Analysis

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Contents

Lecture 1: Limits

Books:

- *A First Course in Mathematical Analysis* -Burkill

- *Calculus* -Spivak

- *Analysis I* -Tao

1 Limits and Convergence

1.1 Review from Numbers and Sets

Notation. We denote sequences by a_n or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

Definition 1.1

We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

Note. $N = N(\epsilon)$ which is dependent on ϵ. That is, if you want to go closer to a, sometimes you need to go higher in N.

Definition 1.2: limit of a sequence

We say that a sequence is a

$$\left. \begin{array}{l} \textit{increasing sequence if } a_n \leq a_{n+1}, \\ \textit{decreasing sequence if } a_n \geq a_{n+1}, \end{array} \right\} \textit{monotone sequence}$$

$$\left. \begin{array}{l} \textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ \textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{array} \right\} \textit{strictly monotone sequence}$$

We also have

Theorem 1.1: Fundamental Axiom of the Real Numbers

If $a_n \in \mathbb{R}$ and a_n is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges*.

Remark. It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.

- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

Definition 1.3: supremum

For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S$,

2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

Note. Supremum is unique, and there is a similar notion of infimum.

Lemma 1.1: Properties of Limits

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.

2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).

3. If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.

4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

6. If $a_n \to a$, then $\frac{1}{a_n} \to \frac{1}{a}$.

7. If $a_n < A$ for all n and $a_n \to a$, then $a \leq A$.

Proof.

1. Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|a_n - b| < \epsilon, \forall n \geq N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \geq N$,

   $$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$

   If $a \neq b$, take $\epsilon = \frac{|a-b|}{3}$, we have

   $$|a - b| < \frac{2}{3}|a - b|. \lightning$$

2. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon, \forall n \geq N$, Since $n_j \geq j$, we know

   $$\left| a_{n_j} - a \right| < \epsilon, \forall j \geq N.$$

   That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have

   $$|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab|$$
   $$= |a_n||b_n - b| + |b||a_n - a|.$$

Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|b_n - b| < \epsilon, \forall n \geq N_2$.

If $n \geq N_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$.

We have

$$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

■

Lemma 1.2

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Proof. $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit a.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma 1.1.}$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma 1.1 $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma 1.1, we have $a = \frac{a}{2} \implies a = 0$. ■

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

Definition 1.4

We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma 1.1 are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.

Lecture 2: Bolzano–Weierstrass theorem

> **Theorem 1.2: Bolzano-Weierstrass Theorem**
>
> f $x_n \in R$ and there exists K such that $|x_n| \leq K$ for all n, then we can find $n_1 < n_2 < n_3 < \ldots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. In other words, every bounded sequence has a convergent subsequence.

Remark. We say nothing about the uniqueness of the limit x.

For example, $x_n = (-1)^n$ has two subsequences tending to -1 and 1 respectively.

Proof. Set $[a_1, b_1] = [-K, K]$. Let c be the mid-point of a_1, b_1, consider the following alternatives,

1. $x_n \in [a_1, c]$ for infinitely many n.

2. $x_n \in [c, a_2]$ for infinitely many n.

Note that (1) and (2) can hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold, and we set $a_2 = c$ and $b_2 = b_1$.

We proceed as above to construct sequences a_n, b_n such that $x_m \in [a_n, b_n]$ for infinitely many values of m. They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

a_n is an increasing sequence and bounded, and b_n is a decreasing sequence and bounded. By Fundamental Axiom, $a_n \to a \in [a_1, b_1], b_n \to b \in [a_1, b_1]$. Using Lemma 1.1, $b - a = \frac{b-a}{2} \implies a = b$.

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j such that $x_{n_j} \in [a_j, b_j]$, that is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$. In other words, there is unlimited supply.

Hence, $a_j \leq x_{n_j} \leq b_j$, so $x_{n_j} \to a$. ∎

1.2 Cauchy Sequences

> **Definition 1.5: Cauchy Sequence**
>
> A sequence $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\epsilon > 0$ $\exists N > 0$ such that $|a_n - a_m| < \epsilon$ $\forall n, m > N$.

Note. N is dependent on ϵ.

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

> **Lemma 1.3**
>
> A convergent sequence is a Cauchy sequence.

Proof. If $a_n \to a$, given $\epsilon > 0$, exists N such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Take $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$

■

> **Lemma 1.4**
>
> Every Cauchy sequence is convergent.

Proof. First we note that if a_n is Cauchy, then it is bounded.

Take $\epsilon = 1$, $N = N(1)$ in the Cauchy property, then

$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$

We have

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$

Let $K = \max\{1 + |a_N|, |a_n| \; n = 1, 2 \ldots, N - 1\}$.

Then $|a_n| \leq K$ for all n. By the Bolzano–Weierstrass theorem, $a_{n_j} \to a$. We must have $a_n \to a$.

Given $\epsilon > 0$, there exists j_0 such that for all $j \geq j_0$, $\left| a_{n_j} - a \right| < \epsilon$.

Also, there exists $N(\epsilon)$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N(\epsilon)$.

Take j such that $n_j \geq \max\{N(\epsilon), n_{j_0}\}$. Then if $n \geq N(\epsilon)$,

$$|a_n - a| \leq \left| a_n - a_{n_j} \right| + \left| a_{n_j} - a \right| < 2\epsilon.$$

$\blacksquare$

Thus, on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

2 Series

> **Definition 2.1**
>
> If $a_n \in \mathbb{R}, \mathbb{C}$ We say that $\sum\limits_{j=1}^{\infty} a_j$ converges to s if the sequence of partial sums
>
> $$S_N = \sum_{j=1}^{N} a_j \to S$$
>
> as $N \to \infty$. We write $\sum\limits_{j=1}^{\infty} a_j = s$. If S_N does not converge, we say that $\sum\limits_{j=1}^{\infty} a_j$ *diverges*.

Remark. Any problem on series is really a problem about the sequence of partial sums.

> ### Lemma 2.1
>
> 1. If $\sum\limits_{j=1}^{\infty} a_j$ and $\sum\limits_{j=1}^{\infty} a_j$ converges, then so does $\sum\limits_{j=1}^{\infty} \lambda a_j + \mu b_j$, when $\lambda, \mu \in \mathbb{C}$;
>
> 2. Suppose there exists N such that $a_i = b_i$ for all $i \geq N$. Then either $\sum\limits_{i=1}^{\infty} a_i$ and $\sum\limits_{i=1}^{\infty} b_i$ both converge or they both diverge. (initial terms do not matter for convergence)

Proof.   1. Exercise.

2. If we have $n \geq N$,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^{n} a_i$$
$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^{n} b_i$$

So $S_n - d_n = \sum\limits_{i=1}^{N-1} a_i - b_i$ which is a constant. So S_n converges if and only if d_n does.

■

Lecture 3

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We have the following important example,

Example (Geometric Series). $x \in \mathbb{R}$, set $a_n = x^{n-1}$ with $n \geq 1$. So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \dfrac{1 - x^n}{1 - x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have $S_n(1 - x) = 1 - x^n$.

If $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$.

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$.

If $x < -1$, S_n does not converge (oscillates).

If $x = -1$, $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$.

Thus, the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta, \delta > 0$, so $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \to 0$ because $(1 + \delta)^n \geq 1 + n\delta$ from binomial expansion.

> **Definition 2.2**
>
> $S_n \to \infty$ if given A, there exists an N such that $S_n > A$ for all $n > N$.
>
> $S_n \to -\infty$ if given A, there exists an N such that $S_n < -A$ for all $n > N$.

> **Lemma 2.2**
>
> If $\sum\limits_{i=1}^{\infty} a_n$ converges, then $\lim\limits_{i \to \infty} a_i = 0$.

Proof. Let $S_n = \sum\limits_{i=1}^{\infty} a_i$, note that $a_n = S_n - s_{n-1}$. If $S_n \to a$, we have $a_n \to 0$ because $S_{n-1} \to a$ also. ∎

Remark. The converse of the preceding lemma is false. One example is $\sum \frac{1}{n}$, the

harmonic series. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$.

So if $S_n \to a$, then $S_{2n} \to a$, also we have $a \geq a + \frac{1}{2}$. Contradiction.

2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

Theorem 2.1: The Comparison Test

Suppose $0 \leq b_n \leq a_n$ for all n. Then if $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} b_n$.

Proof. Let $s_N = \sum_{n=1}^{N} a_n$, $d_N = \sum_{n=1}^{N} b_n$. Because $b_n \leq a_n$, we know $d_N \leq s_N$. But $s_N \to s$, then $d_N \leq s_N \leq s$ for all n, and d_N is an increasing sequence bounded above. So d_N converges. ∎

Example. We consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that $\sum_{n=1}^{\infty} a_n$ converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, we get $\sum_{n=1}^{\frac{1}{n^2}} \leq 1 + 1 = 2$.

For the rest of the lecture, we establish two more tests.

> **Theorem 2.2: Root test/ Cauchy's Test for Convergence**
>
> Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum_{n=1}^{\infty} a_n$ converges; if $a > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. Nothing can be said if $a = 1$.

. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists N such that $\forall n \geq N$,

$$a_n^{1/n} < r \implies a_n < r^n.$$

But since $r < 1$, the geometric series converges, and by comparison test, the series $\sum a_n$ converges as well.

To prove the second part of the theorem, if $a > 1$, for $n \geq N$,

$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus, $\sum_{n=1}^{\infty} a_n$ diverges, since a_n does not tend to zero. ∎

> **Theorem 2.3: Ratio Test/ D'Alembert's Test**
>
> Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum_{n=1}^{\infty} a_n$ converges. If $\ell > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. As before, nothing can be said for $\ell = 1$.

Proof. Supposed $\ell < 1$ and choose r with $\ell < r < 1$. Then $\exists N$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r.$$

Therefore,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \ n > N.$$

So, $a_n < kr^n$ with k independent of n. Since $\sum\limits_{n=1}^{\infty} r^n$ converges, so does $\sum\limits_{n=1}^{\infty} a_n$ by Comparison Test.

If $\ell > 1$, choose $1 < r < \ell$. Then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \ n > N.$$

So the series diverges. ∎

Lecture 4

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Example. To determine the convergence of $\sum\limits_{n=1}^{\infty} a_n = \frac{n}{2^n}$.

By ratio test,
$$\frac{n+1}{2^n} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$
So we have convergence by ratio test.

However, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, and ratio test gives limit 1, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, and ratio test gives limit 1. So ratio test is inconclusive if the limit is 1.

Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, so root test is also inconclusive when the limit is 1.

To see this limit, write
$$n^{\frac{1}{n}} = 1 + \delta_n, \ \delta_n > 0.$$
So
$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2} \delta_n^2.$$
And $\delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$.

Remark. Use the root test when there is a nth power in the series.

> **Theorem 2.4: Cauchy's Condensation Test**
>
> Let a_n be a decreasing sequence of positive terms. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Proof. First we observe that if a_n is decreasing

$$a_{2^k} \leq a_{2^{k-1}+i} \leq a_{2^{k-1}}$$

for all $k \geq 1$ and $1 \leq i \leq 2^{k-1}$.

