Analysis

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Contents

Lecture 1: Limits

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Books:

- *A First Course in Mathematical Analysis* -Burkill
- *Calculus* -Spivak
- *Analysis I* -Tao

1 Limits and Convergence

1.1 Review from Numbers and Sets

Notation. We denote sequences by a_n or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

Definition 1.1

We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

Note. $N = N(\epsilon)$ which is dependent on ϵ. That is, if you want to go closer to a, sometimes you need to go higher in N.

Definition 1.2: limit of a sequence

We say that a sequence is a

$$\left.\begin{aligned} &\textit{increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{decreasing sequence if } a_n \geq a_{n+1}, \end{aligned}\right\} \textit{monotone sequence}$$

$$\left.\begin{aligned} &\textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{aligned}\right\} \textit{strictly monotone sequence}$$

We also have

Theorem 1.1: Fundamental Axiom of the Real Numbers

If $a_n \in \mathbb{R}$ and a_n is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges*.

Remark. It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.
- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

Definition 1.3: supremum

For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S,$
2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

Note. Supremum is unique, and there is a similar notion of infimum.

Lemma 1.1: Properties of Limits

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.
2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).
3. If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.
4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.
6. If $a_n \to a$, then $\frac{1}{a_n} \to \frac{1}{a}$.
7. If $a_n < A$ for all n and $a_n \to a$, then $a \leq A$.

Proof.

1. Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|a_n - b| < \epsilon, \forall n \geq N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \geq N$,
   $$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$
   If $a \neq b$, take $\epsilon = \frac{|a-b|}{3}$, we have
   $$|a - b| < \frac{2}{3}|a - b|. \lightning$$

2. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon, \forall n \geq N$, Since $n_j \geq j$, we know
   $$\left|a_{n_j} - a\right| < \epsilon, \forall j \geq N.$$
   That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have
   $$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|. \end{aligned}$$

Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|b_n - b| < \epsilon, \forall n \geq N_2$.

If $n \geq N_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$.

We have

$$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

■

Lemma 1.2

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Proof. $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit a.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma 1.1.}$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma 1.1 $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma 1.1, we have $a = \frac{a}{2} \implies a = 0$. ■

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

Definition 1.4

We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma 1.1 are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.

Lecture 2: Bolzano–Weierstrass theorem

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Theorem 1.2: Bolzano-Weierstrass Theorem

f $x_n \in R$ and there exists K such that $|x_n| \leq K$ for all n, then we can find $n_1 < n_2 < n_3 < \dots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. In other words, every bounded sequence has a convergent subsequence.

Remark. We say nothing about the uniqueness of the limit x.

For example, $x_n = (-1)^n$ has two subsequences tending to -1 and 1 respectively.

Proof. Set $[a_1, b_1] = [-K, K]$. Let c be the mid-point of a_1, b_1, consider the following alternatives,

1. $x_n \in [a_1, c]$ for infinitely many n.

2. $x_n \in [c, a_2]$ for infinitely many n.

Note that (1) and (2) can hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold, and we set $a_2 = c$ and $b_2 = b_1$.

We proceed as above to construct sequences a_n, b_n such that $x_m \in [a_n, b_n]$ for infinitely many values of m. They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

a_n is an increasing sequence and bounded, and b_n is a decreasing sequence and bounded. By Fundamental Axiom, $a_n \to a \in [a_1, b_1], b_n \to b \in [a_1, b_1]$. Using Lemma 1.1, $b - a = \frac{b-a}{2} \implies a = b$.

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j such that $x_{n_j} \in [a_j, b_j]$, that is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$. In other words, there is unlimited supply.

Hence, $a_j \leq x_{n_j} \leq b_j$, so $x_{n_j} \to a$. ■

1.2 Cauchy Sequences

Definition 1.5: Cauchy Sequence

A sequence $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\epsilon > 0$ $\exists N > 0$ such that $|a_n - a_m| < \epsilon$ $\forall n, m > N$.

Note. N is dependent on ϵ.

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

Lemma 1.3

A convergent sequence is a Cauchy sequence.

Proof. If $a_n \to a$, given $\epsilon > 0$, exists N such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Take $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$

■

Lemma 1.4

Every Cauchy sequence is convergent.

Proof. First we note that if a_n is Cauchy, then it is bounded.

Take $\epsilon = 1$, $N = N(1)$ in the Cauchy property, then

$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$

We have

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$

Let $K = \max\{1 + |a_N|, |a_n| \ n = 1, 2 \ldots, N - 1\}$.

Then $|a_n| \leq K$ for all n. By the Bolzano–Weierstrass theorem, $a_{n_j} \to a$. We must have $a_n \to a$.

Given $\epsilon > 0$, there exists j_0 such that for all $j \geq j_0$, $\left|a_{n_j} - a\right| < \epsilon$.

Also, there exists $N(\epsilon)$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N(\epsilon)$.

Take j such that $n_j \geq \max\{N(\epsilon), n_{j_0}\}$. Then if $n \geq N(\epsilon)$,

$$|a_n - a| \leq \left|a_n - a_{n_j}\right| + \left|a_{n_j} - a\right| < 2\epsilon.$$

■

Thus, on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

2 Series

Definition 2.1

If $a_n \in \mathbb{R}, \mathbb{C}$ We say that $\sum\limits_{j=1}^{\infty} a_j$ converges to s if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to S$$

as $N \to \infty$. We write $\sum\limits_{j=1}^{\infty} a_j = s$. If S_N does not converge, we say that $\sum\limits_{j=1}^{\infty} a_j$ *diverges*.

Remark. Any problem on series is really a problem about the sequence of partial sums.

Lemma 2.1

1. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} a_j$ converges, then so does $\sum_{j=1}^{\infty} \lambda a_j + \mu b_j$, when $\lambda, \mu \in \mathbb{C}$;
2. Suppose there exists N such that $a_i = b_i$ for all $i \geq N$. Then either $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge or they both diverge. (initial terms do not matter for convergence)

Proof.
1. Exercise.
2. If we have $n \geq N$,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^{n} a_i$$
$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^{n} b_i$$

So $S_n - d_n = \sum_{i=1}^{N-1} a_i - b_i$ which is a constant. So S_n converges if and only if d_n does.

■

Lecture 3

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We have the following important example,

Example (Geometric Series)**.** $x \in \mathbb{R}$, set $a_n = x^{n-1}$ with $n \geq 1$. So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \dfrac{1-x^n}{1-x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have $S_n(1-x) = 1 - x^n$.

If $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$.

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$.

If $x < -1$, S_n does not converge (oscillates).

If $x = -1$, $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$.

Thus, the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta, \delta > 0$, so $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \to 0$ because $(1+\delta)^n \geq 1 + n\delta$ from binomial expansion.

Definition 2.2

$S_n \to \infty$ if given A, there exists an N such that $S_n > A$ for all $n > N$.

$S_n \to -\infty$ if given A, there exists an N such that $S_n < -A$ for all $n > N$.

Lemma 2.2

If $\sum\limits_{i=1}^{\infty} a_n$ converges, then $\lim\limits_{i\to\infty} a_i = 0$.

Proof. Let $S_n = \sum\limits_{i=1}^{\infty} a_i$, note that $a_n = S_n - s_{n-1}$. If $S_n \to a$, we have $a_n \to 0$ because $S_{n-1} \to a$ also. ■

Remark. The converse of the preceding lemma is false. One example is $\sum \frac{1}{n}$, the *harmonic series*. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty}$$
$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$.

So if $S_n \to a$, then $S_{2n} \to a$, also we have $a \geq a + \frac{1}{2}$. Contradiction.

2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

Theorem 2.1: The Comparison Test

Suppose $0 \leq b_n \leq a_n$ for all n. Then if $\sum\limits_{n=1}^{\infty} a_n$ converges, so does $\sum\limits_{n=1}^{\infty} b_n$.

Proof. Let $s_N = \sum\limits_{n=1}^{N} a_n$, $d_N = \sum\limits_{n=1}^{N} b_n$. Because $b_n \leq a_n$, we know $d_N \leq s_N$. But $s_N \to s$, then $d_n \leq s_n \leq 2$ for all n, and d_N is a increasing sequence bounded above. So d_N converges. ■

Example. We consider $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that $\sum\limits_{n=1}^{\infty} a_n$ converges, so $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, we get $\sum\limits_{n=1}^{\frac{1}{n^2}} \leq 1 + 1 = 2$.

For the rest of the lecture, we establish two more tests.

Theorem 2.2: Root test/ Cauchy's Test for Convergence

Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum_{n=1}^{\infty} a_n$ converges; if $a > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. Nothing can be said if $a = 1$.

. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists N such that $\forall n \geq N$,

$$a_n^{1/n} < r \implies a_n < r^n.$$

But since $r < 1$, the geometric series converges, and by comparison test, the series $\sum a_n$ converges as well.

To prove the second part of the theorem, if $a > 1$, for $n \geq N$,

$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus, $\sum_{n=1}^{\infty} a_n$ diverges, since a_n does not tend to zero. ∎

Theorem 2.3: Ratio Test/ D'Alembert's Test

Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum_{n=1}^{\infty} a_n$ converges. If $\ell > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. As before, nothing can be said for $\ell = 1$.

Proof. Supposed $\ell < 1$ and choose r with $\ell < r < 1$. Then $\exists N$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r.$$

Therefore,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \ n > N.$$

So, $a_n < kr^n$ with k independent of n. Since $\sum\limits_{n=1}^{\infty} r^n$ converges, so does $\sum\limits_{n=1}^{\infty} a_n$ by Comparison Test.

If $\ell > 1$, choose $1 < r < \ell$. Then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N},\ n > N.$$

So the series diverges. ■

Lecture 4

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Example. To determine the convergence of $\sum\limits_{n=1}^{\infty} a_n = \frac{n}{2^n}$.

By ratio test,

$$\frac{n+1}{2^n}\frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$

So we have convergence by ratio test.

However, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, and ratio test gives limit 1, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, and ratio test gives limit 1. So ratio test is inconclusive if the limit is 1.

Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, so root test is also inconclusive when the limit is 1.

To see this limit, write

$$n^{\frac{1}{n}} = 1 + \delta_n,\ \delta_n > 0.$$

So

$$n = (1+\delta_n)^n > \frac{n(n-1)}{2}\delta_n^2.$$

And $\delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$.

Remark. Use the root test when there is a nth power in the series.

Theorem 2.4: Cauchy's Condensation Test

Let a_n be a decreasing sequence of positive terms. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Proof. First we observe that if a_n is decreasing

$$a_{2^k} \leq a_{2^{k-1}+i} \leq a_{2^{k-1}}$$

for all $k \geq 1$ and $1 \leq i \leq 2^{k-1}$.

Assume that $\sum\limits_{n=1}^{\infty} a_n$ converges with sum A. Then

$$\begin{aligned} 2^{n-1}a_{2^n} &= \underbrace{a_{2^n} + \cdots a_{2^n}}_{2^{n-1} \text{ times}} \\ &\leq a_{2^{n-1}+1} + \cdots + a_{2^n} \\ &= \sum_{m=2^{n-1}+1}^{2^n} a_m. \end{aligned}$$

Thus, $\sum\limits_{n=1}^{N} 2^{n-1}a_{2^n} \leq \sum\limits_{n=1}^{N} \sum\limits_{m=2^{n-1}+1}^{2^n} a_m = \sum\limits_{m=2}^{2^N} a_m$. So

$$\sum_{n=1}^{N} 2^n a_{2^n} \leq 2 \sum_{m=2}^{2^N} a_m \leq 2(A - a_1).$$

Thus, $\sum\limits_{n=1}^{N} 2^n a_{2^n}$ being increasing and bounded above, converges.

