

Virtual homological torsion in low dimensions

BGU Math Colloquium, Dec. 9 2025

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Overview

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- Virtual homological torsion: an introduction

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- A 2-dimensional setting

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-  Detour: an application

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- What's next?

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topological complexity of finite covers grows at a rate reflecting geometry



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Conjecture (Bergeron–Venkatesh)

Let M be a closed, hyperbolic 3-manifold and let

$$M = M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

be a cofinal tower of finite-sheeted normal covers of M , that is,

$$\bigcap_n \pi_1(M_n) = 1.$$

Then

$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}(H_1(M_n; \mathbb{Z}))|)}{[\pi_1(M) : \pi_1(M_n)]} = \frac{\text{vol}(M)}{6\pi}.$$



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BV Conjecture: $\forall p,$
$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}_p(H_1(M_n; \mathbb{Z}))|)}{[\pi_1(M): \pi_1(M_n)]} = 0,$$

so torsion must involve larger and larger sporadic primes.

Beyond 3-manifolds

Open problem

Is there a finitely presented residually finite group G , and a residual normal chain

$$G \triangleright G_1 \triangleright G_2 \triangleright \cdots$$

(i.e. $\bigcap_n G_n = \{1\}$), such that

$$\lim_{n \rightarrow \infty} \frac{\log(|\text{Tor}(G_n^{\text{ab}})|)}{[G : G_n]} > 0 ?$$

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- **Finite presentation:** (Kar–Kropholler–Nikolov '17) For every $f : \mathbb{N} \longrightarrow \mathbb{N}$ there is a fg G and an exhausting normal chain $G \triangleright G_1 \triangleright G_2 \triangleright \dots$ such that $|\text{Tor}(G^{\text{ab}})| > f([G : G_n])$.

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- **Dimension:** (Avramidi–Okun–Schreve '21) Right-angled Artin groups where $\log |\text{Tor}(H_2(–; \mathbb{Z}))|$ grows exponentially in the index.
- **Normality:** (Liu '19) Cofinal towers of finite-sheeted *non-normal* covers of closed, hyperbolic M^3 with exponential growth.

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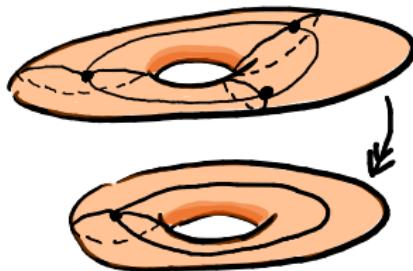
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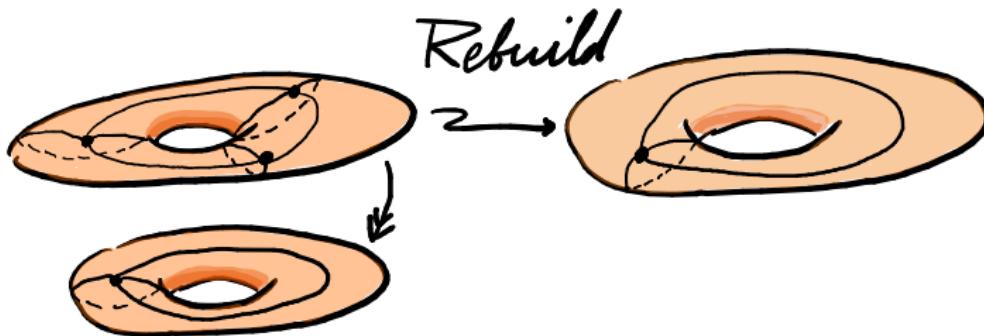
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Theorem (Sun '15)

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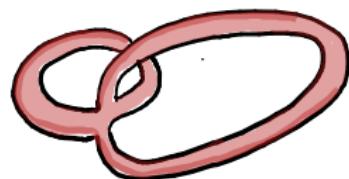
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In graph manifolds:

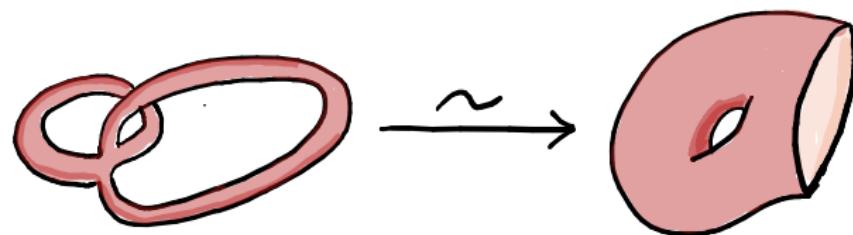
- Solv or Seifert-fibred – restrictions on torsion.
- Non-trivial JSJ – any number can divide $|\text{Tor}(H_1(\widehat{M}; \mathbb{Z}))|$ (F–Hughes–Valiuunas).

Virtual homological torsion in 3-manifolds: a recipe

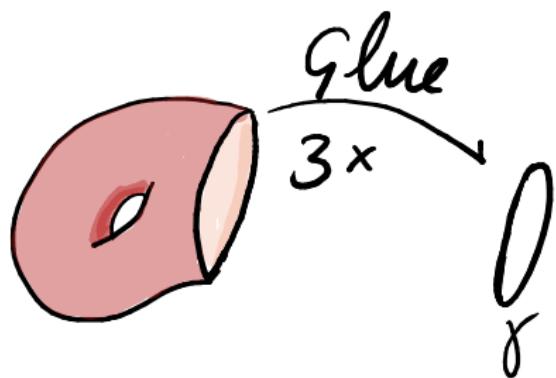
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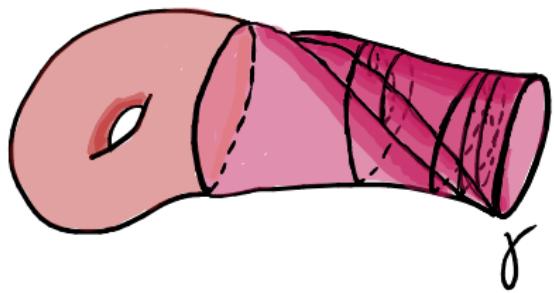
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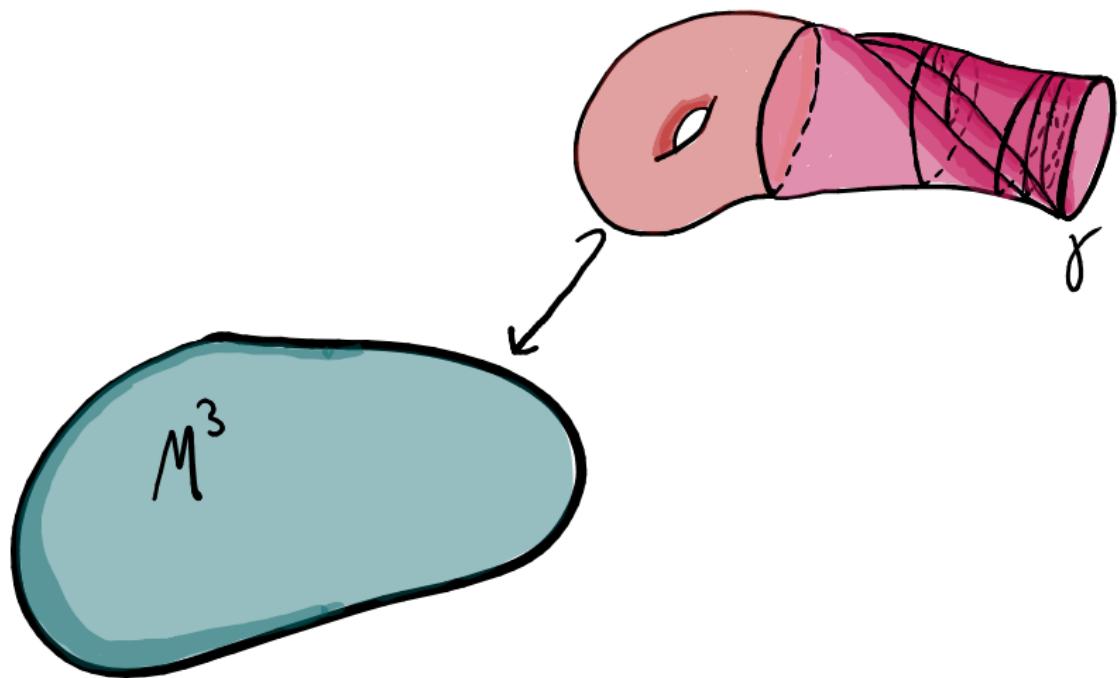
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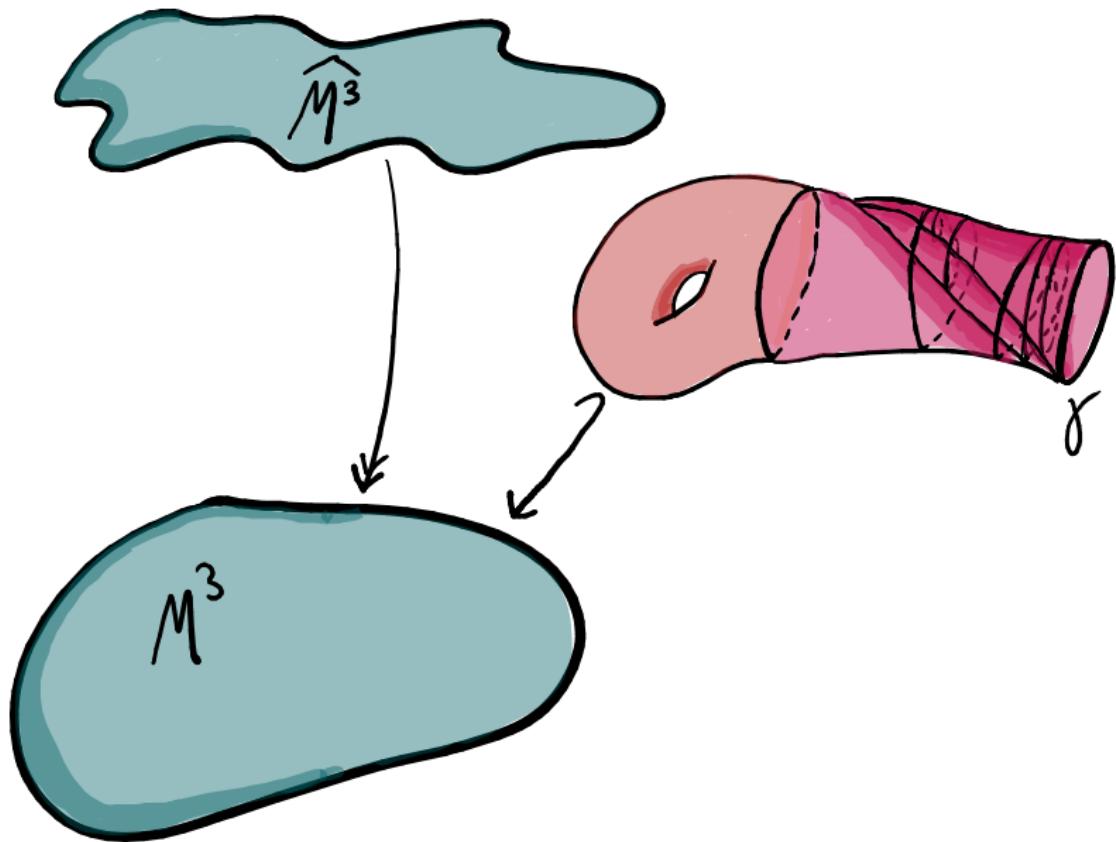
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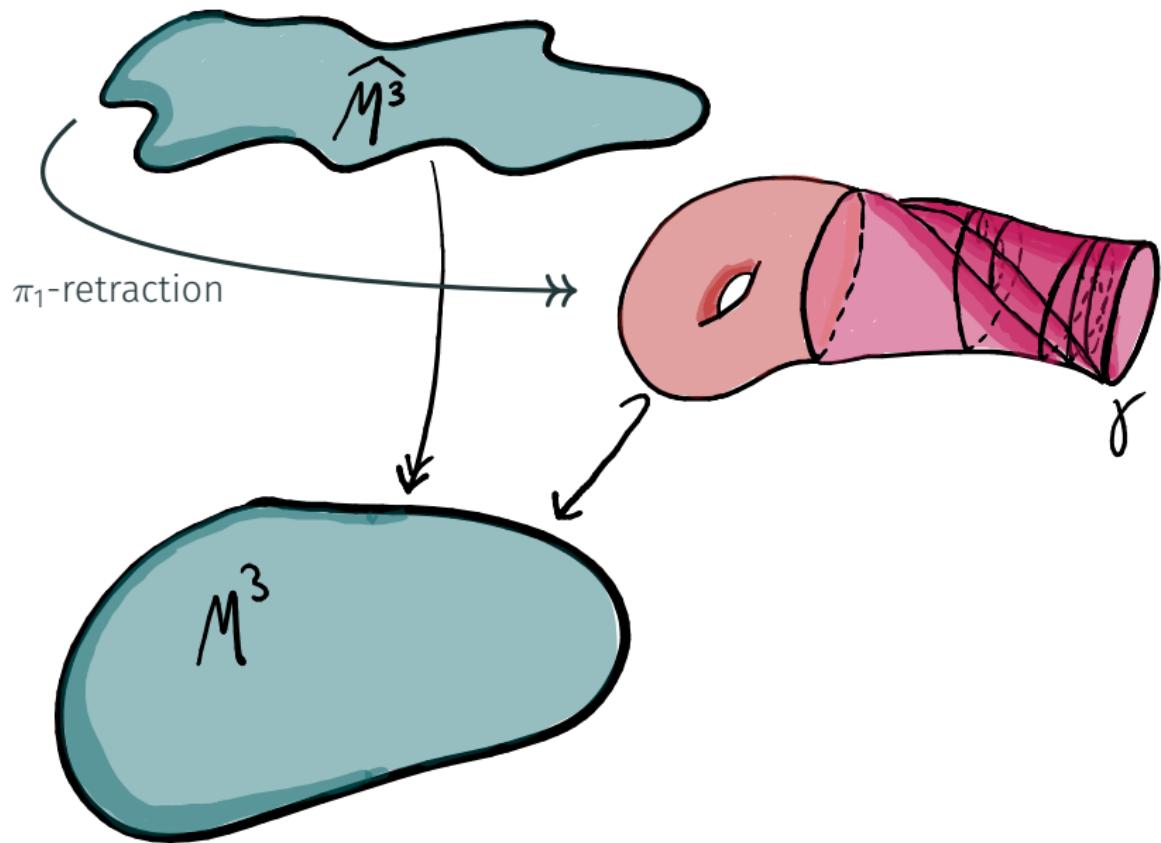
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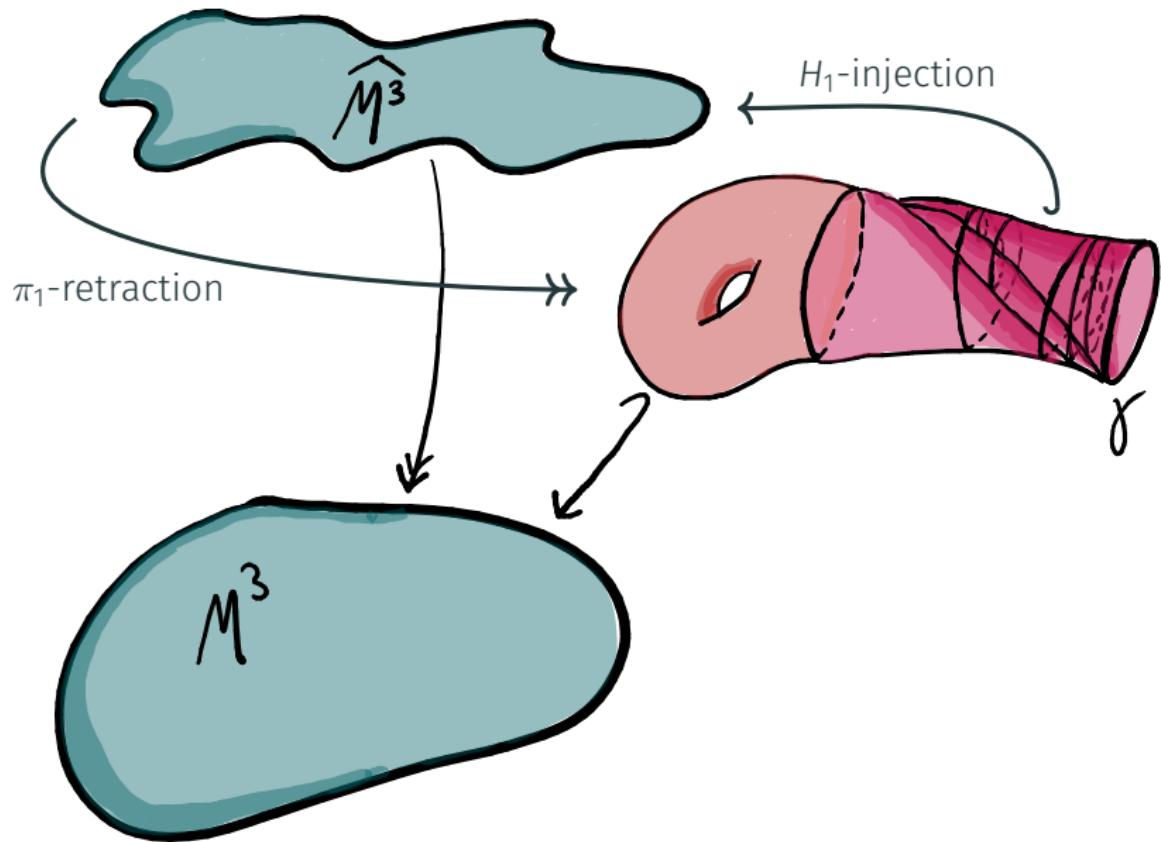
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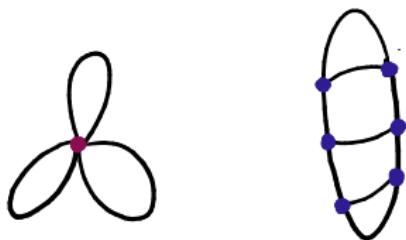


(Hyperbolic) Graphs of free groups with cyclic edges

(Fundamental groups of) graphs joined together by cylinders.

