kernel, and therefore $\mathcal{G}_n$ can be identified with the set of normal subgroups of F_n (which is a subset of 2^{F_n}), which we denote by $\mathcal{N}_n$.

$\mathcal{N}_n$ inherits the product topology from 2^{F_n}; a sub-basis for this topology is given by:

$$U_g = \{K \in \mathcal{N}_n | g \in K\} \quad \text{and} \quad V_g = \{K \in \mathcal{N}_n | g \notin K\}.$$

$\mathcal{N}_n$ also comes equipped with the metric that we previously defined on $\mathcal{G}$. This metric can also be described as follows: given $K, K' \in \mathcal{N}_n$, let

$$v_{\mathcal{N}}(K, K') = \max\{N \in \mathbb{N} \cup \infty | K \cap B_N(F_n, \{x_1, \ldots, x_n\}) = K' \cap B_N(F_n, \{x_1, \ldots, x_n\})\}.$$

We now define

$$d_{\mathcal{N}_n}(K, K') = e^{-v_{\mathcal{N}}(K,K')}.$$

Note that indeed $v((G, S), (G', S')) = v_{\mathcal{N}}(\ker(q_{(G,S)}), \ker(q_{(G',S')}))$.

Exercise VII. Prove Claim II.1.7: the topology that $\mathcal{N}_n$ inherits as a subspace of 2^{F_n} coincides with the topology induced by the metric $d_{\mathcal{N}_n}$ defined above.

Exercise VIII. Prove Claim II.1.8: $\mathcal{N}_n$ is a closed subset of 2^{F_n}. In other words, consider a sequence $(K_i)_{i \in \mathbb{N}}$ that converges to $K \in 2^{F_n}$ and prove that K must be a normal subgroup of F_n.

II.1.2 Algebraic limit groups

We continue by giving another description of limit groups. Recall that limit groups were defined as a set of points that compactify the collection of homomorphisms from finitely generated groups to a free group. The following definition gives another limiting process, and highlights the existence of a *limiting homomorphism* from a finitely generated group to a limit group. The process described in this subsection captures some of the aspects of taking limits in the *equivariant Gromov-Hausdorff topology*; we will see at a later point (in Section II.3.1 that every limit group comes equipped with an action on an $\mathbb{R}$-*tree*, obtained as an equivariant Gromov-Hausdorff limit of a sequence of group actions.

Definition II.1.9. Let G be a finitely generated group and let $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{Hom}(G, F)^{\mathbb{N}}$ be a sequence of homomorphisms from G to a non-abelian free group F. The sequence called *stable* if for every $g \in G$, the sequence $(\varphi_n(g))_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ is eventually always 1, or eventually never 1. The *stable kernel* of $(\varphi_n)_{n\in\mathbb{N}}$ is defined as

$$\underleftarrow{\ker}((\varphi_n)_{n\in\mathbb{N}}) = \left\{g \in G \middle| \text{ the sequence } \ (\varphi_n(g))_{n\in\mathbb{N}} \text{ is eventually always } \ 1\right\}.$$

Remark II.1.10. Note that $\underleftarrow{\ker}((\varphi_n)_{n\in\mathbb{N}})$ is a normal subgroup of G. In addition, we remark that stable sequences of homomorphisms are by no means special: a standard diagonalization argument shows that every sequence of homomorphisms $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{Hom}(G, F)^{\mathbb{N}}$ has a stable subsequence (as long as G is countable, which is always the case since we assume that G is finitely generated).

Definition II.1.11. An *(algebraic) limit group* is a quotient of the form $L = G/\underleftarrow{\ker}((\varphi_n)_{n\in\mathbb{N}})$ for some stable sequence $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{Hom}(G, F)^{\mathbb{N}}$. We denote the quotient map by $\varphi_\infty : G \twoheadrightarrow L$ and call it the *limit map* associated to $(\varphi_n)_{n\in\mathbb{N}}$.

A careful inspection of Definition II.1.11 shows that it coincides with Definition II.1.5:

Lemma II.1.12. *Let L be a group. Then L is a limit group if and only if it is an algebraic limit group.*

Proof. Suppose first that L is a limit group; in other words, there is $n \in \mathbb{N}$, a generating tuple S of L of size n and a sequence of marked groups $(G_i, S_i) \in \mathcal{G}_n$ such that G_i is free for every $i \in \mathbb{N}$ and

$$(G_i, S_i) \xrightarrow[i\to\infty]{} (L, S).$$

We will show that L is an (algebraic) limit group by constructing a stable sequence of homomorphisms $(\varphi_i : F_n \to F)_{i\in\mathbb{N}} \in \mathrm{Hom}(F_n, F)^{\mathbb{N}}$ for which $L = F_n/\underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}})$.

Define $\varphi_i : F_n \to G$ by mapping the standard generating tuple of F_n to the tuple $(S_i) \in {G_i}^n \subset G^n$. To show that this sequence is stable, let $w \in F_n$. If $w(S) = 1$ in L, then for every (G_i, S_i) that is close enough to (L, S) in $\mathcal{G}_n$ we have that $w(S_i) = 1$, so $\varphi_i(w) = 1$. Similarly, if $w(S) \neq 1$ in L, $\varphi_i(w)$ is eventually never trivial. This also shows that the stable kernel $\underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}})$ coincides with $\{w \in F_n | w(S) = 1\}$, and hence $L = F_n/\underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}})$.

For the converse, suppose that there is a group G generated by a tuple $(g_1, \dots, g_n)$ and a stable sequence of homomorphisms $(\varphi_i : G \to F_n)_{i\in\mathbb{N}} \in \mathrm{Hom}(G, F_n)^{\mathbb{N}}$ such that $L = G/\underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}})$. We will show that the marking

$$\left(L = G/\underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}}), S = \left(g_1 \cdot \underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}}), \dots, g_n \cdot \underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}})\right)\right)$$

of L is the limit of the sequence $(G_i = \varphi_i(G), S_i = (\varphi_i(g_1), \dots, \varphi_i(g_n)))_{i\in\mathbb{N}}$ in $\mathcal{G}_n$.

Note that for every $w \in F_n$, there exists $i_w \in \mathbb{N}$ such that for $i > i_w$,

$$w(S_i) = \varphi_i(w(g_1, \dots, g_n)) = 1 \text{ in } G_i \text{ if and only if } w(g_1, \dots, g_n) \in \underleftarrow{\ker}((\varphi_i)_{i\in\mathbb{N}}),$$

or equivalently if $w(S) = 1$ in L.

To finish, we will show that $d_n((L, S), (G_i, S_i)) \leq e^{-N}$ for every N; therefore, given some N, we need to find i_N such that for every $i > i_N$ and every $w \in F_n$ of length at most N, $w(S_i) = 1$ in G_i if and only if $w(S) = 1$ in L. Since the number of words $w \in F_n$ of length up to N is finite, choosing $i_N = \max\{i_w | |w| \leq N\}$ completes the proof. □

II.2 Residual properties and Noetherianity

Recall that a group G is called *residually finite* if every $g \in G$ survives as a "non-trivial residue" in a finite quotient of G (see Definition I.2.6. A simple observation is the following: if G is residually finite, then it is *fully residually finite*, meaning that for every finite subset $S \subset G$ there is a finite quotient $q : G \twoheadrightarrow Q$ such that Q is finite and $q|_S$ is injective.

Definition II.2.1. A group G is *fully residually free* if for every finite subset S *subsetG* there is a free quotient $q : G \twoheadrightarrow F$ such that F is free and $q|_S$ is injective.

Our next goal is to establish that the class of limit groups coincides with the class of finitely generated, residually free groups, that is:

Theorem II.2.2. *A group G is a limit group if and only if it is finitely generated and fully residually free.*

One direction is easier than the other.

Lemma II.2.3. *Every finitely generated, fully residually free group is a limit group.*

Proof. Let G be a finitely generated, fully residually free group and let S be a generating tuple of G of size n. Let $X_1 \subset X_2 \subset \cdots$ be an exhaustion of G by finite sets. For each i, there is a homomorphism $f_i : G \to F$ such that f_i is injective on X_i. Let $(G_i, S_i) = (f_i(G), f_i(S))$. We claim that

$$(G_i, S_i) \xrightarrow{i\to\infty} (G, S).$$

Given $N \in \mathbb{N}$, for every i large enough f_i is injective on $B_N(X(G,S))$. It follows that $B_N(X(G,S)) \cong B_N(X(G_i,S_i))$ and $d_n((G,S),(G_i,S_i)) \le e^{-N}$. $\square$

The converse is a bit trickier to prove, and it requires us to show that free groups are *equationally Noetherian*, a term first coined by G. Baumslag, Miasnikov and Remeslennikov [2]. Recall that a ring R is *Noetherian* if every increasing sequence $I_1 \subset I_2 \subset \cdots$ of left ideals eventually stabilizes (or equivalently if every left ideal is finitely generated). We will prove that a similar phenomenon holds in free groups, where instead of ideals we consider solutions of systems of equations.

Definition II.2.4. A group G is called *equationally Noetherian* if the following holds: every system of equations $\Sigma \subset G * F(\overline{x})$ with finitely many variables $\overline{x} = (x_1, \dots, x_n)$ (and parameters from G) admits a finite subsystem $\Sigma_0 \subset \Sigma$, such that the sets of solutions of Σ and of Σ_0 coincide.

The fact that free groups are Equationally Noetherian relies on Hilbert's basis theorem, which states that polynomial rings over Noetherian rings are Noetherian.

Lemma II.2.5. *Countable free groups are equationally Noetherian.*

Proof. Equational Noetherianity is inherited by subgroups; we will therefore prove the lemma for a free group of rank 2, $F_2 = \langle x_1, x_2 \rangle$, since every countable free group embeds in F_2.

F_2 embeds in $\mathrm{SL}_2(\mathbb{Z})$ by mapping

$$x_1 \longmapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad x_2 \longmapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

We denote this embedding by $\varphi : F_2 \to \mathrm{SL}_2(\mathbb{Z})$. The fact that the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by the two matrices above is a free group of rank 2 is standard, and can be easily verified by considering the action of $\mathrm{SL}_2(\mathbb{Z})$ on the real plane $\mathbb{R}^2$ and using the Ping Pong lemma with the sets $\{(x, y) \in \mathbb{R}^2 \| x| > |y|\}$ and $\{(x, y) \in \mathbb{R}^2 \| x| < |y|\}$ (a good explanation appears in the Ping-Pong Lemma Wikipedia article).

