

# Geometry

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## Lecture 1: Introduction

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## 1 Surfaces

### 1.1 Topological Surfaces

We start with some definitions.

**Definition 1.1.** A *topological surface* is a topological space  $\Sigma$  such that

1. **T1:**  $\forall p \in \Sigma$  there is an open neighborhood  $p \in U \subseteq \Sigma$  such that  $U$  is homeomorphic to  $\mathbb{R}^2$ , or a disc  $D^2 \subseteq \mathbb{R}^2$  with its usual Euclidean topology.
2. **T2:**  $\Sigma$  is Hausdorff and second countable.

**Remark.** We have the following remarks.

1.  $\mathbb{R} \cong D(0, 1)$ , so homeomorphic to a disc is enough as stated in the definition.
2. A space  $X$  is *Hausdorff* if for  $p \neq q \in X$ , there exists disjoint open sets  $p \in U$  and  $q \in V$  in  $X$ .
3. A space  $X$  is *second countable* if it has a countable base i.e.  $\exists \{u_i\}_{i \in \mathbb{N}}$  open sets s.t. every open set is a union of some  $u$ .
4. **T1** is the point and **T2** is for technical honesty.
5. If  $X$  is Hausdorff/ second countable, so are subspaces of  $X$ . In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

**Example.** Here we present some examples of topological surfaces.

1.  $\mathbb{R}^2$ , the plane.

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2. Any open subset of  $\mathbb{R}^2$ , i.e.  $\mathbb{R}^2 \setminus Z$  where  $Z$  is closed:

- $Z = \{0\}$ ,
- $Z = \{(0, 0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \dots)\}$ .

3. Graphs:

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function. The graph  $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^3$  (subspace topology).

Recall that if  $X, Y$  are spaces, the product topology on  $X \times Y$  has basic open sets  $U \times V$  with  $U$  open and  $V$  open.

It has the feature that  $f : Z \rightarrow X \times Y$  is continuous if and open if the two projective maps are continuous.

Application:  $\Gamma_f \subseteq X \times Y$ , if  $f : X \rightarrow Y$  is continuous, if homeomorphic to  $X$ .

So  $\Gamma_f \cong \mathbb{R}^2$  for any  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  that is continuous, so  $\Gamma_f$  is a topological surface.

**Note.** As a topological surface,  $\Gamma_f$  is independent of  $f$ , but later on as a geometric object, it will reflect features of  $f$ .

4. The sphere (subspace topology):

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Stereographic projection

$$\begin{aligned} \pi_+ : S^2 \setminus \{(0, 0, 1)\} &\rightarrow \mathbb{R}^2 \\ (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \end{aligned}$$

**Note.** The map is continuous and has an inverse,  $\pi_+$  is a continuous bijection with continuous inverse, and hence a homeomorphism.

Stereographic projection from the South Pole is also a homeomorphism from  $S^2 \setminus \{(0, 0, -1)\} \rightarrow \mathbb{R}^2$ .

So  $S^2$  is a topological surface:

$\forall p \in S^2$ , either  $p$  lies in the domain of  $\pi_+$  or of  $\pi_-$  (or both) and so it lies in an open set homeomorphic to  $\mathbb{R}^2$ . (And Hausdorff and second countable from  $\mathbb{R}^2$ ).

**Remark.**  $S^2$  has a global property as it is compact as a topological space, since it is a closed bounded set in  $\mathbb{R}^3$ .

5. The real projective plane:

The group  $\mathbb{Z}/2$  acts on  $S^2$  by homeomorphism via the *antipodal map*  $a : S^2 \rightarrow S^2$ .

$$a(x, y, z) = (-x, -y, -z).$$

i.e. There exists a homomorphism  $\mathbb{Z}/2\mathbb{Z} \rightarrow \text{Homeo}(S^2)$ , such that it maps the non-identity element to the antipodal map.

Commutative diagram

Stereographic projection graph

Explicit formula for inverse

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**Definition 1.2.** The *real projective plane* is the quotient space of  $S^2$  given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2 / \mathbb{Z}/2\mathbb{Z}.$$

**Lemma 1.1.** As a set,  $\mathbb{RP}^2$  is naturally in bijection with the set of straight lines in  $\mathbb{R}^3$  through the origin.