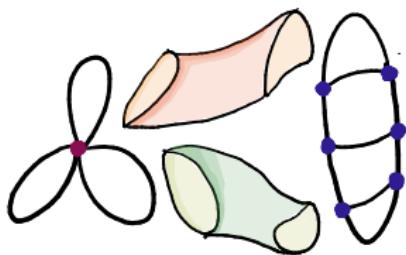
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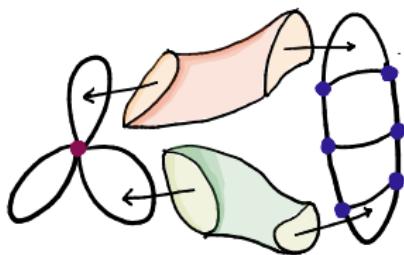
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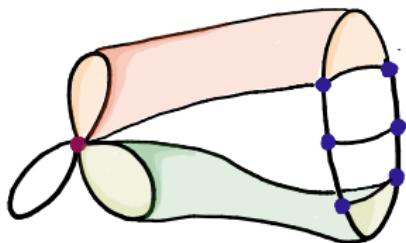
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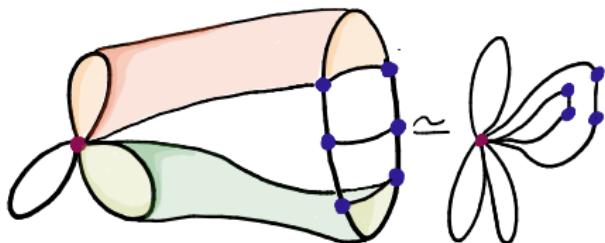
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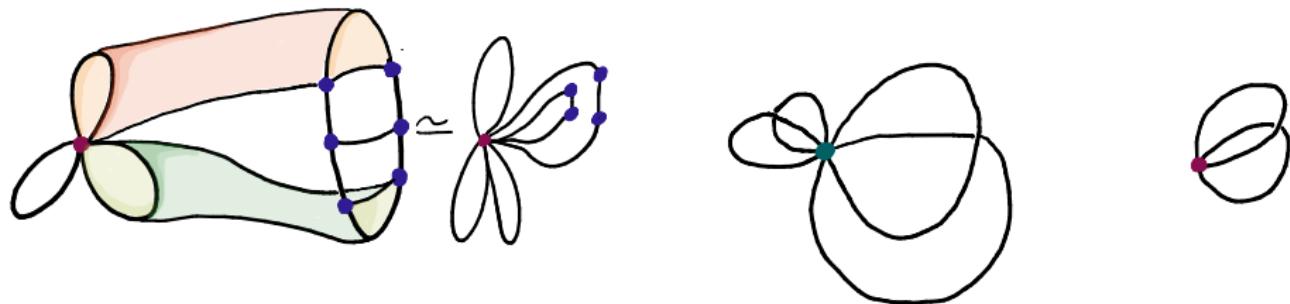
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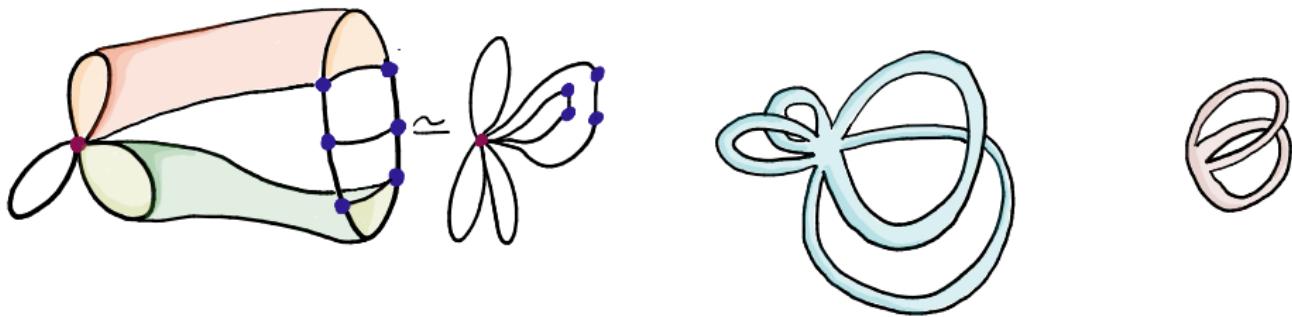
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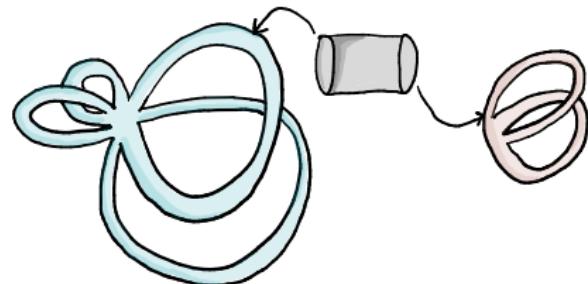
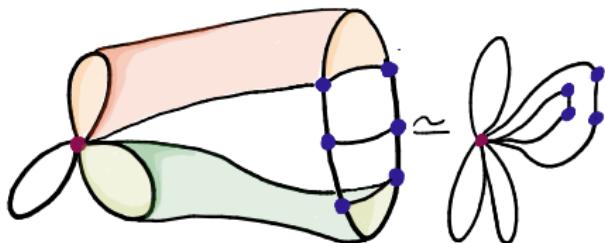
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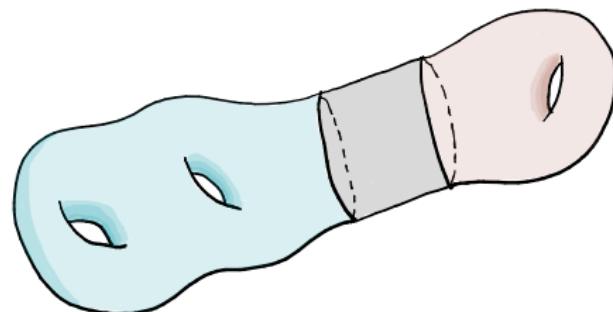
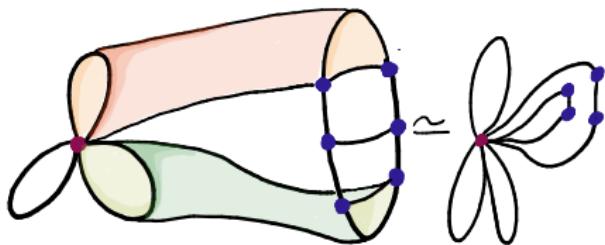
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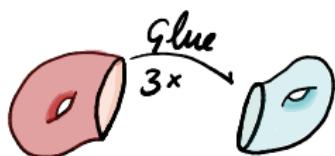
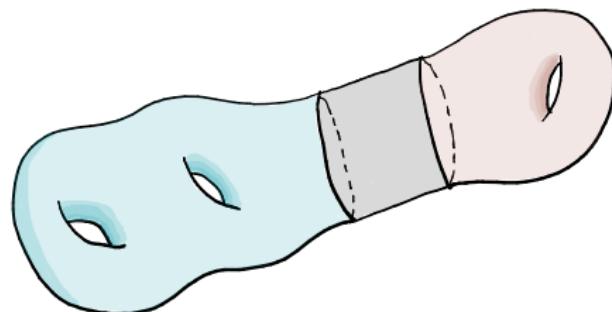
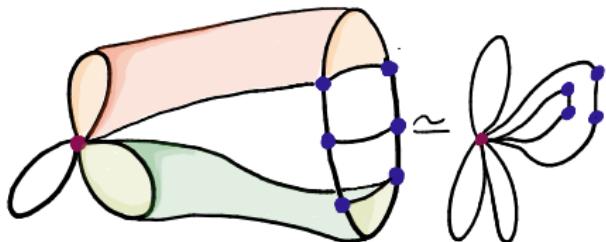
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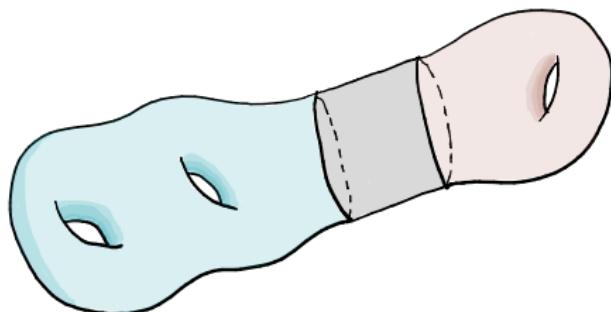
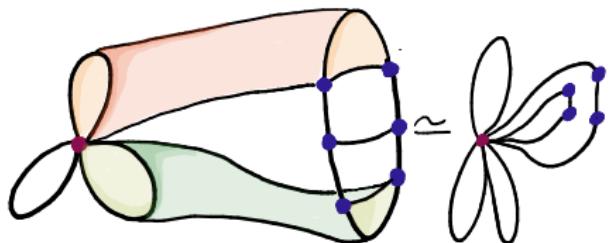
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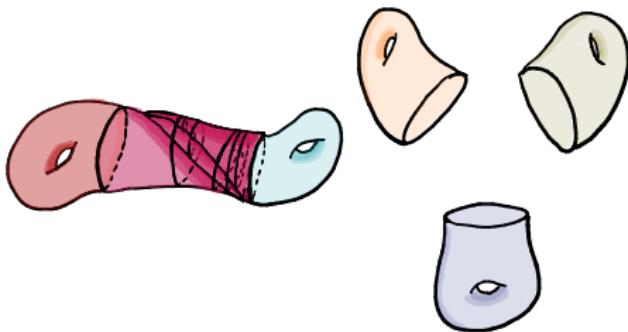
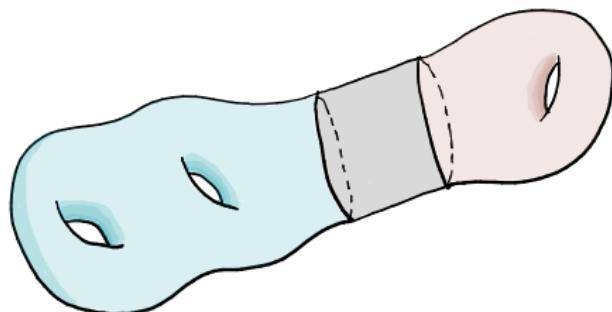
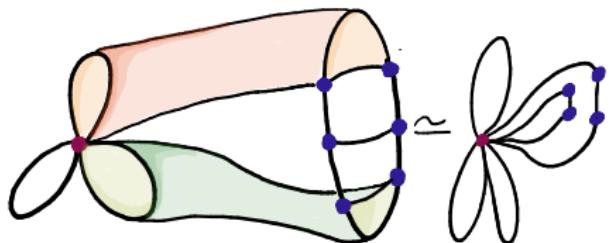
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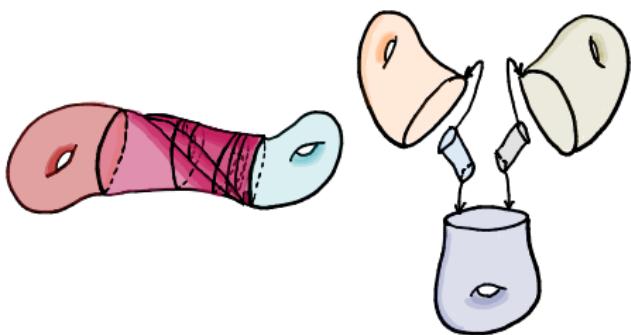
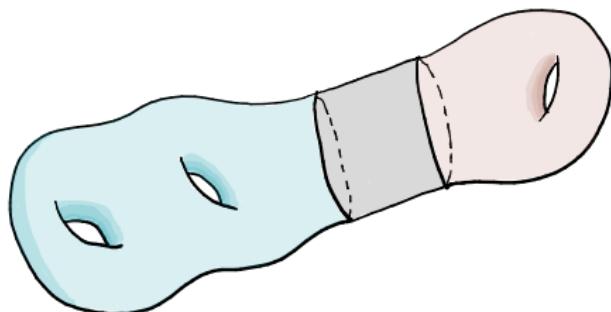
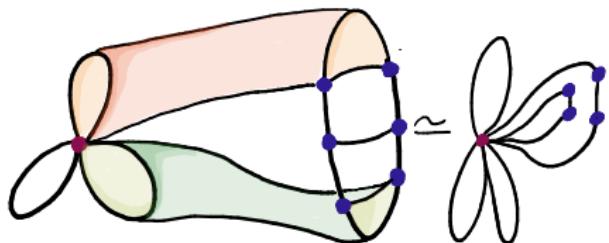
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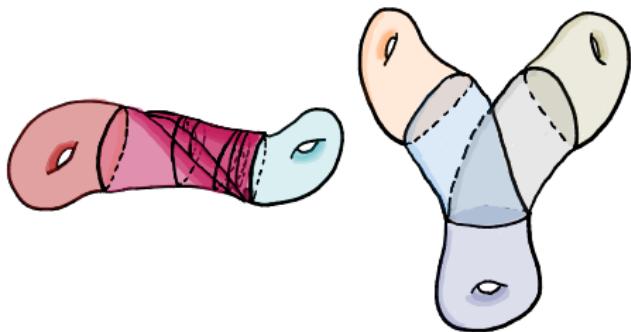
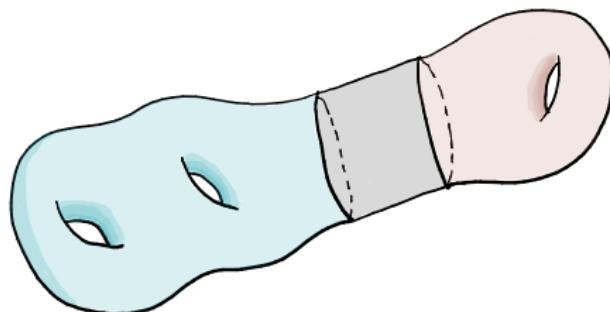