Let $\Sigma \subset F_2 * F_n$ be a system of equations in F_2 with variables $\overline{y} = (y_1, \ldots, y_n)$. Let $\Sigma' \subset \mathrm{SL}_2(\mathbb{Z}) * F_n$ be the corresponding system of equations in $\mathrm{SL}_2(\mathbb{Z})$, that is for every $\sigma \in \Sigma$ define $\sigma' \in \Sigma'$ to be the same equation where the parameters from F_2 are replaced with their image under φ in $\mathrm{SL}_2(\mathbb{Z})$. Note that $\sigma(\overline{g}) = 1$ in F_2 (for $\overline{g} \in F_2^n$) if and only if

$$\sigma'(\varphi(\overline{g})) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Regarding the n variables in each σ' as matrices with four entires, we get that σ' gives rise to four polynomial equations (with coefficients in $\mathbb{Z}$) in $4n$ variables $\overline{z} = (z_1^1, z_2^1, z_3^1, z_4^1, \ldots, z_1^n, z_2^n, z_3^n, z_4^n)$, obtained by comparing the different entries with those of the identity matrix. Therefore, the system of equations Σ gives rise to a system of polynomial equations Ψ over $\mathbb{Z}$.

Consider the ideal (Ψ) in $\mathbb{Z}[\overline{z}]$; by Hilbert's basis theorem, $\mathbb{Z}[\overline{z}]$ is Noetherian, and therefore the ideal (Ψ) is finitely generated. Write $(\Psi) = (\psi_1, \ldots, \psi_k)$. Each ψ_i was obtained from some $\sigma_i' \in \Sigma'$. It follows that $\{\sigma_1', \ldots, \sigma_k'\} \subset \Sigma'$ is equivalent to Σ', and therefore $\{\sigma_1, \ldots, \sigma_k\} \subset \Sigma$ is equivalent to Σ. □

Corollary II.2.6. *Every limit group is fully residually free.*

Proof. Let L be a limit group, that is there is a generating tuple S of L with $|S| = n$, and a sequence $(G_i, S_i)_{i \in \mathbb{N}}$ in $\mathcal{G}_n$ such that every G_i is free and

$$(G_i, S_i) \xrightarrow{i \to \infty} (G, S).$$

Our strategy will be to show that for large enough i, the map $S \to S_i$ extends to a homomorphism $f_i : L \to G_i$. This is enough in order to show that L is fully residually free: let X be a finite subset of L, then X is contained in some $B_N(X(L, S))$, and for sufficiently large i the homomorphism f_i will be injective on $B_N(X(L, S))$ and therefore on X.

Let $f : F_n \twoheadrightarrow L$ be a surjection and let $\Sigma = \ker f$. It is enough to show that for i large enough, $\sigma(S_i) =_{G_i} 1$ for every $\sigma \in \Sigma$. Recall that every G_i is a subgroup of F_n; in addition, since F_n is equationally Noetherian, there is a finite $\Sigma_0 \subset \Sigma$ admitting the same set of solutions as Σ. It follows that the sets of solutions to Σ_0 and Σ coincide in every G_i.

Given $\sigma \in \Sigma_0$, let N_σ be such that σ lies in the N_σ ball in the Cayley graph of F_n. Note that $\sigma(S) =_L 1$, and this can be seen in $B_{N_\sigma}(X(L,S))$. Therefore, there exists i_σ such that for every $i > i_\sigma$, $\sigma(S_i) =_{G_i} 1$. Now for $i > \max\{i_\sigma : \sigma \in \Sigma_0\}$ we have that $\sigma(S_i) =_{G_i} 1$, so S_i is a solution to Σ_0 in G_i. Therefore S_i is a solution to Σ as required. $\square$

The proof above, also implies the following statement; we leave it as an exercise to convince oneself that it is indeed true.

Exercise IX. Let L be a limit group, let $(\varphi_n : G \to F)_{n\in\mathbb{N}}$ be the corresponding stable sequence of homomorphisms and write $L = G/\underleftarrow{\ker}(\varphi_n)$. Denote the limit map by $\varphi_\infty : G \twoheadrightarrow L$. Then for n large enough, the homomorphism φ_n factors via L, that is there exists $\phi_n : L \to F$ such that $\varphi_n = \phi_n \circ \varphi_\infty$. In particular, one obtains a stable sequence of homomorphisms $(\phi_n : L \to F)$ with a trivial stable kernel (compare this to a stable sequence of homomorphisms obtained from being fully residually free).

Naively, the discussion above seems to imply that limit groups are finitely presented. This is indeed the case, but the proof of that fact is a lot more complicated and delicate. We invite the reader to think why the proof above does *not* imply that limit groups are finitely presented. To seal the discussion, we give another equivalent characterization of equational Noetherianity in the following guided exercise.

Exercise X. Let G be a countable group. In this exercise we will prove that G is equationally Noetherian if and only if the following holds: for every finitely generated group H, and every stable sequence of homomorphisms $(\varphi_n : H \to G)_{n\in\mathbb{N}}$, φ_i factors through $\varphi_\infty : H \to H/\underleftarrow{\ker}(\varphi_i)$ for *some* i (equivalently, i large enough). We remark that a group of the form $\varphi_\infty(H)$ is called a *limit group over* G.

1. Prove the direction $\Longrightarrow$ by modifying the proof of Corollary II.2.6 and Exercise IX. In this case, the G_i in Corollary II.2.6 are not free groups, but subgroups of G.

2. For the other direction, let $\Sigma \subset G * F_n$ be a system of equations over G. Explain why Σ must be countable, and consider an exhaustion of Σ by nested finite subsets $\Sigma_1 \subset \Sigma_2 \subset \cdots$. Use each Σ_i to construct $\varphi_i : G * F_n \to G$ such that $\varphi_i|_G = \mathrm{Id}_G$ and $\varphi_i(\sigma) = 1$ for every $\sigma \in \Sigma_i$.

3. Prove that $\Sigma \subset \ker(\varphi_\infty)$.

4. Note that G is not necessarily finitely generated. Consider the sequence $(\varphi_i|_{F_n})$ and note that some element in this sequence factors via $\varphi_\infty|_{F_n}$. Prove that φ_i must factor via φ_∞ and derive a contradiction, showing that G is equationally Noetherian.

Using this new characterization of limit groups, we deduce:

Lemma II.2.7. *Let L be a limit group. Then*

1. *L is torsion-free,*

2. *2-generated subgroups of L are either free or free abelian,*

3. *L is* commutative transitive*, meaning that for $g, h, k \in L$, if $[g, h] = [h, k] = 1$ then $[g, k] = 1$.*

Proof. 1. Every $1 \neq g \in L$ survives in a free quotient, and is therefore not a torsion element.

2. Let $g, h \in L$. If $\langle g, h\rangle$ is abelian, then since L is torsion-free we have that $\langle g, h\rangle$ is one of $\{1\}, \mathbb{Z}$ or $\mathbb{Z}^2$. If $\langle g, h\rangle$ is non-abelian, then there is a free quotient $q : L \twoheadrightarrow F$ which is injective on $\{g, h, [g, h]\}$. It follows that $q(g)$ and $q(h)$ generate a non-abelian subgroup of F, so $\langle q(g), q(h)\rangle \cong F_2$. Since F_2 is not a quotient of any 2-generated group other than F_2, we obtain that $\langle g, h\rangle \cong F_2$.

3. Suppose for a contradiction that $[g, k] \neq 1$ and let $q : L \twoheadrightarrow F$ be a free quotient of L which is injective on $\{g, h, k, [g, k]\}$. We have that $[q(g), q(h)] = [(q(h), q(k)] = 1$ so $q(g), q(h)$ and $q(k)$ are all powers of the same element $x \in F$. It follows that $[q(g), q(k)] = 1$, a contradiction.

□

Exercise XI. Prove a stronger version of *3.* in Lemma II.2.7 above: limit groups are *CSA* (Conjugacy Separated Abelian), meaning that every maximal abelian subgroup M of a limit group L is malnormal: for every $g \in L$, if $gMg^{-1} \cap M \neq \{1\}$ then $g \in L$.

One can also use these results to give a first-order characterization of limit groups. We remind the reader that the *universal theory* of a group G is the set $\mathrm{Th}_\forall(G)$ of all sentences involving only universal quantifiers (after being put in disjunctive normal form), that is the set of all sentences of the form

$$\forall \overline{x} \bigvee_{i=1}^{k} \Sigma_i(\overline{x}) = 1 \wedge \Psi_i(\overline{x}) \neq 1,$$

where $\Sigma(\overline{x})$ is a system of equations and $\Psi(\overline{x})$ is a system of inequalities.

Theorem II.2.8. *A finitely generated group G is a non-abelian limit group if and only if* $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(F)$ *(where F is a non-abelian free group).*

Remark II.2.9. A priori, it is not clear that the universal theories of all non-abelian free groups coincide. However, it easily follows from the following fact: if $H \leq G$ then $\mathrm{Th}_\forall(G) \subset \mathrm{Th}_\forall(H)$. Last, note that for every $n \geq 2$ we have that $F_2 \leq F_n \leq F_n$.

Proof of Theorem II.2.8. Suppose first that G is a non-abelian limit group. By Lemma Lemma II.2.7, $F_2 \leq G$ so $\mathrm{Th}_\forall(G) \leq \mathrm{Th}_\forall(F_2)$. To show that there is also inclusion in the other direction, let $\varphi \in \mathrm{Th}_\forall(F_2)$ and write

$$\varphi = \forall \overline{x} \chi(\overline{x}) = \forall \overline{x} \bigvee_{i=1}^{k} \Sigma_i(\overline{x}) = 1 \wedge \Psi_i(\overline{x}) \neq 1.$$

We will show that $G \vDash \varphi$, or in other words for every tuple $\overline{g}$ of elements from G,

$$G \vDash \chi(\overline{g}).$$

Let E be the set

$$E = \{1\} \cup \{\overline{g}\} \cup \{\sigma(\overline{g}) | \sigma \in \Sigma_i\} \cup \{\psi(g) | \psi \in \Psi_i\},$$

that is E is the set of all words participating in the sentence φ evaluated at $\overline{g}$. The set E is finite, so there is a free quotient $q_E : G \twoheadrightarrow F \leq F_2$ which is injective on q_E. Therefore, for every word σ participating in φ we have that

$$\sigma(\overline{g}) =_G 1 \iff \sigma(q_E(\overline{g})) =_F 1.$$

Indeed, if $\sigma(q_E(\overline{g})) =_F 1$ then the injectivity of $q_E|_E$ implies that $\sigma(\overline{g}) =_G 1$, and if $\sigma(q_E(\overline{g})) \neq_F 1$ then necessarily $\sigma(\overline{g}) \neq_G 1$ since $q_E(\sigma(\overline{g})) = \sigma(q_E(\overline{g}))$. It follows that $G \vDash \chi(\overline{g}) \iff F \vDash \chi(q_E(\overline{G}))$, and since $F \vDash \varphi$ it must be that $G \vDash \chi(\overline{g}))$. Repeating this argument for every tuple $\overline{g}$ from G we obtain that $G \vDash \varphi$.

