Analysis

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February 25, 2022

Contents

Lecture 1: Limits

21 Jan. 11:00

Books:

- *A First Course in Mathematical Analysis* -Burkill
- *Calculus* -Spivak
- *Analysis I* -Tao

1 Limits and Convergence

1.1 Review from Numbers and Sets

Notation. We denote sequences by a_n or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

Definition 1.1. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

Note. $N = N(\epsilon)$ which is dependent on ϵ. That is, if you want to go closer to a, sometimes you need to go higher in N.

Definition 1.2 (limit of a sequence). We say that a sequence is a

$$\left.\begin{aligned} &\textit{increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{decreasing sequence if } a_n \geq a_{n+1}, \end{aligned}\right\} \textit{monotone sequence}$$

$$\left.\begin{aligned} &\textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{aligned}\right\} \textit{strictly monotone sequence}$$

We also have

Theorem 1.1 (Fundamental Axiom of the Real Numbers). If $a_n \in \mathbb{R}$ and a_n is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges*.

Remark. It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.
- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

Definition 1.3 (supremum). For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S$,
2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

Note. Supremum is unique, and there is a similar notion of infimum.

Lemma 1.1 (Properties of Limits).

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.
2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).
3. If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.
4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.
6. If $a_n \to a$, then $\frac{1}{a_n} \to \frac{1}{a}$.
7. If $a_n < A$ for all n and $a_n \to a$, then $a \leq A$.

Proof.

1. Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|a_n - b| < \epsilon, \forall n \geq N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \geq N$,

   $$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$

   If $a \neq b$, take $\epsilon = \frac{|a-b|}{3}$, we have

   $$|a - b| < \frac{2}{3}|a - b|. \lightning$$

2. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon, \forall n \geq N$, Since $n_j \geq j$, we know

   $$\left|a_{n_j} - a\right| < \epsilon, \forall j \geq N.$$

   That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have

   $$\begin{aligned}|a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|.\end{aligned}$$

   Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|b_n - b| < \epsilon, \forall n \geq N_2$.

   If $n \geq N_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$.

   We have

   $$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

$\blacksquare$

Lemma 1.2.

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Proof. $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit a.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma (1.1).}$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma (1.1) $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma (1.1), we have $a = \frac{a}{2} \implies a = 0$. ■

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

Definition 1.4. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma (1.1) are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.

Lecture 2: Bolzano–Weierstrass theorem

24 Jan. 11:00

Theorem 1.2 (Bolzano-Weierstrass Theorem). If $x_n \in R$ and there exists K such that $|x_n| \leq K$ for all n, then we can find $n_1 < n_2 < n_3 < \dots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. In other words, every bounded sequence has a convergent subsequence.

Remark. We say nothing about the uniqueness of the limit x.

For example, $x_n = (-1)^n$ has two subsequences tending to -1 and 1 respectively.

Proof. Set $[a_1, b_1] = [-K, K]$. Let c be the mid-point of a_1, b_1, consider the following alternatives,

1. $x_n \in [a_1, c]$ for infinitely many n.
2. $x_n \in [c, a_2]$ for infinitely many n.

Note that (1) and (2) can hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold, and we set $a_2 = c$ and $b_2 = b_1$.

We proceed as above to construct sequences a_n, b_n such that $x_m \in [a_n, b_n]$ for infinitely many values of m. They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

a_n is an increasing sequence and bounded, and b_n is a decreasing sequence and bounded. By Fundamental Axiom, $a_n \to a \in [a_1, b_1], b_n \to b \in [a_1, b_1]$. Using Lemma (1.1), $b - a = \frac{b-a}{2} \implies a = b$.

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j such that $x_{n_j} \in [a_j, b_j]$, that is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$. In other words, there is unlimited supply.

Hence, $a_j \leq x_{n_j} \leq b_j$, so $x_{n_j} \to a$. ■

1.2 Cauchy Sequences

Definition 1.5 (Cauchy Sequence). $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\epsilon > 0$ $\exists N > 0$ such that $|a_n - a_m| < \epsilon$ $\forall n, m > N$.

Note. N is dependent on ϵ.

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

Lemma 1.3. A convergent sequence is a Cauchy sequence.

Proof. If $a_n \to a$, given $\epsilon > 0$, exists N such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Take $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$

■

Lemma 1.4. Every Cauchy sequence is convergent.

