Geometry

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Contents

Lecture 1: Introduction

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1 Surfaces

1.1 Topological Surfaces

We start with some definitions.

Definition 1.1. A *topological surface* is a topological space Σ such that

1. **T1:** $\forall p \in \Sigma$ there is an open neighborhood $p \in U \subseteq \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subseteq \mathbb{R}^2$ with its usual Euclidean topology.
2. **T2:** Σ is Hausdorff and second countable.

Remark. We have the following remarks.

1. $\mathbb{R} \cong D(0,1)$, so homeomorphic to a disc is enough as stated in the definition.
2. A space X is *Hausdorff* if for $p \neq q \in X$, there exists disjoint open sets $p \in U$ and $q \in V$ in X.
3. A space X is *second countable* if it has a countable base i.e. $\exists \{u_i\}_{i \in \mathbb{N}}$ open sets s.t. every open set is a union of some u.
4. **T1** is the point and **T2** is for technical honesty.
5. If X is Hausdorff/ second countable, so are subspaces of X. In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

Example. Here we present some examples of topological surfaces.

1. $\mathbb{R}^2$, the plane.

2. Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where Z is closed:
   - $Z = \{0\}$,
   - $Z = \{(0,0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \dots)\}$.
3. Graphs:

   Let $f : \mathbb{R}^2 \to R$ be a continuous function. The graph $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$ (subspace topology).

   Recall that if X, Y are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with U open and V open.

   It has the feature that $f : Z \to X \times Y$ is continuous if and open if the two projective maps are continuous.

   Commu-tative dia-gram

   Application: $\Gamma_f \subseteq X \times Y$, if $f : X \to Y$ is continuous, if homeomorphic to X.

   So $\Gamma_f \cong \mathbb{R}^2$ for any $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous, so Γ_f is a topological surface.

   Note. As a topological surface, Γ_f is independent of f, but later on as a geometric object, it will reflect features of f.
4. The sphere (subspace topology):

   $$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

   Stereographic projection

   Stereo-graphic projection graph

   $$\begin{aligned} \pi_+ : S^2 \setminus \{(0,0,1)\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \end{aligned}$$

   Note. The map is continuous and has an inverse, π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

   Explicit formula for inverse

   Stereographic projection from the South Pole is also a homeomorphism from $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$.

   So S^2 is a topological surface:

   $\forall p \in S^2$, either p lies in the domain of π_+ or of π_- (or both) and so it lies in an open set homeomorphic to $\mathbb{R}^2$. (And Hausdorff and second countable from $\mathbb{R}^2$).

   Remark. S^2 has a global property as it is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.
5. The real projective place:

   The group $\mathbb{Z}/2$ acts on S^2 by homeomorphism via the *antipodal map* $a : S^2 \to S^2$.

   $$a(x, y, z) = (-x, -y, -t).$$

   i.e. There exists a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \text{Homeo}(S^2)$, such that it maps the non-identity element to the antipodal map.

Definition 1.2. The *real projective plane* is the quotient space of S^2 given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}/2\mathbb{Z}.$$

Lemma 1.1. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ∎

Graph of the sphere

Lemma 1.2. $\mathbb{RP}^2$ is a topological surface.

Proof. We check that it is Hausdorff:

Recall if X is a space and $q : X \to Y$ is a quotient map, $V \subseteq Y$ is open $\iff q^{-1}V \subseteq X$ open.

More balls

If $[p], [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in S^2$ are distinct antipodal pairs. Take small open discs around p, q and their antipodal images, as in the picture. ∎