For the other direction, suppose that $\mathrm{Th}_\forall(G) = \mathrm{Th}_\forall(F_2)$. Since every universal sentence is the negation of an existential sentence and vice-versa, we have that $\mathrm{Th}_\exists(G) = \mathrm{Th}_\exists(F_2)$. We will use this to show that G is the limit of free groups in the space of marked groups.

Let S be a generating tuple of S of size n. It is enough to find, for every $N \in \mathbb{N}$, a free group G_N and a generating tuple S_N such that $B_N(X(G,S))$ and $B_N(X(G_N,S_N))$ are isomorphic. To do so, one only needs to note (see Exercise XII) that there is a sentence of the form

$$\varphi_N = \exists x_1 \cdots \exists x_n \Sigma(\overline{x}) = 1 \wedge \Psi(\overline{x}) \neq 1.$$

such that for any group H and any tuple $\overline{h}$ in H, $H \vDash \Sigma(\overline{h}) = 1 \wedge \Psi(\overline{h}) \neq 1$ if and only if

$$B_N(\langle \overline{h} \rangle, (\overline{h})) \cong B_N(G,S).$$

Now the tuple S asserts that $G \vDash \varphi_N$, so $F_2 \vDash \varphi_N$. This gives rise to a tuple S_N in F_2 and a free marked group $(G_N = \langle S_N \rangle, S_N)$ as desired. □

Exercise XII. Construct the sentence φ_N mentioned in the proof of Theorem II.2.8 above.

We seal this section by giving a few examples of limit groups, other than free and free abelian groups which were already discussed. The first example that we give is due to Gilbert Baumslag Baumslag.

Theorem II.2.10. *Let Σ be a closed, orientable surface of even genus $2n$. Then $\pi_1\Sigma$ is a limit group.*

The proof relies on the following lemma, which is an easy consequence of the Ping Pong lemma.

Lemma II.2.11 (*cf.* [1, Proposition 1])**.** *Let F be a free group and let $g_1, \dots, g_n, u \in F$. If $[u, g_i] \neq 1$ for every i, then there exists $N \in \mathbb{N}$ such that for every $|k_1|, \dots, |k_n| \geq N$,*

$$g_1 \cdot u^{k_1} \cdots g_n \cdot u^{k_n} \neq_F 1.$$

Proof of Theorem II.2.10. Write

$$\pi_1\Sigma = \langle x_1, y_1, \dots, x_2n, y_2n \mid [x_1,y_1]\cdots[x_{2n}, y_{2n}] = 1 \rangle$$

and let $f : \pi_1\Sigma \to F_{2n} = \langle s_1, \dots, s_2n \rangle$ by mapping $x_1, y_1, \dots x_n, y_n$ to $s_1, \dots, s_2n$, and $x_{n+1}, y_{n+1}, \dots, x_{2n}, y_{2n}$ to $s_1, \dots, s_2n$. Topologically, f admits the following description:

let γ be the loop on Σ which separates Σ into two identical parts (note that in the level of fundamental groups, $[\gamma] = [x_1, y_1] \cdots [x_n, y_n]$). The map f is the map induced by the retraction of Σ onto one half-surface. Let τ be the automorphism of $\pi_1 \Sigma$ induced by a Dehn twist along γ (a nice visualization of τ appears in the following Wikipedia article). On the level of fundamental groups, τ restricts to the identity on one half surface, and to conjugation by $[\gamma]$ on the other one. We will use f and τ to construct a sequence of homomorphisms $\varphi_n : \pi_1 \Sigma \to F_{2n}$ such that given $g \in \pi_1 \Sigma$, $\varphi_n(g) \neq 1$ for n large enough.

Define $\varphi_n : f \circ \tau^n$ and let $g \in \pi_1 \Sigma$. g can be written as an alternating product $g_1 h_1 \cdots g_n h_n$, such that $g_i, h_i \notin \langle [\gamma] \rangle$ (and they are non-trivial, except for, perhaps, g_1 or h_n). In particular, g_i and h_i do not commute with $[\gamma]$, as the only elements commuting with $[\gamma]$ are its powers. One should also note that $f(h_i)$ and $f(g_i)$ do not commute with $f([\gamma])$. Applying φ_k to g we obtain

$$\varphi_k(g) = f(g_1) \cdot f([\gamma])^k \cdot f(h_1) \cdot f([\gamma])^{-k} \cdots f(g_n) \cdot f([\gamma])^k \cdot f(h_n) \cdot f([\gamma])^{-k}.$$

By Lemma II.2.11, $\varphi_k(g) \neq 1$ for k large enough. It follows that $\pi_1(\Sigma)$ is fully residually free, and therefore a limit group. $\square$

Exercise XIII. A *double* of a group G along a subgroup H is the amalgam $G *_H G$ obtained by identifying the two copies of H inside the two copies of G. Let L be a limit group, and let C be a maximal cyclic subgroup of L. Prove that the double $L *_C L$ is a limit group.

The doubling construction gives us a plethora of examples of limit groups. It turns out that every limit group can be obtained in such a manner:

Definition II.2.12. A *generalized double* of a group G is a group H admitting a splitting of the form $A *_C B$ or $A *_C$ satisfying:

1. A and B are finitely generated,
2. C is a non-trivial abelian group, which is maximal abelian in A (and B),
3. there is a surjective homomorphism $H \to G$ whose restriction to A (and B) is injective.

Remark II.2.13. Similar to Exercise XIII above, every generalized double of a limit group is itself a limit group.

Theorem II.2.14 ([4, Theorem 4.6]). *A group L is a limit group if and only if it can be obtained as an* iterated *generalized double over free groups (that is, starting with free groups and repeatedly taking generalized doubles and free products).*

Another family of examples of limit groups comes from a similar construction:

Definition II.2.15. Let G be a group, let $g \in G$ and let C be the centraliser of g. The *free extension of the centraliser* C of g is the group obtained as follows:

Let C' be another copy of C, and let $H = C' \times \mathbb{Z}$. Now identify C and C' to obtain a new group $G(g)$. In other words,

$$G(g) = G *_C (C \times \mathbb{Z}).$$

Lemma II.2.16. *Let F be a free group, let $g \in L$ and let C be the centraliser of g. The free extension of the centraliser C, $F(g)$, is a limit group.*

Sketch of proof. Let g' be a generator of $C_F(g) \cong \mathbb{Z}$ and write $F(g) = F(g') = \langle F, t \mid [g', t] \rangle$. Define $\varphi_n : F(g) \to F$ by $\varphi_n|_F = \mathrm{Id}$ and $\varphi_n(t) = g^n$. The same argument used in Theorem II.2.10 shows that $F(g)$ is a limit group. □

Exercise XIV. Show that every free extension of a centraliser of a limit group is again a limit group.

Note that subgroups of limit groups are limit groups themselves; this gives us another way of seeing that fundamental groups of closed, orientable surfaces of even genus are limit groups:

Example II.2.17. Let Σ be a closed, orientable surface of genus $2n$ and let F_{2n} be a free group with $2n$ generators, which we denote by $x_1, y_1, \ldots, x_n, y_n$. Write $g = [x_1, y_1] \cdots [x_n, y_n]$. Note that F_{2n} is the fundamental group of an orientable surface Σ' of genus n, with a single boundary component g. Consider the group $F_{2n}(g)$; it is the fundamental group of the space X obtained by taking Σ' and a 2-torus T, and identifying g with a coordinate circle of T. Write $F_{2n}(g) = \langle F_{2n}, t \mid [t, g] = 1 \rangle$ and define $f : F_{2n}(g) \to \mathbb{Z}/2$ by mapping F_{2n} to 0 and t to 1. $\ker f$ is an index-2 subgroup of $F_{2n}(g)$. Topologically, it is the fundamental group of a double cover X' of X which can be thought of as taking two copies of Σ', and gluind their boundaries by two annuli. Van-Kampen's theorem tells us that $\pi_1 X' = \ker f = \pi_1(\Sigma) *_{\mathbb{Z}}$, and in particular F_{2n} has $\pi_1 \Sigma$ as a subgroup. This shows that $\pi_1 \Sigma$ is a limit group. A figure will be added later

Again, a more general phenomenon holds in this case as well:

Theorem II.2.18 ([7, Theorem 4]). *A group L is a limit group if and only if it is a finitely generated subgroup of an iterated centraliser extension of a free group.*

II.3 Makanin-Razborov Diagrams, Factor Sets and Shortening Quotients

In the last section of this chapter we will construct the *Makanin-Razborov* diagram of a finitely generated group. This diagram, which takes the shape of a finite rooted tree, describes all homomorphisms from a finitely generated group G to a free group. The root vertex of the tree is labelled by G, and its other vertices are labelled by limit groups which are quotients of G; the leaves are labelled by free groups and each edge comes equipped with a quotient map between its adjacent vertices. Each *branch* of this diagram, namely a path between the root vertex G and a leaf F, describes a collection of homomorphisms from G to F, which differ from each other by *twisting* by an automorphism at the vertices along the path. Each $f : G \to F$ corresponds to a branch in the Makanin-Razborov Diagram. We remark that the diagram is *not* canonical, and one could construct different Makanin-Razborov diagrams with G as their root vertex. A drawing will be added later

We begin by describing the *first level* of a Makanin-Razborov diagram of a finitely generated group G, namely the collection of neighbors of the root vertex. We will first deal with the case where G is *not* a limit group. This is one of the rare occasions where it is easier to prove a theorem for a group that is not a limit group than it is for a limit group.

Definition II.3.1. Let G be a group. A *factor set* of G is a collection of quotients of G $\{q_i : G \twoheadrightarrow Q_i \mid i \in I\}$ such that every homomorphism from G to a free group F must factor via some q_i.