Conversely, assume $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges to B, then

$$\begin{aligned} \sum_{m=2^{n-1}+1}^{2^n} a_m &= a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} \\ &\leq \underbrace{a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} times} = 2^{n-1}a_{2^{n-1}}. \end{aligned}$$

Similarly, we have

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^{N} 2^{n-1}a_{2^{n-1}} \leq B.$$

Therefore, $\sum\limits_{m=1}^{N} a_m$ is a bounded increasing sequence and thus it converges. ■

Example. $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ for $k > 0$ converges if and only if $k > 1$. First we note that $\frac{1}{n^k}$ is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)k} < \frac{1}{n^k} \iff (\frac{n}{n+1})^k < 1 \iff \frac{n}{n+1} < 1.$$

We use Cauchy condensation test, and we have

$$\begin{aligned}2^n a_{2^n} &= 2^n \left(\frac{1}{2^n}\right)^k \\ &= 2^{n-nk} = (2^{1-k})^n.\end{aligned}$$

Which is a geometric series with the ratio 2^{1-k}. So $\sum\limits_{n=1}^{\infty} \frac{1}{n^k}$ converges if and only if $2^{1-k} < 1 \iff k > 1$.

2.2 Alternating Series

Theorem 2.5: Alternating Series Test

If a_n decreases and tends to 0 as $n \to \infty$, then the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Proof. The partial sum is

$$\begin{aligned}S_n &= a_1 - a_2 + \cdots + (-1)^{n+1} a_n \\ S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq S_{2n-1} \\ S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1\end{aligned}$$

So S_{2n} is increasing and bounded above, implying that $S_{2n} \to S$. The odd terms satisfy

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S + 0 = S.$$

This implies that S_n converges to S as well. Given ϵ, there exists N_1 such that $\forall n \geq N_1$, $|S_{2n} - S| < \epsilon$. We also know that there exists N_2 such that $\forall n \geq N_2$, $|S_{2n+1} - S| < \epsilon$. Take $N = 2\max\{N_1, N_2\} + 1$, then if $n \geq N$, $|S_k - S| < \epsilon$. So $S_k \to S$. ■

Lecture 5

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2.3 Absolute Convergence

Definition 2.3

Take $a_n \in \mathbb{C}$. If $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, then the series is called *absolutely convergent*.

Note. Since $|a_n| \geq 0$. We can use the previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

Theorem 2.6

If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose first $a_n \in \mathbb{R}$. Let

$$v_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$w_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

We have $v_n = \frac{|a_n|+a_n}{2}, w_n = \frac{|a_n|-a_n}{2}$. Clearly, $v_n, w_n \geq 0$. We also have $|a_n| = v_n + w_n \geq v_n, w_n$.

So by comparison test, if $\sum\limits_{n=1}^{\infty} |a_n|$ converges, $\sum\limits_{-=1}^{\infty} v_n$, $\sum\limits_{n=1}^{\infty} w_n$ also converges.

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$. We have $|x_n|, |y_n| \leq |a_n|$. So $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are absolutely convergent, so they are convergent. And $\sum\limits_{n=1}^{\infty} a_n$ converges as well. ∎

Example.

1. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely convergent.

2. $\sum\limits_{n=1}^{\infty} \frac{z^n}{2^n}$ for $z \in \mathbb{C}$. We check for absolute convergence first, $\sum\limits_{n=1}^{\infty} \left(\frac{|z|}{2}\right)^n$. So if $|z| < 2$, the series is convergent by absolute convergence.

   Otherwise, if $|z| \geq 2$, $\left|\frac{z}{2}\right| \geq 1$. a_n does not tend to zero, hence the series diverge.

Notation. If $\sum\limits_{n=1}^{\infty} a_n$ converges but not absolutely convergent, it is sometimes called *conditional convergent*.

It is called conditional because the sum to which the series converges is conditional on the order in which elements of the sequence are taken.

Example (Example Sheet 1, Q7)**.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ are two series with different sums. Let s_n be the partial sum of the first series, and t_n be the partial sum of the second series, then $s_n \to s$ and $t_n \to \frac{3s}{2}$.

Definition 2.4

Let σ be a bijection of the positive integers, $a'_n = a_{\sigma(n)}$ is a *rearrangement*.

Theorem 2.7

If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

Proof. Again we do the proof first for $a_n \in \mathbb{R}$. Let $\sum\limits_{n=1}^{\infty} a'_n$ be a rearrangement of $\sum\limits_{n=1}^{\infty} a_n$. Let $s_n = \sum\limits_{i=1}^{n} a_i$ and $t_n = \sum\limits_{i=1}^{n} a'_i$, $S = \sum\limits_{n=1}^{\infty} a_n$. Suppose first that $a_n \geq 0$. Given n, we can find q such that s_q contains every term of t_n. Because $a_n \geq 0$, we have

$$t_n \leq s_n \leq S.$$

So t_n is an increasing sequence bounded above so $t_n \to t$, and from the inequality above, $t \leq s$. By symmetry, we have $s \leq t \implies s = t$. If a_n has any negative term, consider v_n and w_n from Theorem 2.6. Consider $\sum\limits_{n=1}^{\infty} a'_n$, $\sum\limits_{n=1}^{\infty} v'_n$, $\sum\limits_{n=1}^{\infty} w'_n$. Since $\sum\limits_{n=1}^{\infty} |a_n|$ converges, both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge. Using the fact that $v_n, w_n \geq 0$, we case above, we have $\sum_{n=1}^{\infty} v_n' = \sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} w_n'$. But $a_n = v_n - w_n$ so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n'$.

For the case $a_n \in \mathbb{C}$, we write $a_n = x_n + iy_n$. Since $|x_i|, |y_i| \leq |a_n|$, $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are absolutely convergent. By the previous case $\sum_{n=1}^{\infty} x_n' = \sum_{n=1}^{\infty} x_n, \sum_{n=1}^{\infty} y_n' = \sum_{n=1}^{\infty} y_n$. Since $a_n' = x_n' + iy_n'$ so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n'$. ■

Lecture 6

2 Feb. 2022

3 Functions

3.1 Continuity

Suppose $E \subseteq \mathbb{C}$ is a non-empty subset, and we have a function $f : E \to \mathbb{C}$and a point $a \in E$. (this includes the case in which f is real-valued and E is a subset of $\mathbb{R}$)

Definition 3.1

f is *continuous* at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$.

Definition 3.2: ϵ-δ Definition

f is *continuous* at $a \in E$, if given $\epsilon > 0$, $\exists \delta > 0$ such that if $|z - a| < \delta, z \in E$, then $|f(z) - f(a)| < \epsilon$.

We prove right away that the two definitions are equivalent.

Theorem 3.1

The two definitions of continuity are equivalent.

Proof. We first prove the second definition implies the first definition. We know that given $\epsilon > 0, \exists \delta > 0$ such that $|z - a| < \delta, z \in E$, then $|f(z) - f(z)| < \epsilon$. Let $z_n \to a$, then $\exists n_0$ such that $\forall n \geq n_0$, we have $|z_n - a| < \delta$. This implies, by the assumption, $|f(z_n) - f(a)| < \epsilon$. That is, $f(z_n) \to f(a)$.

Next, we prove the other direction. Assume $f(z_n) \to f(a)$ whenever $z_n \to a, z_n \in E$. Suppose f is not continuous at a according to Definition 2.

$\exists \epsilon > 0$, s.t. $\forall \delta > 0$, there exists $z \in E$ s.t. $|z - a| < \delta$ and $|f(z) - f(a)| \geq \epsilon$.

Let $\delta = \frac{1}{n}$ from non-continuity defined above, we get z_n such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \epsilon$. Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \epsilon$. Contradiction. ∎

Proposition 3.1

$a \in E$, and $g, f : E \to \mathbb{C}$ are both continuous at a. So are the functions $f(z) + g(z)$, $f(z)g(z)$ and $\lambda f(z)$ for any constant λ. In addition, if $f(z) \neq 0\ \forall z \in E$, then $\frac{1}{f(z)}$ is continuous at a.

Proof. Using Definition 1 of continuity, this is obvious, using the analogous results for sequences. (Lemma 1.1)

For example,

$$z_n \to a \implies f(z_n) \to f(a), g(z_n) \to g(a) \implies f(z_n) + g(z_n) \to f(a) + g(a).$$

∎

The function $f(z) = z$ is continuous, so by using the proposition, we get that every polynomial is continuous at every point in $\mathbb{C}$.

Note. We say that f is *continuous on* E if it is continuous at every $a \in E$.

Remark. Still it is instructive to prove Proposition 3.1 directly from the ϵ-δ definition.

Next we look at compositions.

Theorem 3.2

Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions such that $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$, then $g \circ f : A \to \mathbb{C}$ is continuous at a.

Proof. Take any sequence $z_n \to a$, by assumption we know $f(z_n) \to f(a)$. Set $w_n = f(z_n) \in B$. By continuity of g, we have $g(w_n) \to g(f(a))$, and we are done. ■

Example.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$
assuming that $\sin x$ is continuous. (to be proved later) If $x \neq 0$, propositions proved above imply that $f(x)$ is continuous at any $x \neq 0$.

   However, it is discontinuous at 0. Consider the sequence satisfying
$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi.$$
We have $f(x_n) \to 1, x_n \to 0$, but $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$
It's continuous at $x \neq 0$ as above, and f is continuous at 0. Take $x_n \to 0$, then $|f(x_n)| \leq |x_n|$ because $\sin\frac{1}{x} \leq 1$, so $f(x_n) \to 0 = f(0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$
It is discontinuous at every point. If $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then $f(x_n) = 0 \not\to f(x) = 1$. Similarly, if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$, we have $f(x_n) = 1 \not\to f(x) = 0$.

Lecture 7

4 Jan. 2022

3.2 Limit of a function

$f : E \subseteq \mathbb{C} \to \mathbb{C}$. We wish to define what is meant by $\lim_{z\to a} f(z)$, even when a might not be in E.

Example. The limit of $\frac{\sin z}{z}$ as $z \to 0$ with $E = \mathbb{C}\{0\}$.

Also, if $E = \{0\} \cup [1, 2]$, it does not make sense to speak about points $z \in E, z \neq 0, z \to 0$.

Definition 3.3

If $E \subseteq \mathbb{C}, a \in \mathbb{C}$, we say that a is a *limit point* of E if for any $\delta > 0, \exists z \in E$ such that $0 < |z - a| < \delta$.

Remark. a is a limit point if and only if there exists a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all n.

Definition 3.4

If $f : E \subseteq \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$ be a limit point of E. We say that $\lim_{z\to a} f(z) = l$ ("f tends to l as z tends to a") if given $\epsilon > 0, \exists\delta > 0$ such that whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \epsilon$.

Equivalently, $f(z_n) \to l$ for every sequence $z_n \in E, z_n \neq a$ and $z_n \to a$.

Remark. Straight from the definitions, we have that if $a \in E$ is limit point, then $\lim_{z\to a} f(z) = f(a)$ if and only if f is continuous at a.

If $a \in E$ is *isolated* (i.e. $a \in E$ is not a limit point), continuity of f at a always holds. The limit of functions has very similar properties to limit of sequences.

