Group homomorphisms

10th June 2022

Todo list

Problem 5

Prove that the groups $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are not isomorphic.

Solution. We know that isomorphism preserves order of elements. $i \in \mathbb{C} \setminus \{0\}$ has order 4, but no element in $\mathbb{R} \setminus \{0\}$ has order 4 because the only solutions to $x^4 = 1$ in $\mathbb{R} \setminus \{0\}$ are ± 1. ■

Problem 6

We have seen that $(\mathbb{R}, +)$ and $(\mathbb{R}^{>0}, \cdot)$ are isomorphic (Example 4.4). Are the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^{>0}, \cdot)$ isomorphic?

Solution. No, the groups cannot be isomorphic. If otherwise, if $\phi : (\mathbb{Q}, +) \to (\mathbb{Q}^{>0}, \cdot)$ is an isomorphism, we have $\phi^{-1}(2) \in \mathbb{Q}$. So $x = {}^{\phi^{-1}(2)}/_{2} \in \mathbb{Q}$, and

$$\phi(x + x) = \phi(\phi^{-1}(2)) = 2 = \phi(x)\phi(x).$$

That is $\phi(x) = \sqrt{2}$, but this is impossible since we know that $\sqrt{2} \notin \mathbb{Q}$. ■

Problem 7

Let G be a group. Prove that the function $G \to G$ defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is Abelian. Prove that $g \mapsto g^2$ is a homomorphism if and only if G is Abelian.

Solution. If $\phi(g) = g^{-1}$ is a homomorphism, for any $g, h \in G$, we have

$$\phi(g^{-1}h^{-1}) = (g^{-1}h^{-1})^{-1} = hg$$
$$\phi(g^{-1})\phi(h^{-1}) = (g^{-1})^{-1}(h^{-1})^{-1} = gh.$$

So G is Abelian. To prove the other direction, if G is Abelian, we have for $g, h \in G$,

$$\phi(gh) = (gh)^{-1} = h^{-1}g^{-1} = \phi(h)\phi(g) = \phi(g)\phi(h).$$

So ϕ is a homomorphism if G is Abelian.

Similarly, if $\psi(g) = g^2$ is a homomorphism, for any $g, h \in G$, we have

$$\phi(gh) = ghgh = \phi(g)\phi(h) = g^2h^2$$

By cancellation, we have $ghgh = g^2h^2 \implies hg = gh$; that is, G is Abelian. To prove the other direction, if G is Abelian, we have for $g, h \in G$,

$$\phi(gh) = ghgh = g^2h^2 = \phi(g)\phi(h).$$

■