One readily sees that every finitely generated group which is not a limit group admits a finite factor set:

Lemma II.3.2. *Let G be a finitely generated group. If G is not a limit group, then it has a finite factor set.*

Proof. Since G is not a limit group, there is a finite subset $X \subset G$ such that every homomorphism $G \to F$ is not injective on X. Up to enlarging X, we can assume that every homomorphism from G to F kills an element of X. It follows that the collection of quotients $\{q_x : G \twoheadrightarrow G/\langle\langle x\rangle\rangle \mid x \in X\}$ is a finite factor set. □

As mentioned earlier, the vertices of Makanin-Razborov diagrams (other than the root vertex) are always labelled by limit groups. We therefore need to strengthen

Lemma II.3.2 above, and show that one can always find a finite factor set of G which consists only of limit groups.

Theorem II.3.3. *Let G be a finitely generated group which is not a limit group. Then G admits a finite factor set consisting only of limit groups.*

Proof. The proof relies on Zorn's lemma. Construct a poset Q_G of quotients of G as follows:

- Every element in the poset is an *equivalence class* of limit quotients of G. We say that two limit quotients $q_1 : G \twoheadrightarrow L_1$ and $q_2 : G \twoheadrightarrow L_2$ of G are equivalent if there is an isomorphism $f : L_1 \to L_2$ such that $f \circ q_1 = q_2$.

- We say that $[q_1 : G \twoheadrightarrow L_1] \leq [q_2 : G \twoheadrightarrow L_2]$ if and only if there is a representative $q_1' \in [q_1]$ which factors via q_2 (and in particular L_1 is a quotient of L_2).

Our goal is to show that the poset Q_G has finitely many maximal elements; these elements will give us the desired factor set (and in fact, every homomorphism from G to a limit group will have to factor through one of them).

Consider a chain $q_1 < q_2 < \cdots$ in Q_G. By Zorn's lemma, it suffices to show that the chain admits an upper bound $[q] \in Q_G$. We construct q using a diagonalization argument. Write $q_i : G \twoheadrightarrow L_i$, and let $(\varphi_n^i : G \to F)_{n \in \mathbb{N}}$ be a stable sequence of homomorphisms which corresponds to L_i. Up to replacing each (φ_n^i) with a subsequence, we may assume that for every i and every n, φ_n^i is injective on the set

$$X_{i,n} = \{g \in G \mid g \notin \ker q_i \text{ and } g \in B_n(G)\}$$

The diagonal sequence $(\varphi_n^n)_{n \in \mathbb{N}}$ gives rise to a limit quotient $q : G \twoheadrightarrow G/\underleftarrow{\ker}\varphi_n^n = L$. Recall that L is finitely presented, which implies that all of the relations of L lie in some $B_N(G)$. It follows that for every i and n large enough, every φ_n^i factors via q and therefore all but finitely many of the q_i factor via q. Since the q_i form a chain, we have that every q_i must factor through q, and hence q is an upper bound for the chain.

We have shown therefore that every limit quotient of G factors through a maximal element of Q_G, and it is left to show that there are only finitely many maximal elements in Q_G. Let $(q_n)_{n \in \mathbb{N}}$ be a sequence of maximal elements in Q_G. Repeating the same diagonalization argument as above, show that all but finitely many of the q_i must factor via some q; but since the q_i are maximal, we obtain that all but finitely many q_i lie in $[q]$, which completes the proof. $\square$

Exercise XV. The proof of Theorem II.3.3 used the fact that limit groups are finitely presented. Prove Theorem II.3.3 without using this fact. *Hint:* Use the fact that free groups are equationally Noetherian.

Corollary II.3.4. *A finitely generated group is residually free if and only if it embeds as a subgroup of a direct product of finitely many limit groups.*

Proof. The fact that every subgroup of a direct product of limit groups is residually free is easy to prove: every non-trivial element in such a group projects non-trivially to some coordinate. Every coordinate is a limit group, and hence residually free.

For the other direction, let G be a finitely generated residually free group and let $\{q_i : G \twoheadrightarrow L_i\}_{1\le i\le n}$ be a finite factor set of G such that every L_i is a limit group. Define $f = q_1 \times \dots \times q_n : G \to L_1 \times \ldots \times L_n$ and note that $\ker f = \bigcap_{i=1}^n \ker q_i$. It is enough therefore to show that $\bigcap_{i=1}^n \ker q_i \neq \{1\}$. Indeed, if $g \in G$ is non-trivial, the fact that G is residually free implies that there is a free quotient $q : G \twoheadrightarrow F$ such that $q(g) \neq 1$. Since the q_i form a factor set, q must factor via some q_i, so $q_i(g) \neq 1$ and $g \notin \ker q_i \supset \bigcap_{i=1}^n \ker q_i$. □

II.3.1 Limiting actions on $\mathbb{R}$-trees

As we have already seen, one often has to pass to subsequences when studying sequences of homomorphisms. To avoid passing to subsequences multiple times (and using nested indices), we introduce the language of *ultrafilters*. Loosely speaking, ultrafilters simply form "strainers" or "filters", in the sense that they give us a precise manner of saying when a subset of a set X is either "small" or "large".

Definition II.3.5. An *ultrafilter* (on $\mathbb{N}$) is a finitely additive probability measure $\omega : 2^{\mathbb{N}} \to \{0,1\}$. Alternatively, we can think of ω as a collection of subsets of $2^{\mathbb{N}}$ which is closed under finite intersections, is closed under taking supersets and is maximal in the sense that it is not a proper subset of any subset of $2^{\mathbb{N}}$ that satisfies these properties.

We say that an ultrafilter ω is *non-principal* if it satisfies $\omega(F) = 0$ for every finite $F \subset \mathbb{N}$. We also define *limits* with respect to ultrafilters: the ω*-limit* of a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is $x \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\omega(\{n \in \mathbb{N} \mid |x - x_n| < \varepsilon\}) = 1.$$

In this case we denote $\lim_\omega(x_n) = x$. We say that $\lim_\omega(x_n) = \infty$ if $\omega(\{n \in \mathbb{N} \mid x_n > N\}) = 1$ holds for every $N \in \mathbb{N}$.

Remark II.3.6. Every sequence of real numbers has a unique ω-limit in $\mathbb{R}\cup\{\pm\infty\}$ (and has a subsequence that converges to this limit). Note that if a sequence is bounded then its ω-limit is always a real number, but the ω-limit of an unbounded sequence can be in $\mathbb{R}$.

Convention II.3.7. Let P be a statement that applies to the elements $(X_n)_{n\in\mathbb{N}}$ (note that the X_n are not necessarily numbers; they can be, for example, topological spaces). We say that P holds ω-*almost-surely* (for the sequence $(X_n)_{n\in\mathbb{N}}$ if

$$\omega(\{n\in\mathbb{N}\mid P \text{ holds for } n\})=1.$$

To practice the use of this convention, we rephrase the (algberaic) definition of limit groups using ultrafilters. Suppose if so that ω is a non-principal ultrafilter on $\mathbb{N}$, and that $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathrm{Hom}(G,F)$, where G is finitely generated. The *stable kernel* of the sequence (with respect to ω), is

$$\underleftarrow{\ker}_\omega(\varphi_n)=\{g\in G\mid \varphi_n(g)=1\ \omega\text{-almost-surely}\},$$

and a group L is a limit group if and only if it is obtained as $G/\underleftarrow{\ker}_\omega(\varphi_n)$.

Our next goal is to show that every limit group comes equipped with a *limiting action* on a topological space called an $\mathbb{R}$-*tree*; we remind the reader that a *tree* is a graph that contains no cycles. Equivalently, it is a graph in which every *triangle* (namely, a collection of three geodesics between three points x, y and z) is in fact a *tripod*: there exists a point c which lies at the intersection of the three geodesics $[x,y]\cap[x,z]\cap[y,z]$. Since a graph is a simplicial complex, such trees are often referred to as “simplicial trees”. We will focus on a family of spaces that generalizes trees:

Definition II.3.8. An $\mathbb{R}$-*tree* (or a *real tree*) is a geodesic metric space (that is, there is a geodesic, which is a segment of length $d(x,y)$, between every two points x,y in the space) in which every triangle is a tripod.

We next give a metric condition for when a space is an $\mathbb{R}$-tree.

Definition II.3.9. Let T be an $\mathbb{R}$-tree and let $x,y,z\in T$. Let c be the central point of the corresponding tripod. We denote by $(x,y)_z$ the distance $d(z,c)$. Note that

$$(x,y)_z=\frac{1}{2}\cdot(d(x,z)+d(y,z)-d(x,y)).$$

A drawing will be added later

More generally, if X is a metric space and x,y and z are points in X, $(x,y)_z$ is defined as the magnitude appearing above, and is called the *Gromov product* of x and y at z.

In the early 20th century, discrete mathematicians were interested in the following question: let X be a finite collection of points, with assigned distances between each pair of points in X. Can one realize X as the leaf set of a finite tree? It turns out that it is enough to verify that every four points in X satisfy a condition. This result is known as Zaretskii's lemma; it was proven for integral distances by Zaretskii in 1965 [19], and generalized to real distances by Pereira in 1969 [18].

Lemma II.3.10 (Zaretskii's Lemma)**.** *A finite set X accompanied with distances as above can be realized as the leaf set of a tree if and only if for every $x, y, z, t \in X$ the following inequality, known as the* four-point condition*, holds:*

$$(x, y)_t \geq \min((x, z)_t, (y, z)_t).$$

Note that a geodesic metric space is an $\mathbb{R}$-tree if and only if every finite subset of it spans a finite tree. Therefore,

Corollary II.3.11. *A geodesic metric space is an $\mathbb{R}$-tree if and only if every four points in the space satisfy the four-point condition.*

Exercise XVI. Prove Zaretskii's Lemma when the distances between points in X are integral. *Hint:* First, rearrange the four-point condition above and show that it is equivalent to

$$d(x, y) + d(z, t) \leq \max((d(x, z) + d(y, t)), (d(x, t) + d(y, z)))$$

Zaretskii's original paper [19] is available online, and it is less than 3 pages long. It is in russian, but it includes figures and mathematical text.

Remark II.3.12. Some readers may be familiar with Ptholemy's theorem, which characterizes when four points in the plane are cocyclic: four points x, y, z, t lie on a circle if and only if

$$d(x, y) \cdot d(z, t) = d(x, z) \cdot d(y, t) + d(x, t) \cdot d(y, z).$$

A drawing will be added

As a matter of fact, Ptholemy proved a stronger result: every four points x, y, z, t in the plane satisfy the following inequality:

$$d(x, y) \cdot d(z, t) \leq d(x, z) \cdot d(y, t) + d(x, t) \cdot d(y, z).$$

We would next like to highlight an unexpected relation between $\mathbb{R}$-trees and *tropical geometry*: the study of polynomials and their geometric properties when addition is replaced with maximization and multiplication is replaced with ordinary addition. A "tropicalization" of Ptholemy's inequality above yields the four-point condition as in Exercise XVI.