Proof. First we note that if a_n is Cauchy, then it is bounded.

Take $\epsilon = 1$, $N = N(1)$ in the Cauchy property, then

$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$

We have

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$

Let $K = \max\{1 + |a_N|, |a_n| \; n = 1, 2 \ldots, N - 1\}$.

Then $|a_n| \leq K$ for all n. By the Bolzano–Weierstrass theorem, $a_{n_j} \to a$. We must have $a_n \to a$.

Given $\epsilon > 0$, there exists j_0 such that for all $j \geq j_0$, $\left|a_{n_j} - a\right| < \epsilon$.

Also, there exists $N(\epsilon)$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N(\epsilon)$.

Take j such that $n_j \geq \max\{N(\epsilon), n_{j_0}\}$. Then if $n \geq N(\epsilon)$,

$$|a_n - a| \leq \left|a_n - a_{n_j}\right| + \left|a_{n_j} - a\right| < 2\epsilon.$$

■

Thus, on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

2 Series

Definition 2.1. If $a_n \in \mathbb{R}, \mathbb{C}$ We say that $\sum_{j=1}^{\infty} a_j$ converges to s if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to S$$

as $N \to \infty$. We write $\sum_{j=1}^{\infty} a_j = s$. If S_N does not converge, we say that $\sum_{j=1}^{\infty} a_j$ *diverges.*

Remark. Any problem on series is really a problem about the sequence of partial sums.

Lemma 2.1.

1. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} a_j$ converges, then so does $\sum_{j=1}^{\infty} \lambda a_j + \mu b_j$, when $\lambda, \mu \in \mathbb{C}$;
2. Suppose there exists N such that $a_i = b_i$ for all $i \geq N$. Then either $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge or they both diverge. (initial terms do not matter for convergence)

Proof.
1. Exercise.
2. If we have $n \geq N$,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^{n} a_i$$
$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^{n} b_i$$

So $S_n - d_n = \sum_{i=1}^{N-1} a_i - b_i$ which is a constant. So S_n converges if and only if d_n does.

■

Lecture 3

26 Jan. 11:00

We have the following important example,

Example (Geometric Series). $x \in \mathbb{R}$, set $a_n = x^{n-1}$ with $n \geq 1$. So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \dfrac{1-x^n}{1-x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have $S_n(1-x) = 1 - x^n$.

If $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$.

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$.

If $x < -1$, S_n does not converge (oscillates).

If $x = -1$, $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$.

Thus, the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta, \delta > 0$, so $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \to 0$ because $(1+\delta)^n \geq 1 + n\delta$ from binomial expansion.

Definition 2.2. $S_n \to \infty$ if given A, there exists an N such that $S_n > A$ for all $n > N$.

$S_n \to -\infty$ if given A, there exists an N such that $S_n < -A$ for all $n > N$.

Lemma 2.2. If $\sum\limits_{i=1}^{\infty} a_n$ converges, then $\lim\limits_{i \to \infty} a_i = 0$.

Proof. Let $S_n = \sum\limits_{i=1}^{\infty} a_i$, note that $a_n = S_n - s_{n-1}$. If $S_n \to a$, we have $a_n \to 0$ because $S_{n-1} \to a$ also. ■

Remark. The converse of the preceding lemma is false. One example is $\sum \frac{1}{n}$, the *harmonic series*. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty}$$
$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$.

So if $S_n \to a$, then $S_{2n} \to a$, also we have $a \geq a + \frac{1}{2}$. Contradiction.

2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

Theorem 2.1 (The Comparison Test). Suppose $0 \leq b_n \leq a_n$ for all n. Then if $\sum\limits_{n=1}^{\infty} a_n$ converges, so does $\sum\limits_{n=1}^{\infty} b_n$.

Proof. Let $s_N = \sum\limits_{n=1}^{N} a_n$, $d_N = \sum\limits_{n=1}^{N} b_n$. Because $b_n \leq a_n$, we know $d_N \leq s_N$. But $s_N \to s$, then $d_n \leq s_n \leq 2$ for all n, and d_N is a increasing sequence bounded above. So d_N converges. ■

Example. We consider $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that $\sum\limits_{n=1}^{\infty} a_n$ converges, so $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, we get $\sum\limits_{n=1}^{\frac{1}{n^2}} \leq 1 + 1 = 2$.