*Proof.* Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ■

Graph of the sphere

**Lemma 1.2.**  $\mathbb{RP}^2$  is a topological surface.

*Proof.* We check that it is Hausdorff:

Recall if  $X$  is a space and  $q : X \rightarrow Y$  is a quotient map,  $V \subseteq Y$  is open  $\iff q^{-1}V \subseteq X$  open.

More balls

If  $[p], [q] \in \mathbb{RP}^2$ , then  $\pm p, \pm q \in S^2$  are distinct antipodal pairs. Take small open discs around  $p, q$  and their antipodal images, as in the picture.

We can then take small balls  $B_{\pm p}(\delta)$ ,  $B_{\pm q}(\delta)$ , which intersects  $S^2$  with open sets around  $\pm q$  and  $\pm p$ .

$\mathbb{RP}^2$  is also second countable.

Let  $\mathcal{U}$  be a countable base for the topology on  $S^2$ , such that for all  $u \in \mathcal{U}$ , the antipodal image is in  $\mathcal{U}$ .

Let  $\bar{\mathcal{U}}$  be the family of open sets in  $\mathbb{RP}^2$  of the form  $q(u) \cup q(-u)$ ,  $u \in \mathcal{U}$ .

Now, if  $v \in \mathbb{RP}^2$  is open, by definition  $q^{-1}v$  is open in  $S^2$ , so  $q^{-1}v$  contains some  $u \in \mathcal{U}$ , and hence contains  $u \cup (-u)$ . So  $\bar{\mathcal{U}}$  is a countable base for the quotient topology on  $\mathbb{RP}^2$  consider all such  $u$  that covers  $q^{-1}v$ .

Finally, let  $p \in S^2$  and  $[p] \in \mathbb{RP}^2$  its image. Let  $\bar{D}$  be a small (contained in an open hemisphere) closed disc neighborhood of  $p \in S^2$ .

If we consider  $q$  restricted to  $\bar{D}$ , it is a continuous map from a compact space to a Hausdorff space.

Also, on  $\bar{D}$ , the map  $q$  is injective. Recall "Topological inverse function theorem": A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. So  $q$  restricted to the disk is a homeomorphism.

It then induces another homeomorphism of  $q$  restricted to  $D$ , and open disk contained in  $\bar{D}$ . So  $[q] \in q(D)$  has an open neighborhood in  $\mathbb{RP}^2$  that is homeomorphic to an open disk, and we are done. ■

## Lecture 2: More Examples

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- Let  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ .

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The *torus*  $S^1 \times S^1$

7. Let  $P$  be a planar Euclidean polygon. Assume the edges are *oriented* and paired, and for simplicity assume the Euclidean length for  $e, \hat{e}$  are equal if they are paired.

If  $\{e, \hat{e}\}$  are paired edges, there is a unique isometry from  $e$  to  $\hat{e}$  respecting their orientations, say  $f_{e\hat{e}} : e \rightarrow \hat{e}$ .

These maps generate an equivalence relation on  $P$  where we identify  $x \in P$  with  $f_{e\hat{e}}(x)$  whenever  $x \in e$ .

**Lemma 1.3.**  $P/\sim$  (with the quotient topology) is a topological surface.

**Example.** The torus as  $[0,1]^2/\sim$ . We consider three different kinds of points.

If  $p$  is in the interior. We can find a small enough neighborhood that is injective, and again by topological inverse function theorem, that small enough disk is homeomorphic to an open disk.

If  $p$  is on the edge. Say  $p = (0, y) \sim (1, y)$  and  $\delta > 0$  is small enough such that a half disk of radius  $\delta$  does not touch vertices. Define a map from the union of the half-disks to  $B(0, \delta) \subseteq \mathbb{R}^2$  by  $(x, y) \mapsto x, y - y_0$  and  $(x, y) \mapsto (x - 1, y - y_0)$  on each part of the half-disk. If  $X = A \cup B$  is a union then they define a continuous map from  $X$  to  $Y$ .