We are finally ready to construct an action of a limit group on an $\mathbb{R}$-tree. In what follows, we assume that all metric spaces are geodesic. A *pointed metric space* is a triplet (X, d, o) where (X, d) is a metric space and $o \in X$.

Definition II.3.13. Let $(X_n, d_n, o_n)_{n\in\mathbb{N}}$ be a sequence of pointed metric spaces. The *ultralimit of* $(X_n, d_n, o_n)_{n\in\mathbb{N}}$ *with respect to an ultrafilter* ω is a triplet $(X_\omega, d_\omega, o_\omega)$ where

$$X_\omega = \Big(\prod_{n\in\mathbb{N}} X_n\Big)/\omega = \frac{\{(x_n)_{n\in\mathbb{N}} \mid \lim_\omega d_n(x_n, o_n) < \infty\}}{(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \iff \lim_\omega d_n(x_n, y_n) = 0},$$

$d_\omega : X_\omega \times X_\omega \to \mathbb{R}_{\geq}0$ is given by

$$d_\omega([(x_n)_{n\in\mathbb{N}}], [(y_n)_{n\in\mathbb{N}}]) = \lim_\omega d_n(x_n, y_n)$$

and o_ω is the equivalence class of the sequence $(o_n)_{n\in\mathbb{N}}$.

Remark II.3.14. We will often abuse notation and refer to the equivalence class of a sequence $(x_n)_{n\in\mathbb{N}}$ in X_ω as $(x_n)_{n\in\mathbb{N}}$ instead of $[(x_n)_{n\in\mathbb{N}}]$. If a sequence $(x_n)_{n\in\mathbb{N}}$ lies in X_ω we call it a *visible sequence*.

It is straightforward (and recommended) to verify that $(X_\omega, d_\omega, o_\omega)$ is a pointed metric space (that is, d_ω is well-defined and satisfies the required conditions). We continue by showcasing a simple instance in which $(X_\omega, d_\omega, o_\omega)$ inherits some properties from the spaces in the sequence $(X_n, d_n, o_n)_{n\in\mathbb{N}}$; this will prove to be of great importance in Chapter III.0.3 Reference will be added later.

Lemma II.3.15. *Let* $(X_n, d_n, o_n)_{n\in\mathbb{N}}$ *be a sequence of pointed metric spaces such that each* (X_n, d_n) *is a tree (or, more generally, an* $\mathbb{R}$*-tree). Then* (X_ω, d_ω) *is an* $\mathbb{R}$*-tree.*

Proof. It suffices to verify that X_ω is a geodesic space that satisfies the four-point condition, that is for every four points $(x_n), (y_n), (z_n), (t_n)$ in X_ω,

$$((x_n), (y_n))_{(z_n)} \geq \min\{((x_n), (z_n))_{(t_n)}, ((y_n), (z_n))_{(t_n)}\}).$$

It is easy to see that (X_ω, d_ω) is a geodesic space: given $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in X_\omega$, let $t_n : [0, d_n(x_n, y_n)] \to X_n$ be a geodesic from x_n to y_n in (X_n, d_n). Define $t : [0, d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})] \to X_\omega$ by

$$s \mapsto t_n\left(\frac{d_n(x_n, y_n)}{d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})} \cdot s\right).$$

It is straightforward to verify that t is the suitable geodesic in (X_ω, d_ω). First, note that for every $s \in [0, d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})]$ with $t(s) = (s_n)_{n\in\mathbb{N}}$ we have that s_n is on the geodesic between x_n and y_n so

$$\lim_\omega d_n(o_n, s_n) \leq \lim_\omega (d_n(o_n, x_n) + d_n(x_n, s_n)) \leq \lim_\omega (d_n(o_n, x_n) + d_n(x_n, y_n)) < \infty$$

and t is well-defined. Furthermore, given $s_1 < s_2 \in [0, d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})]$ we have that

$$\begin{aligned} d_\omega(t(s_1), t(s_2)) &= \lim_\omega d_n\left(\left(\frac{d_n(x_n, y_n)}{d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})} \cdot s_1\right), \left(\frac{d_n(x_n, y_n)}{d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})} \cdot s_2\right)\right) \\ &= \lim_\omega \left(\frac{d_n(x_n, y_n)}{d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})} \cdot (s_2 - s_1)\right) \\ &= \frac{\lim_\omega (d_n(x_n, y_n))}{d_\omega((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}})} \cdot (s_2 - s_1) = s_2 - s_1 \end{aligned}$$

and t minimizes distances.

Finally, given $x_n, y_n, z_n, t_n \in X_n$ for every $n \in \mathbb{N}$ we have that

$$(x_n, y_n)_{z_n} \geq \{(x_n, z_n)_{t_n}, (y_n, z_n)_{t_n}\}$$

and the inequality is preserved in the limit. □

Remark II.3.16. The proof above also shows that if geodesics in (X_ω, d_ω) are unique (e.g. when (X_ω, d_ω) is an $\mathbb{R}$-tree), then every geodesic in (X_ω, d_ω) can be approximated by geodesics in the spaces (X_n, d_n).

Suppose now that G is a finitely generated group, and let $S = (s_1, \dots, s_n)$ be a finite generating tuple of G. Let (X_n, d_n, o_n) be a sequence of pointed metric spaces, and suppose in addition that G acts on each (X_n, d_n, o_n). Under a mild assumption, we get that G acts on X_ω:

Lemma II.3.17. *If* $\lim_\omega d_n(o_n, s.o_n) < \infty$ *for every* $s \in S$*, then* G *acts on* X_ω *by* $g.(x_n)_{n\in\mathbb{N}} = (g.x_n)_{n\in\mathbb{N}}$.

Proof. We just need to verify that for every $(x_n)_{n\in\mathbb{N}}$ and $g \in G$ we have that $(g.x_n)_{n\in\mathbb{N}} \in X_\omega$, that is $\lim_\omega d_n(o_n, g.x_n) < \infty$. The other properties that define a group action easily follow.

Write $g = s_1 \cdots s_k$ where $s_i \in S$ for $1 \leq i \leq k$ and note that

$$\begin{aligned} d_n(o_n, g.o_n) &= d_n(o_n, (s_1 \cdots s_k).o_n) \\ &\leq d_n(o_n, s_1.o_n) + d_n(s_1.o_n, (s_1 \cdot s_2).o_n) + \cdots + d_n((s_1 \cdots s_{k-1}).o_n, (s_1 \cdots s_k).o_n) \\ &= \sum_{i=1}^k d_n(o_n, s_i.o_n). \end{aligned}$$

and therefore

$$\begin{aligned}\lim_\omega d_n(o_n, g.x_n) &\leq \lim_\omega d_n(o_n, g.o_n) + \lim_\omega d_n(g.o_n, g.x_n)\\ &= d_n(o_n, g.o_n) + \lim_\omega d_n(o_n, x_n)\\ &\leq \lim_\omega d_n(o_n, x_n) + \sum_{i=1}^{k} \lim_\omega d_n(o_n, s_i.o_n) < \infty\end{aligned}$$

as desired. □

Corollary II.3.18. *Let H be a group acting on a sequence of pointed metric spaces $(X_n, d_n, o_n)_{n\in\mathbb{N}}$, let G be a group generated by a finite tuple S and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{Hom}(G, H)$. For each n, we get an action $G \curvearrowright X_n$ by setting $g.x = \varphi_n(g).x$ for $g \in G$ and $x \in X_n$. Therefore, by Lemma II.3.17, if $\lim_\omega d_n(o_n, s.o_n) < \infty$ for every $s \in S$ then G acts on X_ω. Furthermore, if H is a free group, then it comes equipped with an action on a (simplicial) tree: its natural action on its Cayley graph. Therefore, under this assumption, we get that G acts on an $\mathbb{R}$-tree.*

The remainder of this subsection will focus on how to extract a (non-trivial) action of a limit group on a limiting $\mathbb{R}$-tree (that is, an action without a global fixed point). We therefore fix a free group F admitting a non-trivial action on its Cayley graph (X, d) which is a simplicial tree, and a limit group $L = G/\underleftarrow{\ker}_\omega((\varphi_n)_{n\in\mathbb{N}})$. We also assume that G is generated by a finite tuple S. The limiting real tree that we obtain will be a limit of a sequence of pointed metric spaces, where each element in the sequence is X equipped with a rescaling of the metric d. We therefore define:

Definition II.3.19. Let o be a choice of a basepoint of (X, d), and let $n \in \mathbb{N}$. The *scaling factor of φ_n at o* is given by

$$\|\varphi_n\|_o = \max_{s\in S} d(o, \varphi_n(s).o).$$

Combining Lemmas II.3.15 and II.3.17 we obtain:

Corollary II.3.20. *Let $(o_n)_{n\in\mathbb{N}}$ be a sequence of points in X. Then the sequence $(X, d/\|\varphi_n\|_{o_n}, o_n)$ converges to a real tree $(X_\omega, d_\omega, o_\omega)$ on which G acts by $g.(x_n)_{n\in\mathbb{N}} = (\varphi_n(h).o_n)_{n\in\mathbb{N}}$. Furthermore, $H \curvearrowright X_\omega$ gives rise to an action $L \curvearrowright X_\omega$.*

Proof. It is enough to show that the boundedness condition of Lemma II.3.17 holds. Indeed, given $s \in S$ we have that

$$\|\varphi_n\|_{o_n}(o_n, \varphi_n(s).o_n) \leq \frac{\max_{s\in S} d(o, \varphi_n(s).o)}{\|\varphi_n\|_o} = 1$$

as desired. Lastly, since $\underleftarrow{\ker}_\omega((\varphi_n)_{n\in\mathbb{N}}$ acts trivially on X_ω, the action $G \curvearrowright X_\omega$ induces an action $L \curvearrowright X_\omega$. □

Corollary II.3.20 is still not enough: there is nothing preventing the action $G \curvearrowright X_\omega$ from having a global fixed point. Bestvina introduced a method for overcoming this problem in [3] (where the spaces considered are hyperbolic n-spaces), which was later generalised by Paulin in [12] (to accommodate any hyperbolic space). This method is often referred to as the "Bestvina-Paulin trick", and it revolves around carefully choosing the sequence basepoints $(o_n)_{n\in\mathbb{N}}$. We amass the relevant definitions for explaining this method, and begin with an absolute version of the scaling factor defined above:

Definition II.3.21. Keeping the notation above, the *scaling factor* of a homomorphism $\varphi_n : G \to H$ is

$$\|\varphi_n\| = \inf_{x\in X} \max_{s\in S} d(x, \varphi_n(s).x) = \inf_{x\in X} \|\varphi_n\|_x.$$

Remark II.3.22. Note that $\|\varphi_n\|$ does not depend on a choice of a basepoint o of X.