For the rest of the lecture, we establish two more tests.

Theorem 2.2 (Root test/ Cauchy's Test for Convergence). Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges; if $a > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Remark. Nothing can be said if $a = 1$.

. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists N such that $\forall n \geq N$,

$$a_n^{1/n} < r \implies a_n < r^n.$$

But since $r < 1$, the geometric series converges, and by comparison test, the series $\sum a_n$ converges as well.

To prove the second part of the theorem, if $a > 1$, for $n \geq N$,

$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus, $\sum\limits_{n=1}^{\infty} a_n$ diverges, since a_n does not tend to zero. ∎

Theorem 2.3 (Ratio Test/ D'Alembert's Test). Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges. If $\ell > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Remark. As before, nothing can be said for $\ell = 1$.

Proof. Supposed $\ell < 1$ and choose r with $\ell < r < 1$. Then $\exists N$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r.$$

Therefore,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \ n > N.$$

So, $a_n < kr^n$ with k independent of n. Since $\sum\limits_{n=1}^{\infty} r^n$ converges, so does $\sum\limits_{n=1}^{\infty} a_n$ by Comparison Test.

If $\ell > 1$, choose $1 < r < \ell$. Then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \ n > N.$$

So the series diverges. ∎

Lecture 4

28 Jan. 2022

Example. To determine the convergence of $\sum\limits_{n=1}^{\infty} a_n = \frac{n}{2^n}$.

By ratio test,

$$\frac{n+1}{2^n} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$

So we have convergence by ratio test.

However, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, and ratio test gives limit 1, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, and ratio test gives limit 1. So ratio test is inconclusive if the limit is 1.

Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, so root test is also inconclusive when the limit is 1. To see this limit, write

$$n^{\frac{1}{n}} = 1 + \delta_n, \ \ \delta_n > 0.$$

So

$$n = (1+\delta_n)^n > \frac{n(n-1)}{2}\delta_n^2.$$

And $\delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$.

Remark. Use the root test when there is a nth power in the series.

Theorem 2.4 (Cauchy's Condensation Test). Let a_n be a decreasing sequence of positive terms. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges.

Proof. First we observe that if a_n is decreasing

$$a_{2^k} \le a_{2^{k-1}+i} \le a_{2^{k-1}}$$

for all $k \ge 1$ and $1 \le i \le 2^{k-1}$.

Assume that $\sum\limits_{n=1}^{\infty} a_n$ converges with sum A. Then

$$\begin{aligned} 2^{n-1}a_{2^n} &= \underbrace{a_{2^n} + \cdots a_{2^n}}_{2^{n-1} \text{ times}} \\ &\le a_{2^{n-1}+1} + \cdots + a_{2^n} \\ &= \sum_{m=2^{n-1}+1}^{2^n} a_m. \end{aligned}$$

Thus, $\sum\limits_{n=1}^{N} 2^{n-1}a_{2^n} \le \sum\limits_{n=1}^{N} \sum\limits_{m=2^{n-1}+1}^{2^n} a_m = \sum\limits_{m=2}^{2^N} a_m$. So

$$\sum_{n=1}^{N} 2^n a_{2^n} \le 2\sum_{m=2}^{2^N} a_m \le 2(A - a_1).$$

Thus, $\sum\limits_{n=1}^{N} 2^n a_{2^n}$ being increasing and bounded above, converges.

Conversely, assume $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges to B, then

$$\begin{aligned} \sum_{m=2^{n-1}+1}^{2^n} a_m &= a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} \\ &\le \underbrace{a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} times} = 2^{n-1}a_{2^{n-1}}. \end{aligned}$$

Similarly, we have

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^{N} 2^{n-1} a_{2^{n-1}} \leq B.$$

Therefore, $\sum_{m=1}^{N} a_m$ is a bounded increasing sequence and thus it converges. ■

Example. $\sum_{n=1}^{\infty} \frac{1}{n^k}$ for $k > 0$ converges if and only if $k > 1$. First we note that $\frac{1}{n^k}$ is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)k} < \frac{1}{n^k} \iff (\frac{n}{n+1})^k < 1 \iff \frac{n}{n+1} < 1.$$

We use Cauchy condensation test, and we have

$$\begin{aligned} 2^n a_{2^n} &= 2^n \left(\frac{1}{2^n}\right)^k \\ &= 2^{n-nk} = (2^{1-k})^n. \end{aligned}$$

Which is a geometric series with the ratio 2^{1-k}. So $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges if and only if $2^{1-k} < 1 \iff k > 1$.