1. It is unique, $f(z) \to A$ and $f(z) \to B$ as $z \to a$, then
$$|A - B| \leq |A - f(z)| + |f(z) - B|.$$
If $z \in E$ is such that $0 < |z - a| < \min\{\delta_1, \delta_2\}$, then $|A - B| < 2\epsilon$. So $A = B$. The existence of such z is a consequence of the condition that a is a limit point of E.

2. $f(z) + g(z) \to A + B$;

3. $f(z)g(z) \to AB$;

4. if $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$. All proved in the same way as before.

3.3 The Intermediate Value Theorem

Theorem 3.3: Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then f takes every value which lies between $f(a)$ and $f(b)$.

Proof. Without loss of generality, suppose $f(a) < f(b)$. Take $f(a) < \eta < f(b)$. Let $S = \{x \in [a, b] \mid f(x) < \eta\}$. We note that $a \in S$, so $S \neq \varnothing$. Clearly S is bounded above by b. Then there is a supremum C where $C \leq b$. By definition of supremum, given n, there exists $x_n \in S$ such that $C - \frac{1}{n} < x_n \leq C$. So $x_n \to C$. Since $x_n \in A$, $f(x_n) < \eta$. By continuity of f, $f(x_n) \to f(C)$. So $f(c) \leq \eta$.

Now observe that $c \neq b$ because $f(b) > \eta$. Then for n large, $C + \frac{1}{n} \in [a, b]$ and $C + \frac{1}{n} \to C$. Again by continuity $f(C + \frac{1}{n}) \to f(C)$. But since $C + \frac{1}{2} > C$, $f(C + \frac{1}{n}) \geq \epsilon$. So $f(c) \geq \eta \implies f(c) = \eta$. ∎

Remark. The theorem is very useful for finding zeroes or fixed points.

Example. Existence of the N-th root of a positive real number. Suppose

$$f(x) = x^N, \quad x \geq 0.$$

Let y be a positive real number. f is continuous on $[0, 1 + y]$, so

$$0 = f(0) < y < (1 + y)^N = f(1 + y).$$

By the IVT, $C \in (0, 1 + y)$ such that $f(c) = y$, i.e. $C^N = y$. C is a positive N-th root of y.

We also have uniqueness. Exercise.

Lecture 8

7 Feb. 2022

3.4 Bounds of a Continuous Function

Theorem 3.4

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists K such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Proof. We argue by contradiction. Suppose the statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j} \to x$. Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By the continuity of f, $f(x_{n_j}) \to f(x)$. But $\left|f(x_{n_j})\right| > n_j \to \infty$ as $j \to \infty$. Contradiction. ■

Theorem 3.5: Extreme Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$.

"A continuous function on a closed bounded interval is bounded and attains its bounds."

Proof. Let $A = \{f(x) \mid x \in [a, b]\} = f([a, b])$. By Theorem 3.4, A is bounded since it is clearly non-empty, it has a supremum M. By definition of supremum, given an integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $M - \frac{1}{n} < f(x_n) \leq M$. From Bolzano-Weierstrass, there exists $x_{n_j} \to x \in [a, b]$. Since $f(x_{n_j}) \to M$, by continuity of f, we get that $f(x) = M$. So $x_2 := x$.

We can prove similarly for the minimum. ■

Proof 2. $A = f([a, b]), M = \sup A$ as before. Suppose $\nexists x_2$ such that $f(x_2) = M$. Let

$$g(x) = \frac{1}{M - f(x)}, x \in [a, b]$$

is defined and continuous on $[a,b]$. By Theorem 3.4 applied to g, $\exists k > 0$ such that $g(x) < K$ for all $x \in [a,b]$. This means that $f(x) \leq M - \frac{1}{k}$ for all $x \in [a,b]$. This is absurd because it contradicts that M is the supremum. ■

Note. Theorems 3.4 and 3.5 are false if the interval is not closed and bounded. For example,

$$f : (0,1] \to \mathbb{R}, x \mapsto \frac{1}{x}.$$

3.5 Inverse Functions

Definition 3.5

f is *increasing* for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 such that $a \leq x_1 < x_2 \leq b$.

If $f(x_1) < f(x_2)$, we say that f is *strictly increasing*.

There are similar definitions for *decreasing* and *strictly decreasing*.

Theorem 3.6

$f : [a,b] \to \mathbb{R}$ is continuous and strictly increasing for $x \in [a,b]$. Let $c = f(a)$ and $d = f(b)$. Then $f : [a,b] \to [c,d]$ is bijective and the inverse $g := f^{-1} : [c,d] \to [a,b]$ is also continuous and strictly increasing.

Remark. There is a similar statement for strictly decreasing function. Take $c < k < d$, from the IVT, $\exists h$ such that $f(h) = k$. Since f is strictly increasing, h is unique. Define $g(k) := h$ and this gives an inverse $g : [c,d] \to [a,b]$ for f.

We first prove that g is strictly increasing. Take $y_1 < y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. If $x_2 \leq x_1$, since f is increasing, $f(x_2) \leq f(x_1) \implies y_2 \leq y_1$. Absurd.

Next we prove continuity. Let $\epsilon > 0$ be given, let $k_1 = f(h - \epsilon)$ and $k_2 = f(h + \epsilon)$. Because f is strictly increasing, we have $k_1 < k < k_2$. If $k_1 < y < k_2$, we have $h - \epsilon < g(y) < h + \epsilon$. So we can just take $\delta = \min\{k_2 - k, k - k_1\}$. So g is continuous at k. Here we took $k \in (c,d)$. A very similar argument establishes continuity at the end points.

Lecture 9

9 Feb. 2022

4 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E = \text{interval} \subseteq \mathbb{R}$.

Definition 4.1

Let $x \in E$ be a point such that $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point), f is said to be *differentiable* at x with derivative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x).$$

If f is differentiable at each $x \in E$, we say f is differentiable on E.

(Think of E as an interval or a disc in the case of $\mathbb{C}$.)

Remark.

1. Other common notations include $\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}f}{\mathrm{d}x}$.
2. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$. ($y = x + h$)
3. Another look at the definition is the following.

   Let $\epsilon(h) := f(x+h) - f(x) - hf'(x)$, then $\lim\limits_{h \to 0} \frac{\epsilon(h)}{h} = 0$. We have also

   $$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear in } h} + \epsilon(h).$$

   Alternative definition of differentiability is f is differentiable at x if $\exists A, E$ such that $f(x+h) = f(x) + hA + \epsilon(h)$ where $\lim\limits_{h \to 0} \frac{\epsilon}{h} = 0$. If such an A exists, then it is unique, since $A = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.
4. If f is differentiable at x, then f is continuous. Since $\epsilon(h) \to 0$, then $f(x+h) \to f(x)$ as $h \to 0$.

5. Alternative ways of writing things:

$$f(x+h) = f(x) + hf'(x) + h\epsilon_f(h) \text{ with } \epsilon_f(h) \to 0 \text{ as } h \to 0.$$

Or,

$$f(x) = f(a) + (x-a)f'(a) + (x-a)\epsilon_f(x) \text{ with } \epsilon_f(x) \to 0 \text{ as } x \to a.$$

Example. If we have $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Clearly, we have $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Take $h_n \downarrow 0$ at $x = 0$, we have

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n\to\infty} \frac{h_n}{h_n} = 1.$$

And take $h_n \uparrow 0$ at $x = 0$, we have

$$\lim_{n\to\infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n\to\infty} \frac{-h_n}{h_n} = -1.$$

So f is not differentiable at $x = 0$.

4.1 Differentiation of Sums, Products, etc

Property.

1. If $f(x) = c$ for all $x \in E$, then f is differentiable with $f'(x) = 0$.
2. f, g are differentiable at x, then so is $f + g$ and
$$(f+g)'(x) = f'(x) + g'(x).$$
3. f, g are differentiable at x, then so is fg and
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$
4. f differentiable at x and $f(x) \neq 0$ for all $x \in E$, then $\frac{1}{f}$ is differentiable at x and
$$(\frac{1}{f})'(x) = \frac{-f'(x)}{[f(x)]^2}.$$

Proof.

1. $\lim\limits_{h\to 0}\frac{c-c}{h}=0.$

2. $$\lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} \quad \text{using properties of limits.}$$
$$=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$
$$=f'(x)+g'(x)$$

3. Let $\phi(x)=f(x)g(x)$, then we have
$$\begin{aligned}\frac{\phi(x+h)-\phi(x)}{h}&=\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\&=f(x+h)[\frac{g(x+h)-g(x)}{h}]+g(x)[\frac{f(x+h)-f(x)}{h}].\end{aligned}$$
So we have $\lim\limits_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}=f(x)g'(x)+f'(x)g(x)$ using standard properties of limits and the fact that f is continuous at x.

4. Define again $\phi(x)=\frac{1}{f(x)}$, then
$$\begin{aligned}\frac{\phi(x+h)-\phi(x)}{h}&=\frac{\frac{1}{f(x+h)}-\frac{1}{f(x)}}{h}\\&=\frac{f(x)-f(x+h)}{hf(x)f(x+h)}.\end{aligned}$$
So we have $\lim\limits_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}=\frac{-f(x)}{[f(x)]^2}$.

■

Remark. From (3) and (4), we get
$$\left(\frac{f(x)}{g(x)}\right)'=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}.$$

Lecture 10

11 Feb. 2022

Example. Consider $f(x)=x^n$ with $n\in\mathbb{Z}, n>0$. When $n=1$, clearly we have $f(x)=x$ and $f'(x)=1$.

We claim that $f'(x) = nx^{n-1}$, and we prove it by induction, $f(x) = xx^n = x^{n+1}$. By product rule and inductive hypothesis,

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Next, we consider $f(x) = x^{-n}$ with $n \in \mathbb{Z}, n > 0$. If $x \neq 0$, use property 4.1, we have

$$f'(x) = \frac{-(x^n)'}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

So we know how to find derivatives of polynomials and rational functions.

We have the following useful result to differentiate a larger class of functions.

Theorem 4.1: Chain Rule

If $f : U \to \mathbb{C}$ is such that $f(x) \in V$ for $x \in U$. If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a)g'(f(a)).$$

Proof. We know

$$f(x) = f(a) + (x-a)f'(a) + \epsilon_f(x)(x-a)$$

such that $\lim_{x \to a} \epsilon_f(x) = 0$, and

$$g(y) = g(b) + (y-b)g'(b) + \epsilon_g(y)(y-b)$$

with $\lim_{y \to b} \epsilon_g(y) = 0$. Let $b = f(a)$, and set $\epsilon_f(a) = 0$ and $\epsilon_g(b) = 0$ to make them continuous at $x = a$ and $f = b$. Now $y = f(x)$ gives

$$\begin{aligned}
g(f(x)) =& g(b) + (f(x) - b)g'(b) + \epsilon_g(f(x))(f(x) - b) \\
=& g(f(a)) + [(x-a)f'(a) + \epsilon_f(x)(x-a)][g'(b) + \epsilon_g(f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + \\
& (x-a)[\epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + (x-a)\sigma(x).
\end{aligned}$$

So it suffices to show $\sigma(x) = \epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))$ tends to 0 as x tends to a. We have clearly $\epsilon_f(x)g'(b) \to 0$, $\epsilon_g(f(x)) \to 0$ and $f'(a) + \epsilon_f(x) \to f'(a)$, so $\lim_{x \to a} \sigma(x) = 0$. ∎

Example.