The scaling factor of a homomorphism will play a crucial role in the study of *shortening quotients* (see Theorem II.3.25). We conclude with the following Theorem:

Theorem II.3.23 (Bestvina-Paulin trick)**.** *If L is a limit group then there exists a choice of basepoints $(o_n)_{n\in\mathbb{N}}$ in X such that the sequence $(X, d/\|\varphi_n\|, o_n)$ converges to a real tree $(X_\omega, d_\omega, o_\omega)$ on which L acts non-trivially.*

Proof. For every $n \in \mathbb{N}$, choose $o_n \in X$ that satisfies $d(o_n, \varphi_n(s).o_n) = \|\varphi_n\|$ for some $s \in S$ (this is possible since X is a simplicial tree, and so the metric d is discrete). As in Corollary II.3.20, we obtain an action $G \curvearrowright X_\omega$ which gives rise to an action $L \curvearrowright X_\omega$; it is enough to show that the action of G on X_ω is non-trivial, that is, no $(x_n)_{n\in\mathbb{N}} \in X_\omega$ is fixed by all of G.

Since S is finite, there exists $s \in S$ such that $d(o_n, \varphi_n(s).o_n) = \|\varphi_n\|$ holds ω-almost surely. Therefore, for every $(x_n)_{n\in\mathbb{N}} \in X_\omega$ we have that FIX HERE based on handwritten

$$\frac{d}{\|\varphi_n\|}(x, \varphi_n(s).x) \geq \frac{d(o_n, \varphi_n(s).o_n)}{\|\varphi_n\|} = 1$$

ω-almost surely, so $d_\omega((x_n)_{n\in\mathbb{K}}, s.(x_n)_{n\in\mathbb{K}}) \geq 1$ and G does not fix $(x_n)_{n\in\mathbb{N}}$. □

Remark II.3.24. Note that G and L are both finitely generated, and therefore countable. If X_ω is not a line, the valence of every vertex of X_ω is uncountable and therefore the action $G \curvearrowright X_\omega$ is not minimal. In other words, G has a proper invariant subtree of X_ω. If needed, one can always reduce to a subtree on which the action is minimal.

II.3.2 The Shortening Argument and generalized factor sets

We are finally ready to construct Makanin-Razborov diagrams; we begin with a simpler construction that yields a diagram with the required properties, but is not *canonical*. In simpler words, the construction of the diagram requires choice, and different choices will yield different diagrams (each of which gives a full description of $\mathrm{Hom}(G, F)$). Refining our arguments, we will then construct a canonical Makanin-Razborov diagram for a finitely generated group G, There is no ambiguity surrounding the diagram obtained with this construction: it is determined (up to equivalence) by a choice of a finite generating set of G.

At the heart of the construction, lies the following important theorem which will also play a key part in the next chapter: the *Shortening Argument*. This theorem admits many versions, the first one of them is due to Rips and Sela [15].

Theorem II.3.25. *Suppose that G does not split as a free product and let (φ_n) be a sequence in $\mathrm{Hom}(G, F)$. If $\underleftarrow{\ker}_\omega(\varphi_n) = 1$ then φ_n is not* short *ω-almost-surely, meaning that there are homomorphisms $\phi_n : G \to F$ such that $\|\phi_n\| < \|\varphi_n\|$ ω-almost-surely.*

Remark II.3.26. The version of the shortening argument stated above does not reveal its full power: the proof of the shortening argument implies that ϕ_n above can always be chosen to be of a specific form:

$$\phi_n = \mathrm{ad}(g) \circ \varphi_n \circ \alpha,$$

where $\alpha \in \mathrm{Aut}(G)$ and $\mathrm{ad}(g)$ is conjugation by $g \in F$. Furthermore, α can be chosen to be an automorphism that lies in the *modular group* $\mathrm{Mod}(G)$ of G. We will discuss, and make use, of $\mathrm{Mod}(G)$ in the next chapter, a reference will be added later.

Recall that in Lemma II.3.2 and Theorem II.3.3 we constructed a factor set for a group G given that it is *not* a limit group. We couldn't adapt our construction to limit groups because we didn't have the means to construct *maximal* quotients of a limit group L. The shortening argument allows us to deal with that problem.

Definition II.3.27. Let G be a group. A *generalized factor set* of G is a collection of quotients of G $\{q_i : G \twoheadrightarrow Q_i \mid i \in I\}$ such that, up to precomposition by some $\alpha \in \mathrm{Aut}(G)$, every homomorphism $f : G \to F$ must factor via some q_i (that is $f \circ \alpha$ factors via q_i).

Corollary II.3.28. *Let L be a limit group which does not admit a free splitting. Then L admits a finite generalized factor set.*

Proof. This proof can be seen as a "limit version" of the proof of Lemma II.3.2. Enumerate the elements of L, that is write $L = \{g_1, g_2, \dots\}$, and let

$$F_n = \{q_i : L \twoheadrightarrow L/\langle\langle g_i \rangle\rangle \mid i \le n\}.$$

We will show that some F_n must be a factor set of L (in fact, F_n is a factor set of L ω-almost-surely).

Suppose not, then for every n there exists $\varphi_n : G \to F$ which is injective on $\{g_1, \dots, g_n\}$. Let ϕ_n be the shortest homomorphism of the form $\mathrm{ad}(g) \circ \varphi_n \circ \alpha$ as in Remark II.3.26. Note that ϕ_n is also injective on $\{g_1, \dots, g_n\}$. Hence the sequence ϕ_n has a trivial stable kernel, contradicting Theorem II.3.25. Therefore, we could not choose such φ_n ω-almost-surely and for some n (in fact, ω-almost-surely), every $f : G \to F$ must factor via some $q_i : G \to G/\langle\langle g_i \rangle\rangle$ where $i \le n$. □

Armed with Corollary II.3.28, we can start constructing a Makanin-Razborov diagram for G by iterating the construction of a factor set. All that's left for us to prove is that the resulting diagram is finite, that is, that the diagram does not admit infinitely-deep branches. Equivalently:

Claim II.3.29. *Every sequence of quotients $L_1 \twoheadrightarrow L_2 \twoheadrightarrow L_3 \twoheadrightarrow \cdots$ where each L_i is a limit group eventually stabilizes.*

Proof. The proof is a simple application of equational Noetherianity. We first note that an epimorphism $q_i : L_i \twoheadrightarrow L_{i+1}$ induces an embedding

$$p_i : \mathrm{Hom}(L_{i+1}, F) \to \mathrm{Hom}(L_i, F)$$

by sending $f : L_{i+1} \to F$ to $p_i(f) = f \circ q_i : L_i \to F$. Furthermore, if q_i is a proper quotient map (that is, q_i is not injective), then p_i is not surjective: indeed, if $g \in \ker(q_i)$ then $f \circ q_i(g) = 1$, but since L_i is a limit group there is a homomorphism $f_g : L_i \to F$ which does not kill g so $f_g \notin \mathrm{Im}(p_i)$. It is therefore enough to prove that for large enough i, p_i is an isomorphism.

For each i, fix a presentation $\langle S \mid R_i \rangle$ where $R_i \subset R_{i+1}$. Let $R = \bigcup_i R_i$ and $L = \langle S \mid R \rangle$. Note that R is a system of equations over a free group, and free groups are equationally Noetherian. Therefore R is equivalent to a finite subsystem R', and $R' \subset R_i$ for some i. Recall that the sets of solutions to R_i in F correspond to homomorphisms in $\mathrm{Hom}(L_i, F)$. It follows that for every $j \ge i$, p_j is a bijection as required. □

II.3.3 Canonization for Makanin-Razborov diagrams

This chapter comes to an end with a detailed description of the *canonical* Makanin-Razborov diagram attached to a finitely generated group G (accompanied by a finite generating set S).

Recall that the construction in Theorem II.3.3 yielded a specific factor set (up to equivalence); this factor set did not depend on anything except for the group G itself. In the following guided exercise, you will prove an analogous version for limit groups, in which the quotients in the (generalized) factor set are *shortening quotients*. This version however *does* depend on the choice of a generating set.

Definition II.3.30. Let L be a limit group which does not admit a free splitting. A quotient $q : L \twoheadrightarrow Q$ is a *shortening quotient* if it is obtained from L by quotienting out the stable kernel of a sequence of homomorphisms $\varphi_n : L \to F$, such that for every $g \in F$ and every $\alpha \in \mathrm{Mod}(G)$, $\|\varphi_n\| \leq \|\mathrm{ad}(g) \circ \varphi_n \circ \alpha\|$.

Remark II.3.31. Note that the shortening argument Theorem II.3.25 implies that every shortening quotient of L is a proper quotient. In addition, the way in which the scaling factor $\|\varphi_n\|$ was defined, implies that the collection of shortening quotients of L depends on a choice of a generating set of L.

Theorem II.3.32. *Let L be a freely indecomposable limit group. Then L admits a generalized factor set consisting of shortening quotients of L, and which depends only on the generating set of L.*

Exercise XVII. Prove Theorem II.3.32. Let L be a freely indecomposable limit group and let S be a generating set of L.

1. Define an equivalence relation on the set $\mathcal{S}(L, S)$ of shortening quotients of L (with respect to S) as in Theorem II.3.3 (note that here one has to allow twisting by elements of $\mathrm{Mod}(L)$). Define a partial order on the set $\mathcal{S}(L, S)$ as in Theorem II.3.3.

2. Prove that every increasing chain in $\mathcal{S}(L, S)$ admits an upper bound in $\mathcal{S}(L, S)$ (that is, the upper bound is itself a shortening quotient).

3. Prove that there are only finitely many (equivalence classes) of maximal shortening quotients in $\mathcal{S}(L, S)$.

After giving informal descriptions of Makanin-Razborov diagrams, we are finally in the position to give a precise description (of the canonical Makanin-Razborov diagram of G with respect to S). We will also make use of the following theorem We will prove this theorem in an appendix that about group actions on trees:

Theorem II.3.33 (Grushko's theorem)**.** *Let G be a finitely generated group. Then G can be decomposed as a (possibly trivial) free product*

$$G = G_1 * \cdots * G_k * F_n$$

where each G_i is freely indecomposable and F_n is free. Furthermore, this decomposition is canonical in the following sense: if $G = G'_1 * \cdots * G'_m * F_\ell$ then $m = k$, $n = \ell$, and up to permuting the factors, each G_i is conjugate to G'_i in G.