Assume that $\sum_{n=1}^{\infty} a_n$ converges with sum A. Then

$$2^{n-1}a_{2^n} = \underbrace{a_{2^n} + \cdots a_{2^n}}_{2^{n-1} \text{ times}}$$

$$\leq a_{2^{n-1}+1} + \cdots + a_{2^n}$$

$$= \sum_{m=2^{n-1}+1}^{2^n} a_m.$$

Thus, $\sum_{n=1}^{N} 2^{n-1}a_{2^n} \leq \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m = \sum_{m=2}^{2^N} a_m.$ So

$$\sum_{n=1}^{N} 2^n a_{2^n} \leq 2 \sum_{m=2}^{2^N} a_m \leq 2(A - a_1).$$

Thus, $\sum_{n=1}^{N} 2^n a_{2^n}$ being increasing and bounded above, converges.

Conversely, assume $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges to B, then

$$\sum_{m=2^{n-1}+1}^{2^n} a_m = a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n}$$

$$\leq \underbrace{a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} \text{times}} = 2^{n-1}a_{2^{n-1}}.$$

Similarly, we have

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^{N} 2^{n-1}a_{2^{n-1}} \leq B.$$

Therefore, $\sum_{m=1}^{N} a_m$ is a bounded increasing sequence and thus it converges. ∎

Example. $\sum_{n=1}^{\infty} \frac{1}{n^k}$ for $k > 0$ converges if and only if $k > 1$. First we note that $\frac{1}{n^k}$ is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)k} < \frac{1}{n^k} \iff (\frac{n}{n+1})^k < 1 \iff \frac{n}{n+1} < 1.$$

We use Cauchy condensation test, and we have

$$2^n a_{2^n} = 2^n \left(\frac{1}{2^n}\right)^k$$
$$= 2^{n-nk} = (2^{1-k})^n.$$

Which is a geometric series with the ratio 2^{1-k}. So $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ converges if and only if $2^{1-k} < 1 \iff k > 1$.

2.2 Alternating Series

> **Theorem 2.5: Alternating Series Test**
>
> If a_n decreases and tends to 0 as $n \to \infty$, then the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Proof. The partial sum is

$$S_n = a_1 - a_2 + \cdots + (-1)^{n+1} a_n$$
$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq S_{2n-1}$$
$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

So S_{2n} is increasing and bounded above, implying that $S_{2n} \to S$. The odd terms satisfy

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S + 0 = S.$$

This implies that S_n converges to S as well. Given ϵ, there exists N_1 such that $\forall n \geq N_1$, $|S_{2n} - S| < \epsilon$. We also know that there exists N_2 such that $\forall n \geq N_2$, $|S_{2n+1} - S| < \epsilon$. Take $N = 2 \max\{N_1, N_2\} + 1$, then if $n \geq N$, $|S_k - S| < \epsilon$. So $S_k \to S$. ∎

Lecture 5

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2.3 Absolute Convergence

> **Definition 2.3**
>
> Take $a_n \in \mathbb{C}$. If $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, then the series is called *absolutely convergent*.

Note. Since $|a_n| \geq 0$. We can use the previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

> **Theorem 2.6**
>
> If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose first $a_n \in \mathbb{R}$. Let

$$v_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$w_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

We have $v_n = \frac{|a_n| + a_n}{2}, w_n = \frac{|a_n| - a_n}{2}$. Clearly, $v_n, w_n \geq 0$. We also have $|a_n| = v_n + w_n \geq v_n, w_n$.

So by comparison test, if $\sum\limits_{n=1}^{\infty} |a_n|$ converges, $\sum\limits_{-=1}^{\infty} v_n, \sum\limits_{n=1}^{\infty} w_n$ also converges.

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$. We have $|x_n|, |y_n| \leq |a_n|$. So $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are absolutely convergent, so they are convergent. And $\sum\limits_{n=1}^{\infty} a_n$ converges as well. ∎

Example.

1. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely convergent.

2. $\sum\limits_{n=1}^{\infty} \frac{z^n}{2^n}$ for $z \in \mathbb{C}$. We check for absolute convergence first, $\sum\limits_{n=1}^{\infty} \left(\frac{|z|}{2}\right)^n$. So if $|z| < 2$, the series is convergent by absolute convergence.

   Otherwise, if $|z| \geq 2$, $\left|\frac{z}{2}\right| \geq 1$. a_n does not tend to zero, hence the series diverge.

Notation. If $\sum\limits_{n=1}^{\infty} a_n$ converges but not absolutely convergent, it is sometimes called *conditional convergent*.

It is called conditional because the sum to which the series converges is conditional on the order in which elements of the sequence are taken.

Example (Example Sheet 1, Q7). $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ are two series with different sums. Let s_n be the partial sum of the first series, and t_n be the partial sum of the second series, then $s_n \to s$ and $t_n \to \frac{3s}{2}$.

Definition 2.4

Let σ be a bijection of the positive integers, $a'_n = a_{\sigma(n)}$ is a *rearrangement*.

Theorem 2.7

If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

Proof. Again we do the proof first for $a_n \in \mathbb{R}$. Let $\sum\limits_{n=1}^{\infty} a'_n$ be a rearrangement of $\sum\limits_{n=1}^{\infty} a_n$.

Let $s_n = \sum\limits_{i=1}^{n} a_i$ and $t_n = \sum\limits_{i=1}^{n} a'_i$, $S = \sum\limits_{n=1}^{\infty} a_n$. Suppose first that $a_n \geq 0$. Given n, we can find q such that s_q contains every term of t_n. Because $a_n \geq 0$, we have

$$t_n \leq s_n \leq S.$$

So t_n is an increasing sequence bounded above so $t_n \to t$, and from the inequality above, $t \leq s$. By symmetry, we have $s \leq t \implies s = t$. If a_n has any negative term, consider v_n and w_n from Theorem 2.6. Consider $\sum\limits_{n=1}^{\infty} a'_n$, $\sum\limits_{n=1}^{\infty} v'_n$, $\sum\limits_{n=1}^{\infty} w'_n$. Since $\sum\limits_{n=1}^{\infty} |a_n|$ converges,

both $\sum\limits_{n=1}^{\infty} v_n$ and $\sum\limits_{n=1}^{\infty} w_n$ converge. Using the fact that $v_n, w_n \geq 0$, we case above, we have

$\sum\limits_{n=1}^{\infty} v'_n = \sum\limits_{n=1}^{\infty} v_n$ and $\sum\limits_{n=1}^{\infty} w_n = \sum\limits_{n=1}^{\infty} w'_n$. But $a_n = v_n - w_n$ so $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a'_n$.

For the case $a_n \in \mathbb{C}$, we write $a_n = x_n + iy_n$. Since $|x_i|, |y_i| \leq |a_n|$, $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are absolutely convergent. By the previous case $\sum\limits_{n=1}^{\infty} x'_n = \sum\limits_{n=1}^{\infty} x_n, \sum\limits_{n=1}^{\infty} y'_n = \sum\limits_{n=1}^{\infty} y_n$. Since $a'_n = x'_n + iy'_n$ so $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a'_n$. ∎

Lecture 6

2 Feb. 2022

3 Functions

3.1 Continuity

Suppose $E \subseteq \mathbb{C}$ is a non-empty subset, and we have a function $f : E \to \mathbb{C}$ and a point $a \in E$. (this includes the case in which f is real-valued and E is a subset of $\mathbb{R}$)

> **Definition 3.1**
>
> f is *continuous* at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$.

> **Definition 3.2: ϵ-δ Definition**
>
> f is *continuous* at $a \in E$, if given $\epsilon > 0$, $\exists \delta > 0$ such that if $|z - a| < \delta, z \in E$, then $|f(z) - f(a)| < \epsilon$.

We prove right away that the two definitions are equivalent.

> **Theorem 3.1**
>
> The two definitions of continuity are equivalent.

Proof. We first prove the second definition implies the first definition. We know that

given $\epsilon > 0, \exists \delta > 0$ such that $|z - a| < \delta, z \in E$, then $|f(z) - f(z)| < \epsilon$. Let $z_n \to a$, then $\exists n_0$ such that $\forall n \geq n_0$, we have $|z_n - a| < \delta$. This implies, by the assumption, $|f(z_n) - f(a)| < \epsilon$. That is, $f(z_n) \to f(a)$.

Next, we prove the other direction. Assume $f(z_n) \to f(a)$ whenever $z_n \to a, z_n \in E$. Suppose f is not continuous at a according to Definition 2.

$\exists \epsilon > 0$, s.t. $\forall \delta > 0$, there exists $z \in E$ s.t. $|z - a| < \delta$ and $|f(z) - f(a)| \geq \epsilon$.

Let $\delta = \frac{1}{n}$ from non-continuity defined above, we get z_n such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \epsilon$. Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \epsilon$. Contradiction. ∎

> **Proposition 3.1**
>
> $a \in E$, and $g, f : E \to \mathbb{C}$ are both continuous at a. So are the functions $f(z) + g(z)$, $f(z)g(z)$ and $\lambda f(z)$ for any constant λ. In addition, if $f(z) \neq 0 \ \forall z \in E$, then $\frac{1}{f(z)}$ is continuous at a.

Proof. Using Definition 1 of continuity, this is obvious, using the analogous results for sequences. (Lemma 1.1)

For example,

$$z_n \to a \implies f(z_n) \to f(a), g(z_n) \to g(a) \implies f(z_n) + g(z_n) \to f(a) + g(a).$$

∎

The function $f(z) = z$ is continuous, so by using the proposition, we get that every polynomial is continuous at every point in $\mathbb{C}$.

Note. We say that f is *continuous on E* if it is continuous at every $a \in E$.

Remark. Still it is instructive to prove Proposition 3.1 directly from the ϵ-δ definition.

Next we look at compositions.

> **Theorem 3.2**
>
> Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions such that $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$, then $g \circ f : A \to \mathbb{C}$ is continuous at a.

Proof. Take any sequence $z_n \to a$, by assumption we know $f(z_n) \to f(a)$. Set $w_n = f(z_n) \in B$. By continuity of g, we have $g(w_n) \to g(f(a))$, and we are done. ∎

Example.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$

   assuming that $\sin x$ is continuous. (to be proved later) If $x \neq 0$, propositions proved above imply that $f(x)$ is continuous at any $x \neq 0$.

   However, it is discontinuous at 0. Consider the sequence satisfying

$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi.$$

   We have $f(x_n) \to 1, x_n \to 0$, but $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

   It's continuous at $x \neq 0$ as above, and f is continuous at 0. Take $x_n \to 0$, then $|f(x_n)| \leq |x_n|$ because $\sin\frac{1}{x} \leq 1$, so $f(x_n) \to 0 = f(0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

   It is discontinuous at every point. If $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then $f(x_n) = 0 \nrightarrow f(x) = 1$. Similarly, if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$, we have $f(x_n) = 1 \nrightarrow f(x) = 0$.

Lecture 7

4 Jan. 2022

3.2 Limit of a function

$f : E \subseteq \mathbb{C} \to \mathbb{C}$. We wish to define what is meant by $\lim\limits_{z \to a} f(z)$, even when a might not be in E.

Example. The limit of $\frac{\sin z}{z}$ as $z \to 0$ with $E = \mathbb{C}\{0\}$.

Also, if $E = \{0\} \cup [1, 2]$, it does not make sense to speak about points $z \in E, z \neq 0, z \to 0$.

> **Definition 3.3**
>
> If $E \subseteq \mathbb{C}, a \in \mathbb{C}$, we say that a is a *limit point* of E if for any $\delta > 0, \exists z \in E$ such that $0 < |z - a| < \delta$.