1. Consider $f(x) = \sin(x^2)$, and we have

$$f'(x) = 2x\cos(x^2).$$

2. Consider $f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. From previous lectures, we know that f is continuous, and it is differentiable at every $x \neq 0$ by the previous theorems. At $x = 0$, take $t \neq 0$ and we have

$$\frac{f(t) - f(0)}{t - 0} = \sin(\frac{1}{t}).$$

Again from previous lecture, we know $\lim\limits_{t \to 0} \frac{f(t)-f(0)}{t-0}$ does not exist, so f is not differentiable at $x = 0$.

4.2 The Mean Value Theorem

Theorem 4.2: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let $M = \max_{x \in [a,b]} f(x)$, and $m = \min_{x \in [a,b]} f(x)$. Theorem 3.5 says that these values are achieved. Let $k = f(a) = f(b)$. If $M = m = k$, then f is constant and $f'(c) = 0$ for all $c \in (a, b)$.

If f not constant, then $M > k$ or $m < k$. Suppose $M > k$. By Theorem 3.5, exist $c \in (a, b)$ such that $f(c) = M$.

If $f'(c) > 0$, then there are values to right of c for which $f(x) > f(c)$ because

$$f(h + c) - f(c) = h(f'(c) + \epsilon_f(h)).$$

Since $\epsilon_f(h) \to 0$ as $h \to 0$, $f'(c) + \epsilon_f(h) > 0$ for h small. This contradicts that M is the maximum. Similarly, if $f'(c) < 0$, there exists x to the left of c for which $f(x) > f(c)$.

So we must have $f'(c) = 0$. ∎

Lecture 11

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Theorem 4.3: Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b), then $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Write $\phi(x) = f(x) - kx$, and choose k such that $\phi(a) = \phi(b)$. So

$$f(b) - bk = f(a) - ak \implies k = \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem applied to ϕ, $\exists c \in (a, b)$ such that $\phi'(c) = 0$. That is, $f'(c) = k$. ■

Remark. We will often write

$$f(a + h) = f(a) + hf'(a + \theta h)$$

with $\theta \in (0, 1)$. We need to be careful, and consider $\theta = \theta(h)$.

Corollary 4.1

$f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b).

1. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

2. If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)

3. If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.

Proof.

1. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) > 0 \implies f(y) > f(x)$.

2. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) \geq 0 \implies f(y) \geq f(x)$.

3. Take $x \in [a, b]$. Then use the MVT in $[a, x]$ to get $c \in (a, x)$ such that $f(x) - f(a) = f'(c)(x - a) = 0$. So $f(x) = f(a)$ and f is constant.

∎

Theorem 4.4: Inverse Function Theorem

If $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b) with $f'(x) > 0$ for all $x \in (a,b)$. Let $f(a) = c$ and $f(b) = d$, then the function $f : [a,b] \to [c,d]$ is bijective and f^{-1} is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Proof. By Corollary 4.1, f is strictly increasing in $[a,b]$. By Theorem 3.6, $\exists g : [c,d] \to [a,b]$ which is a continuous strictly increasing inverse of f. We want to show that g is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$ and $x \in (a,b)$.

If $k \neq 0$ is given, let h be given by $y + k = f(x+h)$. That is, $g(y+k) = x + h$ for $h \neq 0$. Then

$$\frac{g(y+k) - g(y)}{k} = \frac{x + h - x}{f(x+h) - f(x)} = \frac{h}{f(x+h) - f(x)}.$$

Let $k \to 0$, then $h \to 0$ because g is continuous. So we have

$$g'(y) = \lim_{k \to 0} \frac{g(y+k) - g(y)}{k} = \frac{1}{f'(x)}.$$

∎

Example. We take $g(x) = x^{\frac{1}{q}}$ with $x > 0$ and q positive integer. So $f(x) = x^q$, with $f'(x) = qx^{q-1}$. g is differentiable and so is g, and by Theorem 4.4,

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q} x^{\frac{1}{q} - 1}.$$

Remark. If $g(x) = x^r$ with $r \in \mathbb{Q}$, then $g'(x) = rx^{r-1}$.

Suppose $f, g : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b) and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b-a)f'(s)}{(b-a)g'(t)} = \frac{f'(s)}{g'(t)}.$$

Cauchy showed that we can take $s = t$.

Theorem 4.5: Cauchy's Mean Value Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then $\exists t \in (a, b)$ such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a)).$$

Remark. We recover the MVT if we take $g(x) = x$.

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Proof. Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}.$$

We have ϕ continuous on $[a, b]$ and differentiable on (a, b). Also, $\phi(a) = \phi(b) = 0$. By Rolle's Theorem, there exists $t \in (a, b)$ such that $\phi'(t) = 0$, and

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)]. \end{aligned}$$

So $\phi'(t) = 0$ gives the result. ■

Note. Good choice of auxiliary function and Rolle's theorem proves the theorem.

Example (L' Hopital's Rule)**.** If we want to find $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$, we have

$$\frac{e^x - 1}{\sin x} = \frac{e^x - x^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

for some $t \in (0, x)$ by Cauchy's Mean Value Theorem. So

$$\frac{e^x - 1}{\sin x} \to 1$$

as $x \to 0$.

Goal: We want to extend the MVT to include higher order derivatives.

Theorem 4.6: Taylor's Theorem with Lagrange's reminder

Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!f''(a)} + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n f^{(n)}(a+\theta h)}{n!}$$

where $\theta \in (0, 1)$.

Note.

1. For $n = 1$, we get back the MVT, so this is an "n-th order MVT".
2. $R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$ is known as Lagrange's form of the remainder.

Proof. Define for $0 \leq t \leq h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} b$$

where we choose b such that $\phi(h) = 0$, and we clearly have $\phi(0) = 0$. (Recall that in the proof of the MVT, we used $f(x) - kx$ and picked k that we can use Rolle's Theorem. We also have that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0.$$

We use Rolle's Theorem n times. Since $\phi(0) = \phi(h) = 0$, $\phi'(h_1) = 0$ for some $0 < h_1 < h$. And since $\phi'(0) = \phi'(h_1) = 0$, we have $\phi''(h_2) = 0$ for some $0 < h_2 < h_1$. Finally, $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$. So $\phi^{(n)}(h_n) = 0$ with $0 < h_n < h_{n-1} < \cdots < h$. So $h_n = \theta h$ for $\theta \in (0, 1)$, now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - b \implies b = f^{(n)}(a + \theta h).$$

Set $t = h$, $\phi(h) = 0$ and put this value of b to the second line in the proof. ∎

Theorem 4.7: Taylor's Theorem with Cauchy's reminder

Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, and if $a = 0$ for simplification, then we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1-\theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!}$$

with $\theta \in (0, 1)$.

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1} f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$. Note that we have

$$\begin{aligned} F'(t) &= -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \cdots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t) \\ &= -\frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t). \end{aligned}$$

Set

$$\phi(t) = F(t) - (\frac{h-t}{h})^p F(0)$$

with $p \in \mathbb{Z}, 1 \leq p \leq n$. Then $\phi(0) = \phi(h) = 0$, and by Rolle's, $\exists \theta \in (0, 1)$ such that $\phi'(\theta h) = 0$. But,

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} F(0) = 0.$$

So

$$0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h} [f(h) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0)].$$

Rearranging the two sides, and we get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n (1-\theta)^{n-1}}{(n-1)! p (1-\theta)^{p-1}} f^{(n)}(\theta h).$$

Taking $p = n$, we get Lagrange's reminder, and taking $p = 1$ gives Cauchy's reminder. ■

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To get a Taylor series for f, one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort".

Remark. Theorems 4.6 and 4.7 work equally well in an interval $[a+h, a]$ with $h < 0$.

Example. The binomial series

$$f(x) = (1+x)^r, r \in \mathbb{Q}.$$

We claim that $|x| < 1$, then

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

Proof. Clearly,

$$f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}.$$

If $r \in \mathbb{Z}_{\geq 0}$, then $f^{(n+1)} = 0$, we have a polynomial of degree r.

In general, by Lagrange's reminder, we have

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \binom{r}{n}\frac{x^n}{(1+\theta x)^{n-r}}.$$

Note that θ depends on both x and n.

For $0 < x < 1$, $(1+\theta x)^{n-r} > 1$ for $n > r$. Now observe that the series $\sum \binom{r}{n}x^n$ is absolutely convergent for $|x| < 1$. Indeed, by the ratio test,

$$\begin{aligned} a_n &= \binom{r}{n}x^n \\ \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{r(r-1)(r-n+1)\cdots(r-n)x^{n+1}}{(n+1)!}\right|\left|\frac{n!}{r(r-1)\cdots(r-n+1)x^n}\right| \\ &= \left|\frac{(r-n)x}{n+1}\right| \end{aligned}$$

which tends to a value less than 1. In particular, $a_n \to 0$ and $\binom{r}{n}x^n \to 0$.

Hence, for $n > r$, and $0 < x < 1$, we have that $|R_n| \leq \left|\binom{r}{n}x^n\right| = |a_n| \to 0$ as $n \to \infty$.

So the claim is proved in the range $0 \leq x < 1$. If $-1 < x < 0$, the argument above breaks, but Cauchy's form for R_n works.

$$\begin{aligned} R_n &= \frac{(1-\theta)^{n-1} r(r-1)\cdots(r-n+1)(1+\theta h)^{r-n} x^n}{(n-1)!} \\ &= \frac{r(r-1)\cdots(r-n+1)}{(n-1)!} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^n \\ &= r\binom{r-1}{n-1} x^n (1+\theta x)^{r-1} (\frac{1-\theta}{1+\theta x})^{n-1}. \end{aligned}$$

So $|R_n| \leq \left|r\binom{r-1}{n-1}x^n\right|(1+\theta x)^{r-1}$. Check that $(1+\theta x)^{r-1} \leq \max\{1, (1+x)^{r-1}\}$. Let $K_r = |r| \max\{1, (1+x)^{r-1}\}$ is independent of n. So we have

$$|R_n| \leq |K_r| \left| \binom{r-1}{n-1} x^n \right| \to 0.$$

So $R_n \to 0$ as $n \to \infty$. ■

4.3 Remarks on Complex Differentiation

Formally, for function $f : E \subseteq \mathbb{C} \to \mathbb{C}$, we have properties for sums, products, chain rule etc. But it is much more restrictive than differentiability on the real line.

Example. $f : \mathbb{C} \to \mathbb{C}$, with $z \mapsto \bar{z}$. We consider the sequence $z_n = z + \frac{1}{n} \to z$.

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1.$$

If we approach it vertically instead, taking $z_n = z + \frac{i}{n} \to z$, we have

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1.$$

So $\lim\limits_{w \to z} \frac{f(w)-f(z)}{w-z}$ does not exist. f is nowhere $\mathbb{C}$-differentiable.

If we consider it as a function on $\mathbb{R}^2$, $f(x,y) = (x, -y)$. It is real differentiable.

In fact, if a function is complex differentiable, it is infinitely complex differentiable. It is discussed in more detail in IB Complex Analysis.

Lecture 14

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5 Power Series

We want to look at

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $z \in \mathbb{C}, a_n \in \mathbb{C}$. The case $\sum\limits_{n=0}^{\infty} a_n (z-z_0)^n$, with z fixed can be reduced to power series around 0 by translation.

Lemma 5.1

If $\sum\limits_{n=0}^{\infty} a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum\limits_{n=0}^{\infty} a_n z^n$ converges absolutely.

Proof. Since $\sum\limits_{n=0}^{\infty} a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus, there exists $K > 0$ such that $|a_n z_1^n| \leq K$ for all n.