2.2 Alternating Series

Theorem 2.5 (Alternating Series Test). If a_n decreases and tends to 0 as $n \to \infty$, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Example. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Proof. The partial sum is

$$\begin{aligned} S_n &= a_1 - a_2 + \cdots + (-1)^{n+1} a_n \\ S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq S_{2n-1} \\ S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1 \end{aligned}$$

So S_{2n} is increasing and bounded above, implying that $S_{2n} \to S$. The odd terms satisfy

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S + 0 = S.$$

This implies that S_n converges to S as well. Given ϵ, there exists N_1 such that $\forall n \geq N_1$, $|S_{2n} - S| < \epsilon$. We also know that there exists N_2 such that $\forall n \geq N_2$, $|S_{2n+1} - S| < \epsilon$. Take $N = 2\max\{N_1, N_2\} + 1$, then if $n \geq N$, $|S_k - S| < \epsilon$. So $S_k \to S$. ■

Lecture 5

31 Jan. 2022

2.3 Absolute Convergence

Definition 2.3. Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is called *absolutely convergent.*

Note. Since $|a_n| \geq 0$. We can use the previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

Theorem 2.6. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose first $a_n \in \mathbb{R}$. Let

$$v_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$w_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

We have $v_n = \frac{|a_n|+a_n}{2}, w_n = \frac{|a_n|-a_n}{2}$. Clearly, $v_n, w_n \geq 0$. We also have $|a_n| = v_n + w_n \geq v_n, w_n$.

So by comparison test, if $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{-=1}^{\infty} v_n$, $\sum_{n=1}^{\infty} w_n$ also converges.

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$. We have $|x_n|, |y_n| \leq |a_n|$. So $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are absolutely convergent, so they are convergent. And $\sum_{n=1}^{\infty} a_n$ converges as well.
■

Example.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely convergent.
2. $\sum_{n=1}^{\infty} \frac{z^n}{2^n}$ for $z \in \mathbb{C}$. We check for absolute convergence first, $\sum_{n=1}^{\infty} \left(\frac{|z|}{2}\right)^n$. So if $|z| < 2$, the series is convergent by absolute convergence.

   Otherwise, if $|z| \geq 2$, $\left|\frac{z}{2}\right| \geq 1$. a_n does not tend to zero, hence the series diverge.

Notation. If $\sum_{n=1}^{\infty} a_n$ converges but not absolutely convergent, it is sometimes called *conditional convergent.*

It is called conditional because the sum to which the series converges is conditional on the order in which elements of the sequence are taken.

Example (Example Sheet 1, Q7). $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ and $1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$ are two series with different sums. Let s_n be the partial sum of the first series, and t_n be the partial sum of the second series, then $s_n \to s$ and $t_n \to \frac{3s}{2}$.

Definition 2.4. Let σ be a bijection of the positive integers, $a'_n = a_{\sigma(n)}$ is a *rearrangement*.

Theorem 2.7. If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

Proof. Again we do the proof first for $a_n \in \mathbb{R}$. Let $\sum_{n=1}^{\infty} a'_n$ be a rearrangement of $\sum_{n=1}^{\infty} a_n$. Let $s_n = \sum_{i=1}^{n} a_i$ and $t_n = \sum_{i=1}^{n} a'_i$, $S = \sum_{n=1}^{\infty} a_n$. Suppose first that $a_n \geq 0$. Given n, we can find q such that s_q contains every term of t_n. Because $a_n \geq 0$, we have

$$t_n \leq s_n \leq S.$$

So t_n is an increasing sequence bounded above so $t_n \to t$, and from the inequality above, $t \leq s$. By symmetry, we have $s \leq t \implies s = t$. If a_n has any negative term, consider v_n and w_n from Theorem (2.6). Consider $\sum_{n=1}^{\infty} a'_n$, $\sum_{n=1}^{\infty} v'_n$, $\sum_{n=1}^{\infty} w'_n$. Since $\sum_{n=1}^{\infty} |a_n|$ converges, both $\sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n$ converge. Using the fact that $v_n, w_n \geq 0$, we case above, we have $\sum_{n=1}^{\infty} v'_n = \sum_{n=1}^{\infty} v_n$ and $\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} w'_n$. But $a_n = v_n - w_n$ so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a'_n$.