Remark. a is a limit point if and only if there exists a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all n.

> **Definition 3.4**
>
> If $f : E \subseteq \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$ be a limit point of E. We say that $\lim\limits_{z \to a} f(z) = l$ ("f tends to l as z tends to a") if given $\epsilon > 0, \exists \delta > 0$ such that whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \epsilon$.
>
> Equivalently, $f(z_n) \to l$ for every sequence $z_n \in E, z_n \neq a$ and $z_n \to a$.

Remark. Straight from the definitions, we have that if $a \in E$ is limit point, then $\lim\limits_{z \to a} f(z) = f(a)$ if and only if f is continuous at a.

If $a \in E$ is *isolated* (i.e. $a \in E$ is not a limit point), continuity of f at a always holds. The limit of functions has very similar properties to limit of sequences.

1. It is unique, $f(z) \to A$ and $f(z) \to B$ as $z \to a$, then
$$|A - B| \leq |A - f(z)| + |f(z) - B|.$$
   If $z \in E$ is such that $0 < |z - a| < \min\{\delta_1, \delta_2\}$, then $|A - B| < 2\epsilon$. So $A = B$. The existence of such z is a consequence of the condition that a is a limit point of E.

2. $f(z) + g(z) \to A + B$;

3. $f(z)g(z) \to AB$;

4. if $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$. All proved in the same way as before.

3.3 The Intermediate Value Theorem

> **Theorem 3.3: Intermediate Value Theorem**
>
> If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then f takes every value which lies between $f(a)$ and $f(b)$.

Proof. Without loss of generality, suppose $f(a) < f(b)$. Take $f(a) < \eta < f(b)$. Let $S = \{x \in [a,b] \mid f(x) < \eta\}$. We note that $a \in S$, so $S \neq \varnothing$. Clearly S is bounded above by b. Then there is a supremum C where $C \leq b$. By definition of supremum, given n, there exists $x_n \in S$ such that $C - \frac{1}{n} < x_n \leq C$. So $x_n \to C$. Since $x_n \in A$, $f(x_n) < \eta$. By continuity of f, $f(x_n) \to f(C)$. So $f(c) \leq \eta$.

Now observe that $c \neq b$ because $f(b) > \eta$. Then for n large, $C + \frac{1}{n} \in [a,b]$ and $C + \frac{1}{n} \to C$. Again by continuity $f(C + \frac{1}{n}) \to f(C)$. But since $C + \frac{1}{2} > C$, $f(C + \frac{1}{n}) \geq \epsilon$. So $f(c) \geq \eta \implies f(c) = \eta$. ∎

Remark. The theorem is very useful for finding zeroes or fixed points.

Example. Existence of the N-th root of a positive real number. Suppose

$$f(x) = x^N, \quad x \geq 0.$$

Let y be a positive real number. f is continuous on $[0, 1+y]$, so

$$0 = f(0) < y < (1+y)^N = f(1+y).$$

By the IVT, $C \in (0, 1+y)$ such that $f(c) = y$, i.e. $C^N = y$. C is a positive N-th root of y.

We also have uniqueness. Exercise.

Lecture 8

3.4 Bounds of a Continuous Function

> **Theorem 3.4**
>
> Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists K such that $|f(x)| \le K$ for all $x \in [a,b]$.

Proof. We argue by contradiction. Suppose the statement is false. Then given any integer $n \ge 1$, there exists $x_n \in [a,b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j} \to x$. Since $a \le x_{n_j} \le b$, we must have $x \in [a,b]$. By the continuity of f, $f(x_{n_j}) \to f(x)$. But $\left|f(x_{n_j})\right| > n_j \to \infty$ as $j \to \infty$. Contradiction. ∎

> **Theorem 3.5: Extreme Value Theorem**
>
> Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in [a,b]$ such that
> $$f(x_1) \le f(x) \le f(x_2)$$
> for all $x \in [a,b]$.
>
> "A continuous function on a closed bounded interval is bounded and attains its bounds."

Proof. Let $A = \{f(x) \mid x \in [a,b]\} = f([a,b])$. By Theorem 3.4, A is bounded since it is clearly non-empty, it has a supremum M. By definition of supremum, given an integer $n \ge 1$, there exists $x_n \in [a,b]$ such that $M - \frac{1}{n} < f(x_n) \le M$. From Bolzano-Weierstrass, there exists $x_{n_j} \to x \in [a,b]$. Since $f(x_{n_j}) \to M$, by continuity of f, we get that $f(x) = M$. So $x_2 := x$.

We can prove similarly for the minimum. ∎

Proof 2. $A = f([a,b]), M = \sup A$ as before. Suppose $\nexists x_2$ such that $f(x_2) = M$. Let

$$g(x) = \frac{1}{M - f(x)}, x \in [a,b]$$

is defined and continuous on $[a,b]$. By Theorem 3.4 applied to g, $\exists k > 0$ such that $g(x) < K$ for all $x \in [a,b]$. This means that $f(x) \leq M - \frac{1}{k}$ for all $x \in [a,b]$. This is absurd because it contradicts that M is the supremum. ∎

Note. Theorems 3.4 and 3.5 are false if the interval is not closed and bounded. For example,

$$f : (0,1] \to \mathbb{R}, x \mapsto \frac{1}{x}.$$

3.5 Inverse Functions

Definition 3.5

f is *increasing* for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 such that $a \leq x_1 < x_2 \leq b$.

If $f(x_1) < f(x_2)$, we say that f is *strictly increasing*.

There are similar definitions for *decreasing* and *strictly decreasing*.

Theorem 3.6

$f : [a,b] \to \mathbb{R}$ is continuous and strictly increasing for $x \in [a,b]$. Let $c = f(a)$ and $d = f(b)$. Then $f : [a,b] \to [c,d]$ is bijective and the inverse $g := f^{-1} : [c,d] \to [a,b]$ is also continuous and strictly increasing.

Remark. There is a similar statement for strictly decreasing function. Take $c < k < d$, from the IVT, $\exists h$ such that $f(h) = k$. Since f is strictly increasing, h is unique. Define $g(k) := h$ and this gives an inverse $g : [c,d] \to [a,b]$ for f.

We first prove that g is strictly increasing. Take $y_1 < y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. If $x_2 \leq x_1$, since f is increasing, $f(x_2) \leq f(x_1) \implies y_2 \leq y_1$. Absurd.

Next we prove continuity. Let $\epsilon > 0$ be given, let $k_1 = f(h - \epsilon)$ and $k_2 = f(h + \epsilon)$. Because f is strictly increasing, we have $k_1 < k < k_2$. If $k_1 < y < k_2$, we have $h - \epsilon < g(y) < h + \epsilon$. So we can just take $\delta = \min\{k_2 - k, k - k_1\}$. So g is continuous at k. Here we took $k \in (c,d)$. A very similar argument establishes continuity at the end points.

Lecture 9

4 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E = \text{interval} \subseteq \mathbb{R}$.

> **Definition 4.1**
>
> Let $x \in E$ be a point such that $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point), f is said to be *differentiable* at x with derivative $f'(x)$ if
>
> $$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x).$$
>
> If f is differentiable at each $x \in E$, we say f is differentiable on E.
>
> (Think of E as an interval or a disc in the case of $\mathbb{C}$.)

Remark.

1. Other common notations include $\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}f}{\mathrm{d}x}$.

2. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$. $(y = x + h)$

3. Another look at the definition is the following.

   Let $\epsilon(h) := f(x+h) - f(x) - hf'(x)$, then $\lim\limits_{h \to 0} \frac{\epsilon(h)}{h} = 0$. We have also

   $$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear in } h} + \epsilon(h).$$

   Alternative definition of differentiability is f is differentiable at x if $\exists A, E$ such that $f(x+h) = f(x) + hA + \epsilon(h)$ where $\lim\limits_{h \to 0} \frac{\epsilon}{h} = 0$. If such an A exists, then it is unique, since $A = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

4. If f is differentiable at x, then f is continuous. Since $\epsilon(h) \to 0$, then $f(x+h) \to f(x)$ as $h \to 0$.

5. Alternative ways of writing things:

$$f(x + h) = f(x) + hf'(x) + h\epsilon_f(h) \text{ with } \epsilon_f(h) \to 0 \text{ as } h \to 0.$$

Or,

$$f(x) = f(a) + (x - a)f'(a) + (x - a)\epsilon_f(x) \text{ with } \epsilon_f(x) \to 0 \text{ as } x \to a.$$

Example. If we have $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Clearly, we have $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Take $h_n \downarrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{h_n}{h_n} = 1.$$

And take $h_n \uparrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{-h_n}{h_n} = -1.$$

So f is not differentiable at $x = 0$.

4.1 Differentiation of Sums, Products, etc

Property.

1. If $f(x) = c$ for all $x \in E$, then f is differentiable with $f'(x) = 0$.

2. f, g are differentiable at x, then so is $f + g$ and

$$(f + g)'(x) = f'(x) + g'(x).$$

3. f, g are differentiable at x, then so is fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

4. f differentiable at x and $f(x) \neq 0$ for all $x \in E$, then $\frac{1}{f}$ is differentiable at x and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{[f(x)]^2}.$$

Proof.

1. $\lim\limits_{h \to 0} \frac{c - c}{h} = 0$.

2. $\lim\limits_{h \to 0} \dfrac{f(x + h) + g(x + h) - f(x) - g(x)}{h}$ using properties of limits.

$= \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} + \lim\limits_{h \to 0} \dfrac{g(x + h) - g(x)}{h}$

$= f'(x) + g'(x)$

3. Let $\phi(x) = f(x)g(x)$, then we have

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$

$$= f(x + h)\left[\frac{g(x + h) - g(x)}{h}\right] + g(x)\left[\frac{f(x + h) - f(x)}{h}\right].$$

So we have $\lim\limits_{h \to 0} \frac{\phi(x + h) - \phi(x)}{h} = f(x)g'(x) + f'(x)g(x)$ using standard properties of limits and the fact that f is continuous at x.

4. Define again $\phi(x) = \frac{1}{f(x)}$, then

$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$

$$= \frac{f(x) - f(x + h)}{h f(x) f(x + h)}.$$

So we have $\lim\limits_{h \to 0} \frac{\phi(x + h) - \phi(x)}{h} = \frac{-f(x)}{[f(x)]^2}$.

∎

Remark. From (3) and (4), we get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Lecture 10

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Example. Consider $f(x) = x^n$ with $n \in \mathbb{Z}, n > 0$. When $n = 1$, clearly we have $f(x) = x$ and $f'(x) = 1$.

We claim that $f'(x) = nx^{n-1}$, and we prove it by induction, $f(x) = xx^n = x^{n+1}$. By product rule and inductive hypothesis,

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Next, we consider $f(x) = x^{-n}$ with $n \in \mathbb{Z}, n > 0$. If $x \neq 0$, use property 4.1, we have

$$f'(x) = \frac{-(x^n)'}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

So we know how to find derivatives of polynomials and rational functions.

We have the following useful result to differentiate a larger class of functions.