Then,

$$|a_n z^n| = |a_n z^n| \frac{|z_1^n|}{|z_1^n|} \leq K \left| \frac{z}{z_1} \right|^n.$$

Since the geometric series $\sum\limits_{n=0}^{\infty} \left| \frac{z}{z_1} \right|^n$ converges, the lemma follows by comparison. ∎

Using this lemma, we will prove that every power series has a *radius of convergence*.

Theorem 5.1

A power series either

1. converges absolutely for all z, or
2. converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outsider it, or
3. converges for $z = 0$ only.

Definition 5.1

The circle $|z| = R$ is called the *circle of convergence* and R the *radius of convergence*.

In (1) of Theorem 5.1, we agree that $R = \infty$, and in (3) $R = 0$, so $R \in [0, \infty]$.

Proof. Let $S = \{x \in \mathbb{R} \mid x \geq 0, \sum a_n x^n \text{ converges}\}$. Clearly $0 \in S$. By (5.1), if $x_1 \in S$, then $[0, x_1] \subseteq S$. If $S = [0, \infty)$, we have case 1.

Otherwise, there exists a finite supremum for S. $R = \sup S < \infty, R \geq 0$. If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely. Pick R_0 such that $|z_1| < R_0 < R$, then $R_0 \in S$ and the series converges for $z = R_0$. By (5.1), $\sum |a_n z_1^n|$ converges.

Finally, we show that if $|z_2| > R$, then the series does not converge for z_2. Pick $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges, by (5.1), $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. ■

The following lemma is useful for computing R.

Lemma 5.2

If $\left|\frac{a_{n+1}}{a_n}\right| \to \ell$, as $n \to \infty$, then $R = \frac{1}{\ell}$.

Proof. By the ratio test, we have absolute convergence if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| = \ell |z| < 1.$$

So if $|z| < \frac{1}{\ell}$, we have absolute convergence. If $|z| > \frac{1}{\ell}$, the series diverges, again by the ratio test. ■

Remark. One can also use the Root Test to get that if $|a_n|^{\frac{1}{n}} \to \ell$, then $R = \frac{1}{\ell}$.

Example.

1. $\sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$.

We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = \ell.$$
So $R = \infty$.

2. Geometric Series, $\sum\limits_{n=0}^{\infty} z^n$.

   By ratio test, we have $R = 1$. Note that at $|z| = 1$, we have divergence.

3. $\sum\limits_{n=0}^{\infty} n!z^n$.

   We have
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{n!} = n+1 \to \infty.$$
   So $R = 0$.

4. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$, and we have $R = 1$. For $z = 1$, it diverges. What happens for $|z| = 1$ and $z \neq 1$?

   Consider $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}(1-z)$, we have the partial sum
$$\begin{aligned} S_N &= \sum_{n=1}^{N} (\frac{z^n - z^{n+1}}{n}) \\ &= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} \\ &= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=2}^{N+1} \frac{z^n}{n-1} \\ &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} z^n (\frac{1}{n} - \frac{1}{n-1}) \\ &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} -\frac{z^n}{n(n-1)}. \end{aligned}$$
   If $|z| = 1$, $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum \frac{1}{n(n-1)}$ converges, so S_N converges.

5. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$, and we have $R = 1$. But it converges for all z with $|z| = 1$.

Conclusion is that, in principle, nothing can be said about $|z| = R$ and each case has to be discussed separately.

Within the radius of convergence, "life is great". Power series behave as if "they were polynomials".

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Theorem 5.2

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius convergence R. Then f is differentiable at all points with $|z| < R$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

Proof (non examinable).

Watch the lecture and finish the proof.

∎

5.1 The Standard Functions

In this section, we will discuss exponential, logarithmic, trigonometric, etc.

We have already seen that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has $R = \infty$. Define $e : \mathbb{C} \to \mathbb{C}$, $z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}$. From Theorem 5.2, e is differentiable, and $e'(z) = e(z)$.

If $F : \mathbb{C} \to \mathbb{C}$ has $F'(z) = 0$ for all $z \in \mathbb{C}$, then F is constant.

Proof. Consider $g(t) = F(tz)$, and chain rule gives $g'(t) = F'(tz)z = 0$. If $g(t) = u(t) + iv(t)$. It is immediate that $g'(t) = u'(t) + iv'(t)$. So $u'(t) = v'(t) = 0$. By previously proved corollary, we have $u(t), v(t)$ constant. Thus, $F(z)$ is constant. ■

Now let $a, b \in \mathbb{C}$. Consider

$$F(z) = e(a + b - z)e(z).$$

We have $F'(z) = 0$, so F is constant.

$$e(a + b - z)e(z) = F(0) = e(a + b).$$

Setting $z = b$ gives

$$e(a)e(b) = e(a + b).$$

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Now we restrict to $\mathbb{R}$, $e : \mathbb{R} \to \mathbb{R}$.

Theorem 5.3

1. $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$.
2. $e(x + y) = e(x)e(y)$.
3. $e(x) > 0$ for all $x \in \mathbb{R}$.
4. e is strictly increasing.
5. $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$.
6. $e : \mathbb{R} \to (0, \infty)$ is a bijection.

Proof. We proved (1) and (2) above.

To prove (3), clearly we have $e(x) > 0$ for $x \geq 0$ and $e(0) = 1$. Also, $e(0) = e(x - x) = e(x)e(-x0 = 1$. So $e(-x) > 0$ for $x > 0$.

To prove (4), we have $e'(x) = e(x) > 0$ so e is strictly increasing.

To prove (5), $e(x) > 1 + x$ for $x > 0$. So if $x \to \infty$, clearly $e(x) \to \infty$. For $x > 0$, since $e(-x) = \frac{1}{e(x)}$, then $e(x) \to 0$ as $x \to -\infty$.

For (6), injectivity follows right away from being strictly increasing. To prove surjectivity, take any $y \in (0, \infty)$. Since $e(x) \to \infty$ as $x \to \infty$ and $e(x) \to 0$ as $x \to -\infty$. We can find a, b such that $e(a) < y < e(b)$. By the IVT, there exists $x \in \mathbb{R}$ such that $e(x) = y$. ■

Remark. $e : (\mathbb{R}, +) \to ((0, \infty), \cdot)$ is a group isomorphism.

Since e is a bijection, we have an inverse $\ell : (0, \infty) \to \mathbb{R}$.

Theorem 5.4

1. $\ell : (0, \infty) \to \mathbb{R}$ is a bijection and $\ell(e(x)) = x$ for all $x \in \mathbb{R}$ and $e(\ell(t)) = t$ for all $t \in (0, \infty)$.

2. ℓ is differentiable and $\ell'(t) = \frac{1}{t}$.

3. $\ell(xy) = \ell(x) + \ell(y)$ for all $x, y \in (0, \infty)$.

Proof.

1. Obvious from the definition of e.

2. Inverse rule gives that ℓ is differentiable and
$$\ell'(t) = \frac{1}{e'(\ell(t))} = \frac{1}{t}.$$

3. From IA Groups, if e is an isomorphism, so is its inverse.

■

Now define for $\alpha \in \mathbb{R}$ and $x > 0$,
$$r_\alpha(x) = e(\alpha \lambda(x)).$$

Theorem 5.5

Suppose $x, y > 0$, and $\alpha, \beta \in \mathbb{R}$, then

1. $r_\alpha(xy) = r_\alpha(x) r_\alpha(y)$;
2. $r_{\alpha+\beta} = r_\alpha(x) r_\beta(x)$;
3. $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$;
4. $r_1(x) = x$ and $r_0(x) = 1$.

Proof.

1. $$\begin{aligned} r_\alpha(xy) &= e(\alpha\ell(xy)) \\ &= e(\alpha\ell(x) + \alpha\ell(y)) \\ &= e(\alpha\ell(x))e(\alpha\ell(y)) \\ &= r_\alpha(x) r_\alpha(y). \end{aligned}$$

2. $$\begin{aligned} r_{\alpha+\beta}(x) &= e((\alpha+\beta)\ell(x)) \\ &= e(\alpha\ell(x))e(\beta\ell(b)) \\ &= r_\alpha(x) r_\beta(x). \end{aligned}$$

3. $$\begin{aligned} r_\alpha(r_\beta(x)) &= r_\alpha(e(\beta\ell(x))) \\ &= e(\alpha\ell \circ e(\beta\ell(x))) \\ &= e(\alpha\beta\ell(x)) \\ &= r_{\alpha\beta}(x). \end{aligned}$$

4. $r_1(x) = e(\ell(x)) = x$ and $r_0(x) = e(0\ell(x)) = e(0) = 1$

■

For some $n \in \mathbb{Z}_{\geq 1}$, then

$$r_n(x) = r_{\underbrace{1 + \cdots + 1}_{n}}(x) = \underbrace{x \cdots x}_{n} = x^n.$$

We also have

$$r_1(x) r_{-1}(x) = r_0(x) = 1 \implies r_{-1}(x) = \frac{1}{x}.$$

So $r_{-n}(x) = \frac{1}{x^n}$. Next we consider for $q \in \mathbb{Z}_{\geq 1}$,

$$(r_{\frac{1}{q}})^q = r_1(x) = x \implies r_{\frac{1}{q}}(x) = x^{\frac{1}{q}}.$$

And thus we also have $r_{\frac{p}{q}}(x) = (r_{\frac{1}{q}}(x))^p = x^{\frac{p}{q}}$. Thus, $r_\alpha(x)$ agrees with x^α when $\alpha \in \mathbb{Q}$ as previously defined.

Now we give the above functions names:

1. $\exp(x) = e(x)$ for $x \in \mathbb{R}$;
2. $\log x = \ell(x)$ for $x \in (0, \infty;$
3. $x^\alpha = r_\alpha(x)$ for $\alpha \in \mathbb{R}, x \in (0, \infty)$.

If $e(x) = e(x \log e)$ where $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$, we have $e(x) = r_x(e) = e^x$. $\exp(x)$ is also a power, which we may as well write as e^x.

Finally, we compute

$$\begin{aligned}(x^\alpha)' &= (e^{\alpha \log x})' \\ &= e^{\alpha \log x} \alpha \frac{1}{x} \\ &= \alpha x^{\alpha - 1}.\end{aligned}$$

And we have

$$(a^x)' = (e^{x \log a})' = e^{x \log a} \log a = a^x \log a.$$

Lecture 17: Trigonometric Functions

28 Feb. 2022

Remark ("Expoentials beat polynomials")**.** We have

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty. \qquad (k > 0)$$

We use power series to prove it. We have

$$e^x = \sum_{i=0}^{\infty} > \frac{x^n}{n!}. \qquad (x > 0)$$

Now we pick $n > k$, and we have

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty \text{ as } x \to \infty.$$

Definition 5.2: Trigonometric Functions

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by Theorem 5.2, we get

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z.$$

Also note that we have

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=1}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{n=1}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \cos z + i \sin z.$$

Similarly,

$$e^{-iz} = \cos z - i \sin z.$$

So we can write $\cos z$ and $\sin z$ as

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

From this, we get many trigonometric identities. For example,

$$\cos z = \cos(-z), \quad \sin(-z) = -\sin z, \quad \cos(0) = 1, \quad \sin(0) = 0.$$

And the addition formulas

1. $\sin(z + w) = \sin z \cos w + \cos z \sin w$;

2. $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

They essentially follow from $e^{a+b} = e^a e^b$. To prove (2), write

$$\cos(z + w) = \frac{1}{2}(e^{i(z+w)} + e^{-i(z+w)}),$$

and expand $\cos z \cos w - \sin z \sin w$ similarly to get the result. We can also get

$$\sin^2 z + \cos^2 z = 1 \qquad \forall z \in \mathbb{C} \tag{1}$$

by direct computation.