For the case $a_n \in \mathbb{C}$, we write $a_n = x_n + iy_n$. Since $|x_i|, |y_i| \leq |a_n|$, $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ are absolutely convergent. By the previous case $\sum_{n=1}^{\infty} x'_n = \sum_{n=1}^{\infty} x_n$, $\sum_{n=1}^{\infty} y'_n = \sum_{n=1}^{\infty} y_n$. Since $a'_n = x'_n + iy'_n$ so $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a'_n$. ■

Lecture 6 2 Feb. 2022

3 Functions

3.1 Continuity

Suppose $E \subseteq \mathbb{C}$ is a non-empty subset, and we have a function $f : E \to \mathbb{C}$and a point $a \in E$. (this includes the case in which f is real-valued and E is a subset of $\mathbb{R}$)

Definition 3.1. f is *continuous* at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$.

Definition 3.2 (ϵ-δ Definition). f is *continuous* at $a \in E$, if given $\epsilon > 0$, $\exists \delta > 0$ such that if $|z - a| < \delta, z \in E$, then $|f(z) - f(a)| < \epsilon$.

We prove right away that the two definitions are equivalent.

Theorem 3.1. The two definitions of continuity are equivalent.

Proof. We first prove the second definition implies the first definition. We know that given $\epsilon > 0, \exists \delta > 0$ such that $|z - a| < \delta, z \in E$, then $|f(z) - f(z)| < \epsilon$. Let $z_n \to a$, then $\exists n_0$ such that $\forall n \geq n_0$, we have $|z_n - a| < \delta$. This implies, by the assumption, $|f(z_n) - f(a)| < \epsilon$. That is, $f(z_n) \to f(a)$.

Next, we prove the other direction. Assume $f(z_n) \to f(a)$ whenever $z_n \to a, z_n \in E$. Suppose f is not continuous at a according to Definition 2.

$\exists \epsilon > 0$, s.t. $\forall \delta > 0$, there exists $z \in E$ s.t. $|z - a| < \delta$ and $|f(z) - f(a)| \geq \epsilon$.

Let $\delta = \frac{1}{n}$ from non-continuity defined above, we get z_n such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \epsilon$. Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \epsilon$. Contradiction. ∎

Proposition 3.1. $a \in E$, and $g, f : E \to \mathbb{C}$ are both continuous at a. So are the functions $f(z) + g(z)$, $f(z)g(z)$ and $\lambda f(z)$ for any constant λ. In addition, if $f(z) \neq 0\ \forall z \in E$, then $\frac{1}{f(z)}$ is continuous at a.

Proof. Using Definition 1 of continuity, this is obvious, using the analogous results for sequences. (Lemma (1.1))

For example,

$$z_n \to a \implies f(z_n) \to f(a), g(z_n) \to g(a) \implies f(z_n) + g(z_n) \to f(a) + g(a).$$

∎

The function $f(z) = z$ is continuous, so by using the proposition, we get that every polynomial is continuous at every point in $\mathbb{C}$.

Note. We say that f is *continuous on E* if it is continuous at every $a \in E$.

Remark. Still it is instructive to prove Proposition (3.1) directly from the ϵ-δ definition.

Next we look at compositions.

Theorem 3.2. Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions such that $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$, then $g \circ f : A \to \mathbb{C}$ is continuous at a.

Proof. Take any sequence $z_n \to a$, by assumption we know $f(z_n) \to f(a)$. Set $w_n = f(z_n) \in B$. By continuity of g, we have $g(w_n) \to g(f(a))$, and we are done. ∎

Example.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$
assuming that $\sin x$ is continuous. (to be proved later) If $x \neq 0$, propositions proved above imply that $f(x)$ is continuous at any $x \neq 0$.