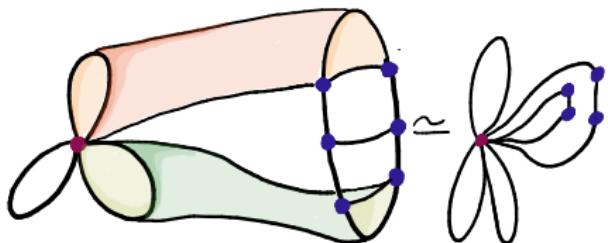
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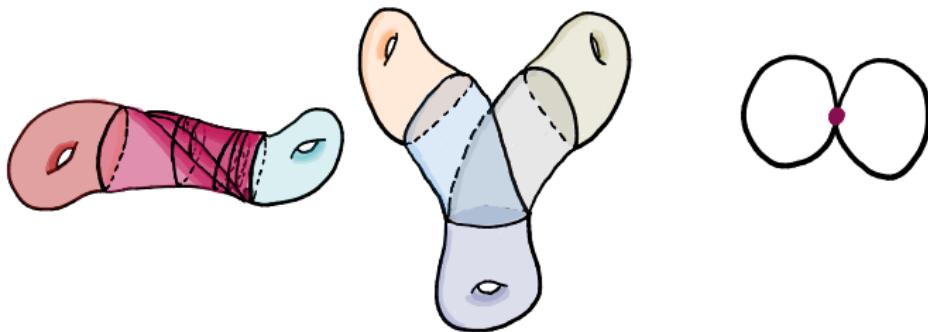
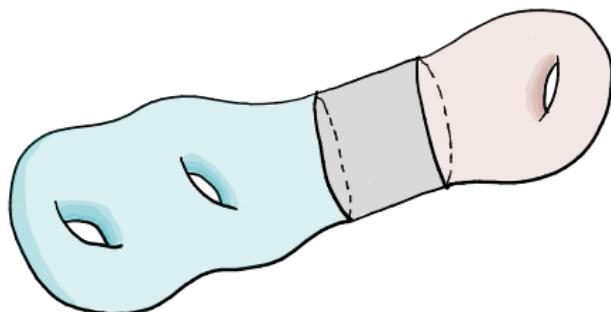
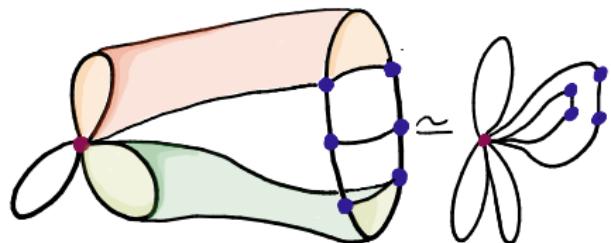
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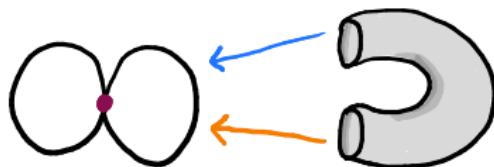
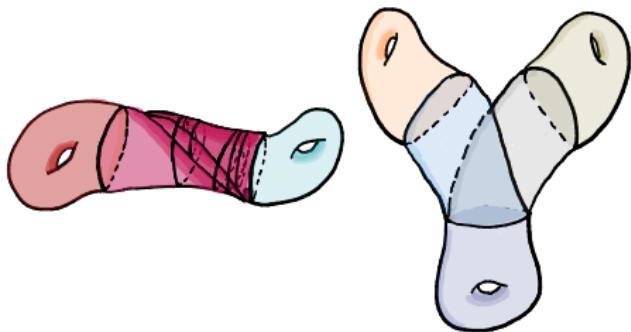
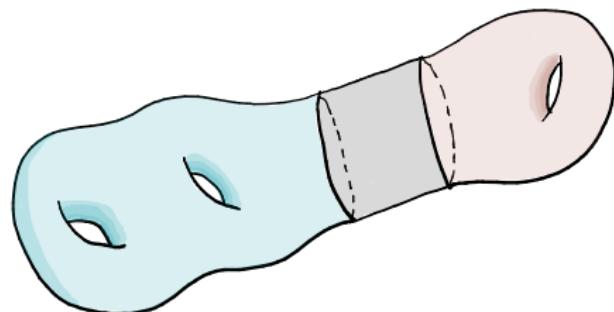
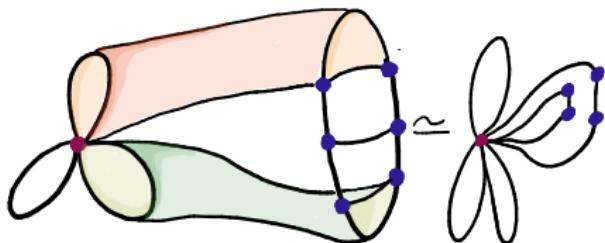
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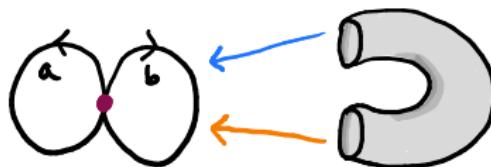
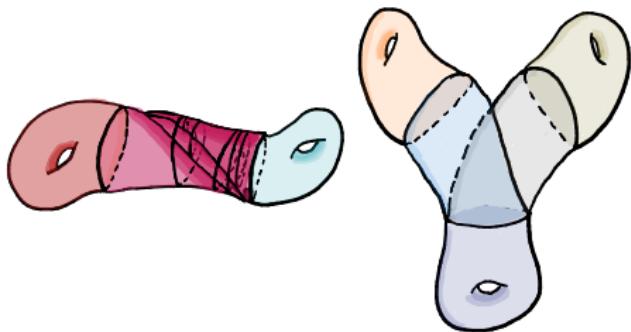
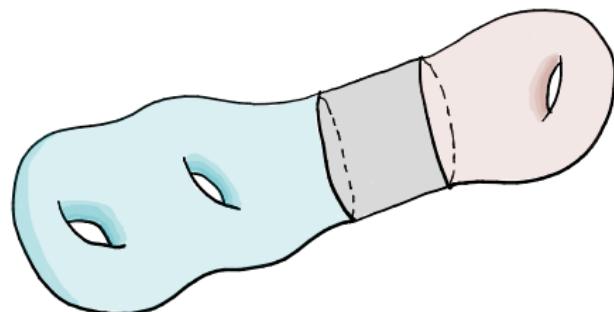
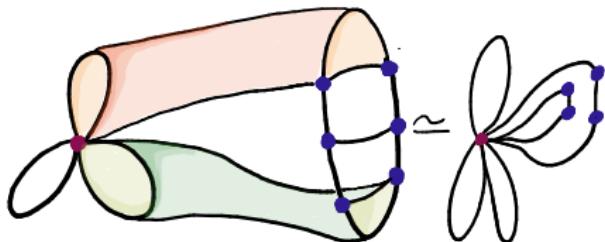
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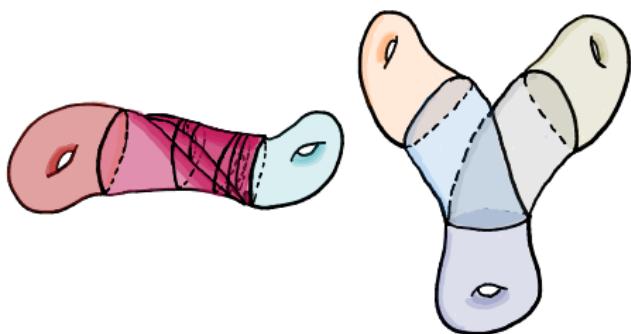
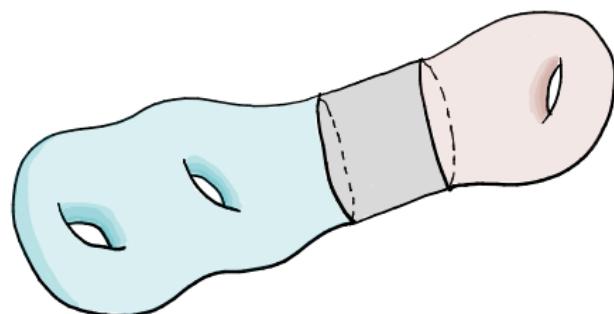
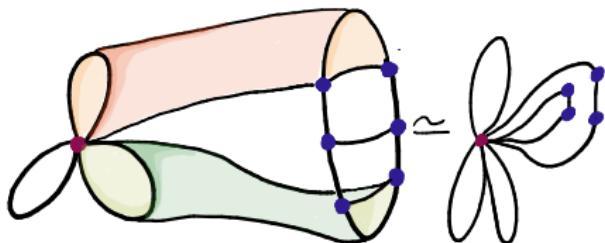
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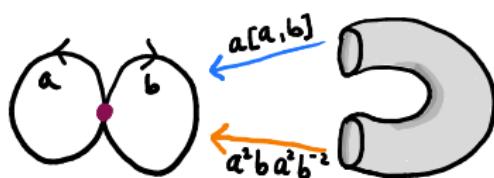
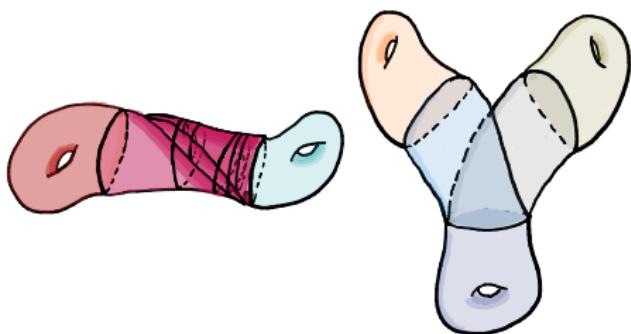
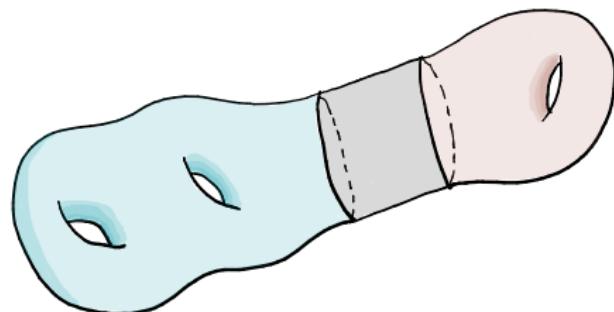
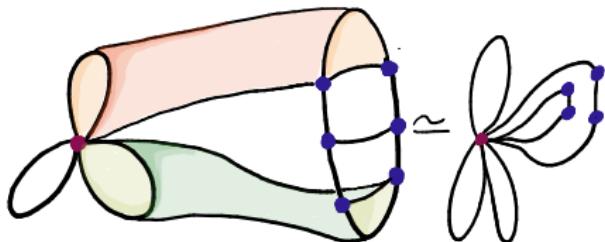
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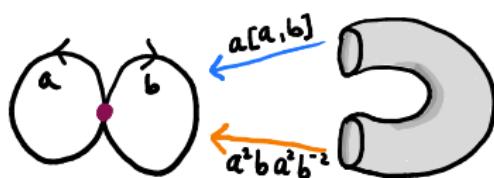
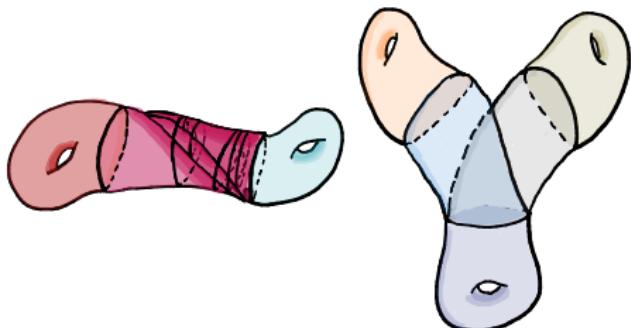
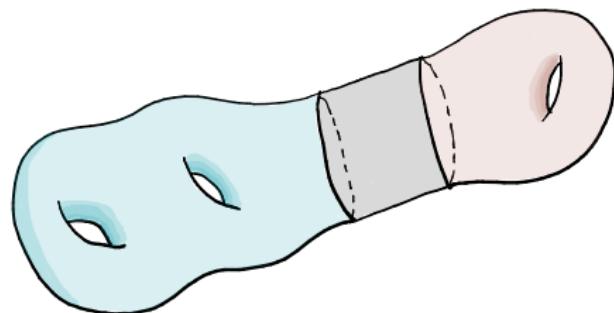
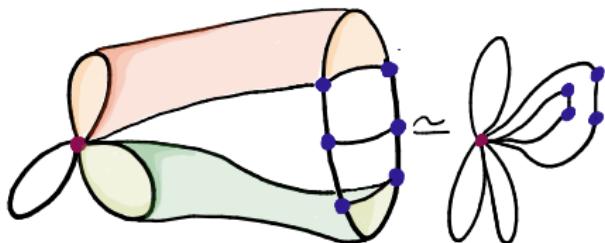
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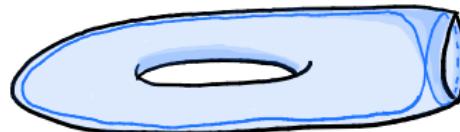
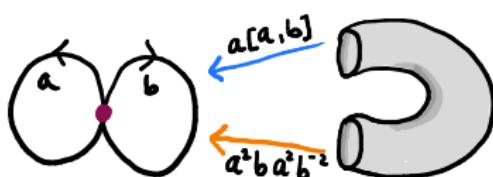
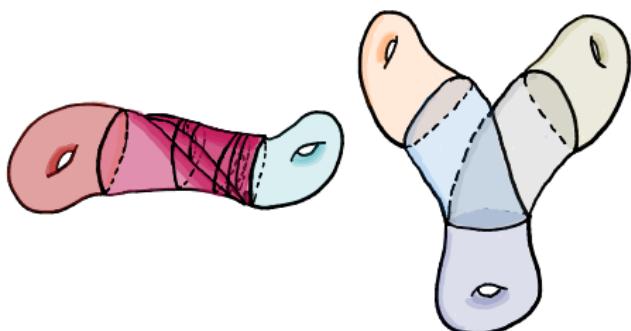
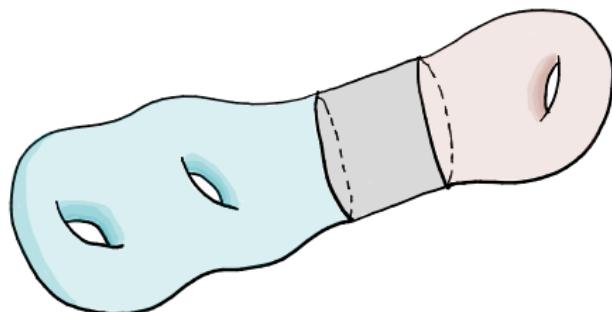
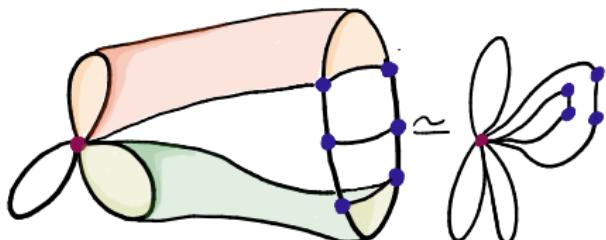
(Hyperbolic) Graphs of free groups with cyclic edges

