Analysis

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Contents

Lecture 1: Limits

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Books:

- *A First Course in Mathematical Analysis* -Burkill
- *Calculus* -Spivak
- *Analysis I* -Tao

1 Limits and Convergence

1.1 Review from Numbers and Sets

Notation. We denote sequences by a_n or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

Definition 1.1. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

Note. $N = N(\epsilon)$ which is dependent on ϵ. That is, if you want to go closer to a, sometimes you need to go higher in N.

Definition 1.2 (limit of a sequence). We say that a sequence is a

$$\left.\begin{aligned} &\textit{increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{decreasing sequence if } a_n \geq a_{n+1}, \end{aligned}\right\}\textit{monotone sequence}$$

$$\left.\begin{aligned} &\textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{aligned}\right\}\textit{strictly monotone sequence}$$

We also have

Theorem 1.1 (Fundamental Axiom of the Real Numbers). If $a_n \in \mathbb{R}$ and a_n is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges*.

Remark. It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.
- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

Definition 1.3 (supremum). For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S$,
2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

Note. Supremum is unique, and there is a similar notion of infimum.

Lemma 1.1 (Properties of Limits).

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.
2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).
3. If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.
4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.
6. If $a_n \to a$, then $\frac{1}{a_n} \to \frac{1}{a}$.
7. If $a_n < A$ for all n and $a_n \to a$, then $a \leq A$.

Proof.

1. Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|a_n - b| < \epsilon, \forall n \geq N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \geq N$,

   $$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$

   If $a \neq b$, take $\epsilon = \frac{|a-b|}{3}$, we have

   $$|a - b| < \frac{2}{3}|a - b|. \lightning$$

2. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon, \forall n \geq N$, Since $n_j \geq j$, we know

$$\left|a_{n_j} - a\right| < \epsilon, \forall j \geq N.$$

That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|. \end{aligned}$$

Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|b_n - b| < \epsilon, \forall n \geq N_2$.

If $n \geq N_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$.

We have

$$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

■

Lemma 1.2.

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Proof. $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit a.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma (1.1).}$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma (1.1) $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma (1.1), we have $a = \frac{a}{2} \implies a = 0$. ■

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

Definition 1.4. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma (1.1) are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.