   However, it is discontinuous at 0. Consider the sequence satisfying
$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi.$$
We have $f(x_n) \to 1, x_n \to 0$, but $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$
It's continuous at $x \neq 0$ as above, and f is continuous at 0. Take $x_n \to 0$, then $|f(x_n)| \leq |x_n|$ because $\sin \frac{1}{x} \leq 1$, so $f(x_n) \to 0 = f(0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be
$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$
It is discontinuous at every point. If $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then $f(x_n) = 0 \not\to f(x) = 1$. Similarly, if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$, we have $f(x_n) = 1 \not\to f(x) = 0$.

Lecture 7

4 Jan. 2022

3.2 Limit of a function

$f : E \subseteq \mathbb{C} \to \mathbb{C}$. We wish to define what is meant by $\lim_{z \to a} f(z)$, even when a might not be in E.

Example. The limit of $\frac{\sin z}{z}$ as $z \to 0$ with $E = \mathbb{C}\{0\}$.

Also, if $E = \{0\} \cup [1, 2]$, it does not make sense to speak about points $z \in E, z \neq 0, z \to 0$.

Definition 3.3. If $E \subseteq \mathbb{C}, a \in \mathbb{C}$, we say that a is a *limit point* of E if for any $\delta > 0, \exists z \in E$ such that $0 < |z - a| < \delta$.

Remark. a is a limit point if and only if there exists a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all n.

Definition 3.4. If $f : E \subseteq \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$ be a limit point of E. We say that $\lim_{z \to a} f(z) = l$ ("f tends to l as z tends to a") if given $\epsilon > 0, \exists \delta > 0$ such that whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \epsilon$.

Equivalently, $f(z_n) \to l$ for every sequence $z_n \in E, z_n \neq a$ and $z_n \to a$.

Remark. Straight from the definitions, we have that if $a \in E$ is limit point, then $\lim_{z \to a} f(z) = f(a)$ if and only if f is continuous at a.

If $a \in E$ is *isolated* (i.e. $a \in E$ is not a limit point), continuity of f at a always holds. The limit of functions has very similar properties to limit of sequences.

1. It is unique, $f(z) \to A$ and $f(z) \to B$ as $z \to a$, then

$$|A - B| \leq |A - f(z)| + |f(z) - B|.$$

   If $z \in E$ is such that $0 < |z - a| < \min\{\delta_1, \delta_2\}$, then $|A - B| < 2\epsilon$. So $A = B$. The existence of such z is a consequence of the condition that a is a limit point of E.

2. $f(z) + g(z) \to A + B$;
3. $f(z)g(z) \to AB$;
4. if $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$. All proved in the same way as before.

3.3 The Intermediate Value Theorem

Theorem 3.3 (Intermediate Value Theorem). If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then f takes every value which lies between $f(a)$ and $f(b)$.

Proof. Without loss of generality, suppose $f(a) < f(b)$. Take $f(a) < \eta < f(b)$. Let $S = \{x \in [a, b] \mid f(x) < \eta\}$. We note that $a \in S$, so $S \neq \varnothing$. Clearly S is bounded above by b. Then there is a supremum C where $C \leq b$. By definition of supremum, given n, there exists $x_n \in S$ such that $C - \frac{1}{n} < x_n \leq C$. So $x_n \to C$. Since $x_n \in A$, $f(x_n) < \eta$. By continuity of f, $f(x_n) \to f(C)$. So $f(c) \leq \eta$.

Now observe that $c \neq b$ because $f(b) > \eta$. Then for n large, $C + \frac{1}{n} \in [a, b]$ and $C + \frac{1}{n} \to C$. Again by continuity $f(C + \frac{1}{n}) \to f(C)$. But since $C + \frac{1}{2} > C$, $f(C + \frac{1}{n}) \geq \epsilon$. So $f(c) \geq \eta \implies f(c) = \eta$. ∎

Remark. The theorem is very useful for finding zeroes or fixed points.

Example. Existence of the N-th root of a positive real number. Suppose

$$f(x) = x^N, \quad x \geq 0.$$

Let y be a positive real number. f is continuous on $[0, 1+y]$, so

$$0 = f(0) < y < (1+y)^N = f(1+y).$$

By the IVT, $C \in (0, 1+y)$ such that $f(c) = y$, i.e. $C^N = y$. C is a positive N-th root of y.

We also have uniqueness. Exercise.

Lecture 8

7 Feb. 2022

3.4 Bounds of a Continuous Function

Theorem 3.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists K such that $|f(x)| \leq K$ for all $x \in [a, b]$.