(Fundamental groups of) graphs joined together by cylinders.



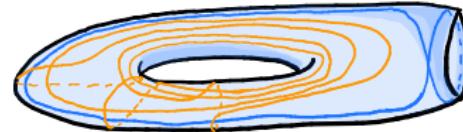
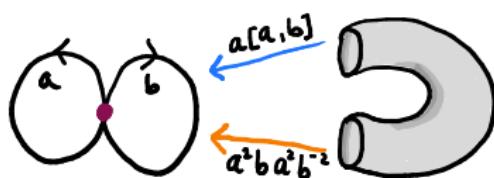
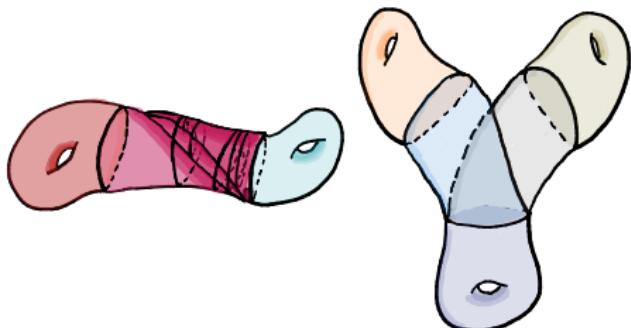
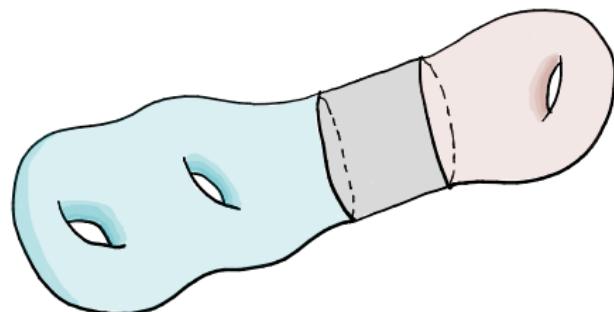
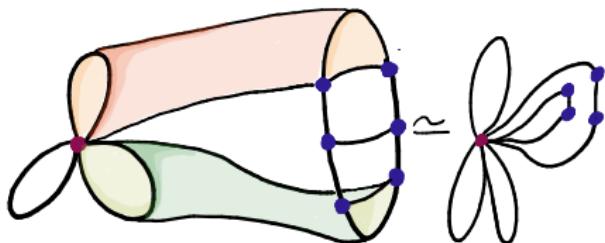
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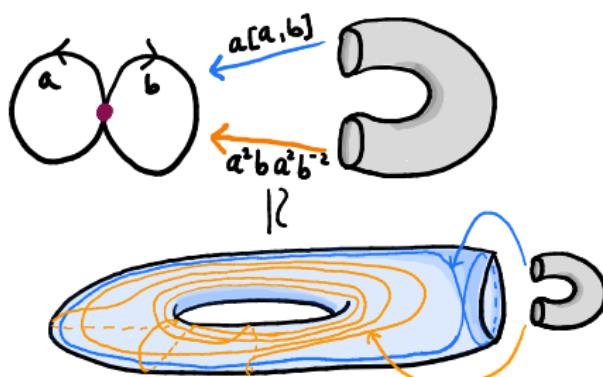
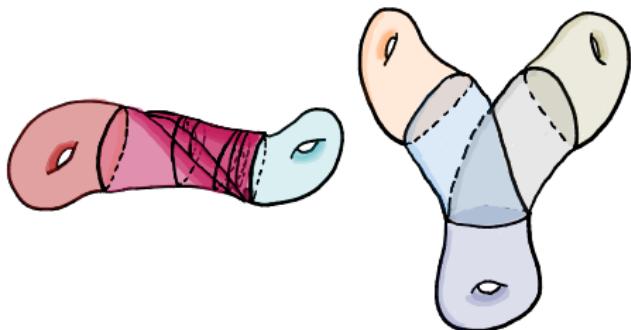
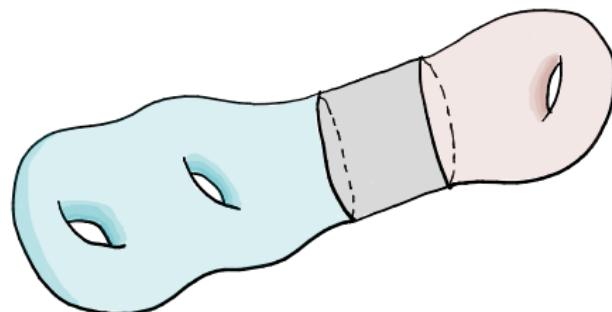
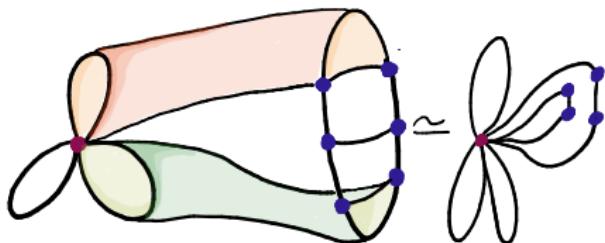
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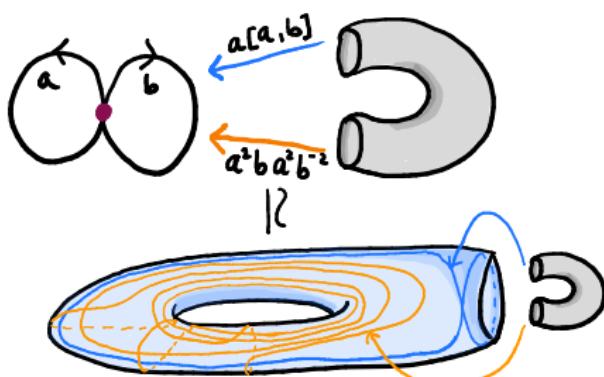
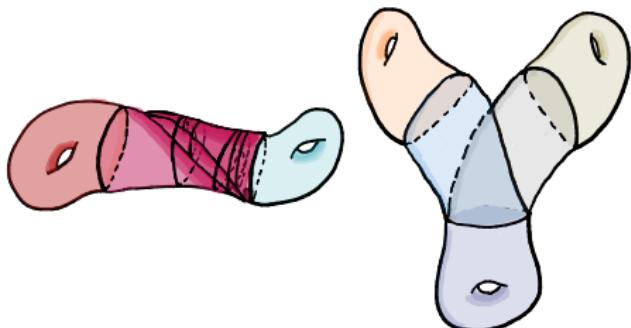
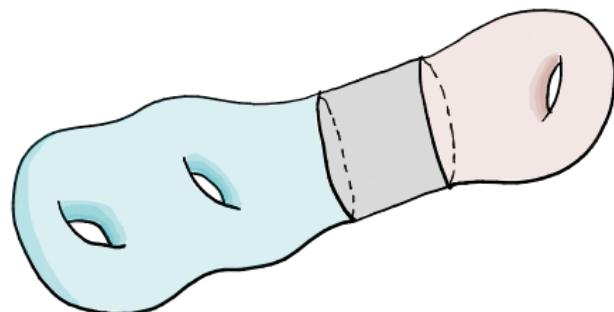
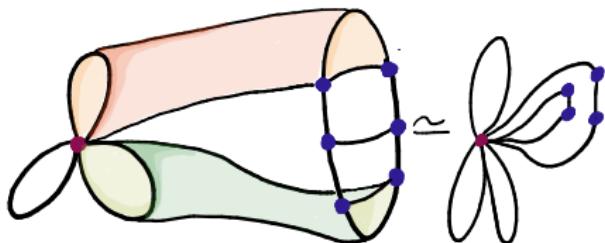
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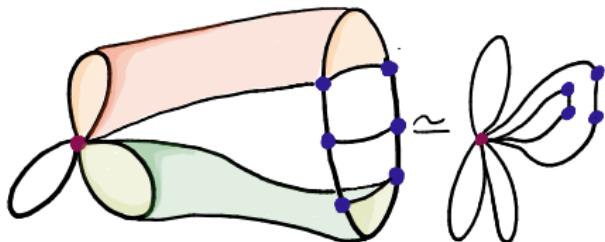
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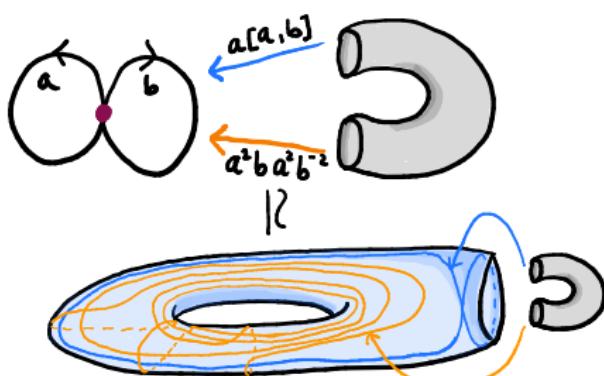
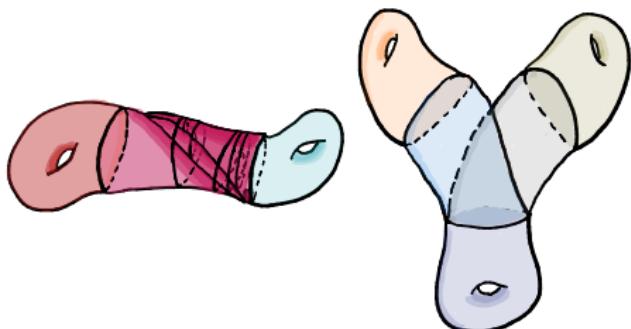
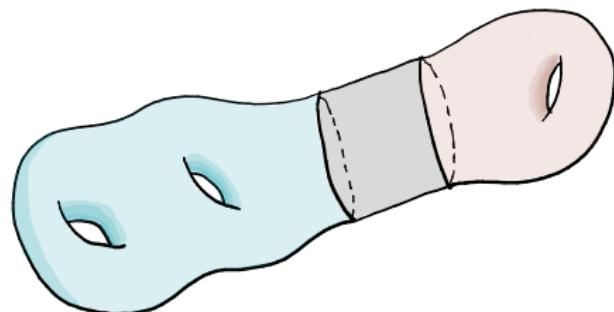


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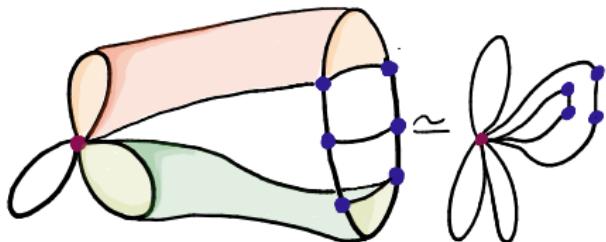


Free and surface groups
(and their free products)

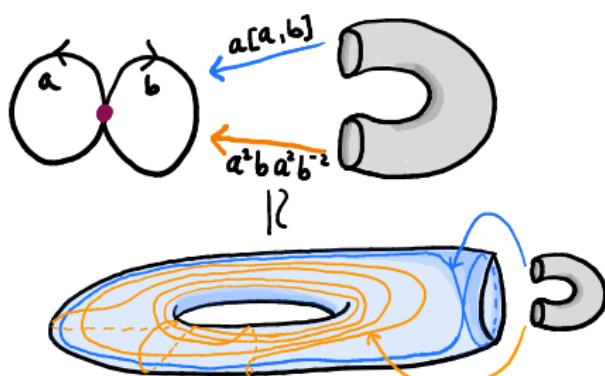
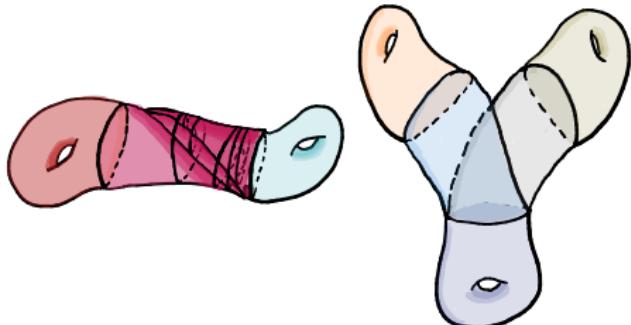
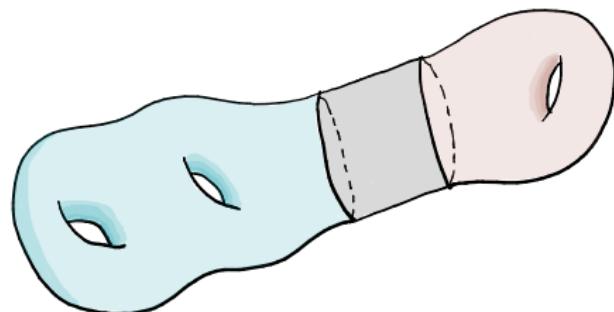


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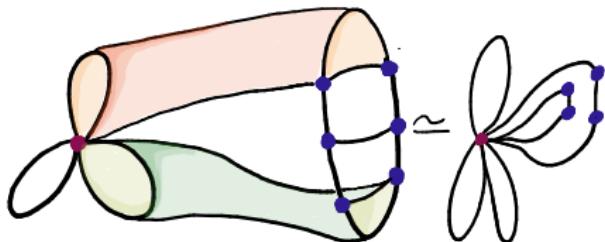


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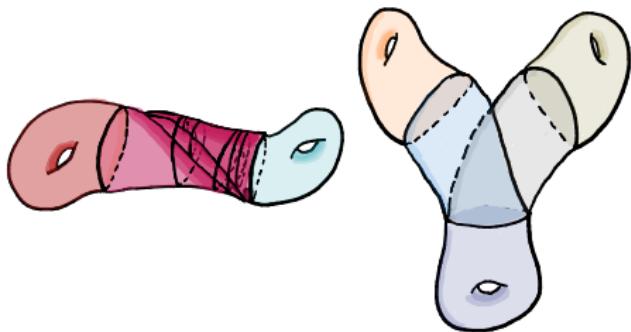
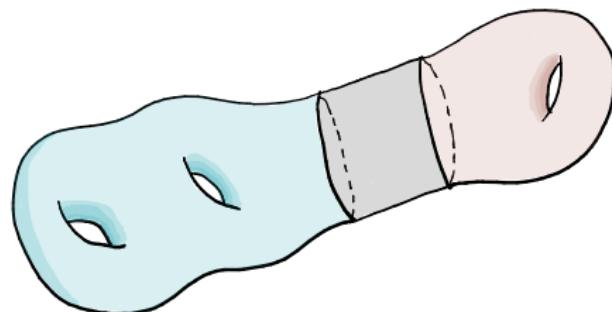


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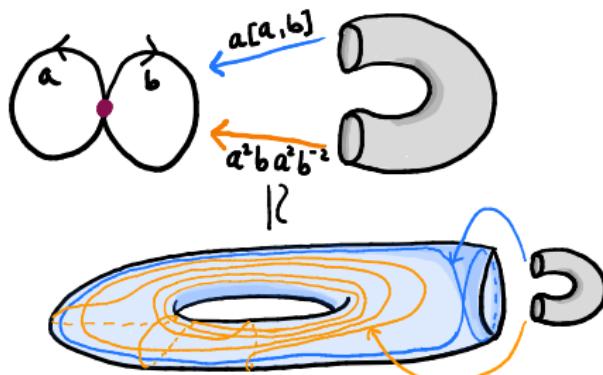
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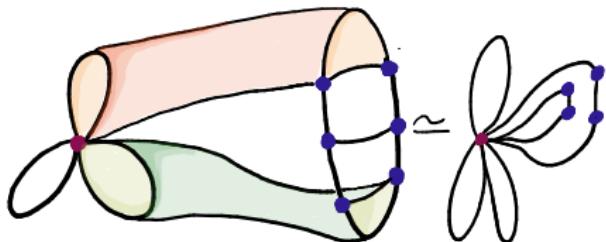


“More than a surface”

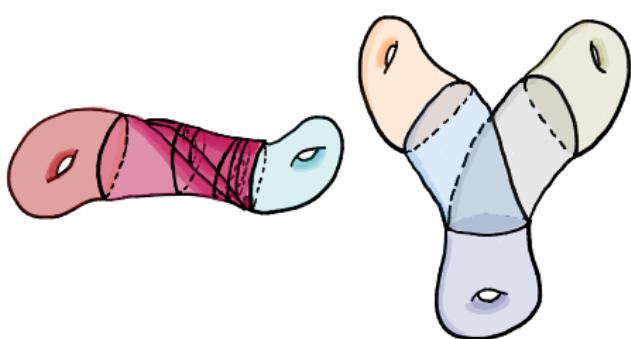
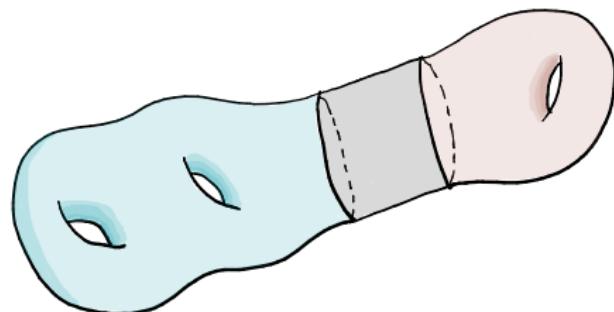


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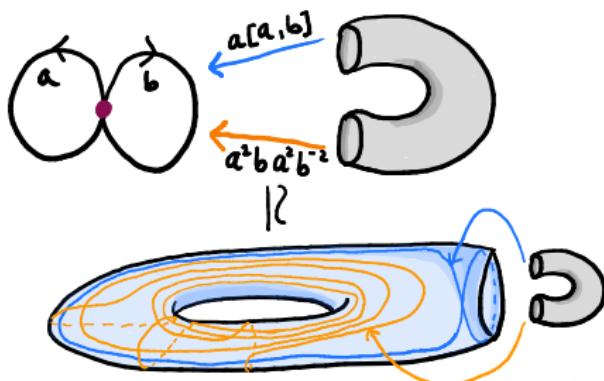
(Fundamental groups of) graphs joined together by cylinders.



Free and surface groups
(and their free products)



“More than a surface”



“Has a rigid vertex”

Homological torsion: Existence

Homological torsion: Existence

Theorem (Ascari-F '25)

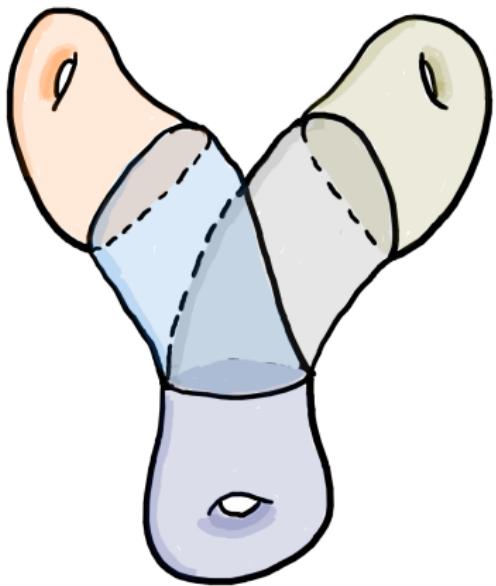
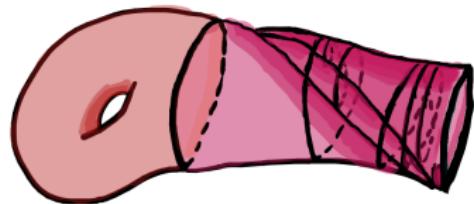
Let G be a hyperbolic graph of free groups with \mathbb{Z} edges, which is not a free product of free and surface groups. Then G has abundant virtual homological torsion, that is,

for every finite abelian group A , there is a finite-index subgroup $G_0 \leq G$ such that

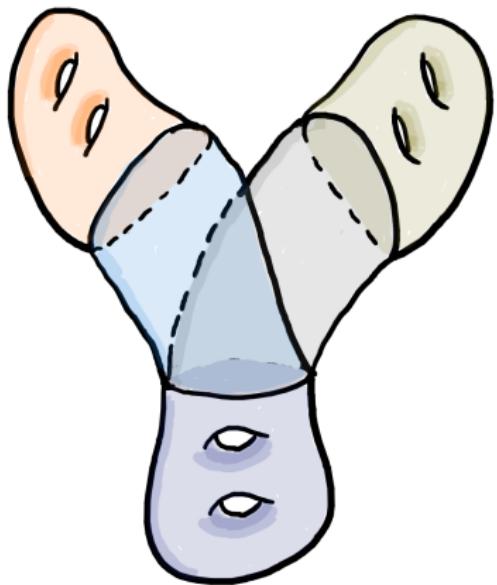
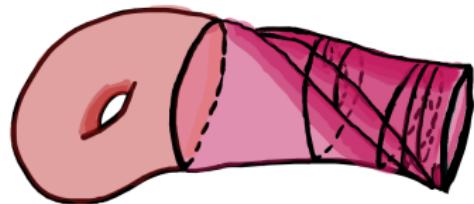
$$G_0^{\text{ab}} = A \oplus \cdots.$$