> **Theorem 4.1: Chain Rule**
>
> If $f : U \to \mathbb{C}$ is such that $f(x) \in V$ for $x \in U$. If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with
>
> $$(g \circ f)'(a) = f'(a)g'(f(a)).$$

Proof. We know
$$f(x) = f(a) + (x - a)f'(a) + \epsilon_f(x)(x - a)$$
such that $\lim_{x \to a} \epsilon_f(x) = 0$, and

$$g(y) = g(b) + (y - b)g'(b) + \epsilon_g(y)(y - b)$$

with $\lim_{y \to b} \epsilon_g(y) = 0$. Let $b = f(a)$, and set $\epsilon_f(a) = 0$ and $\epsilon_g(b) = 0$ to make them continuous at $x = a$ and $f = b$. Now $y = f(x)$ gives

$$\begin{aligned}
g(f(x)) =& g(b) + (f(x) - b)g'(b) + \epsilon_g(f(x))(f(x) - b) \\
=& g(f(a)) + [(x-a)f'(a) + \epsilon_f(x)(x-a)][g'(b) + \epsilon_g(f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + \\
& (x-a)[\epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + (x-a)\sigma(x).
\end{aligned}$$

So it suffices to show $\sigma(x) = \epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))$ tends to 0 as x tends to a. We have clearly $\epsilon_f(x)g'(b) \to 0$, $\epsilon_g(f(x)) \to 0$ and $f'(a) + \epsilon_f(x) \to f'(a)$, so $\lim_{x \to a} \sigma(x) = 0$. ∎

Example.

1. Consider $f(x) = \sin(x^2)$, and we have

$$f'(x) = 2x\cos(x^2).$$

2. Consider $f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. From previous lectures, we know that f is continuous, and it is differentiable at every $x \neq 0$ by the previous theorems. At $x = 0$, take $t \neq 0$ and we have

$$\frac{f(t) - f(0)}{t - 0} = \sin(\frac{1}{t}).$$

Again from previous lecture, we know $\lim_{t\to 0} \frac{f(t)-f(0)}{t-0}$ does not exist, so f is not differentiable at $x = 0$.

4.2 The Mean Value Theorem

> **Theorem 4.2: Rolle's Theorem**
>
> Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let $M = \max_{x\in[a,b]} f(x)$, and $m = \min_{x\in[a,b]} f(x)$. Theorem 3.5 says that these values are achieved. Let $k = f(a) = f(b)$. If $M = m = k$, then f is constant and $f'(c) = 0$ for all $c \in (a, b)$.

If f not constant, then $M > k$ or $m < k$. Suppose $M > k$. By Theorem 3.5, exist $c \in (a, b)$ such that $f(c) = M$.

If $f'(c) > 0$, then there are values to right of c for which $f(x) > f(c)$ because

$$f(h + c) - f(c) = h(f'(c) + \epsilon_f(h)).$$

Since $\epsilon_f(h) \to 0$ as $h \to 0$, $f'(c) + \epsilon_f(h) > 0$ for h small. This contradicts that M is the maximum. Similarly, if $f'(c) < 0$, there exists x to the left of c for which $f(x) > f(c)$.

So we must have $f'(c) = 0$. ∎

Lecture 11

Theorem 4.3: Mean Value Theorem

Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b), then $\exists c \in (a,b)$ such that
$$f(b) - f(a) = f'(c)(b-a).$$

Proof. Write $\phi(x) = f(x) - kx$, and choose k such that $\phi(a) = \phi(b)$. So

$$f(b) - bk = f(a) - ak \implies k = \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem applied to ϕ, $\exists c \in (a,b)$ such that $\phi'(c) = 0$. That is, $f'(c) = k$. ∎

Remark. We will often write

$$f(a + h) = f(a) + h f'(a + \theta h)$$

with $\theta \in (0,1)$. We need to be careful, and consider $\theta = \theta(h)$.

Corollary 4.1

$f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b).

1. If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

2. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)

3. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof.

1. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) > 0 \implies f(y) > f(x)$.

2. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) \geq 0 \implies f(y) \geq f(x)$.

3. Take $x \in [a,b]$. Then use the MVT in $[a,x]$ to get $c \in (a,x)$ such that $f(x) - f(a) = f'(c)(x - a) = 0$. So $f(x) = f(a)$ and f is constant.

■

> ### Theorem 4.4: Inverse Function Theorem
>
> If $f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$. Let $f(a) = c$ and $f(b) = d$, then the function $f : [a, b] \to [c, d]$ is bijective and f^{-1} is differentiable with
>
> $$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Proof. By Corollary 4.1, f is strictly increasing in $[a, b]$. By Theorem 3.6, $\exists g : [c, d] \to [a, b]$ which is a continuous strictly increasing inverse of f. We want to show that g is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$ and $x \in (a, b)$.

If $k \neq 0$ is given, let h be given by $y + k = f(x + h)$. That is, $g(y + k) = x + h$ for $h \neq 0$. Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)} = \frac{h}{f(x + h) - f(x)}.$$

Let $k \to 0$, then $h \to 0$ because g is continuous. So we have

$$g'(y) = \lim_{k \to 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}.$$

■

Example. We take $g(x) = x^{\frac{1}{q}}$ with $x > 0$ and q positive integer. So $f(x) = x^q$, with $f'(x) = qx^{q-1}$. g is differentiable and so is g, and by Theorem 4.4,

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q}x^{\frac{1}{q}-1}.$$

Remark. If $g(x) = x^r$ with $r \in \mathbb{Q}$, then $g'(x) = rx^{r-1}$.

Suppose $f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b - a)f'(s)}{(b - a)g'(t)} = \frac{f'(s)}{g'(t)}.$$

Cauchy showed that we can take $s = t$.

> ### Theorem 4.5: Cauchy's Mean Value Theorem
>
> Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then $\exists t \in (a, b)$ such that
> $$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a)).$$

Remark. We recover the MVT if we take $g(x) = x$.

Lecture 12

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Proof. Let
$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}.$$
We have ϕ continuous on $[a, b]$ and differentiable on (a, b). Also, $\phi(a) = \phi(b) = 0$. By Rolle's Theorem, there exists $t \in (a, b)$ such that $\phi'(t) = 0$, and

$$\phi'(x) = f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x)$$
$$= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)].$$

So $\phi'(t) = 0$ gives the result. $\blacksquare$

Note. Good choice of auxiliary function and Rolle's theorem proves the theorem.

Example (L' Hopital's Rule). If we want to find $\lim\limits_{x \to 0} \frac{e^x - 1}{\sin x}$, we have

$$\frac{e^x - 1}{\sin x} = \frac{e^x - x^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

for some $t \in (0, x)$ by Cauchy's Mean Value Theorem. So

$$\frac{e^x - 1}{\sin x} \to 1$$

as $x \to 0$.

Goal: We want to extend the MVT to include higher order derivatives.

> ### Theorem 4.6: Taylor's Theorem with Lagrange's reminder
>
> Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, then
>
> $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!f''(a)} + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n f^{(n)}(a+\theta h)}{n!}$$
>
> where $\theta \in (0, 1)$.

Note.

1. For $n = 1$, we get back the MVT, so this is an "n-th order MVT".

2. $R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$ is known as Lagrange's form of the remainder.

Proof. Define for $0 \le t \le h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} b$$

where we choose b such that $\phi(h) = 0$, and we clearly have $\phi(0) = 0$. (Recall that in the proof of the MVT, we used $f(x) - kx$ and picked k that we can use Rolle's Theorem. We also have that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0.$$

We use Rolle's Theorem n times. Since $\phi(0) = \phi(h) = 0$, $\phi'(h_1) = 0$ for some $0 < h_1 < h$. And since $\phi'(0) = \phi'(h_1) = 0$, we have $\phi''(h_2) = 0$ for some $0 < h_2 < h_1$. Finally, $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$. So $\phi^{(n)}(h_n) = 0$ with $0 < h_n < h_{n-1} < \cdots < h$. So $h_n = \theta h$ for $\theta \in (0, 1)$, now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - b \implies b = f^{(n)}(a + \theta h).$$

Set $t = h$, $\phi(h) = 0$ and put this value of b to the second line in the proof. ∎

Theorem 4.7: Taylor's Theorem with Cauchy's reminder

Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, and if $a = 0$ for simplification, then we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1-\theta)^{n-1}f^{(n)}(\theta h)h^n}{(n-1)!}$$

with $\theta \in (0, 1)$.

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$. Note that we have

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$= -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t).$$

Set

$$\phi(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0)$$

with $p \in \mathbb{Z}, 1 \le p \le n$. Then $\phi(0) = \phi(h) = 0$, and by Rolle's, $\exists \theta \in (0, 1)$ such that $\phi'(\theta h) = 0$. But,

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h}F(0) = 0.$$

So

$$0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}[f(h) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)].$$

Rearranging the two sides, and we get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!p(1-\theta)^{p-1}}f^{(n)}(\theta h).$$

Taking $p = n$, we get Lagrange's reminder, and taking $p = 1$ gives Cauchy's reminder. ∎

Lecture 13

To get a Taylor series for f, one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort".

Remark. Theorems 4.6 and 4.7 work equally well in an interval $[a+h, a]$ with $h < 0$.

Example. The binomial series

$$f(x) = (1+x)^r, r \in \mathbb{Q}.$$

We claim that $|x| < 1$, then

$$(1+x)^r = 1 + \binom{r}{1} x + \cdots + \binom{r}{n} x^n + \cdots$$

where

$$\binom{r}{n} = \frac{r(r-1) \cdots (r-n+1)}{n!}.$$

Proof. Clearly,

$$f^{(n)}(x) = r(r-1) \cdots (r-n+1)(1+x)^{r-n}.$$

If $r \in \mathbb{Z}_{\geq 0}$, then $f^{(n+1)} = 0$, we have a polynomial of degree r.

In general, by Lagrange's reminder, we have

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}}.$$

Note that θ depends on both x and n.

For $0 < x < 1$, $(1+\theta x)^{n-r} > 1$ for $n > r$. Now observe that the series $\sum \binom{r}{n} x^n$ is absolutely convergent for $|x| < 1$. Indeed, by the ratio test,

$$a_n = \binom{r}{n} x^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{r(r-1)(r-n+1) \cdots (r-n) x^{n+1}}{(n+1)!} \right| \left| \frac{n!}{r(r-1) \cdots (r-n+1) x^n} \right|$$

$$= \left| \frac{(r-n)x}{n+1} \right|$$

which tends to a value less than 1. In particular, $a_n \to 0$ and $\binom{r}{n} x^n \to 0$.

Hence, for $n > r$, and $0 < x < 1$, we have that $|R_n| \leq \left|\binom{r}{n}x^n\right| = |a_n| \to 0$ as $n \to \infty$.

So the claim is proved in the range $0 \leq x < 1$. If $-1 < x < 0$, the argument above breaks, but Cauchy's form for R_n works.

$$
\begin{aligned}
R_n &= \frac{(1-\theta)^{n-1}r(r-1)\cdots(r-n+1)(1+\theta h)^{r-n}x^n}{(n-1)!} \\
&= \frac{r(r-1)\cdots(r-n+1)}{(n-1)!}\frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}}x^n \\
&= r\binom{r-1}{n-1}x^n(1+\theta x)^{r-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}.
\end{aligned}
$$

So $|R_n| \leq \left|r\binom{r-1}{n-1}x^n\right|(1+\theta x)^{r-1}$. Check that $(1+\theta x)^{r-1} \leq \max\{1, (1+x)^{r-1}\}$. Let $K_r = |r|\max\{1, (1+x)^{r-1}\}$ is independent of n. So we have

$$
|R_n| \leq |K_r|\left|\binom{r-1}{n-1}x^n\right| \to 0.
$$

So $R_n \to 0$ as $n \to \infty$. ∎

4.3 Remarks on Complex Differentiation

Formally, for function $f : E \subseteq \mathbb{C} \to \mathbb{C}$, we have properties for sums, products, chain rule etc. But it is much more restrictive than differentiability on the real line.