Proof. We argue by contradiction. Suppose the statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, x_n has a convergent subsequence $x_{n_j} \to x$. Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By the continuity of f, $f(x_{n_j}) \to f(x)$. But $\left|f(x_{n_j})\right| > nj \to \infty$ as $j \to \infty$. Contradiction. ∎

Theorem 3.5 (Extreme Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in [a, b]$ such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all $x \in [a, b]$.

"A continuous function on a closed bounded interval is bounded and attains its bounds."

Proof. Let $A = \{f(x) \mid x \in [a, b]\} = f([a, b])$. By Theorem (3.4), A is bounded since it is clearly non-empty, it has a supremum M. By definition of supremum, given an integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $M - \frac{1}{n} < f(x_n) \leq M$. From Bolzano-Weierstrass, there exists $x_{n_j} \to x \in [a, b]$. Since $f(x_{n_j}) \to M$, by continuity of f, we get that $f(x) = M$. So $x_2 \coloneqq x$.

We can prove similarly for the minimum. ∎

Proof 2. $A = f([a, b]), M = \sup A$ as before. Suppose $\nexists x_2$ such that $f(x_2) = M$. Let

$$g(x) = \frac{1}{M - f(x)}, x \in [a, b]$$

is defined and continuous on $[a,b]$. By Theorem (3.4) applied to g, $\exists k > 0$ such that $g(x) < K$ for all $x \in [a,b]$. This means that $f(x) \leq M - \frac{1}{k}$ for all $x \in [a,b]$. This is absurd because it contradicts that M is the supremum. ■

Note. Theorems (3.4) and (3.5) are false if the interval is not closed and bounded. For example,

$$f : (0,1] \to \mathbb{R}, x \mapsto \frac{1}{x}.$$

3.5 Inverse Functions

Definition 3.5. f is *increasing* for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 such that $a \leq x_1 < x_2 \leq b$.

If $f(x_1) < f(x_2)$, we say that f is *strictly increasing*.

There are similar definitions for *decreasing* and *strictly decreasing*.

Theorem 3.6. $f : [a,b] \to \mathbb{R}$ is continuous and strictly increasing for $x \in [a,b]$. Let $c = f(a)$ and $d = f(b)$. Then $f : [a,b] \to [c,d]$ is bijective and the inverse $g \coloneqq f^{-1} : [c,d] \to [a,b]$ is also continuous and strictly increasing.

Remark. There is a similar statement for strictly decreasing function. Take $c < k < d$, from the IVT, $\exists h$ such that $f(h) = k$. Since f is strictly increasing, h is unique. Define $g(k) \coloneqq h$ and this gives an inverse $g : [c,d] \to [a,b]$ for f.

We first prove that g is strictly increasing. Take $y_1 < y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. If $x_2 \leq x_1$, since f is increasing, $f(x_2) \leq f(x_1) \implies y_2 \leq y_1$. Absurd.

Next we prove continuity. Let $\epsilon > 0$ be given, let $k_1 = f(h-\epsilon)$ and $k_2 = f(h+\epsilon)$. Because f is strictly increasing, we have $k_1 < k < k_2$. If $k_1 < y < k_2$, we have $h - \epsilon < g(y) < h + \epsilon$. So we can just take $\delta = \min\{k_2 - k, k - k_1\}$. So g is continuous at k. Here we took $k \in (c,d)$. A very similar argument establishes continuity at the end points.

Lecture 9

9 Feb. 2022

4 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E = \text{interval} \subseteq \mathbb{R}$.

Definition 4.1. Let $x \in E$ be a point such that $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point), f is said to be *differentiable* at x with derivative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x).$$

If f is differentiable at each $x \in E$, we say f is differentiable on E.

(Think of E as an interval or a disc in the case of $\mathbb{C}$.)

Remark.

1. Other common notations include $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}f}{\mathrm{d}x}$.
2. $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$. $(y = x + h)$
3. Another look at the definition is the following.