Homological torsion: By the book

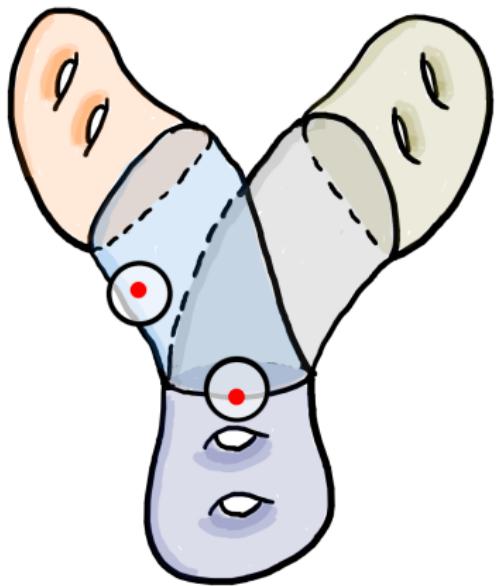
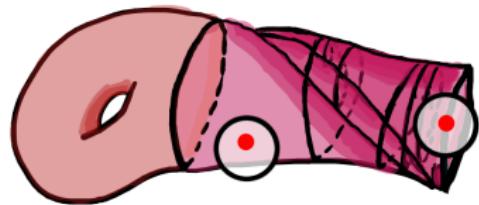
Homological torsion: By the book



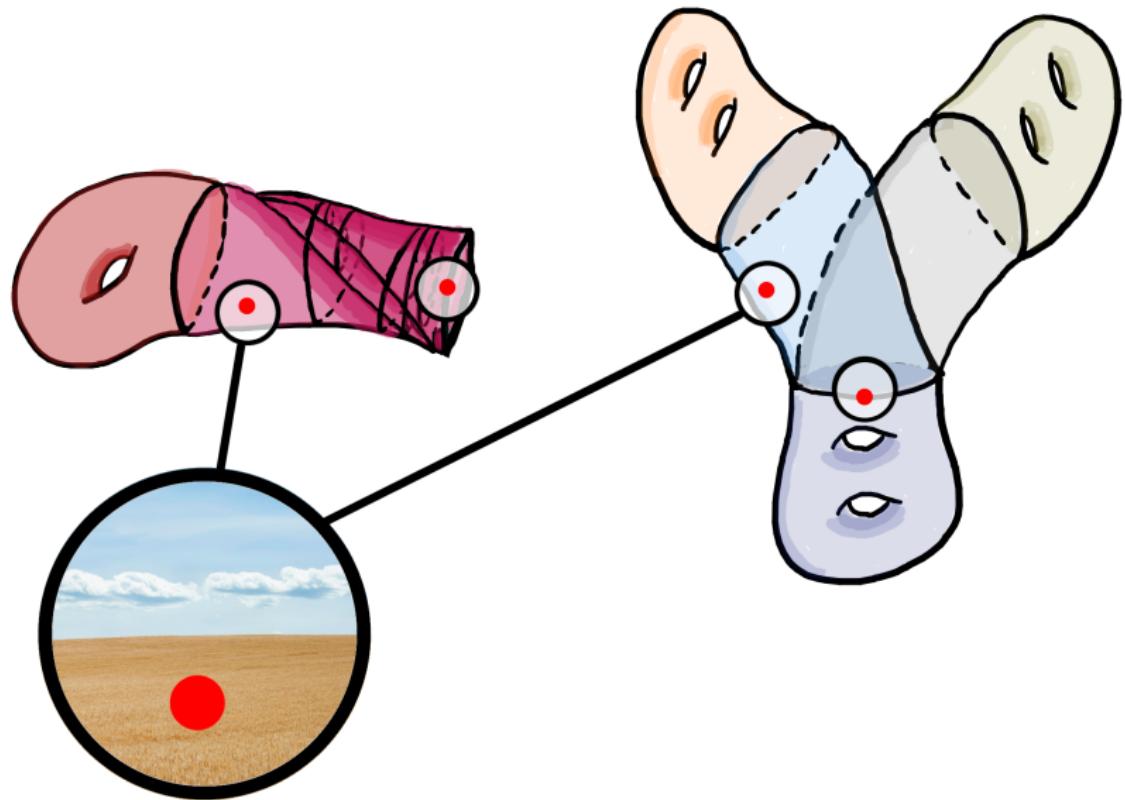
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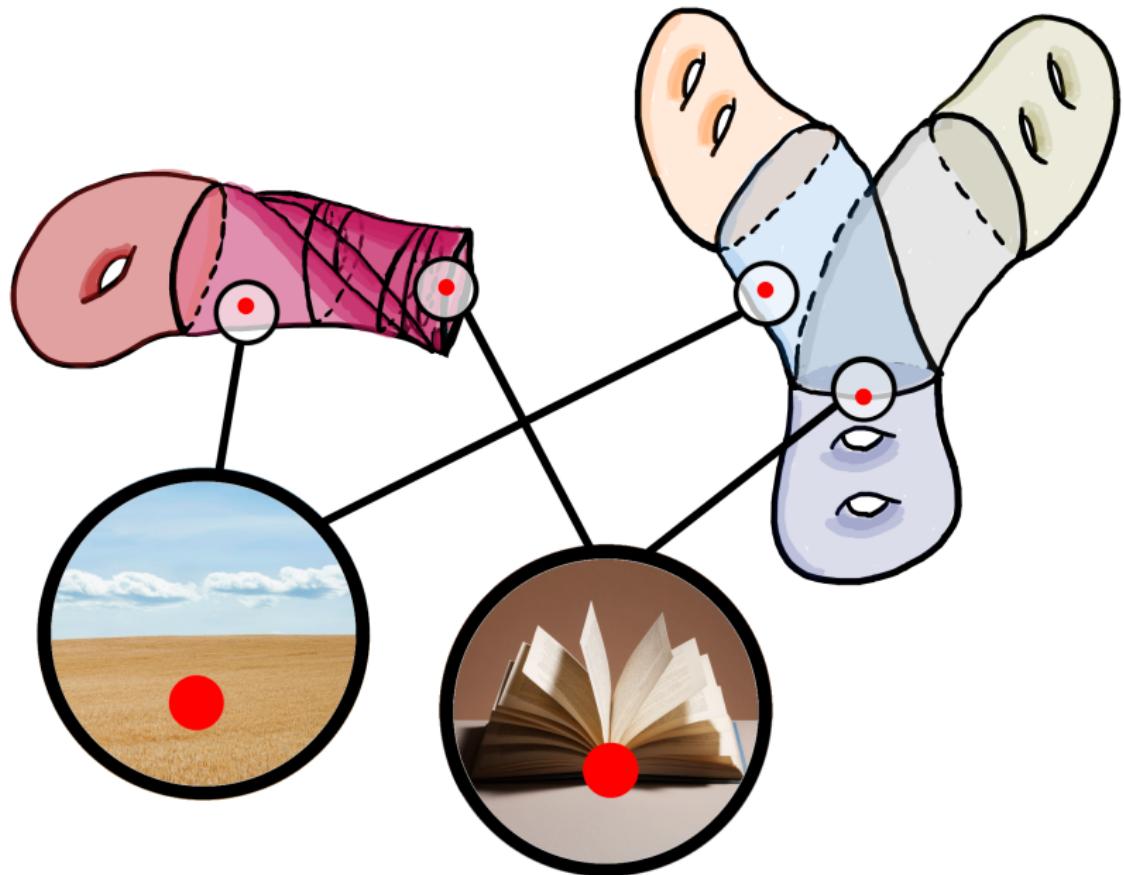
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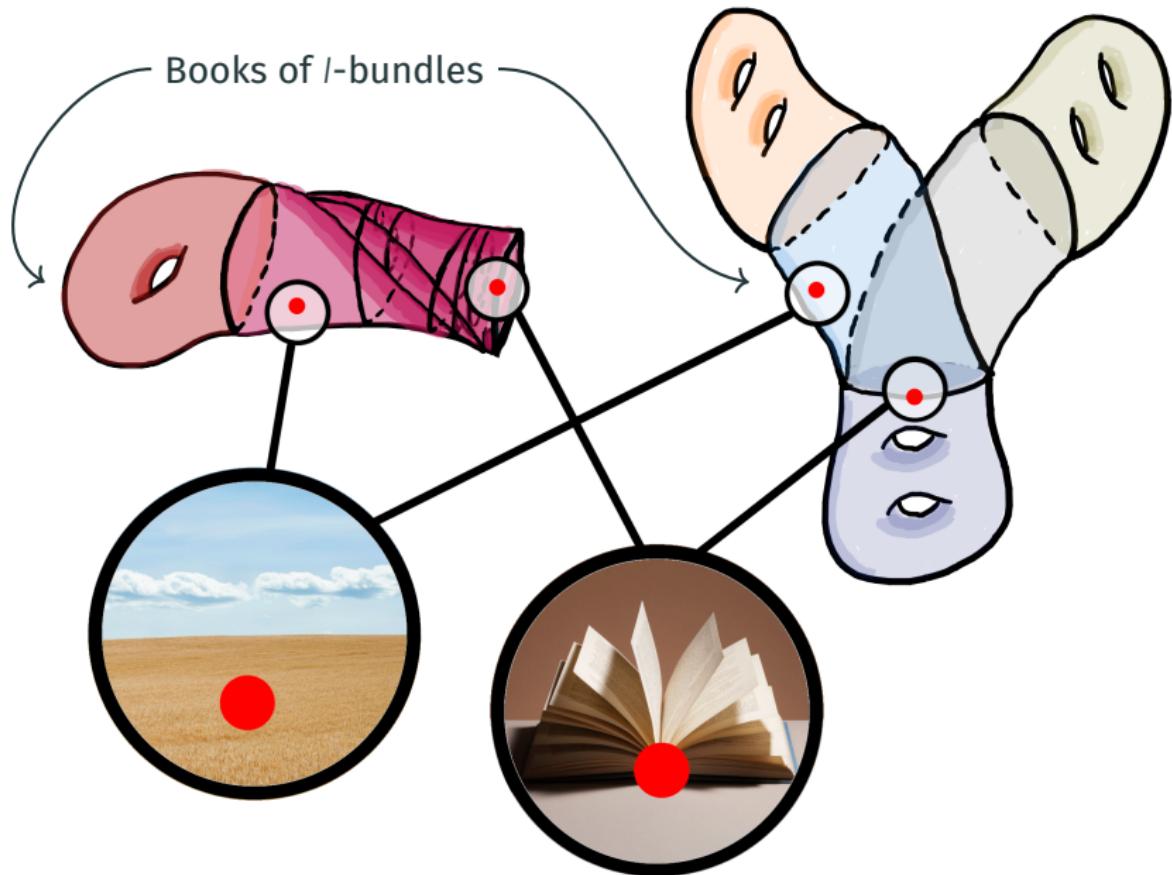
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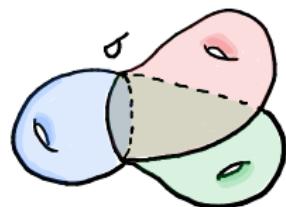


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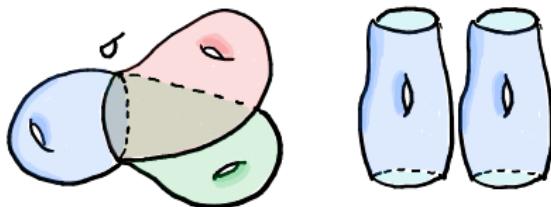


Homological torsion: Systems of surfaces

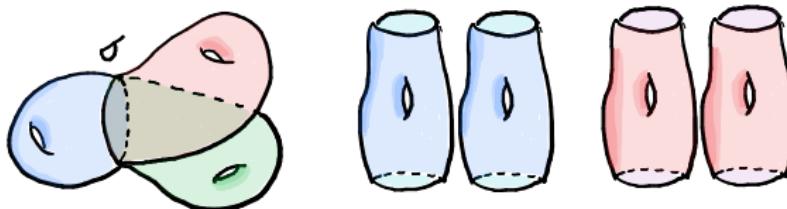
Homological torsion: Systems of surfaces



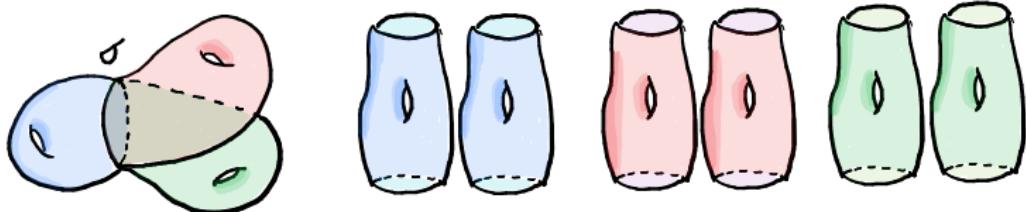
Homological torsion: Systems of surfaces



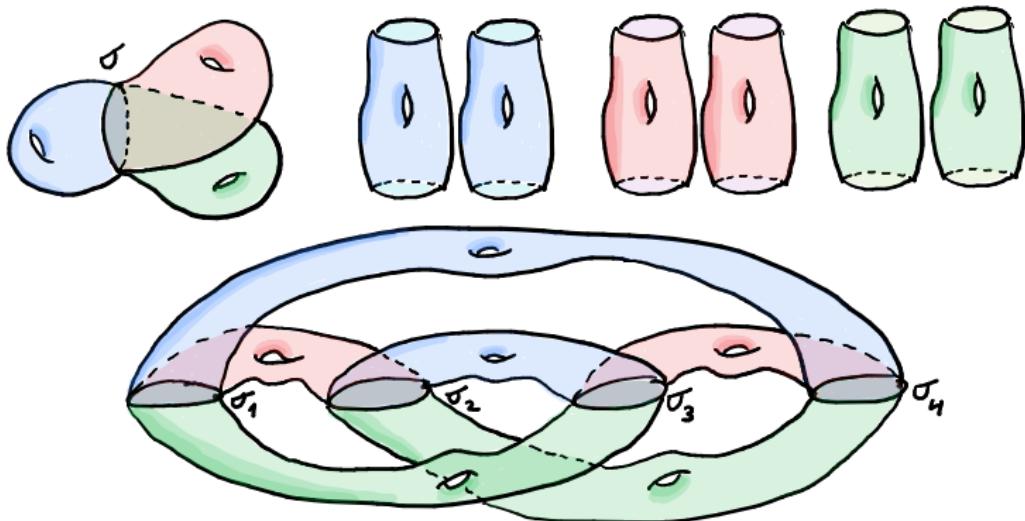
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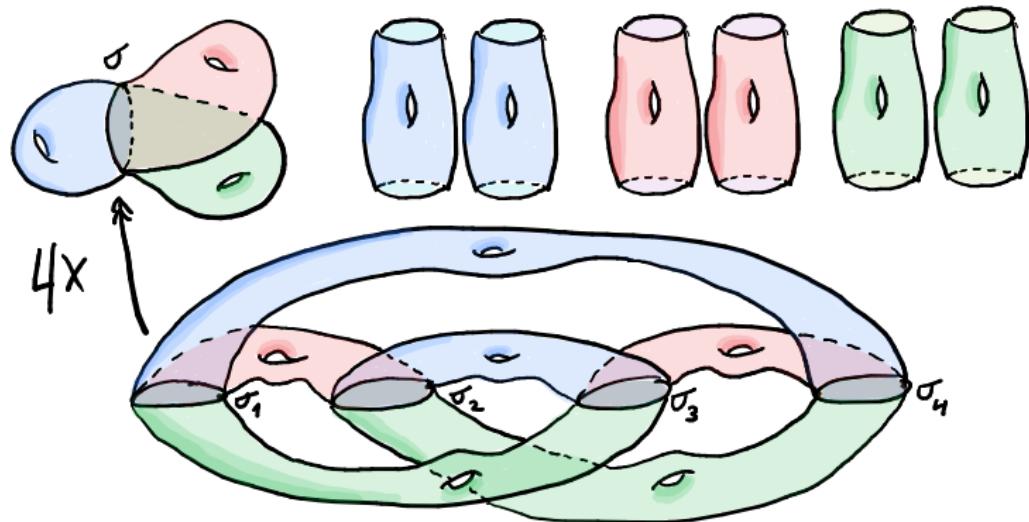
Homological torsion: Systems of surfaces