Now if $x \in \mathbb{R}$, then $\sin x, \cos x \in \mathbb{R}$, and eq. (1) gives $|\sin x|, |\cos x| \leq 1$. We should be careful that they don't have to be bounded when z is not real. For example

$$\cos(iy) = \frac{1}{e^{-y} + e^{y}}. \qquad y \in \mathbb{R}$$

So $\cos(iy) \to \infty$ as $y \to \infty$.

Proposition 5.1

There is a smallest positive number w (where $\sqrt{2} < \frac{w}{2} < \sqrt{3}$) such that

$$\cos \frac{w}{2} = 0.$$

Proof. If $0 < x < 2$,

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

since $0 < x < 2 \implies \frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!}$, and $(\cos x)' = -\sin x < 0$ for $0 < x < 2$. So $\cos x$ is strictly decreasing on $(0, 2)$. We will show that $\cos\sqrt{2} > 0$ and $\cos\sqrt{3} < 0$. Then by the intermediate value theorem, the existence of w follows. Now we prove that.

$$\cos\sqrt{2} = \left(\frac{(\sqrt{2})^2}{4!} - \frac{(\sqrt{2})^6}{6!}\right) + \cdots > 0$$

$$\cos\sqrt{3} = 1 - \frac{3}{2} + \frac{9}{4!} - \left(\frac{x^6}{6!} - \frac{x^8}{8!}\right) - \cdots = -\frac{1}{8} - \cdots < 0,$$

and we are done. ■

Corollary 5.1

$$\sin \frac{w}{2} = 1.$$

Proof. We have $\sin^2 \frac{w}{2} + \cos^2 \frac{w}{2} = 0$, and we know that sin is positive on $(0, 2)$. ■

Now we define $\pi = w$.

Theorem 5.6

1. $\sin(z + \frac{\pi}{2}) = \cos z$, $\cos(z + \frac{\pi}{2}) = -\sin z$;
2. $\sin(z + \pi) = -\sin z$, $\cos(z + \pi) = -\cos z$;
3. $\sin(z + 2\pi) = \sin z$, $\cos(z + 2\pi) = \cos z$.

Proof. It is immediate from addition formulas and $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$. ■

This implies

$$\begin{aligned} e^{iz+2\pi i} &= \cos(z + 2\pi) + i \sin(z + 2\pi) \\ &= \cos z + i \sin z \\ &= e^{iz}. \end{aligned}$$

So e^z is periodic with period $2\pi i$.

Remark ("Relation with geometry"). Given two vectors $x, y \in \mathbb{R}^2$, define $x \cdot y$ as in Part IA Vector and Matrices,

$$x \cdot y = x_1 y_1 + x_2 y_2.$$

Cauchy-Schwarz gives $|x \cdot y| \leq \|x\| \|y\|$. So, for $x \neq 0$, $y \neq 0$,

$$-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1.$$

Define the angle between x and y as the unique $\theta \in [0, \pi]$ with $\cos \theta = \frac{x \cdot y}{\|x\| \|y\|}$. And we recover the unit circle picture by defining angle this way.

Lecture 18: Hyperbolic Functions

2 Mar. 2022

Definition 5.3

$$\begin{aligned} \cosh z &= \frac{1}{2}(e^z + e^{-z}) \\ \sinh z &= \frac{1}{2}(e^z - e^{-z}) \end{aligned}$$

We clearly have the relationships

$$\begin{aligned}
\cosh z &= \cos(iz) \\
\sinh z &= -i\sin(iz) \\
(\cosh z)' &= \sinh z \\
(\sinh z)' &= \cosh z \\
\cosh^2 z - \sinh^2 z &= 1.
\end{aligned}$$

The rest of the trigonometric functions (tan, cot, sec, csc) are defined in the usual way.

6 Integration

Suppose we have a function $f : [a, b] \to \mathbb{R}$ bounded. That is $\exists K$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Definition 6.1

A *dissection* (or *partition*) $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points a and b. We write

$$\mathcal{D} = \{x_0, x_1, \ldots, x_n\}$$

with $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

A picture of the function

Definition 6.2

We define the *upper sum* and *lower sum* associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(upper)}$$

$$\mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \qquad \text{(lower)}$$

Clearly, $\mathscr{S}(f, \mathcal{D}) \leq S(f, \mathcal{D})$ for any $\mathcal{D}$.

Lemma 6.1

If $\mathcal{D}, \mathcal{D}'$ are dissections with $\mathcal{D}' \supseteq \mathcal{D}$, then

$$S(f,\mathcal{D}) \geq S(f,\mathcal{D}') \geq \mathscr{S}(f,\mathcal{D}') \geq \mathscr{S}(f,\mathcal{D}).$$

Proof. $S(f,\mathcal{D}') \geq \mathscr{S}(f,\mathcal{D}')$ is obvious.

Suppose $\mathcal{D}'$ contains an extra point than $\mathcal{D}$, let's say $y \in (x_{r_1}, x_r)$. Clearly,

$$\sup_{x\in[x_{r-1},y]} f(x), \sup_{x\in[y,x_r]} f(x) \leq \sup_{x\in[x_{r-1},x_r]} f(x).$$

So we have

$$(x_r - x_{r-1}) \sup_{x\in[x_{r-1},x_r]} f(x) \geq (y - x_{r-1}) \sup_{x\in[x_{r-1},y]} f(x) + (x_r - y) \sup_{x\in[y,x_r]} f(x).$$

And the same argument goes for $\mathscr{S}$ and the same if $\mathcal{D}'$ has more extra points than $\mathcal{D}$. ■

Lemma 6.2

Suppose $\mathcal{D}_1, \mathcal{D}_2$ are two arbitrary dissections. Then

$$S(f,\mathcal{D}_1) \geq S(f,\mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathscr{S}(f,\mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathscr{S}(f,\mathcal{D}_2).$$

So $S(f,\mathcal{D}_1) \geq \mathscr{S}(f,\mathcal{D}_2)$.

Proof. Take $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$ and apply the previous lemma. ■

Definition 6.3

The *upper integral* of f is $I^*(f) = \inf_{\mathcal{D}} S(f,\mathcal{D})$.

The *lower integer* of f is $I_*(f) = \sup_{\mathcal{D}} \mathscr{S}(f,\mathcal{D})$.

The supremum, infimum always exists by Lemma 6.2 and picking an arbitrary $\mathcal{D}_1, \mathcal{D}_2$ respectively.

And by Lemma 6.2,

$$I^*(f) \geq I_*(f)$$

because

$$\begin{aligned} S(f,\mathcal{D}_1) &\geq \mathscr{S}(f,\mathcal{D}_2) \\ I^*(f) = \inf_{\mathcal{D}_1} S(f,\mathcal{D}_1) &\geq \mathscr{S}(f,\mathcal{D}_2) \\ I^*(f) &\geq \sup_{\mathcal{D}_2} \mathscr{S}(f,\mathcal{D}_2) = I_*(f). \end{aligned}$$

Definition 6.4

A bounded function $f : [a,b] \to \mathbb{R}$ is said to be *Riemann integrable* (or just *integrable*) if $I^*(f) = I_*(f)$.

And we set

$$\int_a^b f(x)\,\mathrm{d}x = I^*(f) = I_*(f) = \int_a^b f.$$

Example. The function $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$ is not Riemann integrable since

$$\begin{aligned} &\sup_{[x_{j-1},x_j]} f(x) = 1, \qquad \inf_{[x_{j-1},x_j]} f(x) = 0 \\ \implies &S(f,\mathcal{D}) = 1, \qquad \mathscr{S}(f,\mathcal{D}) = 1 \qquad \forall \mathcal{D} \end{aligned}$$

So $I^*(f) = 1$, but $I_*(f) = 0$.

Lecture 19

4 Mar. 2022

We first develop a useful criterion for integrability.

Theorem 6.1

A bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if given $\epsilon > 0$, $\exists \mathcal{D}$ such that

$$S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) < \epsilon.$$

Proof. For every dissection $\mathcal{D}$, we have

$$0 \leq I^*(f) - I_*(f) \leq S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}).$$

If the given condition holds, then for all $\epsilon > 0$, we can find $\mathcal{D}$ such that

$$0 \leq I^*(f) - I_*(f) \leq S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) < \epsilon.$$

So $I^*(f) = I_*(f)$.

Conversely, if f is integrable, by definition of sup and inf, there are partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

$$\int_a^b f\,\mathrm{d}x - \frac{\epsilon}{2} = I_*(f) - \frac{\epsilon}{2} < \mathscr{S}(f,\mathcal{D}_1)$$
$$\int_a^b f\,\mathrm{d}x + \frac{\epsilon}{2} = I^*(f) + \frac{\epsilon}{2} > S(f,\mathcal{D}_2)$$

By Lemma 6.2,

$$S(f,\mathcal{D}_1 \cup \mathcal{D}_2) - \mathscr{S}(f,\mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f,\mathcal{D}_2) - \mathscr{S}(f,\mathcal{D}_1) < \epsilon.$$

∎

We now use this criterion to show that monotone and continuous functions are integrable.

Remark. Monotone and continuous functions are bounded.

Theorem 6.2

Let $f : [a,b] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.

Proof. Suppose without loss of generality that f is increasing (same proof for f decreasing), then $\sup_{x\in[x_{j-1},x_j]} f(x) = f(x_j)$ and $\inf_{x\in[x_{j-1},x_j]} f(x) = f(x_{j-1})$. Thus,

$$S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})((f(x_j) - f(x_{j-1})).$$

Now choose

$$\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}.$$

In other words, $x_j = a + \frac{(b-a)j}{n}$ for $0 \leq j \leq n$. So

$$S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) = \frac{b-a}{n}(f(b) - f(a)).$$

Take n large enough such that $\frac{b-a}{n}(f(b) - f(a)) < \epsilon$ and use Theorem 6.1. ∎

To prove that of continuous functions, we need an auxiliary lemma.

Lemma 6.3

For $f : [a, b] \to \mathbb{R}$ continuous. Given $\epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. *(uniform continuity)*

The point is that δ works for all x, y as long as $|x - y| < \delta$. (in the definition of continuity of f at x, the δ depends on x)

Proof. Suppose the claim is false. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$, we can find $x, y \in [a, b]$ such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$.

Take $\delta = \frac{1}{n}$ to get $x_n, y_n \in [a, b]$ with $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \epsilon$. By Bolzano-Weierstrass theorem, we know there exists $x_{n_k} \to c \in [a, b]$, and

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \to 0.$$

So $y_{n_k} \to c$. But $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$. Let $k \to \infty$, by continuity of f,

$$|f(c) - f(c)| \geq \epsilon.$$

Contradiction. ■

Theorem 6.3: L

t $f : [a, b] \to \mathbb{R}$ be continuous, then f is Riemann integrable.