Example. $f : \mathbb{C} \to \mathbb{C}$, with $z \mapsto \bar{z}$. We consider the sequence $z_n = z + \frac{1}{n} \to z$.

$$
\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1.
$$

If we approach it vertically instead, taking $z_n = z + \frac{i}{n} \to z$, we have

$$
\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1.
$$

So $\lim_{w \to z} \frac{f(w) - f(z)}{w - z}$ does not exist. f is nowhere $\mathbb{C}$-differentiable.

If we consider it as a function on $\mathbb{R}^2$, $f(x, y) = (x, -y)$. It is real differentiable.

In fact, if a function is complex differentiable, it is infinitely complex differentiable. It is discussed in more detail in IB Complex Analysis.

Lecture 14

5 Power Series

We want to look at

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $z \in \mathbb{C}, a_n \in \mathbb{C}$. The case $\sum_{n=0}^{\infty} a_n(z - z_0)^n$, with z fixed can be reduced to power series around 0 by translation.

Lemma 5.1

If $\sum_{n=0}^{\infty} a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

Proof. Since $\sum_{n=0}^{\infty} a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus, there exists $K > 0$ such that $|a_n z_1^n| \le K$ for all n.

Then,

$$|a_n z^n| = |a_n z^n| \frac{|z_1^n|}{|z_1^n|} \le K \left| \frac{z}{z_1} \right|^n.$$

Since the geometric series $\sum_{n=0}^{\infty} \left| \frac{z}{z_1} \right|^n$ converges, the lemma follows by comparison. ∎

Using this lemma, we will prove that every power series has a *radius of convergence*.

Theorem 5.1

A power series either

1. converges absolutely for all z, or

2. converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outsider it, or

3. converges for $z = 0$ only.

> ### Definition 5.1
>
> The circle $|z| = R$ is called the *circle of convergence* and R the *radius of convergence*.
>
> In (1) of Theorem 5.1, we agree that $R = \infty$, and in (3) $R = 0$, so $R \in [0, \infty]$.

Proof. Let $S = \{x \in \mathbb{R} \mid x \geq 0, \sum a_n x^n \text{ converges}\}$. Clearly $0 \in S$. By (5.1), if $x_1 \in S$, then $[0, x_1] \subseteq S$. If $S = [0, \infty)$, we have case 1.

Otherwise, there exists a finite supremum for S. $R = \sup S < \infty, R \geq 0$. If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely. Pick R_0 such that $|z_1| < R_0 < R$, then $R_0 \in S$ and the series converges for $z = R_0$. By (5.1), $\sum |a_n z_1^n|$ converges.

Finally, we show that if $|z_2| > R$, then the series does not converge for z_2. Pick $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges, by (5.1), $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. ∎

The following lemma is useful for computing R.

> ### Lemma 5.2
>
> If $\left| \frac{a_{n+1}}{a_n} \right| \to \ell$, as $n \to \infty$, then $R = \frac{1}{\ell}$.

Proof. By the ratio test, we have absolute convergence if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| = \ell |z| < 1.$$

So if $|z| < \frac{1}{\ell}$, we have absolute convergence. If $|z| > \frac{1}{\ell}$, the series diverges, again by the ratio test. ∎

Remark. One can also use the Root Test to get that if $|a_n|^{\frac{1}{n}} \to \ell$, then $R = \frac{1}{\ell}$.

Example.

1. $\sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = \ell.$$

So $R = \infty$.

2. Geometric Series, $\sum\limits_{n=0}^{\infty} z^n$.

   By ratio test, we have $R = 1$. Note that at $|z| = 1$, we have divergence.

3. $\sum\limits_{n=0}^{\infty} n!z^n$.

   We have

   $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{n!} = n+1 \to \infty.$$

   So $R = 0$.

4. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$, and we have $R = 1$. For $z = 1$, it diverges. What happens for $|z| = 1$ and $z \neq 1$?

   Consider $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}(1-z)$, we have the partial sum

   $$\begin{aligned}
   S_N &= \sum_{n=1}^{N}\left(\frac{z^n - z^{n+1}}{n}\right) \\
   &= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} \\
   &= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=2}^{N+1} \frac{z^n}{n-1} \\
   &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} z^n\left(\frac{1}{n} - \frac{1}{n-1}\right) \\
   &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} -\frac{z^n}{n(n-1)}.
   \end{aligned}$$

   If $|z| = 1$, $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum \frac{1}{n(n-1)}$ converges, so S_N converges.

5. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$, and we have $R = 1$. But it converges for all z with $|z| = 1$.

Conclusion is that, in principle, nothing can be said about $|z| = R$ and each case has to be discussed separately.

Within the radius of convergence, "life is great". Power series behave as if "they were polynomials".

Lecture 15

> **Theorem 5.2**
>
> $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ has radius convergence R. Then f is differentiable at all points with $|z| < R$ with
> $$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof (non examinable).

Watch the lecture and finish the proof.

∎

5.1 The Standard Functions

In this section, we will discuss exponential, logarithmic, trigonometric, etc.

We have already seen that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has $R = \infty$. Define $e : \mathbb{C} \to \mathbb{C}$, $z \mapsto \sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$. From Theorem 5.2, e is differentiable, and $e'(z) = e(z)$.

If $F : \mathbb{C} \to \mathbb{C}$ has $F'(z) = 0$ for all $z \in \mathbb{C}$, then F is constant.

Proof. Consider $g(t) = F(tz)$, and chain rule gives $g'(t) = F'(tz)z = 0$. If $g(t) = u(t) + iv(t)$. It is immediate that $g'(t) = u'(t) + iv'(t)$. So $u'(t) = v'(t) = 0$. By

previously proved corollary, we have $u(t), v(t)$ constant. Thus, $F(z)$ is constant. ∎

Now let $a, b \in \mathbb{C}$. Consider

$$F(z) = e(a + b - z)e(z).$$

We have $F'(z) = 0$, so F is constant.

$$e(a + b - z)e(z) = F(0) = e(a + b).$$

Setting $z = b$ gives

$$e(a)e(b) = e(a + b).$$

Lecture 16

Now we restrict to $\mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$.

> ### Theorem 5.3
>
> 1. $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$.
>
> 2. $e(x + y) = e(x)e(y)$.
>
> 3. $e(x) > 0$ for all $x \in \mathbb{R}$.
>
> 4. e is strictly increasing.
>
> 5. $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$.
>
> 6. $e : \mathbb{R} \to (0, \infty)$ is a bijection.

Proof. We proved (1) and (2) above.

To prove (3), clearly we have $e(x) > 0$ for $x \geq 0$ and $e(0) = 1$. Also, $e(0) = e(x - x) = e(x)e(-x0 = 1$. So $e(-x) > 0$ for $x > 0$.

To prove (4), we have $e'(x) = e(x) > 0$ so e is strictly increasing.

To prove (5), $e(x) > 1 + x$ for $x > 0$. So if $x \to \infty$, clearly $e(x) \to \infty$. For $x > 0$, since $e(-x) = \frac{1}{e(x)}$, then $e(x) \to 0$ as $x \to -\infty$.

For (6), injectivity follows right away from being strictly increasing. To prove surjectivity, take any $y \in (0, \infty)$. Since $e(x) \to \infty$ as $x \to \infty$ and $e(x) \to 0$ as $x \to -\infty$. We can find a, b such that $e(a) < y < e(b)$. By the IVT, there exists $x \in \mathbb{R}$ such that $e(x) = y$. ∎

Remark. $e : (\mathbb{R}, +) \to ((0, \infty), \cdot)$ is a group isomorphism.

Since e is a bijection, we have an inverse $\ell : (0, \infty) \to \mathbb{R}$.

Theorem 5.4

1. $\ell : (0, \infty) \to \mathbb{R}$ is a bijection and $\ell(e(x)) = x$ for all $x \in \mathbb{R}$ and $e(\ell(t)) = t$ for all $t \in (0, \infty)$.

2. ℓ is differentiable and $\ell'(t) = \frac{1}{t}$.

3. $\ell(xy) = \ell(x) + \ell(y)$ for all $x, y \in (0, \infty)$.

Proof.

1. Obvious from the definition of e.

2. Inverse rule gives that ℓ is differentiable and

$$\ell'(t) = \frac{1}{e'(\ell(t))} = \frac{1}{t}.$$

3. From IA Groups, if e is an isomorphism, so is its inverse.

∎

Now define for $\alpha \in \mathbb{R}$ and $x > 0$,

$$r_\alpha(x) = e(\alpha \lambda(x)).$$

> **Theorem 5.5**
>
> Suppose $x, y > 0$, and $\alpha, \beta \in \mathbb{R}$, then
>
> 1. $r_\alpha(xy) = r_\alpha(x)r_\alpha(y)$;
>
> 2. $r_{\alpha+\beta} = r_\alpha(x)r_\beta(x)$;
>
> 3. $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$;
>
> 4. $r_1(x) = x$ and $r_0(x) = 1$.

Proof.

1. $r_\alpha(xy) = e(\alpha\ell(xy))$
$$= e(\alpha\ell(x) + \alpha\ell(y))$$
$$= e(\alpha\ell(x))e(\alpha\ell(y))$$
$$= r_\alpha(x)r_\alpha(y).$$

2. $r_{\alpha+\beta}(x) = e((\alpha + \beta)\ell(x))$
$$= e(\alpha\ell(x))e(\beta\ell(b))$$
$$= r_\alpha(x)r_\beta(x).$$

3. $r_\alpha(r_\beta(x)) = r_\alpha(e(\beta\ell(x)))$
$$= e(\alpha\ell \circ e(\beta\ell(x)))$$
$$= e(\alpha\beta\ell(x))$$
$$= r_{\alpha\beta}(x).$$

4. $r_1(x) = e(\ell(x)) = x$ and $r_0(x) = e(0\ell(x)) = e(0) = 1$

■

For some $n \in \mathbb{Z}_{\geq 1}$, then

$$r_n(x) = r_{\underbrace{1 + \cdots + 1}_{n}}(x) = \underbrace{x \cdots x}_{n} = x^n.$$

We also have

$$r_1(x)r_{-1}(x) = r_0(x) = 1 \implies r_{-1}(x) = \frac{1}{x}.$$

So $r_{-n}(x) = \frac{1}{x^n}$. Next we consider for $q \in \mathbb{Z}_{\geq 1}$,

$$(r_{\frac{1}{q}})^q = r_1(x) = x \implies r_{\frac{1}{q}}(x) = x^{\frac{1}{q}}.$$

And thus we also have $r_{\frac{p}{q}}(x) = (r_{\frac{1}{q}}(x))^p = x^{\frac{p}{q}}$. Thus, $r_\alpha(x)$ agrees with x^α when $\alpha \in \mathbb{Q}$ as previously defined.

Now we give the above functions names:

1. $\exp(x) = e(x)$ for $x \in \mathbb{R}$;

2. $\log x = \ell(x)$ for $x \in (0, \infty$;

3. $x^\alpha = r_\alpha(x)$ for $\alpha \in \mathbb{R}, x \in (0, \infty)$.