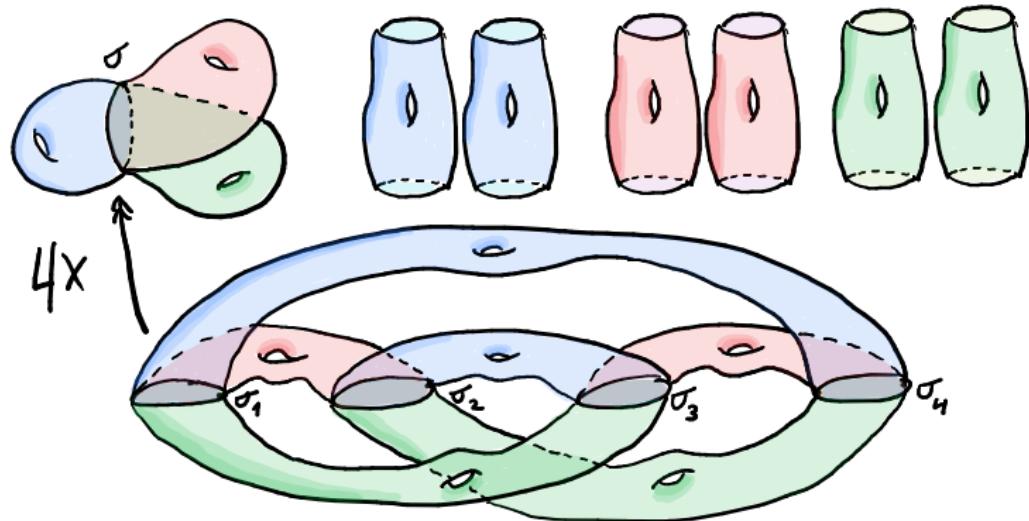
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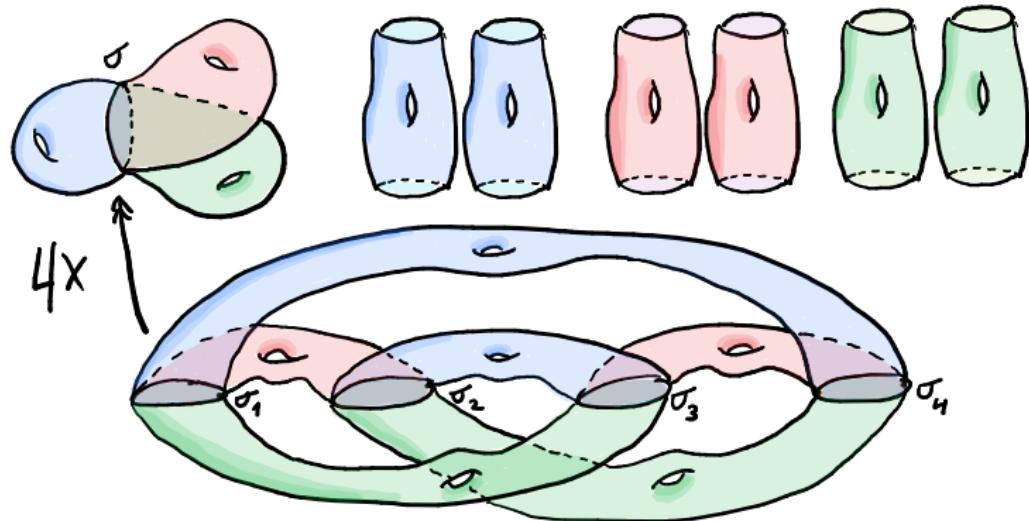


Homological torsion: Systems of surfaces



In first homology:

Homological torsion: Systems of surfaces



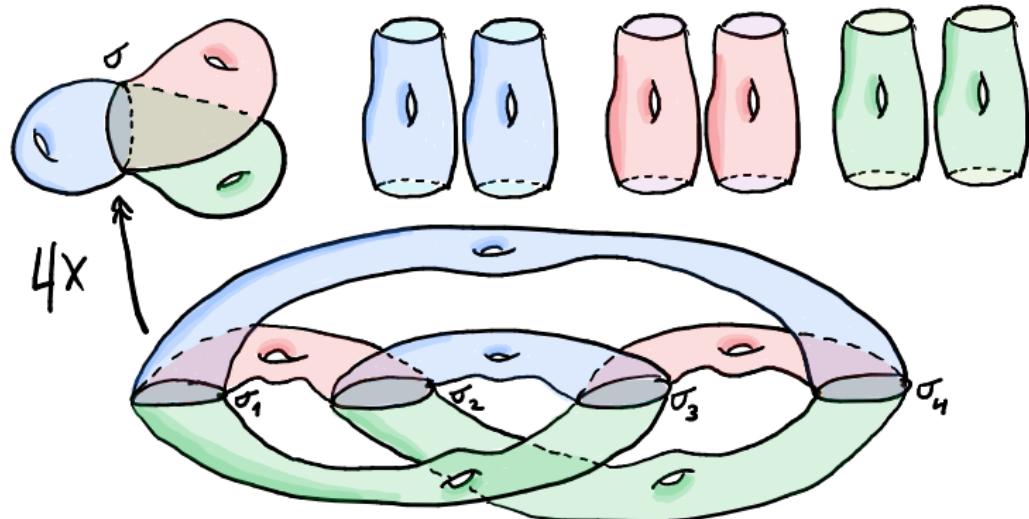
In first homology:

$$\sigma_1 + \sigma_4 = 0 \quad \sigma_2 + \sigma_3 = 0$$

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Homological torsion: Systems of surfaces



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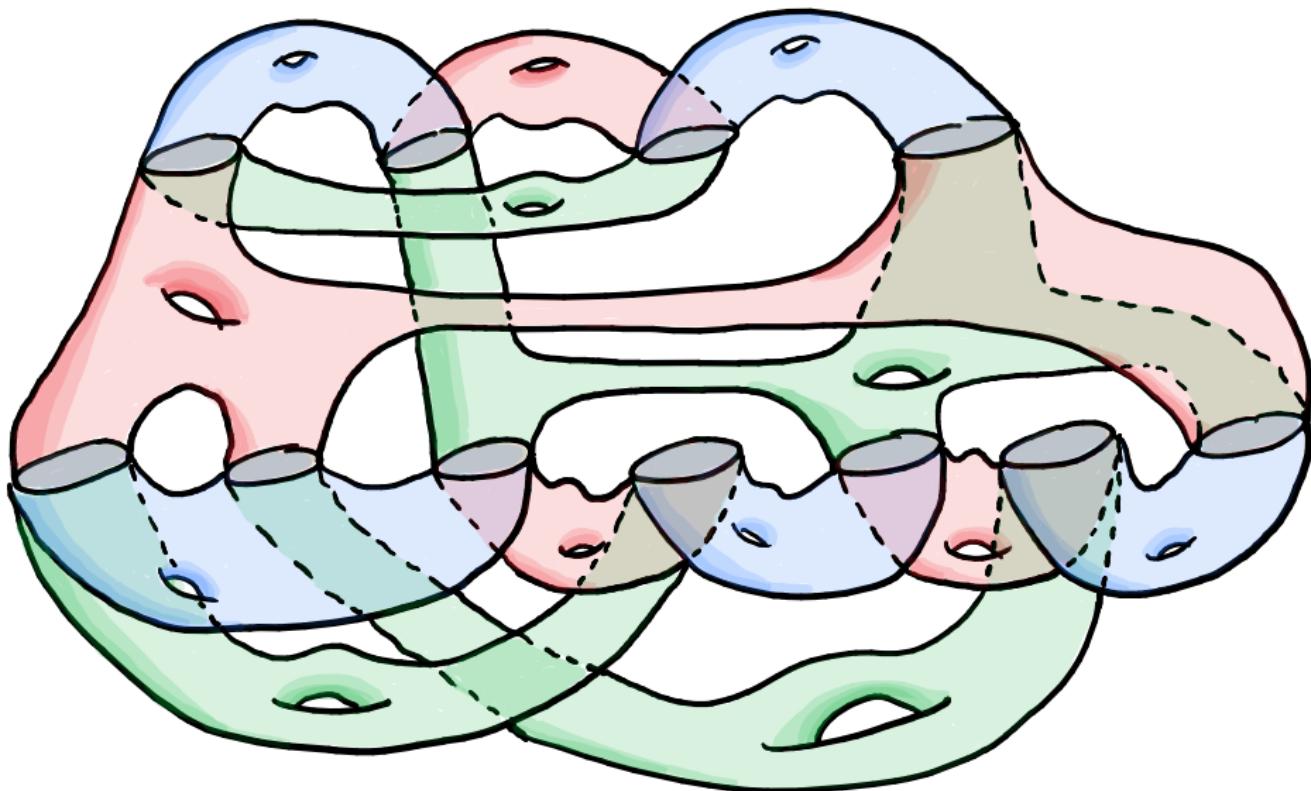
$$\Rightarrow$$

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$$

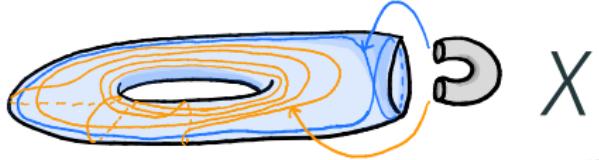
$$2 \cdot \sigma_1 = 0$$

Homological torsion: Two and three

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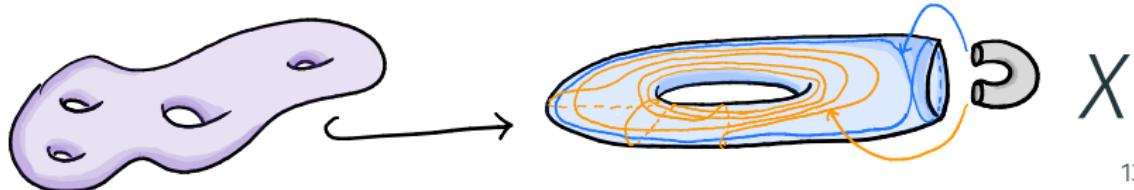
Enter a rigid vertex



Enter a rigid vertex

Theorem (Wilton '18)

If G is not free then G has a surface subgroup.

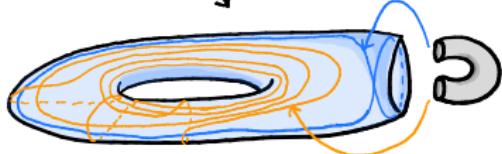
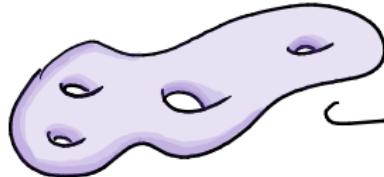
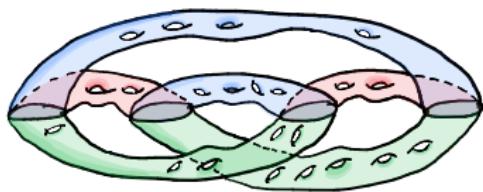


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B



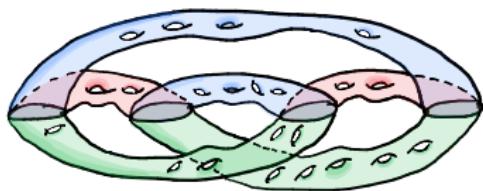
X

Enter a rigid vertex

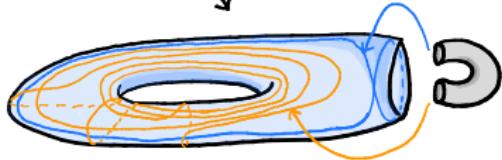
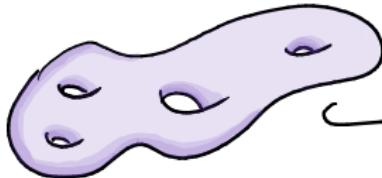
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B



Just a map (not π_1 -injective)



X

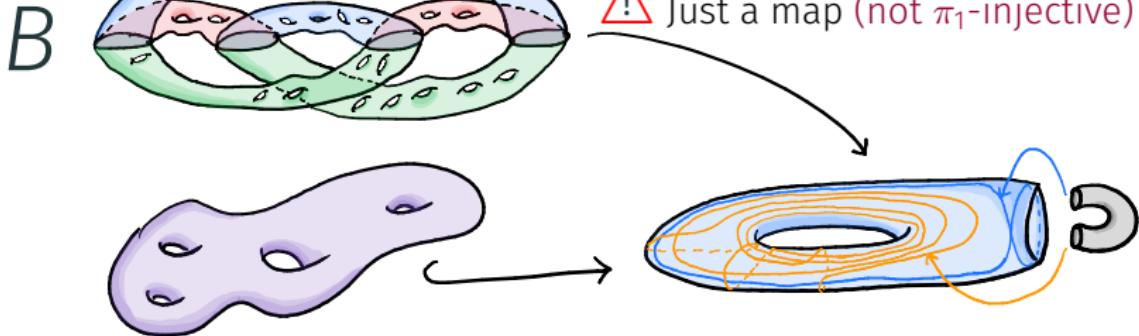
Enter a rigid vertex

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Theorem (Calegari '09)

Let F be a free group and let $g \in [F, F]$. Recall
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Enter a rigid vertex

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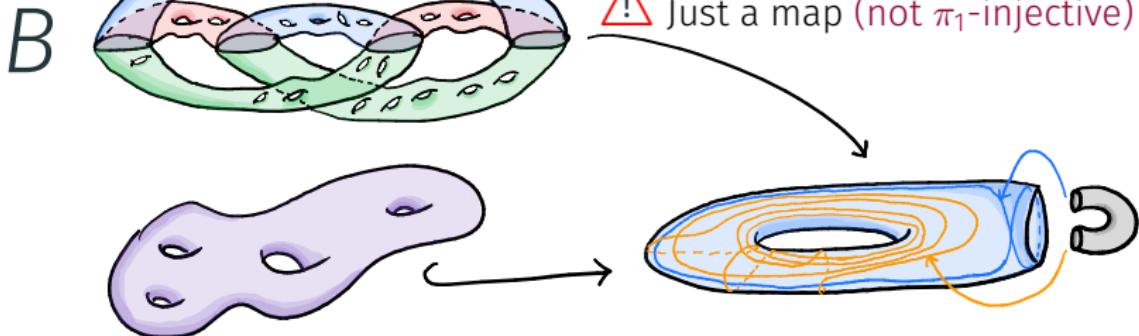
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$$\text{scl}(g) = \lim_{n \rightarrow \infty} \frac{\text{cl}(g^n)}{n} \in \mathbb{Q}.$$



Enter a rigid vertex

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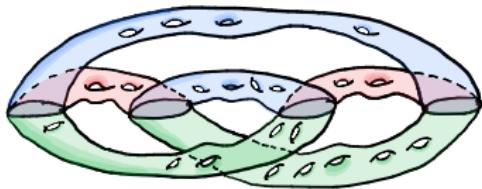
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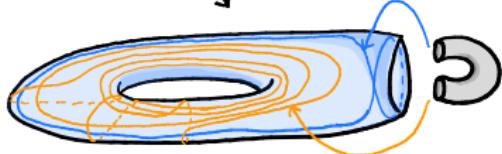
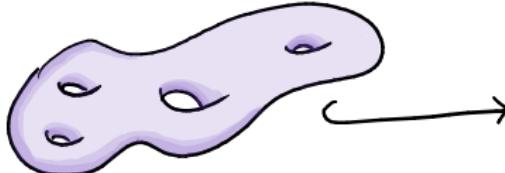
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\rightsquigarrow has a topological interpretation in terms of surfaces

B



Just a map (not π_1 -injective)



X

Enter a rigid vertex

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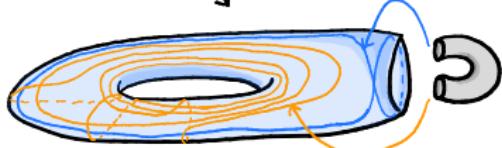
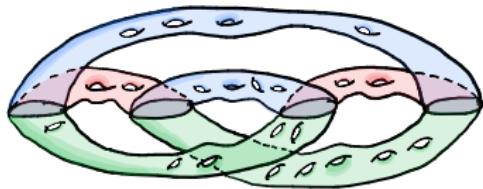
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X

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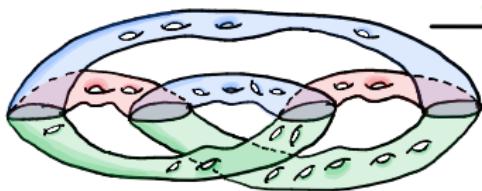
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⚠ $\text{Tor}(H_1)$

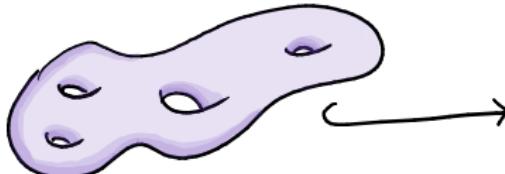
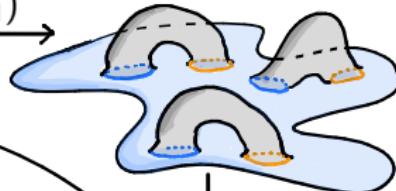
injective

B



↷ has a topological interpretation in terms of surfaces

\widehat{X}

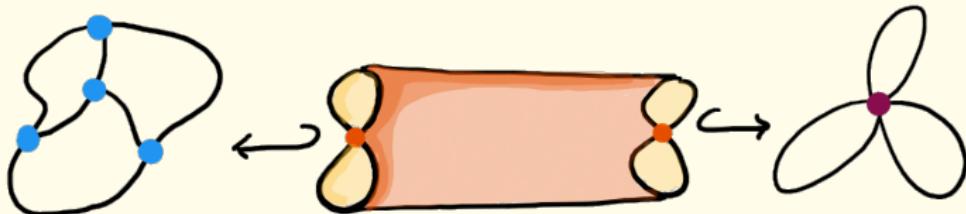


X

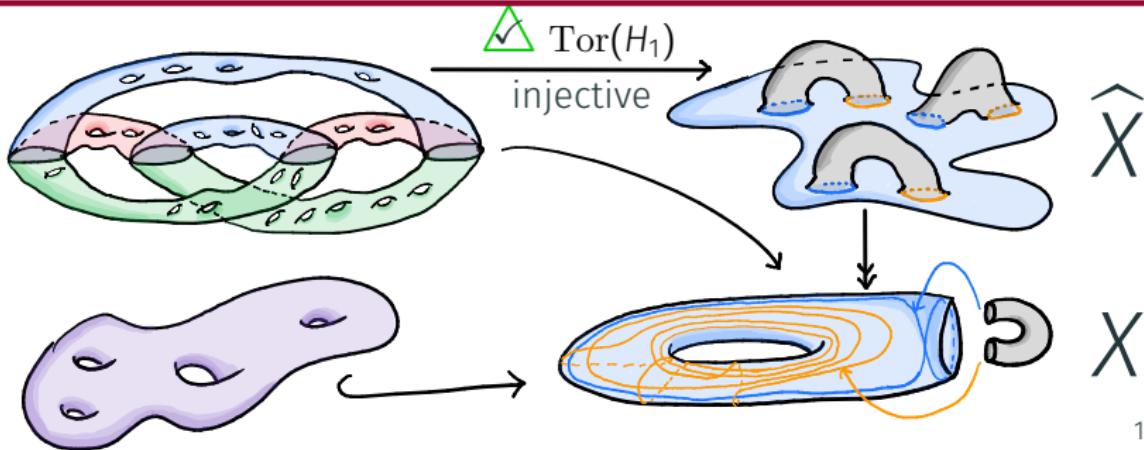
Enter a rigid vertex

Applicable to other settings?

e.g. graphs of graphs.