As we have already mentioned, the Makanin-Razborov diagram $\mathrm{MR}(G, S)$ is a *rooted* tree. Starting with the root vertex, which is labelled by G, $\mathrm{MR}(G, S)$ can be constructed inductively as follows:

1. The edges coming out of G connect it to vertices labeled by the different factors in its canonical factor set from Theorem II.3.3; we label these edges by the corresponding quotient maps. If G is a limit group, we connect it to vertices labelled by the different factors in its Grushko decomposition (so if G is freely indecomposable, there is a single edge coming out of G and ending at another vertex labeled by G). These edges are all unlabeled.

2. Let H be a vertex connected to G. Since H is a quotient of G, it comes equipped with a generating set which is the image of S. Then by Theorem II.3.3 and Theorem II.3.32, H admits a finite (generalized) factor set. Connect H by edges to vertices labeled by the quotients appearing in this factor set, and label each edge by the corresponding quotient map.

3. Let K be a vertex connected to H. If K is free, there are no more edges attached to K. Otherwise, repeat (1) for K (and the generating set that it inherits from G).

4. By Claim II.3.29, our construction must terminate after finitely many steps.

A drawing will be added later

To finish, we describe how the factorization of a homomorphism $f : G \to F$ can be read from the diagram. To each such f, we associate a *subdiagram* T_f of the Makanin-Razborov diagram, constructed as follows:

1. The root vertex of T_f is labeled by (G, f).

2. Since the vertices adjacent to G in the diagram are a factor set, f must factor via an edge corresponding to a quotient $q : G \twoheadrightarrow Q$; write $f = f' \circ q$. Add an edge labeled by q connecting (G, f) to a vertex labeled by (Q, f').

3. The vertex (Q, f') is now a limit group. Let $Q = Q_1 * \cdots * Q_n * F_m$ be its Grushko decomposition. Then

$$\mathrm{Hom}(Q, F) = \prod_{i=1}^{n} \mathrm{Hom}(Q_i, F) \times F^m,$$

   and f' decomposes as $(f_1, \ldots, f_n, f^m)$ in this product. Attach vertices labelled by (Q_i, f_i) (and (F_m, f^m)) to (Q, f') by unlabeled edges. The vertex (F_m, f^m) is a leaf of T_f.

4. Now, continue with each (Q_i, f_i) as follows: the vertices adjacent to Q_i in the Makanin-Razborov diagram form a generalized factor set of Q_i. Therefore f_i is equivalent to some $\widehat{f_i}$, and $\widehat{f_i}$ factors via a shortening quotient $q : Q_i \twoheadrightarrow Q'$ of Q_i. Write $\widehat{f_i} = \widehat{f'} \circ q$ and add an edge labeled by q connecting (Q_i, f_i) to a vertex labeled by $(Q', \widehat{f'})$.

5. Go back to step 3. and repeat the construction for $(Q', \widehat{f'})$.

III

Formal solutions and $\mathrm{Th}^+_{\forall\exists}(F)$

MERZLYAKOV studied *positive* formulas in free groups and came up with an algorithm that, given an $\forall\exists$-sentence φ which holds in F, yields a proof that $F \models \varphi$. Merzlyakov's proof implies that all non-abelian free groups share the same positive $\forall\exists$ theory. In this chapter we will prove Merzlyakov's theorem using modern techniques (which are quite different to Merzlyakov's original combinatorial argument).

We begin by laying the groundwork which will allow us to state (and outline a rough strategy for the proof of) Merzlyakov's theorem. Let φ be a sentence in the language of groups, and assume that φ is in *disjunctive normal form* (definition will be added to prelims later), that is

$$\varphi = \forall \overline{x}_1 \exists \overline{x}_2 \cdots \bigvee_{i=1}^{k} \bigwedge_{j=1}^{m_i} w_j(\overline{x}_1, \dots, \overline{x}_n) \underset{\neq}{\overset{=}{}} 1.$$

The sentence φ is called *positive* if $\neq$ does not appear in φ. We denote the *positive theory* of a group G, which consists of all the positive sentences which are true in G, by $\mathrm{Th}^+(G)$.

Remark III.0.1. Note that some non-positive sentences may look positive upon first inspection. For example, the sentence $\forall x \forall y \forall z \;\; xy = xz \to y = z$ does not contain the symbol $\neq$, but it is logically equivalent to the following sentence in DNF form

$$\forall x \forall y \forall z \;\; xy \neq xz \vee y = z,$$

which is not a positive sentence.

The following simple observation, which is left as an exercise, is crucial for our strategy:

Exercise XVIII. Let φ be a positive sentence and suppose that $G \models \varphi$. Let $f : G \to H$ be a surjective homomorphism. Prove that $H \models \varphi$.

We remark (although we will not use this result) that in 1959, Lyndon proved the converse:

Theorem III.0.2 ([9, Corollary 5.3])**.** *Let φ be a sentence such that whenever $G \models \varphi$, then φ holds in every homomorphic image of G. Then φ is logically equivalent to a positive sentence.*

The observation from Exercise XVIII turns the problem of showing that all free groups share the same positive $\forall\exists$-theory into a *lifting probelm*: let φ be a positive $\forall\exists$-sentence. For expositional purposes, we assume for now that φ takes the following form,

$$\varphi = \forall \overline{x} \exists \overline{y} \;\; \Sigma(\overline{x}, \overline{y}) = 1,$$

where $\Sigma(\overline{x}, \overline{y})$ is a system of equations in the variables $\overline{x}, \overline{y}$ (so that φ is a disjunction of a single statement). Fix a non-abelian free group F_n, and suppose that $F_n \models \varphi$. Consider the finitely generated (in fact, finitely presented) group

$$G_\Sigma = \langle \overline{x}, \overline{y} \mid \Sigma(\overline{x}, \overline{y}) \rangle.$$

Note that the subgroup $\langle \overline{x} \rangle$ of G_Σ must be isomorphic to the free group on $\overline{x}$: otherwise, there would be some $\sigma \in \Sigma(\overline{x}, \overline{y})$ which involves only the variables $\overline{x}$, which contradicts the fact that $F_n \models \varphi$. In order to show that $F_m \models \varphi$, we need to solve the following lifting problem:

$$\begin{tikzcd}
G_\Sigma \arrow[dr, dashed, "\exists \widehat{f}"] & \\
\langle \overline{x} \rangle \cong F(\overline{x}) \arrow[u, "i"] \arrow[r, "f"] & F_m
\end{tikzcd}$$

where i is the inclusion map $\langle \overline{x} \rangle \hookrightarrow G_\Sigma$ and $f : \langle \overline{x} \rangle \to F_m$ is an arbitrary homomorphism. Indeed, given any tuple $\overline{g}$ in F_m of the same arity as $\overline{x}$, we can define f by mapping $\overline{x}$ to $\overline{g}$ and $F_m \models \Sigma(\overline{g}, \widehat{f}(\overline{y}))$.

Merzlyakov solved this lifting problem in the strongest possible way, namely by constructing the following diagram (note that in such a diagram we can replace F_m with any group G)

$$\begin{tikzcd} & G_\Sigma \arrow[d, dashed, bend right, "\exists r"'] \\ \langle \overline{x} \rangle \cong & F(\overline{x}) \arrow[u, "i"'] \arrow[r, "f"] & G \end{tikzcd}$$

where $r : G_\Sigma \to \langle \overline{x} \rangle$ is a *retraction* (the restriction $r|_{\langle \overline{x} \rangle}$ is the identity map). Such a retraction is called a *formal solution* for the system of equations $\Sigma(\overline{x}, \overline{y}) = 1$. Simply put, a formal solution converts the "relations" between $\overline{x}$ and $\overline{y}$ given by $\Sigma(\overline{x}, \overline{y})$ into a mechanism, that given a choice of values $f(\overline{x})$ for $\overline{x}$ in G, outputs suitable values for $\overline{y}$ ($f \circ r(\overline{y})$, which can be written as words in $\overline{x}$). In other words,

Theorem III.0.3 ([10]). *Let $\varphi = \forall \overline{x} \exists \overline{y} \ \ \Sigma(\overline{x}, \overline{y}) = 1$ and let F be a non-abelian free group. If $F \models \varphi$ then there is a formal solution for $\Sigma(\overline{x}, \overline{y}) = 1$.*

We immediately deduce:

Corollary III.0.4. *All non-abelian free groups share the same positive $\forall\exists$-theory.*

Remark III.0.5. Note that the existence of a formal solution implies that if φ is a positive $\forall\exists$-sentence which holds in a (non-abelian) free group, then φ is satisfied by every group. Exercise XVIII also implies this fact, under the assumption that the positive $\forall\exists$-theories of all non-abelian free groups coincide. In other words, $\mathrm{Th}^+_{\forall\exists}(F) \subset \mathrm{Th}^+_{\forall\exists}(G)$ for every G.

This is the origin of the following terminology: if $\mathrm{Th}^+_{\forall\exists}(G)$ coincides with that of F, we say that G has *trivial positive $\forall\exists$-theory*.

Remark III.0.6. In a recent paper, Casals-Ruiz, Garreta, and de la Nuez González proved that if a group G satisfies a positive sentence which is not satisfied by F, then G satisfies a positive $\forall\exists$-sentence which is not satisfied by F. Therefore, if G has trivial positive $\forall\exists$-theory then it has *trivial positive theory*: every positive sentence satisfied by G is satisfied by all groups.

Lastly, we would like to bring the reader's attention to the following connection:

Remark III.0.7. Recall the *implicit function theorem*: Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be a continuously differentiable function, and suppose that $F(x_0, y_0) = 0$ for some point $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$. If the Jacobian matrix $\frac{\partial F}{\partial x}(x_0, y_0)$ is invertible (i.e., it has full rank), then we can solve the equation $F(x, y) = 0$ locally for y as a function of x. Specifically, there exists a neighborhood U of x_0 in $\mathbb{R}^n$ and a continuously differentiable function $g : U \to \mathbb{R}^m$ such that:

$$F(x, g(x)) = 0 \quad \text{for all} \quad x \in U,$$

with $g(x_0) = y_0$. In other words, the implicit relation between x and y given by $F(x, y) = 0$ can be converted into an explicit function $y = g(x)$ in a neighborhood of x_0.

This statement bears a great resemblence to Merzlyakov's Theorem III.0.3; for this reason, some authors refer to Merzlyakov's theorem as an implicit function theorem for free groups..