If $e(x) = e(x \log e)$ where $e = \sum_{n=0}^{\infty} \frac{1}{n!}$, we have $e(x) = r_x(e) = e^x$. $\exp(x)$ is also a power, which we may as well write as e^x.

Finally, we compute

$$\begin{aligned}
(x^\alpha)' &= (e^{\alpha \log x})' \\
&= e^{\alpha \log x} \alpha \frac{1}{x} \\
&= \alpha x^{\alpha - 1}.
\end{aligned}$$

And we have

$$(a^x)' = (e^{x \log a})' = e^{x \log a} \log a = a^x \log a.$$

Lecture 17: Trigonometric Functions

28 Feb. 2022

Remark ("Expoentials beat polynomials"). We have

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty. \qquad (k > 0)$$

We use power series to prove it. We have

$$e^x = \sum_{i=0}^{\infty} > \frac{x^n}{n!}. \qquad (x > 0)$$

Now we pick $n > k$, and we have

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty \text{ as } x \to \infty.$$

Definition 5.2: Trigonometric Functions

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by Theorem 5.2, we get

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z.$$

Also note that we have

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=1}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{n=1}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \cos z + i \sin z.$$

Similarly,

$$e^{-iz} = \cos z - i \sin z.$$

So we can write $\cos z$ and $\sin z$ as

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

From this, we get many trigonometric identities. For example,

$$\cos z = \cos(-z), \quad \sin(-z) = -\sin z, \quad \cos(0) = 1, \quad \sin(0) = 0.$$

And the addition formulas

1. $\sin(z + w) = \sin z \cos w + \cos z \sin w$;

2. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

They essentially follow from $e^{a+b} = e^a e^b$. To prove (2), write

$$\cos(z + w) = \frac{1}{2}(e^{i(z+w)} + e^{-i(z+w)}),$$

and expand $\cos z \cos w - \sin z \sin w$ similarly to get the result. We can also get

$$\sin^2 z + \cos^2 z = 1 \qquad \forall z \in \mathbb{C} \tag{1}$$

by direct computation.

Now if $x \in \mathbb{R}$, then $\sin x, \cos x \in \mathbb{R}$, and eq. (1) gives $|\sin x|, |\cos x| \leq 1$. We should be careful that they don't have to be bounded when z is not real. For example

$$\cos(iy) = \frac{1}{e^{-y} + e^y}. \qquad y \in \mathbb{R}$$

So $\cos(iy) \to \infty$ as $y \to \infty$.

Proposition 5.1

There is a smallest positive number w (where $\sqrt{2} < \frac{w}{2} < \sqrt{3}$) such that

$$\cos \frac{w}{2} = 0.$$

Proof. If $0 < x < 2$,

$$\sin x = \left(x - \frac{x^3}{3!} \right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!} \right) + \cdots > 0$$

since $0 < x < 2 \implies \frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!}$, and $(\cos x)' = -\sin x < 0$ for $0 < x < 2$. So $\cos x$ is strictly decreasing on $(0,2)$. We will show that $\cos \sqrt{2} > 0$ and $\cos \sqrt{3} < 0$. Then by the intermediate value theorem, the existence of w follows. Now we prove that.

$$\cos \sqrt{2} = \left(\frac{(\sqrt{2})^2}{4!} - \frac{(\sqrt{2})^6}{6!} \right) + \cdots > 0$$

$$\cos \sqrt{3} = 1 - \frac{3}{2} + \frac{9}{4!} - \left(\frac{x^6}{6!} - \frac{x^8}{8!} \right) - \cdots = -\frac{1}{8} - \cdots < 0,$$

and we are done. $\blacksquare$

Corollary 5.1

$$\sin \frac{w}{2} = 1.$$

Proof. We have $\sin^2 \frac{w}{2} + \cos^2 \frac{w}{2} = 0$, and we know that $\sin$ is positive on $(0,2)$. $\blacksquare$

Now we define $\pi = w$.

> **Theorem 5.6**
>
> 1. $\sin(z + \frac{\pi}{2}) = \cos z$, $\cos(z + \frac{\pi}{2}) = -\sin z$;
>
> 2. $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$;
>
> 3. $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

Proof. It is immediate from addition formulas and $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$. ∎

This implies

$$
\begin{aligned}
e^{iz + 2\pi i} &= \cos(z + 2\pi) + i\sin(z + 2\pi) \\
&= \cos z + i\sin z \\
&= e^{iz}.
\end{aligned}
$$

So e^z is periodic with period $2\pi i$.

Remark ("Relation with geometry"). Given two vectors $x, y \in \mathbb{R}^2$, define $x \cdot y$ as in Part IA Vector and Matrices,

$$
x \cdot y = x_1 y_1 + x_2 y_2.
$$

Cauchy-Schwarz gives $|x \cdot y| \leq \|x\|\|y\|$. So, for $x \neq 0$, $y \neq 0$,

$$
-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1.
$$

Define the angle between x and y as the unique $\theta \in [0, \pi]$ with $\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$. And we recover the unit circle picture by defining angle this way.

Lecture 18: Hyperbolic Functions

2 Mar. 2022

> **Definition 5.3**
>
> $$
> \cosh z = \frac{1}{2}(e^z + e^{-z})
> $$
>
> $$
> \sinh z = \frac{1}{2}(e^z - e^{-z})
> $$

We clearly have the relationships

$$\cosh z = \cos(iz)$$
$$\sinh z = -i \sin(iz)$$
$$(\cosh z)' = \sinh z$$
$$(\sinh z)' = \cosh z$$
$$\cosh^2 z - \sinh^2 z = 1.$$

The rest of the trigonometric functions (tan, cot, sec, csc) are defined in the usual way.

6 Integration

Suppose we have a function $f : [a, b] \to \mathbb{R}$ bounded. That is $\exists K$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Definition 6.1

A *dissection* (or *partition*) $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points a and b. We write
$$\mathcal{D} = \{x_0, x_1, \ldots, x_n\}$$
with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

A picture of the function

Definition 6.2

We define the *upper sum* and *lower sum* associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(upper)}$$

$$\mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(lower)}$$

Clearly, $\mathscr{S}(f, \mathcal{D}) \leq S(f, \mathcal{D})$ for any $\mathcal{D}$.

Lemma 6.1

If $\mathcal{D}, \mathcal{D}'$ are dissections with $\mathcal{D}' \supseteq \mathcal{D}$, then

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D}') \geq \mathscr{S}(f, \mathcal{D}') \geq \mathscr{S}(f, \mathcal{D}).$$

Proof. $S(f, \mathcal{D}') \geq \mathscr{S}(f, \mathcal{D}')$ is obvious.

Suppose $\mathcal{D}'$ contains an extra point than $\mathcal{D}$, let's say $y \in (x_{r_1}, x_r)$. Clearly,

$$\sup_{x \in [x_{r-1}, y]} f(x), \sup_{x \in [y, x_r]} f(x) \leq \sup_{x \in [x_{r-1}, x_r]} f(x).$$

So we have

$$(x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x) \geq (y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} f(x).$$

And the same argument goes for $\mathscr{S}$ and the same if $\mathcal{D}'$ has more extra points than $\mathcal{D}$. ∎

Lemma 6.2

Suppose $\mathcal{D}_1, \mathcal{D}_2$ are two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathscr{S}(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathscr{S}(f, \mathcal{D}_2).$$

So $S(f, \mathcal{D}_1) \geq \mathscr{S}(f, \mathcal{D}_2)$.

Proof. Take $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$ and apply the previous lemma. ∎

Definition 6.3

The *upper integral* of f is $I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$.

The *lower integer* of f is $I_*(f) = \sup_{\mathcal{D}} \mathscr{S}(f, \mathcal{D})$.

The supremum, infimum always exists by Lemma 6.2 and picking an arbitrary $\mathcal{D}_1, \mathcal{D}_2$ respectively.

And by Lemma 6.2,

$$I^*(f) \geq I_*(f)$$

because

$$S(f, \mathcal{D}_1) \geq \mathscr{S}(f, \mathcal{D}_2)$$
$$I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_1) \geq \mathscr{S}(f, \mathcal{D}_2)$$
$$I^*(f) \geq \sup_{\mathcal{D}_2} \mathscr{S}(f, \mathcal{D}_2) = I_*(f).$$

Definition 6.4

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* (or just *integrable*) if $I^*(f) = I_*(f)$.

And we set

$$\int_a^b f(x)\,\mathrm{d}x = I^*(f) = I_*(f) = \int_a^b f.$$

Example. The function $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$ is not Riemann integrable since

$$\sup_{[x_{j-1}, x_j]} f(x) = 1, \qquad \inf_{[x_{j-1}, x_j]} f(x) = 0$$
$$\implies S(f, \mathcal{D}) = 1, \qquad \mathscr{S}(f, \mathcal{D}) = 1 \qquad \forall \mathcal{D}$$

So $I^*(f) = 1$, but $I_*(f) = 0$.

Lecture 19

4 Mar. 2022

We first develop a useful criterion for integrability.

Theorem 6.1

A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if given $\epsilon > 0$, $\exists \mathcal{D}$ such that
$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) < \epsilon.$$

Proof. For every dissection $\mathcal{D}$, we have

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}).$$

If the given condition holds, then for all $\epsilon > 0$, we can find $\mathcal{D}$ such that

$$0 \le I^*(f) - I_*(f) \le S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) < \epsilon.$$

So $I^*(f) = I_*(f)$.

Conversely, if f is integrable, by definition of sup and inf, there are partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\int_a^b f \, dx - \frac{\epsilon}{2} = I_*(f) - \frac{\epsilon}{2} < \mathscr{S}(f, \mathcal{D}_1)$$
$$\int_a^b f \, dx + \frac{\epsilon}{2} = I^*(f) + \frac{\epsilon}{2} > S(f, \mathcal{D}_2)$$

By Lemma 6.2,

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) - \mathscr{S}(f, \mathcal{D}_1 \cup \mathcal{D}_2) \le S(f, \mathcal{D}_2) - \mathscr{S}(f, \mathcal{D}_1) < \epsilon.$$

$\blacksquare$

We now use this criterion to show that monotone and continuous functions are integrable.

Remark. Monotone and continuous functions are bounded.

Theorem 6.2

Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.

Proof. Suppose without loss of generality that f is increasing (same proof for f decreasing), then $\sup_{x \in [x_{j-1}, x_j]} f(x) = f(x_j)$ and $\inf_{x \in [x_{j-1}, x_j]} f(x) = f(x_{j-1})$. Thus,

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1})((f(x_j) - f(x_{j-1})).$$

Now choose

$$\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b\}.$$

In other words, $x_j = a + \frac{(b-a)j}{n}$ for $0 \le j \le n$. So

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \frac{b-a}{n}(f(b) - f(a)).$$

Take n large enough such that $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$ and use Theorem 6.1. $\blacksquare$

To prove that of continuous functions, we need an auxiliary lemma.

> ## Lemma 6.3
>
> For $f : [a, b] \to \mathbb{R}$ continuous. Given $\epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. (*uniform continuity*)

The point is that δ works for all x, y as long as $|x - y| < \delta$. (in the definition of continuity of f at x, the δ depends on x)

Proof. Suppose the claim is false. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, we can find $x, y \in [a, b]$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$.