B



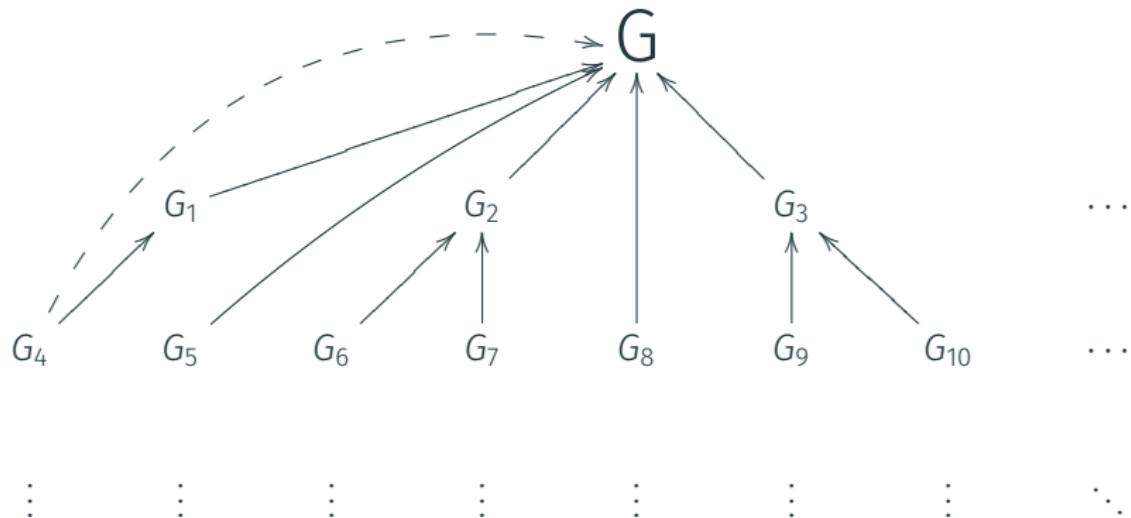
Profinite rigidity 101

Let G be a finitely generated *residually finite* group (every $g \in G$ survives in some finite quotient $q : G \twoheadrightarrow Q$).

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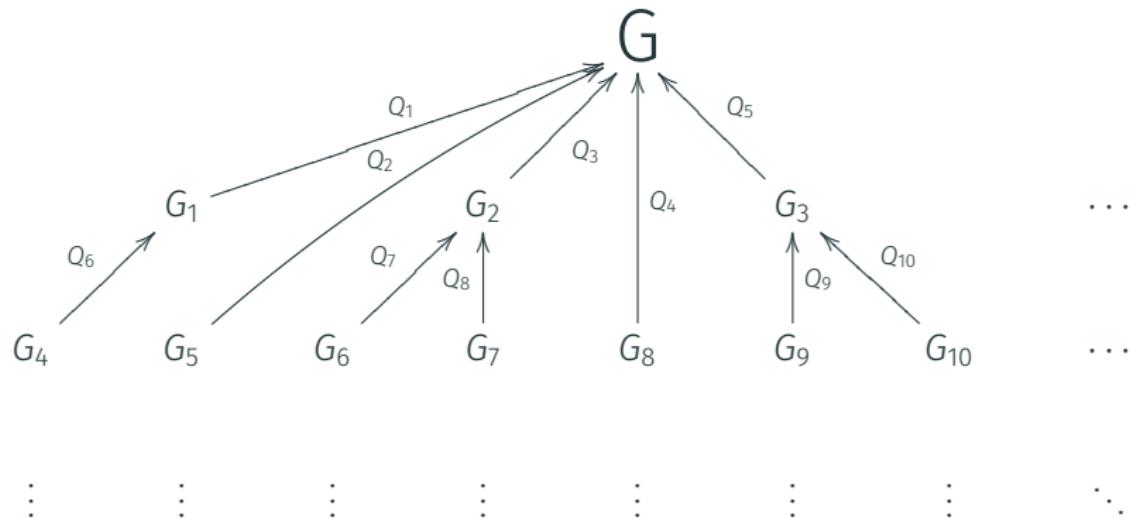
Lattice/poset of finite-index normal subgroups of G



Profinite rigidity 101

Let G be a finitely generated *residually finite* group (every $g \in G$ survives in some finite quotient $q : G \twoheadrightarrow Q$).

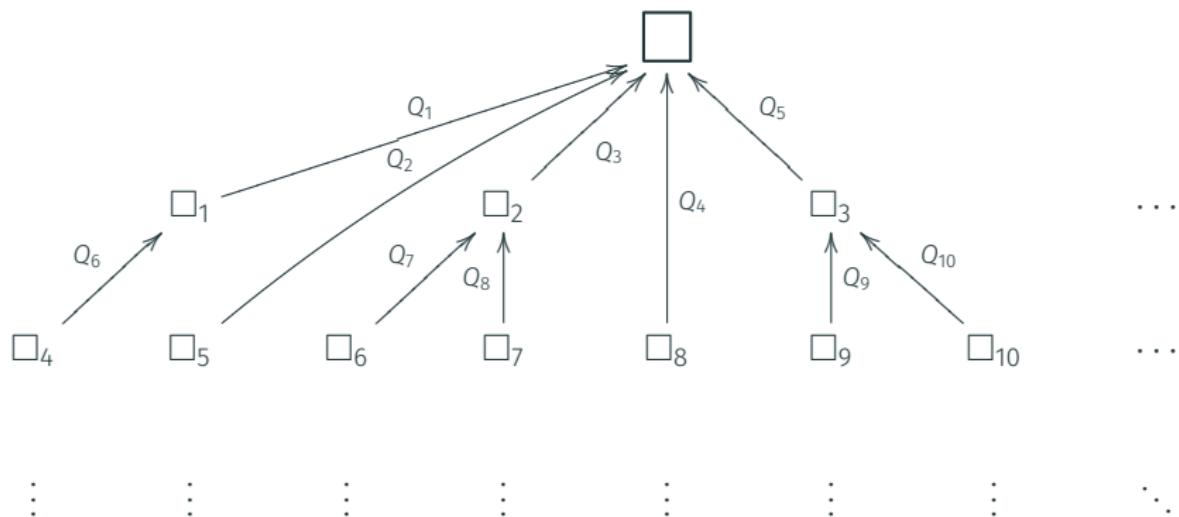
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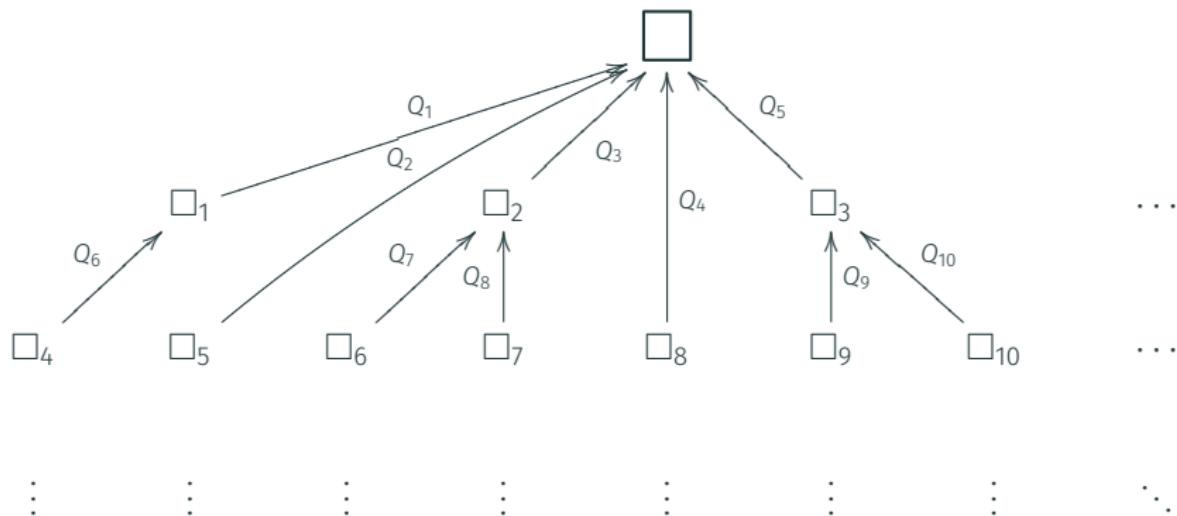
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Lattice/poset of finite-index normal subgroups of G



Question

Which fg residually finite groups H can fill the empty \square 's?

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The basics

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Question (Remeslennikov)

Are finitely generated (non-abelian) free groups profinitely rigid?

Profinite rigidity 201

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Theorem (Ascari-F '25)

Free products of free and surface groups are profinitely rigid amongst hyperbolic graphs of free groups with \mathbb{Z} edges.

Growth: non-normal towers

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Theorem (Ascari-F '25)

There exists a tower of finite-index non-normal subgroups

$$G \geq G_1 \geq G_2 \geq \cdots, \quad \text{with} \quad \bigcap_n G_n = \{1\},$$

such that simultaneously for every prime p ,

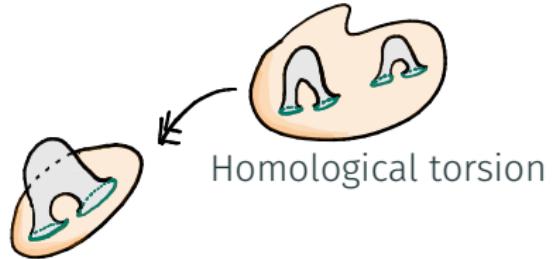
$$\lim_{n \rightarrow \infty} \frac{\log(|\mathrm{Tor}_p(G_n^{\mathrm{ab}}))|)}{[G : G_n]} > 0.$$