Lastly, we would like to mention that Merzlyakov's theorem admits many generalizations. These include the introduction of inequalities, a restriction of $\overline{x}$ to a variety, and versions for other classes of groups (hyperolic groups, acylindrically hyperbolic groups, π-groups).

A

Group actions on simplicial trees

JEAN-PIERRE Serre developed in the 1970s a theory which allows one to describe the structure of a group based on its action on a simplicial tree. A detailed account of this theory appears in [17], and is commonly known as *Bass-Serre theory* (Hyman Bass was the editor for an earlier, French, monograph of Serre's). In this appendix we give a brief overview of this theory.

We begin with a motivating example:

Example A.0.1. Let T be a bi-infinite line whose vertices are indexed by the integers, and let G_1 and G_2 be two groups that act on it as follows:

1. $G_1 = \mathbb{Z} = \langle g \rangle$ where $g.x = x + 1$.

2. $G_2 = D_\infty = \langle g, h \mid g^2 = h^2 = 1 \rangle$, where g acts by reflection around 0 (that is, $g.x = -x$) and h acts by reflection around 1 (that is, $h.x = 1 - x$).

Consider the two quotient spaces $X_1 = T/G_1$ and $X_2 = T/G_2$. We have that $X_1 \cong S^1$, and the fundamental group of the circle is $\mathbb{Z}$. In this case, one could recover the group G_1 just by looking at the action on T (or rather, its quotient). The tree T can also be recovered as the universal cover of X_1. On the other hand, $X_2 \cong [0, 1]$ and the fundamental group of an interval is trivial.

The key idea behind Bass-Serre theory is the following: if we keep track of the stabilizers of the vertices and edges of T (as well as how the edge stabilizers embed into the corresponding vertex stabilizers) in the quotient T/G_2, we can reconstruct

G_2 (as well as T) from the quotient. In this case, our quotient graph T/G_2 has two vertices connected by a single edge. One of these vertices is the image of 0, which is stabilized by $\langle g\rangle \cong \mathbb{Z}/2$, and the other is the image of the vertex 1 (stabilized by $\langle h\rangle$). The edge between 0 and 1 is not stabilized by any element of G_2 (except for, of course, the trivial element). Bass-Serre theory tells us that looking at the "marked quotient" (or "graph of groups")

$$\langle g\rangle \cong \mathbb{Z}/2 \bullet \xrightarrow{\quad\{1\}\quad} \bullet \langle h\rangle \cong \mathbb{Z}/2$$

we can reconstruct G_2 and T.

Remark A.0.2. If two vertices (or edges) in T are in the same orbit, then their vertices are conjugate and therefore isomorphic. Therefore, the choice of a preimage of the vertices (or edges) does not affect the "marked quotient".

Definition A.0.3. A *(simplicial) tree* is a 1-dimensional simplicial complex (or, in other words, a discrete graph) that contains no cycles. Let G be a group acting on a tree T and assume that the action does not invert any edge e of T. A *graph of groups* is a quotient of the form $X = T/G$, along with labels on the vertices and edges of the quotient graph X such that:

1. for every $v \in \mathrm{V}(X)$, choose $\tilde{v} \in \mathrm{V}(T)$ that's mapped to v by the quotient map and label v by $\mathrm{Stab}(\tilde{v})$,

2. for every $e \in \mathrm{E}(X)$ choose $\tilde{e} \in \mathrm{E}(T)$. Write $e = \{v, u\}$ and $\tilde{e} = \{\tilde{v}, \tilde{u}\}$ where $\tilde{v}, \tilde{u}$ are mapped to v, u respectively in the quotient. Note that $\mathrm{Stab}(\tilde{e})$ is a subgroup of $\mathrm{Stab}(\tilde{v})$ and of $\mathrm{Stab}(\tilde{u})$ and denote the inclusion homomorphisms by i_v and i_u respectively. We label the edge e by $\mathrm{Stab}(\tilde{e})$, and record the *edge maps* i_v and i_u.

We can also define graphs of groups without a group action on a tree:

Definition A.0.4. A *(Serre) graph* X consists of a vertex set V and an edge set E; X also comes equipped with

1. an *edge reversal* map $\bar{\cdot} : E \to E$ satisfying $\overline{e} \neq e$ and $\overline{\overline{e}} = e$, and

2. an *initial vertex* map $\iota : E \to V$ which maps e to its initial vertex (so that the edge e connects the vertices $\iota(e)$ and $\iota(\overline{e})$). We will sometimes use $\tau : E \to V$ to denote the *terminal vertex* map which sends e to its terminal vertex (that is, $\tau(e) = \iota(\overline{e})$.

A *graph of groups* $\mathcal{X}$ is comprised of the following data:

1. a connected graph X,
2. a *vertex group* X_v for every $v \in V$,
3. an *edge group* X_e for every $e \in E$, and
4. an injective *edge map* $i_e : X_e \to X_{\iota(e)}$ for every $e \in E$.

We can attach a *fundamental group* to a graph of groups. Just like in topological spaces, we can look at *paths* and *loops* in a graph of groups $\mathcal{X}$. In this context, a path from $v \in V$ to $u \in V$ is a finite sequence of the form

$$(a_0, e_1, a_1, e_2, \dots, e_n, a_n),$$

where $(e_1, \dots, e_n)$ is a path from v to u, $a_i \in X_{\iota(e_{i+1})}$ for $i < n$ and $a_n \in X_{\tau(e_n)}$; a loop is a path from a vertex $v \in V$ to itself. Note that each path is an element of $(*_{v \in V} X_v) * F(E)$. Much like in the case of the standard fundamental group of a topological space, we want to consider loops up to "homotopy". In the case of a graph of groups, instead of homotopy we define an equivalence relation on paths. We say that two paths p_1 and p_2 are *$\mathcal{X}$-equivalent* if p_2 can be obtained from p_1 by a finite sequence of replacements according to the following rule:

$$(e, i_{\overline{e}}(g), \overline{e}) \longleftrightarrow (i_e(g)).$$

We denote the equivalence class of a path p by $[p]$. Finally, we define the *fundamental group of $\mathcal{X}$ at $v \in V$* by

$$\pi_1(\mathcal{X}, v) = \{[p] \mid p \text{ is a loop based at } v\}.$$

One easily sees that $\pi_1(\mathcal{X}, v)$ has a group structure (where multiplication is given by path concatenation), and that its definition does not depend on v. We will therefore write $\pi_1(\mathcal{X})$ without specifying a base point.

The fundamental group of $\mathcal{X}$ can also be described by the following presentation: fix a *spanning tree* T of X (that is, a subtree T of X which contains every $v \in V$), and fix a presentation $\langle S_v \mid R_v \rangle$ for every vertex group X_v. Then the fundamental group of $\mathcal{X}$ (with respect to T) has the presentation:

$$\pi_1(\mathcal{X}, T) = \left\langle \bigcup_{v \in V} S_v \cup \{t_e \mid e \in E\} \;\middle|\; \bigcup_{v \in V} R_v \cup R \right\rangle$$

where R contains the following relations:

- t_e for every $e \in T$,
- $t_e t_{\overline{e}}$ for every $e \in E$, and
- $i_e(g) = t_e \cdot i_{\overline{e}}(g) \cdot {t_e}^{-1}$ for every $e \in E$ and $g \in X_e$.

Again, it is not hard to see that the resulting group does not depend on the spanning tree T (and we will therefore refer to it as $\pi_1(\mathcal{X})$).

Exercise XIX. Show that the two definitions of $\pi_1(\mathcal{X})$ coincide.

To appear: fundamental group of a graph of groups, fundamental theorem of Bass-Serre theory, classification of elements acting on T, "splittings", free products+amalgamated products+HNN extensions (+ topological description examples), normal forms, Grushko theorem

Bibliography

[1] G. Baumslag. On generalised free products. *Math. Z.*, 78:423–438, 1962.

[2] G. Baumslag, A. Myasnikov, and V. Remeslennikov. Algebraic geometry over groups. I. Algebraic sets and ideal theory. *J. Algebra*, 219(1):16–79, 1999.

[3] Mladen Bestvina. Degenerations of the hyperbolic space. *Duke Mathematical Journal*, 56(1):143–161, February 1988.

[4] C. Champetier and V. Guirardel. Limit groups as limits of free groups. *Israel J. Math.*, 146:1–75, 2005.

[5] R. I. Grigorchuk. Degrees of growth of finitely generated groups, and the theory of invariant means. *Mathematics of the USSR-Izvestiya*, 25(2):259–300, April 1985.

[6] M. Gromov. Groups of polynomial growth and expanding maps. *Publications mathématiques de l'IHÉS*, 53(1):53–78, December 1981.

[7] O. Kharlampovich and A. Myasnikov. Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups. *J. Algebra*, 200(2):517–570, 1998.

[8] O. Kharlampovich and A. Myasnikov. Tarski's problem about the elementary theory of free groups has a positive solution. *Electronic Research Announcements of the American Mathematical Society*, 4(14):101–108, December 1998.

[9] Roger Lyndon. Properties preserved under homomorphism. *Pacific Journal of Mathematics*, 9(1):143–154, March 1959.

[10] J. Merzljakov. Positive formulae on free groups. *Algebra i Logika Sem.*, 5(4):25–42, 1966.

[11] A. Nies. Aspects of free groups. *J. Algebra*, 263(1):119–125, 2003.

[12] F. Paulin. Outer automorphisms of hyperbolic groups and small actions on $\mathbf{R}$-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.

[13] C. Perin. *Elementary embeddings in torsion-free hyperbolic groups*. Theses, Université de Caen, October 2008. Thèse rédigée en anglais, avec une introduction détaillée en français.

[14] Chloé Perin and Rizos Sklinos. Homogeneity in the free group. *Duke Mathematical Journal*, 161(13), October 2012.

[15] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geom. Funct. Anal.*, 4(3):337–371, 1994.

[16] Z. Sela. Diophantine geometry over groups I: Makanin-razborov diagrams. *Publications Mathématiques de l'IHÉS*, 93:31–105, 2001.

[17] Jean-Pierre Serre. *Trees*. Springer Berlin Heidelberg, 1980.

[18] J.M.S. Simões Pereira. A note on the tree realizability of a distance matrix. *Journal of Combinatorial Theory*, 6(3):303–310, April 1969.

[19] K. A. Zaretskii. Constructing a tree on the basis of a set of distances between the hanging vertices. *Uspekhi Mat. Nauk*, 20(6(126)):90–92, 1965.