Take $\delta = \frac{1}{n}$ to get $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \epsilon$. By Bolzano-Weierstrass theorem, we know there exists $x_{n_k} \to c \in [a, b]$, and

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \to 0.$$

So $y_{n_k} \to c$. But $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$. Let $k \to \infty$, by continuity of f,

$$|f(c) - f(c)| \geq \epsilon.$$

Contradiction. ∎

> ## Theorem 6.3
>
> Let $f : [a, b] \to \mathbb{R}$ be continuous, then f is Riemann integrable.

Proof. By Lemma 6.3, given $\epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Let $\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b\}$. Choose n large enough such that $\frac{b-a}{n} < \delta$, then for $x, y \in [x_{j-1}, x_j]$, $|f(x) - f(y)| < \epsilon$, since $|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$. Observe that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j). \qquad p_j, q_j \in [x_{j-1}, x_j]$$

Minimum and maximum are achieved due to continuity. So

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})(f(p_j) - f(q_j)) < \epsilon(b - a).$$

And we are done by Theorem 6.1. ∎

Lecture 20

Functions more complicated than monotone or continuous can be Riemann integrable.

Example. Consider $f : [0,1] \longrightarrow \mathbb{R}$. Clearly,
$$x \longmapsto \begin{cases} 1/q, & \text{if } x = p/q \in (0,1] \text{ in its lowest term} \\ 0, & \text{otherwise} \end{cases}$$
$\mathscr{S}(f, \mathcal{D}) = 0$ for all $\mathcal{D}$. We will show that given $\epsilon > 0$, $\exists \mathcal{D}$ such that $S(f, \mathcal{D}) < \epsilon$. This implies that f is integrable with $\int_0^1 f \, dx = 0$.

Consider the set

$$\{ x \in [0,1] \mid f(x) \geq \frac{1}{N} \} = \{ \frac{p}{q} \mid 1 \leq q \leq N, 1 \leq p \leq q \}.$$

Take $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. This is a finite set with

$$0 < t_1 < t_2 < \cdots < t_R = 1.$$

Consider a dissection $\mathcal{D}$ of $[a, b]$ such that

1. Each t_k with $1 \leq k \leq R$ is in some (x_{j-1}, x_j);

2. For all k, the unique interval containing t_k has length at most $\epsilon/2R$.

Such dissection clearly exists. Note that $f \leq 1$ everywhere, and

$$S(f, \mathcal{D}) \leq \frac{1}{N} + \frac{\epsilon}{2} < \epsilon.$$

The function is integrable but has countable many discontinuities.

6.1 Elementary Properties of the Integral

Let f, g be bounded and integrable functions on $[a, b]$.

1. If $f \leq g$ on $[a, b]$, then
$$\int_a^b f \leq \int_a^b g.$$

2. $f + g$ is integrable on $[a, b]$ and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

3. For any constant k, kg is integrable and

$$\int_a^b kf = k \int_a^b f.$$

4. $|f|$ is integrable and

$$\left| \int_a^b f \right| \le \int_a^b |f|.$$

5. The product fg is integrable.

Proof.   1. If $f \le g$, then

$$\int_a^b f = I^*(f) \le S(f, \mathcal{D}) \le S(g, \mathcal{D})$$

$$\implies \int_a^b f = I^*(f) \le I^*(g) = \int_a^b g.$$

2. We have $\sup_{[x_{j-1}, x_j]} (f + g) \le \sup_{[x_{j-1}, x_j]} f + \sup_{[x_{j-1}, x_j]} g$, so

$$S(f + g, \mathcal{D}) \le S(f, \mathcal{D}) + S(g, \mathcal{D}).$$

Now for dissections $\mathcal{D}_1$ and $\mathcal{D}_2$,

$$\begin{aligned} I^*(f + g) &\le S(f + g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\le S(f, \mathcal{D}_1 \cup \mathcal{D}_2) + S(g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\le S(f, \mathcal{D}_1) + S(g, \mathcal{D}_2). \end{aligned}$$

Fix $\mathcal{D}_1$ and take inf over $\mathcal{D}_2$ to get

$$I^*(f + g) \le S(f, \mathcal{D}_1) + I^*(g).$$

Again, take inf over all $\mathcal{D}_1$, we have

$$I^*(f + g) \le I^*(f) + I^*(g) = \int_a^b f + \int_a^b g.$$

Similarly, $\int_a^b f + \int_a^b g \le I_*(f + g)$, so $f + g$ is integrable with the integral equal to the sum of integrals.

3. Exercise.

4. Consider the function $f_+(x) = \max(f(x), 0)$, we have

$$\sup_{[x_{j-1}, x_j]} f_+ - \inf_{[x_{j-1}, x_j]} f_+ \leq \sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f.$$

We know that given $\epsilon > 0$, there exists $\mathcal{D}$ such that

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) (\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f) < \epsilon.$$

By inequality above, we have

$$S(f_+, \mathcal{D}) - \mathscr{S}(f_+, \mathcal{D}) \leq S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) < \epsilon.$$

Note $|f| = 2f_+ - f$, and by (2) and (3), $|f|$ is integrable. Since $-|f| \leq f \leq |f|$, and by property (1), $\left| \int_a^b f \right| \leq \int_a^b |f|$.

5. Take f integrable and $f \geq 0$. Then

$$\sup_{[x_{j-1}, x_j]} f^2 = (\sup_{[x_{j-1}, x_j]} f)^2 = (M_j)^2$$
$$\inf_{[x_{j-1}, x_j]} f^2 = (\inf_{[x_{j-1}, x_j]} f)^2 = (m_j)^2.$$

Note $M_j + m_j < 2K$ for some K since f is bounded. Thus,

$$S(f^2, \mathcal{D}) - \mathscr{S}(f^2, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})(M_j^2 - m_j^2)$$
$$= \sum_{j=1}^{n} (x_j - x_{j-1})(M_j + m_j)(M_j - m_j)$$
$$\leq 2K(S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}))$$

Using Theorem 6.1, we deduce that f^2 is integrable. Now take any f, then $|f| \geq 0$. Since $f^2 = |f|^2$, we deduce that f^2 is integrable for any f. Finally, for fg, note

$$4fg = (f + g)^2 - (f - g)^2.$$

And we are done since the right-hand side is integrable.

$\blacksquare$

Lecture 21

We have an additional property.

6. If f is integrable on $[a,b]$. If $a < c < b$, then f is integrable over $[a,c]$ and $c,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Conversely, if f is integrable over $[a,c]$ and $[c,b]$, then f is integrable over $[a,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. We first make two observations. If $\mathcal{D}_1$ is a dissection of $[a,c]$ and $\mathcal{D}_2$ is a dissection of $[c,b]$, then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a dissection of $[a,b]$ and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2).$$

Also, if $\mathcal{D}$ is a dissection of $[a,b]$, then

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D} \cup \{c\}) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2)$$

where $\mathcal{D}_1$ dissects $[a,c]$ and $\mathcal{D}_2$ dissects $[c,b]$. From the first inequality, we have

$$I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

From the second inequality, we have

$$I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

So, we have

$$I^*(f) = I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

Similarly, the lower integral

$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]}).$$

Thus,

$$0 \leq I^*(f) - I_*(f) = (I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})) + (I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})).$$

From this (6) follows right away. ∎

We have the convention that if $a > b$, then

$$\int_a^b f = -\int_b^a f;$$

If $a = b$, we agree that its value is zero. With this convention, if $|f| \leq K$,

$$\left| \int_a^b f \right| \leq K|b - a|.$$

6.2 The Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be a bounded and integrable function, and we write

$$F(x) = \int_a^x f(t) \, dt. \qquad x \in [a, b]$$

Theorem 6.4

F is continuous.

Proof. Consider the difference $F(x + h) - F(x) = \int_x^{x+h} f(t) \, dt$. We have

$$|F(x + h) - f(x)| = \left| \int_x^{x+h} f(t) \, dt \right|$$
$$\leq K|h|$$

if $|f(t)| \leq K$ for all $t \in [a, b]$. Now let $h \to 0$ and we are done. (In fact, F is Lipschitz continuous) ∎

Theorem 6.5: FTC

If in addition f is continuous at x, then F is differentiable at x, and

$$F'(x) = f(x).$$

Proof. We consider

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| \qquad x + h \in [a, b], h \neq 0$$
$$= \frac{1}{|h|} \left| \int_x^{x+h} f(t) \, dt - hf(x) \right|$$
$$= \frac{1}{|h|} \left| \int_x^{x+h} f(t) - f(x) \, dt \right|.$$

Since f is continuous at x, given $\epsilon > 0$, there exists $\delta > 0$ such that $|t - x| < \delta \implies |f(t) - f(x)| < \epsilon$. So if $|h| < \delta$, we have

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| \leq \frac{1}{|h|} \epsilon |h| = \epsilon.$$

That is,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

∎

Example. Consider the function $f(x) = \begin{cases} -1, & \text{if } x \in [-1,0] \\ 1, & \text{if } x \in (0,1] \end{cases}$. Taking the integral of f,
and we have $F(x) = -1 + |x|$. It is differentiable everywhere except for $x = 0$ where there is a discontinuity on f.

Corollary 6.1: Integration is the inverse of differentiation

If $f = g'$ is continuous on $[a,b]$, then

$$\int_a^x f(t)\,dt = g(x) - g(a). \qquad \forall x \in [a,b].$$

Proof. From Theorem 6.5, $F - g$ has zero derivative in $[a,b]$. So $F - g$ is constant, and since $F(a) = 0$, we have $F(x) = g(x) - g(a)$. ∎

Every continuous function has an *indefinite integral* or *antiderivative* written $\int f(x)\,dx$ which is determined up to a constant.

Remark. We have solved the ODE

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}.$$

Lecture 22

Corollary 6.2: Integration by Parts

Suppose f' and g' exist and are continuous on $[a,b]$, then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

Proof. By the product rule,
$$(fg)' = f'g + fg'.$$

By Theorem 6.5,
$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'.$$

∎

Corollary 6.3: Integration by Substitution

Let $g : [\alpha, \beta] \to [a, b]$ be a continuous function with $g(\alpha) = a$ and $g(\beta) = b$, and g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then
$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(g(t))g'(t) \, dt.$$

Proof. Set $F(x) = \int_a^x f(t) \, dt$, and let $h(t) = F(g(t))$. (defined as g takes values in $[a, b]$) Then we have

$$\begin{aligned}
\int_\alpha^\beta f(g(t))g'(t) \, dt &= \int_\alpha^\beta F'(g(t))g'(t) \, dt \\
&= \int_\alpha^\beta h'(t) \, dt \\
&= h(\beta) - h(\alpha) \\
&= F(b) - F(a) \\
&= \int_a^b f(x) \, dx.
\end{aligned}$$

∎

Theorem 6.6: Taylor's theorem with remainder an integral

Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1 - t)^{n-1} f^{(n)}(th) \, dt.$$

Proof. Substitute $u = th$,

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u) \, du.$$

Integrating by parts, we get the following

$$R_n = \frac{-h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u) \, du.$$

If we integrate by parts $n - 1$ times, we arrive at

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - h f'(0) + \int_0^h f'(u) \, du$$

$$= -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - h f'(0) + f(h) - f(0).$$

∎

Now we can get the Cauchy and Lagrange form of the remainder. However, note that the proof above uses continuity of $f^{(n)}$ not just the existence as we proved before.