Fractal-like towers

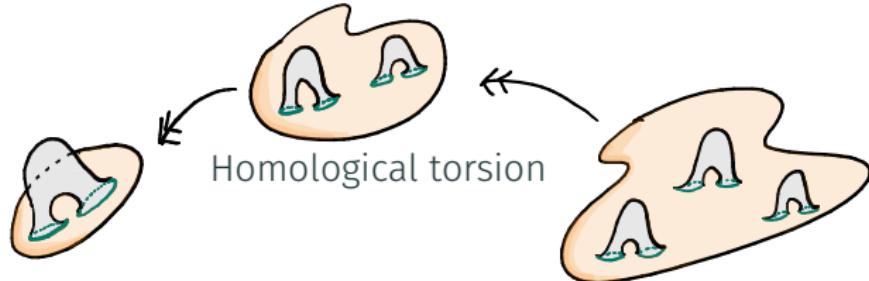
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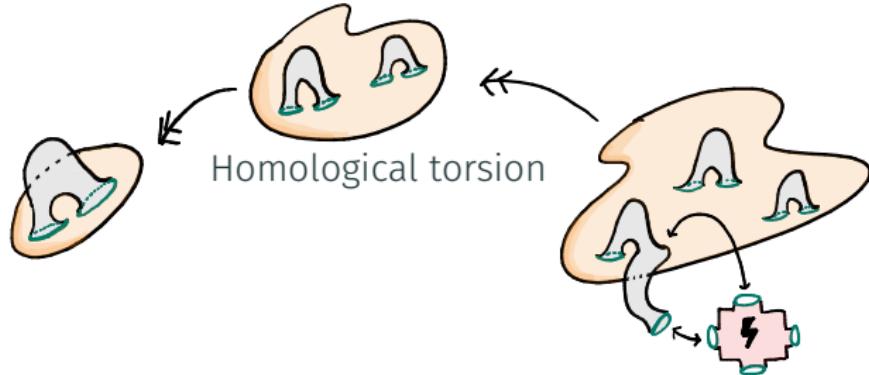
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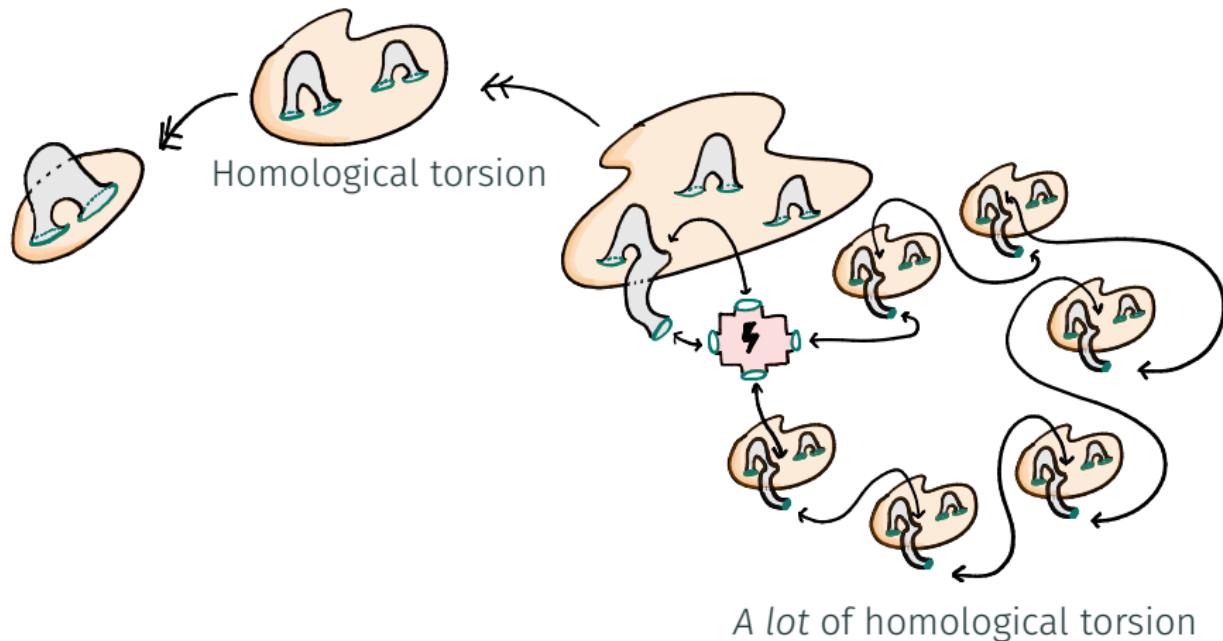
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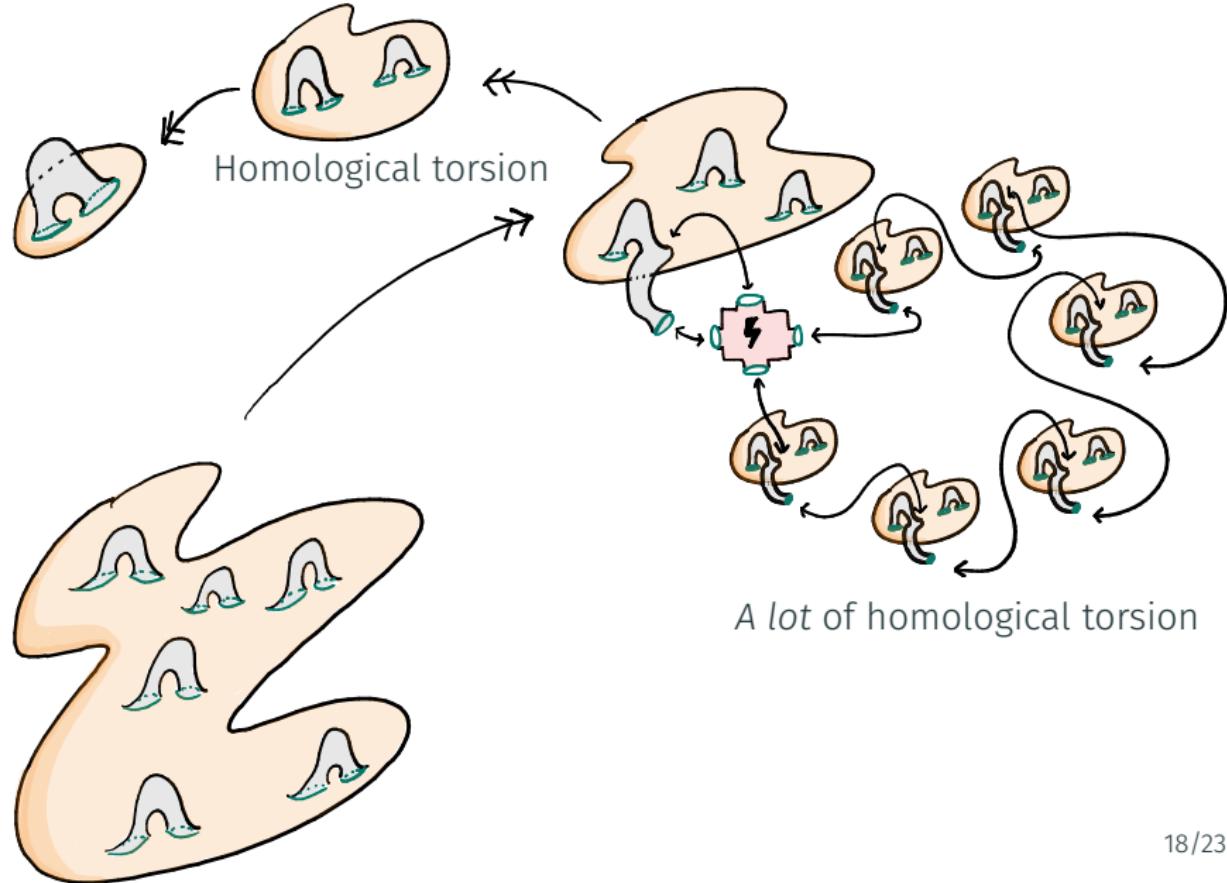
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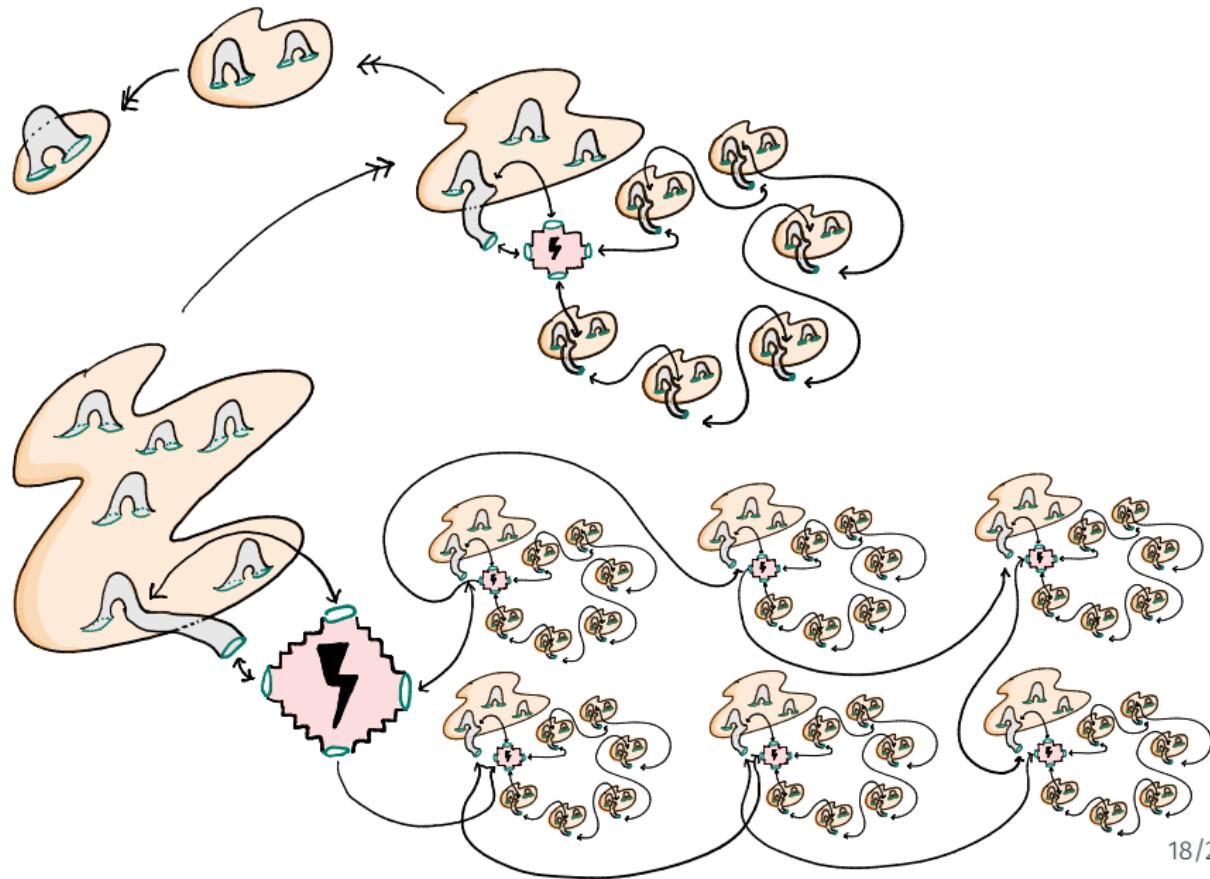
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Growth: normal towers

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Theorem (F '25)

In books of \mathbb{Z} -bundles, homological torsion grows subexponentially along any exhausting tower of finite-index normal subgroups.

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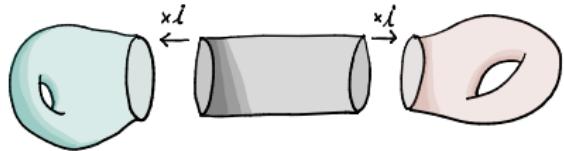
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- Bergeron-Venkatesh: such behaviour expected in dimension ≥ 4 .

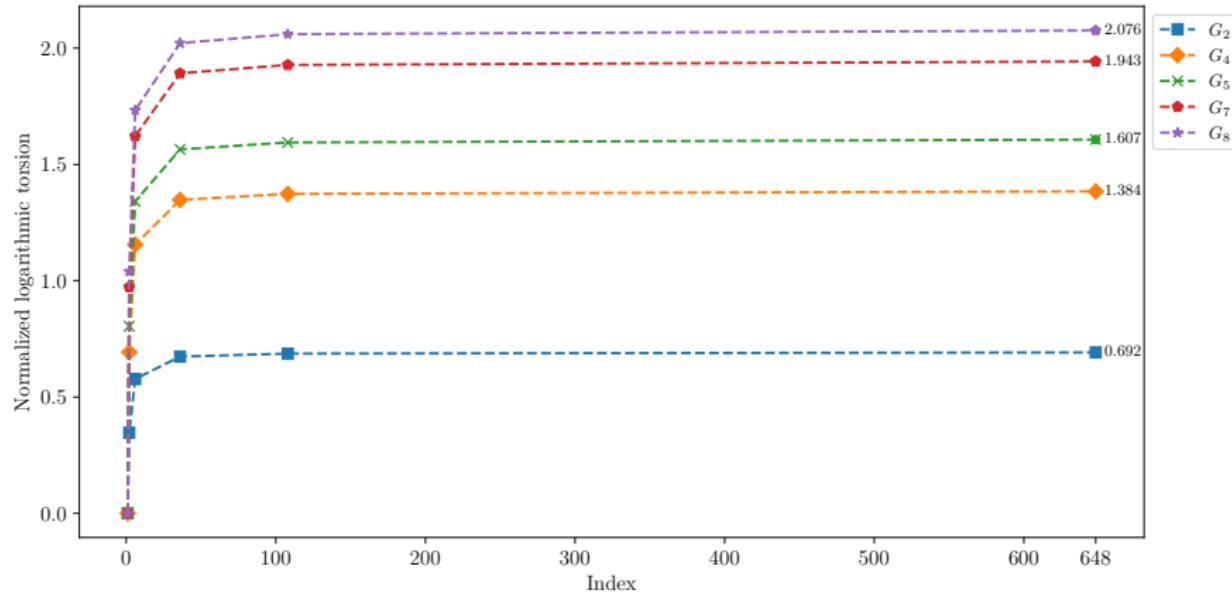
A computation: unwrapping cylinders

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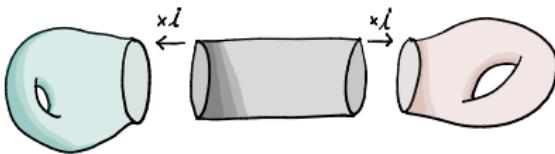
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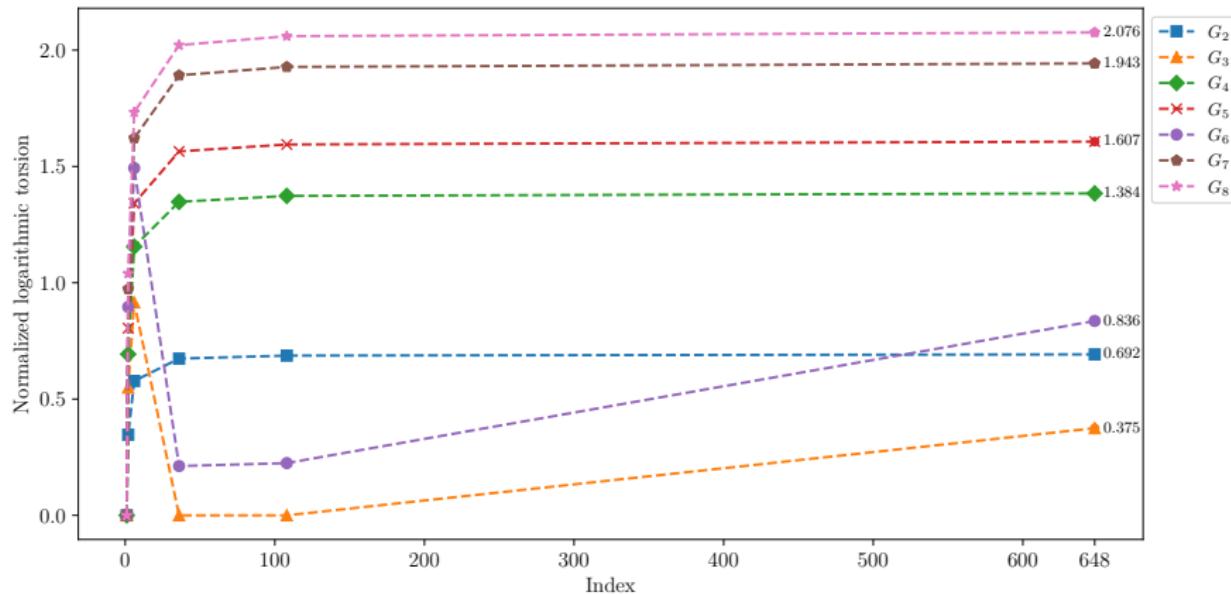
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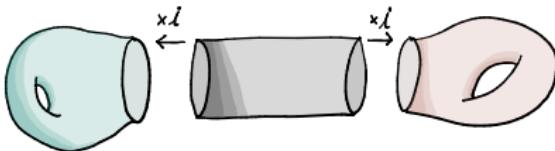
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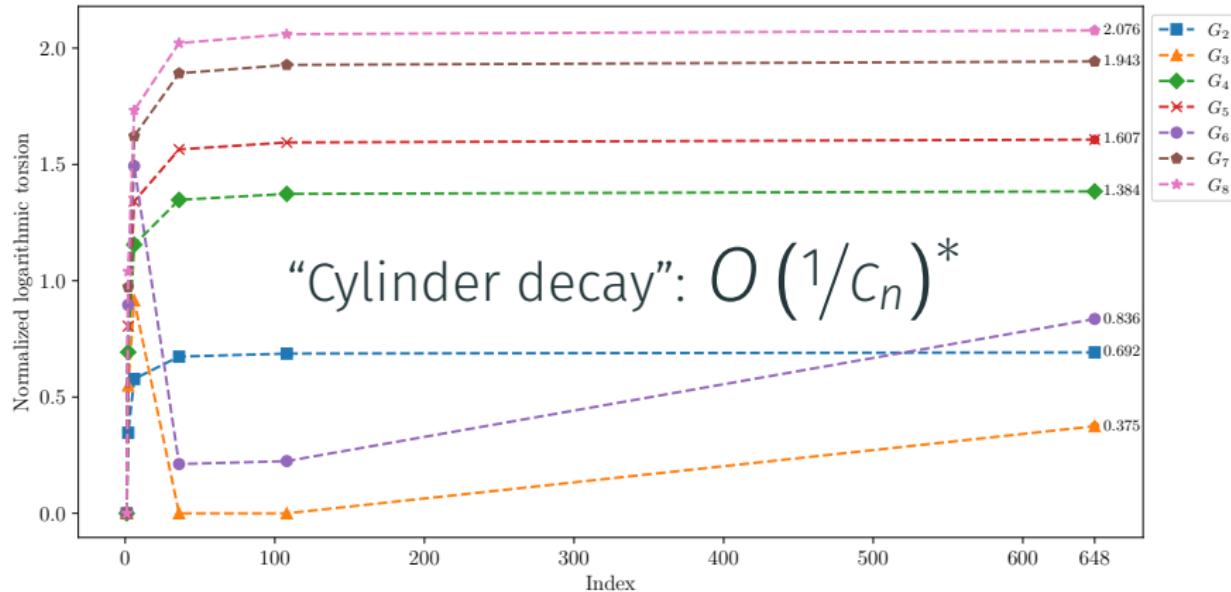
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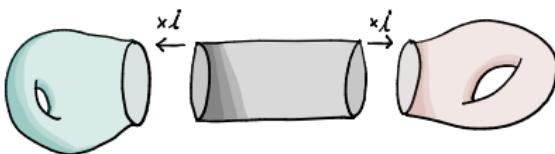


A computation: unwrapping cylinders



* C_n measures the wrapping degree of the cylinder in the n^{th} cover.

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Enter a rigid vertex: revisited

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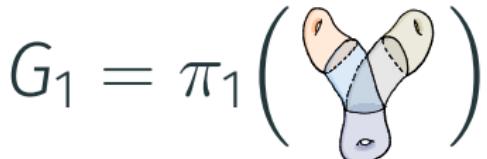
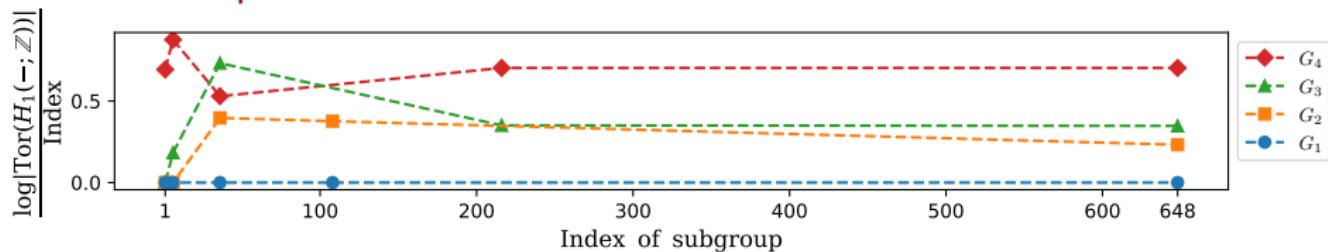
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- **Computations:**



G_{i+1} obtained from G_i by adding a single letter to its presentation

The landscape

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| $\text{Tor}(H_1)$ | (Hyp) 3-manifolds | Surface groups | Books of I -bundles | Rigid vertex |
|-------------------------------|-------------------|----------------|-----------------------|----------------|
| Abundance | (Sun '15) | | (Ascari-F '25) | (Ascari-F '25) |
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| "Cylinder decay" rate | | | $O(1/c_n)$ | $O(\log(c_n)/c_n)$ |

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\rightsquigarrow cross the threshold?



Thank you for listening!