Proof. By Lemma 6.3, given $\epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Let $\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}$. Choose n large enough such that $\frac{b-a}{n} < \delta$, then for $x, y \in [x_{j-1}, x_j]$, $|f(x) - f(y)| < \epsilon$, since $|x - y| \leq \left|x_j - x_{j-1}\right| = \frac{b-a}{n} < \delta$. Observe that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j). \qquad p_j, q_j \in [x_{j-1}, x_j]$$

Minimum and maximum are achieved due to continuity. So

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})(f(p_j) - f(q_j)) < \epsilon(b - a).$$

And we are done by Theorem 6.1. ■

Lecture 20

7 Mar. 2022

Functions more complicated than monotone or continuous can be Riemann integrable.

Example. Consider $f\colon[0,1] \longrightarrow \mathbb{R}$ $\qquad x \longmapsto \begin{cases} 1/q, & \text{if } x = p/q \in (0,1] \text{ in its lowest term} \\ 0, & \text{otherwise} \end{cases}$. Clearly, $\mathscr{S}(f,\mathcal{D}) = 0$ for all $\mathcal{D}$. We will show that given $\epsilon > 0$, $\exists\mathcal{D}$ such that $S(f,\mathcal{D}) < \epsilon$. This implies that f is integrable with $\int_0^1 f \,\mathrm{d}x = 0$.

Consider the set

$$\{\, x \in [0,1] \mid f(x) \geq \frac{1}{N} \,\} = \{\, \frac{p}{q} \mid 1 \leq q \leq N, 1 \leq p \leq q \,\}.$$

Take $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. This is a finite set with

$$0 < t_1 < t_2 < \cdots < t_R = 1.$$

Consider a dissection $\mathcal{D}$ of $[a,b]$ such that

1. Each t_k with $1 \leq k \leq R$ is in some (x_{j-1}, x_j);
2. For all k, the unique interval containing t_k has length at most $\epsilon/2R$.

Such dissection clearly exists. Note that $f \leq 1$ everywhere, and

$$S(f,\mathcal{D}) \leq \frac{1}{N} + \frac{\epsilon}{2} < \epsilon.$$

The function is integrable but has countable many discontinuities.

6.1 Elementary Properties of the Integral

Let f, g be bounded and integrable functions on $[a,b]$.

1. If $f \leq g$ on $[a,b]$, then
$$\int_a^b f \leq \int_a^b g.$$

2. $f+g$ is integrable on $[a,b]$ and
$$\int_a^b f+g = \int_a^b f + \int_a^b g.$$

3. For any constant k, kg is integrable and
$$\int_a^b kf = k\int_a^b f.$$

4. $|f|$ is integrable and
$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

5. The product fg is integrable.

Proof. 1. If $f \leq g$, then
$$\begin{aligned}\int_a^b f = I^*(f) &\leq S(f,\mathcal{D}) \leq S(g,\mathcal{D})\\ \implies \int_a^b f = I^*(f) &\leq I^*(g) = \int_a^b g.\end{aligned}$$

2. We have $\sup\limits_{[x_{j-1},x_j]}(f+g) \leq \sup\limits_{[x_{j-1},x_j]} f + \sup\limits_{[x_{j-1},x_j]} g$, so
$$S(f+g,\mathcal{D}) \leq S(f,\mathcal{D}) + S(g,\mathcal{D}).$$
Now for dissections $\mathcal{D}_1$ and $\mathcal{D}_2$,
$$\begin{aligned}I^*(f+g) &\leq S(f+g,\mathcal{D}_1\cup\mathcal{D}_2)\\ &\leq S(f,\mathcal{D}_1\cup\mathcal{D}_2) + S(g,\mathcal{D}_1\cup\mathcal{D}_2)\\ &\leq S(f,\mathcal{D}_1) + S(g,\mathcal{D}_2).\end{aligned}$$
Fix $\mathcal{D}_1$ and take inf over $\mathcal{D}_2$ to get
$$I^*(f+g) \leq S(f,\mathcal{D}_1) + I^*(g).$$
Again, take inf over all $\mathcal{D}_1$, we have
$$I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g.$$
Similarly, $\int_a^b f + \int_a^b g \leq I_*(f+g)$, so $f+g$ is integrable with the integral equal to the sum of integrals.

3. Exercise.

4. Consider the function $f_+(x) = \max(f(x), 0)$, we have

$$\sup_{[x_{j-1},x_j]} f_+ - \inf_{[x_{j-1},x_j]} f_+ \leq \sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f.$$

We know that given $\epsilon > 0$, there exists $\mathcal{D}$ such that

$$S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) = \sum_{j=1}^{n}(x_j - x_{j-1})(\sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f) < \epsilon.$$

By inequality above, we have

$$S(f_+,\mathcal{D}) - \mathscr{S}(f_+,\mathcal{D}) \leq S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D}) < \epsilon.$$

Note $|f| = 2f_+ - f$, and by (2) and (3), $|f|$ is integrable. Since $-|f| \leq f \leq |f|$, and by property (1), $\left|\int_a^b f\right| \leq \int_a^b |f|$.

5. Take f integrable and $f \geq 0$. Then

$$\sup_{[x_{j-1},x_j]} f^2 = (\sup_{[x_{j-1},x_j]} f)^2 = (M_j)^2$$
$$\inf_{[x_{j-1},x_j]} f^2 = (\inf_{[x_{j-1},x_j]} f)^2 = (m_j)^2.$$

Note $M_j + m_j < 2K$ for some K since f is bounded. Thus,

$$\begin{aligned} S(f^2,\mathcal{D}) - \mathscr{S}(f^2,\mathcal{D}) &= \sum_{j=1}^{n}(x_j - x_{j-1})(M_j^2 - m_j^2) \\ &= \sum_{j=1}^{n}(x_j - x_{j-1})(M_j + m_j)(M_j - m_j) \\ &\leq 2K(S(f,\mathcal{D}) - \mathscr{S}(f,\mathcal{D})) \end{aligned}$$

Using Theorem 6.1, we deduce that f^2 is integrable. Now take any f, then $|f| \geq 0$. Since $f^2 = |f|^2$, we deduce that f^2 is integrable for any f. Finally, for fg, note

$$4fg = (f+g)^2 - (f-g)^2.$$

And we are done since the right-hand side is integrable.

■

Lecture 21

9 Mar. 2022

We have an additional property.

6. If f is integrable on $[a, b]$. If $a < c < b$, then f is integrable over $[a, c]$ and $c, b]$, and
$$\int_a^b f = \int_a^c f + \int_c^b f.$$
Conversely, if f is integrable over $[a, c]$ and $[c, b]$, then f is integrable over $[a, b]$, and
$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. We first make two observations. If $\mathcal{D}_1$ is a dissection of $[a, c]$ and $\mathcal{D}_2$ is a dissection of $[c, b]$, then $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ is a dissection of $[a, b]$ and
$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2).$$
Also, if $\mathcal{D}$ is a dissection of $[a, b]$, then
$$S(f, \mathcal{D}) \geq S(f, \mathcal{D} \cup \{c\}) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2)$$
where $\mathcal{D}_1$ dissects $[a, c]$ and $\mathcal{D}_2$ dissects $[c, b]$. From the first inequality, we have
$$I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$
From the second inequality, we have
$$I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$
So, we have
$$I^*(f) = I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$
Similarly, the lower integral
$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]}).$$
Thus,
$$0 \leq I^*(f) - I_*(f) = (I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})) + (I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})).$$
From this (6) follows right away. ■

We have the convention that if $a > b$, then
$$\int_a^b f = -\int_b^a f;$$
If $a = b$, we agree that its value is zero. With this convention, if $|f| \leq K$,
$$\left| \int_a^b f \right| \leq K|b - a|.$$

6.2 The Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be a bounded and integrable function, and we write

$$F(x) = \int_a^x f(t)\,\mathrm{d}t. \qquad x \in [a, b]$$

Theorem 6.4

F is continuous.

Proof. Consider the difference $F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t$. We have

$$\begin{aligned} |F(x+h) - f(x)| &= \left| \int_x^{x+h} f(t)\,\mathrm{d}t \right| \\ &\leq K|h| \end{aligned}$$

if $|f(t)| \leq K$ for all $t \in [a, b]$. Now let $h \to 0$ and we are done. (In fact, F is Lipschitz continuous) ■

Theorem 6.5: FTC

If in addition f is continuous at x, then F is differentiable at x, and

$$F'(x) = f(x).$$

Proof. We consider

$$\begin{aligned} &\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \qquad x + h \in [a, b], h \neq 0 \\ =&\frac{1}{|h|} \left| \int_x^{x+h} f(t)\,\mathrm{d}t - hf(x) \right| \\ =&\frac{1}{|h|} \left| \int_x^{x+h} f(t) - f(x)\,\mathrm{d}t \right|. \end{aligned}$$

Since f is continuous at x, given $\epsilon > 0$, there exists $\delta > 0$ such that $|t - x| < \delta \implies |f(t) - f(x)| < \epsilon$. So if $|h| < \delta$, we have

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{|h|}\epsilon|h| = \epsilon.$$

That is,

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

■

Example. Consider the function $f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in (0, 1] \end{cases}$. Taking the integral of f, and we have $F(x) = -1 + |x|$. It is differentiable everywhere except for $x = 0$ where there is a discontinuity on f.

Corollary 6.1: Integration is the inverse of differentiation

If $f = g'$ is continuous on $[a, b]$, then

$$\int_a^x f(t)\,\mathrm{d}t = g(x) - g(a). \qquad \forall x \in [a, b].$$

Proof. From Theorem 6.5, $F - g$ has zero derivative in $[a, b]$. So $F - g$ is constant, and since $F(a) = 0$, we have $F(x) = g(x) - g(a)$. ■

Every continuous function has an *indefinite integral* or *antiderivative* written $\int f(x)\,\mathrm{d}x$ which is determined up to a constant.

Remark. We have solved the ODE

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}.$$

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Corollary 6.2: Integration by Parts

Suppose f' and g' exist and are continuous on $[a, b]$, then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

Proof. By the product rule,
$$(fg)' = f'g + fg'.$$
By Theorem 6.5,
$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'.$$
■

Corollary 6.3: Integration by Substitution

Let $g : [\alpha, \beta] \to [a, b]$ be a continuous function with $g(\alpha) = a$ and $g(\beta) = b$, and g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then
$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t.$$

Proof. Set $F(x) = \int_a^x f(t)\,\mathrm{d}t$, and let $h(t) = F(g(t))$. (defined as g takes values in $[a, b]$) Then we have
$$\begin{aligned}\int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t &= \int_\alpha^\beta F'(g(t))g'(t)\,\mathrm{d}t \\ &= \int_\alpha^\beta h'(t)\,\mathrm{d}t \\ &= h(\beta) - h(\alpha) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)\,\mathrm{d}x.\end{aligned}$$
■

Theorem 6.6: Taylor's theorem with remainder an integral

Let $f^(n)(x)$ be continuous for $x \in [0, h]$. Then
$$f(h) = f(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_n$$
where
$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,\mathrm{d}t.$$

Proof. Substitute $u = th$,

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u)\,\mathrm{d}u.$$

Integrating by parts, we get the following

$$R_n = \frac{-h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u)\,\mathrm{d}u.$$

If we integrate by parts $n-1$ times, we arrive at

$$\begin{aligned} R_n &= -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - hf'(0) + \int_0^h f'(u)\,\mathrm{d}u \\ &= -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - hf'(0) + f(h) - f(0). \end{aligned}$$

■

Now we can get the Cauchy and Lagrange form of the remainder. However, note that the proof above uses continuity of $f^{(n)}$ not just the existence as we proved before.