> **Theorem 6.7**
>
> Suppose $f, g : [a, b] \to \mathbb{R}$ be continuous functions with $g(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that
>
> $$\int_a^b f(x) g(x) \, dx = f(c) \int_a^b g(x) \, dx.$$

Note. If we take $g(x) = 1$, we get

$$\int_a^b f(x) \, dx = f(c)(b - a).$$

Proof. We are going to use Cauchy's MVT. (Theorem 4.5) Let

$$F(x) = \int_a^x fg, \quad G(x) = \int_a^x g.$$

Theorem 4.5 proves the existence of $c \in (a, b)$ such that

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$$

$$(\int_a^b fg)g(c) = f(c)g(c) \int_a^b g$$

Since $g(c) \neq 0$, we simplify and get the required form. ∎

Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) \, dt.$$

First we use Theorem 6.7 with $g(x) = 1$, to get

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$$

with some $\theta \in (0,1)$ which is the Cauchy's form of remainder. To get Lagrange's form, we use Theorem 6.7 with $g(t) = (1-t)^{n-1}$ which is greater than 0 for $t \in (0,1)$. We know there exists $\theta \in (0,1)$ such that

$$\begin{aligned} R_n &= \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \int_0^1 (1-t)^{n-1} \, dt \\ &= \frac{h^n}{n!} f^{(n)}(\theta h) \end{aligned}$$

which is the Lagrange's form of the remainder.

Lecture 23

6.3 Improper Integrals

> **Definition 6.5**
>
> Suppose $f : [a, \infty) \to \mathbb{R}$ is integrable (and bounded) on every interval $[a, R]$ and that as $R \to \infty$,
> $$\int_a^R f(x) \, dx \to \ell.$$
> Then we say that $\int_a^\infty f(x) \, dx$ *exists* or *converges* and that its value is ℓ. If $\int_a^R f(x) \, dx$ does not tend to a limit, we say that $\int_a^\infty f(x) \, dx$ *diverges*.
>
> A similar definition applies to $\int_{-\infty}^a f(x) \, dx$
>
> If $\int_a^\infty f = \ell_1$ and $\int_{-\infty}^a f = \ell_2$, we write $\int_{-\infty}^\infty f = \ell_1 + \ell_2$. (independent of the particular value of a)

Remark. The last part is not the same as saying that $\lim_{R \to \infty} \int_{-R}^R f(x) \, dx$ exists. It is stronger since $\int_{-R}^R x \, dx = 0$.

Example. The integral $\int_1^\infty \frac{dx}{x^k}$ converges if and only if $k > 1$.

Indeed, if $k \neq 1$,

$$\int_1^R \frac{dx}{x^k} = \frac{x^{1-k}}{1-k}\bigg|_1^R = \frac{R^{1-k} - 1}{1 - k}$$

and as $R \to \infty$, this limit is finite if and only if $k > 1$. If $k = 1$,

$$\int_1^R \frac{dx}{x} = \log R \to \infty.$$

Remark.   1. $1/\sqrt{x}$ is continuous on $[\delta, 1]$ for any $\delta > 0$, and

$$\int_\delta^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\big|_\delta^1 = 2 - 2\sqrt{\delta} \to 2$$

as $\delta \to 0$. The function is unbounded on $(0, 1]$, but the limit of the integral exists.

However,

$$\int_0^1 \frac{dx}{x} = \lim_{\delta \to 0} \int_\delta^1 \frac{dx}{x} = \lim_{\delta \to 0} \log x\big|_\delta^1 = \lim_{\delta \to 0}(\log 1 - \log \delta)$$

does not exist.

2. If $f \geq 0$ and $g \geq 0$ for $x \geq a$, and

$$f(x) \leq Kg(x). \qquad \forall x \geq a$$

Then $\int_a^\infty g$ converges implies $\int_a^\infty f$ converges and $\int_a^\infty f \leq K \int_a^\infty g$.

Just note that $\int_a^R f \leq K \int_a^R g$. The function $R \mapsto \int_a^R f$ is increasing because $f \geq 0$, and bounded above because $\int_a^\infty g$ converges. Take $\ell = \sup_{R \geq a} \int_a^R f < \infty$. Now check that

$$\lim_{R \to \infty} \int_a^R f = \ell.$$

Given $\epsilon > 0$, there exists R_0 such that

$$\int_a^{R_o} f \geq \ell - \epsilon.$$

Thus, if $R \geq R_0$,

$$\int_a^R f \geq \int_a^{R_0} f \geq \ell - \epsilon.$$

And the limit follows.

Example. Consider the integral $\int_0^\infty e^{-x^2/2}\,\mathrm{d}x$. We have

$$e^{-x^2/2} \le e^{-x/2}. \qquad x \ge 1$$

Since

$$\int_1^R e^{-x/2}\,\mathrm{d}x = \frac{1}{2}(e^{-1/2} - e^{-R/2}) \to \frac{e^{-1/2}}{2},$$

the integral $\int_0^\infty e^{-x^2/2}$ converges.

3. We know that if $\sum a_n$ converges, then $a_n \to 0$. However, $\int_a^\infty f$ converges may not imply that $f \to 0$.

   For example, consider a function of smaller and smaller bumps to the highest point with value 1.

> ### Theorem 6.8: Integral test
>
> Let $f(x)$ be a positive decreasing function for $x \ge 1$. Then
>
> 1. the integral $\int_1^\infty f(x)\,\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ both converge or diverge;
>
> 2. As $n \to \infty$, $\sum_{r=1}^n f(r) - \int_1^n f(x)\,\mathrm{d}x$ tends to a limit ℓ such that $0 \le \ell \le f(1)$.

Proof. Note that f deceasing implies that f is integrable on every bounded subinterval by Theorem 6.2.

If $n - 1 \le x \le n$, then $f(n-1) \ge f(x) \ge f(n)$, so

$$f(n-1) \ge \int_{n-1}^n f(x)\,\mathrm{d}x \ge f(n).$$

Adding the terms together, we have

$$\sum_1^{n-1} f(r) \ge \int_1^n f(x)\,\mathrm{d}x \ge \sum_2^n f(r).$$

From this, claim (1) is clear. For the proof of (2), set $\phi(n) = \sum_1^n f(r) - \int_1^n f(x)\,\mathrm{d}x$, then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x)\,\mathrm{d}x \le 0.$$

So $\phi(n)$ is decreasing. We also have $0 < \phi(n) \le f(1)$. Thus, $\phi(n)$ tends to a limit ℓ such that $0 \le \ell < f(1)$. ∎

Lecture 24

Example.   1. The series $\sum_1^\infty \frac{1}{n^k}$ converges if and only if $k > 1$.

We proved last lecture that $\int_1^\infty \frac{1}{x^k}$ converges if and only if $k > 1$, and the convergence follows from integral test.

2. The series $\sum_2^\infty \frac{1}{n \log n}$ is divergent by considering $f(x) = \frac{1}{x \log x}$ for $x \geq 2$.

$$\int_2^R \frac{dx}{x \log x} = \log(\log x)\Big|_2^R = \log(\log R) - \log(\log 2) \to \infty$$

as $R \to \infty$. So divergence follows from integral test.

Corollary 6.4: Euler's constant

As $n \to \infty$, $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \to \gamma$ with $0 \leq \gamma \leq 1$.

Proof. Set $f(x) = 1/x$, and use Theorem 6.8. ■

Remark. It is still an open question whether γ is irrational. ($\gamma \approx 0.577$)

We have seen that monotone functions and continuous function are Riemann integrable. We can generalize this a bit and say that piece-wise continuous functions are integrable.

Definition 6.6: A

unction $f : [a, b] \to \mathbb{R}$ is said to be piece-wise continuous if there is a dissection $\mathcal{D} = \{a = x_0, x_1, \ldots, x_n = b\}$ such that

1. f is continuous on (x_{j-1}, x_j) for all j;

2. the one-sided limits $\lim_{x \to x_{j-1}^+} f(x), \lim_{x \to x_j^-} f(x)$ exist.

It is now an exercise to check that f is Riemann integrable; just check that $f|_{[x_{j-1}, x_j]}$ is integrable for each j (the values of f at the end points won't really matter) and use additivity of domains.

Problem. How large can the discontinuity set of f be while f is still Riemann integrable?

Recall the example $f(x) = \begin{cases} 1/q, & x = p/q \\ 0, & \text{otherwise} \end{cases}$ which has a countably infinite set of discontinuities.

———————————*non-examinable*———————————

The question is answered by Henri Lebesgue.

Bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points has *measure zero*.

> **Definition 6.7**
>
> Let $\ell(I)$ be the length of an interval I. A subset $A \subseteq \mathbb{R}$ is said to have *measure zero* if for each $\epsilon > 0$, there exists a countable collection of intervals I_j such that $A \subseteq \cup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} \ell(I_j) < \epsilon$.

> **Lemma 6.4**
>
> 1. Every countable set has measure zero.
>
> 2. If B has measure zero and $A \subseteq B$, then A has measure zero.
>
> 3. If A_k has measure zero for all $k \in \mathbb{N}$, then $\cup_{k \in \mathbb{N}} A_k$ also has measure zero.

We use the *oscillation* of f, which is for I an interval

$$\omega_f(I) = \sup_I f - \inf_I f.$$

And the oscillation of f at a point is

$$\omega_f(x) = \lim_{\epsilon \to 0} \omega_f(x - \epsilon, x + \epsilon).$$

> **Lemma 6.5**
>
> f is continuous at x if and only if $\omega_f(x) = 0$.

Brief Sketch of proof of Lebesgue's Criterion. We consider the set

$$D = \{\, x \in [a, b] \mid f \text{ discontinuous at } x \,\} = \{\, x \mid \omega_f(x) > 0 \,\}.$$

Let $N(\alpha) = \{ x \mid \omega_f(x) \geq \alpha \}$, then $D = \cup_1^\infty N(1/k)$. We want to show that D has measure zero.

Let $\epsilon > 0$ be given, there exists $\mathcal{D}$ such that

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\epsilon \alpha}{2}.$$

Consider $F = \{ j \mid (x_{j-1}, x_j) \cap N(\alpha) \neq \varnothing \}$, then for each $j \in F$,

$$\omega_f([x_{j-1}, x_j]) \geq \alpha.$$

So

$$\alpha \sum_{j \in F}(x_j - x_{j-1}) \leq S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) < \frac{\epsilon \alpha}{2}.$$

And $\sum_{j \in F}(x_j - x_{j-1}) < \epsilon/2$. These cover $N(\alpha)$ except perhaps for $\mathcal{D}$. But these can be covered by intervals of total length less than $\epsilon/2$. So $N(\alpha)$ can be covered by intervals of total length less than ϵ. That is, $N(\alpha)$ has measure zero.

For the other direction, let $\epsilon > 0$ be given, consider $N(\epsilon) \subseteq D$, so $N(\epsilon)$ has measure zero. $N(\epsilon)$ is closed and bounded, so it can be covered by finitely many open intervals of total length less than ϵ because it has measure zero. $N(\epsilon) = \cup_{i=1}^m U_i$, and $K = [a,b]/\cup_i^m U_i$ compact. So it can be covered by finitely meany intervals J_j such that $\omega_f(J_j) < \epsilon$. And we are done by considering the partition with these intervals. $\blacksquare$