Theorem 6.7

Suppose $f, g : [a,b] \to \mathbb{R}$ be continuous functions with $g(x) \neq 0$ for all $x \in (a,b)$. Then there exists $c \in (a,b)$ such that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c) \int_a^b g(x)\,\mathrm{d}x.$$

Note. If we take $g(x) = 1$, we get

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a).$$

Proof. We are going to use Cauchy's MVT. (Theorem 4.5) Let

$$F(x) = \int_a^x fg, \quad G(x) = \int_a^x g.$$

Theorem 4.5 proves the existence of $c \in (a,b)$ such that

$$\begin{gathered} (F(b) - F(a))G'(c) = F'(c)(G(b) - G(a)) \\ (\int_a^b fg)g(c) = f(c)g(c) \int_a^b g \end{gathered}$$

Since $g(c) \neq 0$, we simplify and get the required form. ■

Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) \, \mathrm{d}t.$$

First we use Theorem 6.7 with $g(x) = 1$, to get

$$R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta h)$$

with some $\theta \in (0,1)$ which is the Cauchy's form of remainder. To get Lagrange's form, we use Theorem 6.7 with $g(t) = (1-t)^{n-1}$ which is greater than 0 for $t \in (0,1)$. We know there exists $\theta \in (0,1)$ such that

$$\begin{aligned} R_n &= \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \int_0^1 (1-t)^{n-1} \, \mathrm{d}t \\ &= \frac{h^n}{n!} f^{(n)}(\theta h) \end{aligned}$$

which is the Lagrange's form of the remainder.

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6.3 Improper Integrals

Definition 6.5

Suppose $f : [a, \infty) \to \mathbb{R}$ is integrable (and bounded) on every interval $[a, R]$ and that as $R \to \infty$,

$$\int_a^R f(x) \, \mathrm{d}x \to \ell.$$

Then we say that $\int_a^\infty f(x) \, \mathrm{d}x$ *exists* or *converges* and that its value is ℓ. If $\int_a^R f(x) \, \mathrm{d}x$ does not tend to a limit, we say that $\int_a^\infty f(x) \, \mathrm{d}x$ *diverges*.

A similar definition applies to $\int_{-\infty}^a f(x) \, \mathrm{d}x$

If $\int_a^\infty f = \ell_1$ and $\int_{-\infty}^a f = \ell_2$, we write $\int_{-\infty}^\infty f = \ell_1 + \ell_2$. (independent of the particular value of a)

Remark. The last part is not the same as saying that $\lim_{R \to \infty} \int_{-R}^R f(x) \, \mathrm{d}x$ exists. It is stronger since $\int_{-R}^R x \, \mathrm{d}x = 0$.

Example. The integral $\int_1^\infty \frac{\mathrm{d}x}{x^k}$ converges if and only if $k > 1$.

Indeed, if $k \neq 1$,

$$\int_1^R \frac{\mathrm{d}x}{x^k} = \frac{x^{1-k}}{1-k}\bigg|_1^R = \frac{R^{1-k}-1}{1-k}$$

and as $R \to \infty$, this limit is finite if and only if $k > 1$. If $k = 1$,

$$\int_1^R \frac{\mathrm{d}x}{x} = \log R \to \infty.$$

Remark.

1. $1/\sqrt{x}$ is continuous on $[\delta, 1]$ for any $\delta > 0$, and

   $$\int_\delta^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\big|_\delta^1 = 2 - 2\sqrt{\delta} \to 2$$

   as $\delta \to 0$. The function is unbounded on $(0, 1]$, but the limit of the integral exists.

   However,

   $$\int_0^1 \frac{\mathrm{d}x}{x} = \lim_{\delta\to 0}\int_\delta^1 \frac{\mathrm{d}x}{x} = \lim_{\delta\to 0} \log x\big|_\delta^1 = \lim_{\delta\to 0}(\log 1 - \log\delta)$$

   does not exist.

2. If $f \geq 0$ and $g \geq 0$ for $x \geq a$, and

   $$f(x) \leq Kg(x). \qquad \forall x \geq a$$

   Then $\int_a^\infty g$ converges implies $\int_a^\infty f$ converges and $\int_a^\infty f \leq K\int_a^\infty g$.

   Just note that $\int_a^R f \leq K\int_a^R g$. The function $R \mapsto \int_a^R f$ is increasing because $f \geq 0$, and bounded above because $\int_a^\infty g$ converges. Take $\ell = \sup_{R\geq a}\int_a^R f < \infty$. Now check that

   $$\lim_{R\to\infty}\int_a^R f = \ell.$$

   Given $\epsilon > 0$, there exists R_0 such that

   $$\int_a^{R_o} f \geq \ell - \epsilon.$$

   Thus, if $R \geq R_0$,

   $$\int_a^R f \geq \int_a^{R_0} f \geq \ell - \epsilon.$$

   And the limit follows.

Example. Consider the integral $\int_0^\infty e^{-x^2/2}\,\mathrm{d}x$. We have

$$e^{-x^2/2} \le e^{-x/2}. \qquad x \ge 1$$

Since

$$\int_1^R e^{-x/2}\,\mathrm{d}x = \frac{1}{2}(e^{-1/2} - e^{-R/2}) \to \frac{e^{-1/2}}{2},$$

the integral $\int_0^\infty e^{-x^2/2}$ converges.

3. We know that if $\sum a_n$ converges, then $a_n \to 0$. However, $\int_a^\infty f$ converges may not imply that $f \to 0$.

   For example, consider a function of smaller and smaller bumps to the highest point with value 1.

Theorem 6.8: Integral test

Let $f(x)$ be a positive decreasing function for $x \ge 1$. Then

1. the integral $\int_1^\infty f(x)\,\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ both converge or diverge;
2. As $n \to \infty$, $\sum_{r=1}^n f(r) - \int_1^n f(x)\,\mathrm{d}x$ tends to a limit ℓ such that $0 \le \ell \le f(1)$.

Proof. Note that f deceasing implies that f is integrable on every bounded subinterval by Theorem 6.2.

If $n-1 \le x \le n$, then $f(n-1) \ge f(x) \ge f(n)$, so

$$f(n-1) \ge \int_{n-1}^n f(x)\,\mathrm{d}x \ge f(n).$$

Adding the terms together, we have

$$\sum_1^{n-1} f(r) \ge \int_1^n f(x)\,\mathrm{d}x \ge \sum_2^n f(r).$$

From this, claim (1) is clear. For the proof of (2), set $\phi(n) = \sum_1^n f(r) - \int_1^n f(x)\,\mathrm{d}x$, then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x)\,\mathrm{d}x \le 0.$$

So $\phi(n)$ is decreasing. We also have $0 < \phi(n) \le f(1)$. Thus, $\phi(n)$ tends to a limit ℓ such that $0 \le \ell < f(1)$. ■

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Example. 1. The series $\sum_1^\infty \frac{1}{n^k}$ converges if and only if $k > 1$.

We proved last lecture that $\int_1^\infty \frac{1}{x^k}$ converges if and only if $k > 1$, and the convergence follows from integral test.

2. The series $\sum_2^\infty \frac{1}{n \log n}$ is divergent by considering $f(x) = \frac{1}{x \log x}$ for $x \geq 2$.

$$\int_2^R \frac{\mathrm{d}x}{x \log x} = \log(\log x)\Big|_2^R = \log(\log R) - \log(\log 2) \to \infty$$

as $R \to \infty$. So divergence follows from integral test.

Corollary 6.4: Euler's constant

As $n \to \infty$, $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \to \gamma$ with $0 \leq \gamma \leq 1$.

Proof. Set $f(x) = 1/x$, and use Theorem 6.8. ∎

Remark. It is still an open question whether γ is irrational. ($\gamma \approx 0.577$)

We have seen that monotone functions and continuous function are Riemann integrable. We can generalize this a bit and say that piece-wise continuous functions are integrable.

Definition 6.6: A

unction $f : [a, b] \to \mathbb{R}$ is said to be piece-wise continuous if there is a dissection $\mathcal{D} = \{a = x_0, x_1, \ldots, x_n = b\}$ such that

1. f is continuous on (x_{j-1}, x_j) for all j;
2. the one-sided limits $\lim_{x \to x_{j-1}^+} f(x), \lim_{x \to x_j^-} f(x)$ exist.

It is now an exercise to check that f is Riemann integrable; just check that $f|_{[x_{j-1}, x_j]}$ is integrable for each j (the values of f at the end points won't really matter) and use additivity of domains.

Problem. How large can the discontinuity set of f be while f is still Riemann integrable?

Recall the example $f(x) = \begin{cases} 1/q, & x = p/q \\ 0, & \text{otherwise} \end{cases}$ which has a countably infinite set of discontinuities.

non-examinable

The question is answered by Henri Lebesgue.

Bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if the set of discontinuity points has *measure zero*.

Definition 6.7

Let $\ell(I)$ be the length of an interval I. A subset $A \subseteq \mathbb{R}$ is said to have *measure zero* if for each $\epsilon > 0$, there exists a countable collection of intervals I_j such that $A \subseteq \cup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} \ell(I_j) < \epsilon$.

Lemma 6.4

1. Every countable set has measure zero.
2. If B has measure zero and $A \subseteq B$, then A has measure zero.
3. If A_k has measure zero for all $k \in \mathbb{N}$, then $\cup_{k \in \mathbb{N}} A_k$ also has measure zero.

We use the *oscillation* of f, which is for I an interval

$$\omega_f(I) = \sup_I f - \inf_I f.$$

And the oscillation of f at a point is

$$\omega_f(x) = \lim_{\epsilon \to 0} \omega_f(x - \epsilon, x + \epsilon).$$

Lemma 6.5

f is continuous at x if and only if $\omega_f(x) = 0$.

Brief Sketch of proof of Lebesgue's Criterion. We consider the set

$$D = \{ x \in [a, b] \mid f \text{ discontinuous at } x \} = \{ x \mid \omega_f(x) > 0 \}.$$

Let $N(\alpha) = \{ x \mid \omega_f(x) \geq \alpha \}$, then $D = \cup_1^\infty N(1/k)$. We want to show that D has measure zero.

Let $\epsilon > 0$ be given, there exists $\mathcal{D}$ such that

$$S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) = \sum_{j=1}^{n} \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\epsilon\alpha}{2}.$$

Consider $F = \{ j \mid (x_{j-1}, x_j) \cap N(\alpha) \neq \varnothing \}$, then for each $j \in F$,

$$\omega_f([x_{j-1}, x_j]) \geq \alpha.$$

So

$$\alpha \sum_{j \in F} (x_j - x_{j-1}) \leq S(f, \mathcal{D}) - \mathscr{S}(f, \mathcal{D}) < \frac{\epsilon\alpha}{2}.$$

And $\sum_{j \in F}(x_j - x_{j-1}) < \epsilon/2$. These cover $N(\alpha)$ except perhaps for $\mathcal{D}$. But these can be covered by intervals of total length less than $\epsilon/2$. So $N(\alpha)$ can be covered by intervals of total length less than ϵ. That is, $N(\alpha)$ has measure zero.

For the other direction, let $\epsilon > 0$ be given, consider $N(\epsilon) \subseteq D$, so $N(\epsilon)$ has measure zero. $N(\epsilon)$ is closed and bounded, so it can be covered by finitely many open intervals of total length less than ϵ because it has measure zero. $N(\epsilon) = \cup_{i=1}^m U_i$, and $K = [a,b]/\cup_i^m U_i$ compact. So it can be covered by finitely meany intervals J_j such that $\omega_f(J_j) < \epsilon$. And we are done by considering the partition with these intervals. ■