   Let $\epsilon(h) := f(x+h) - f(x) - hf'(x)$, then $\lim\limits_{h \to 0} \frac{\epsilon(h)}{h} = 0$. We have also

   $$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear in } h} + \epsilon(h).$$

   Alternative definition of differentiability is f is differentiable at x if $\exists A, E$ such that $f(x+h) = f(x) + hA + \epsilon(h)$ where $\lim\limits_{h \to 0} \frac{\epsilon}{h} = 0$. If such an A exists, then it is unique, since $A = \lim\limits_{h \to 0} \frac{f(x+h)-f(x)}{h}$.
4. If f is differentiable at x, then f is continuous. Since $\epsilon(h) \to 0$, then $f(x+h) \to f(x)$ as $h \to 0$.
5. Alternative ways of writing things:

   $$f(x+h) = f(x) + hf'(x) + h\epsilon_f(h) \text{ with } \epsilon_f(h) \to 0 \text{ as } h \to 0.$$

   Or,

   $$f(x) = f(a) + (x-a)f'(a) + (x-a)\epsilon_f(x) \text{ with } \epsilon_f(x) \to 0 \text{ as } x \to a.$$

Example. If we have $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Clearly, we have $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Take $h_n \downarrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{h_n}{h_n} = 1.$$

And take $h_n \uparrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{-h_n}{h_n} = -1.$$

So f is not differentiable at $x = 0$.

4.1 Differentiation of Sums, Products, etc

Property.

1. If $f(x) = c$ for all $x \in E$, then f is differentiable with $f'(x) = 0$.
2. f, g are differentiable at x, then so is $f + g$ and
$$(f+g)'(x) = f'(x) + g'(x).$$
3. f, g are differentiable at x, then so is fg and
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$
4. f differentiable at x and $f(x) \neq 0$ for all $x \in E$, then $\frac{1}{f}$ is differentiable at x and
$$(\frac{1}{f})'(x) = \frac{-f'(x)}{[f(x)]^2}.$$

Proof.

1. $\lim\limits_{h \to 0} \frac{c-c}{h} = 0$.
2. $$\lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \qquad \text{using properties of limits.}$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$
3. Let $\phi(x) = f(x)g(x)$, then we have
$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= f(x+h)[\frac{g(x+h) - g(x)}{h}] + g(x)[\frac{f(x+h) - f(x)}{h}]. \end{aligned}$$
So we have $\lim\limits_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = f(x)g'(x) + f'(x)g(x)$ using standard properties of limits and the fact that f is continuous at x.
4. Define again $\phi(x) = \frac{1}{f(x)}$, then
$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} \\ &= \frac{f(x) - f(x+h)}{hf(x)f(x+h)}. \end{aligned}$$
So we have $\lim\limits_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \frac{-f(x)}{[f(x)]^2}$.

■

Remark. From (3) and (4), we get
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Lecture 10

11 Feb. 2022

Example. Consider $f(x) = x^n$ with $n \in \mathbb{Z}, n > 0$. When $n = 1$, clearly we have $f(x) = x$ and $f'(x) = 1$.

We claim that $f'(x) = nx^{n-1}$, and we prove it by induction, $f(x) = xx^n = x^{n+1}$. By product rule and inductive hypothesis,

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Next, we consider $f(x) = x^{-n}$ with $n \in \mathbb{Z}, n > 0$. If $x \neq 0$, use Proposition (4.1), we have

$$f'(x) = \frac{-(x^n)'}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

So we know how to find derivatives of polynomials and rational functions.

We have the following useful result to differentiate a larger class of functions.

Theorem 4.1 (Chain Rule). If $f : U \to \mathbb{C}$ is such that $f(x) \in V$ for $x \in U$. If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a)g'(f(a)).$$

Proof. We know

$$f(x) = f(a) + (x-a)f'(a) + \epsilon_f(x)(x-a)$$

such that $\lim_{x\to a} \epsilon_f(x) = 0$, and

$$g(y) = g(b) + (y-b)g'(b) + \epsilon_g(y)(y-b)$$

with $\lim_{y\to b} \epsilon_g(y) = 0$. Let $b = f(a)$, and set $\epsilon_f(a) = 0$ and $\epsilon_g(b) = 0$ to make them continuous at $x = a$ and $f = b$. Now $y = f(x)$ gives

$$\begin{aligned}
g(f(x)) =& g(b) + (f(x) - b)g'(b) + \epsilon_g(f(x))(f(x) - b) \\
=& g(f(a)) + [(x-a)f'(a) + \epsilon_f(x)(x-a)][g'(b) + \epsilon_g(f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + \\
& (x-a)[\epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))] \\
=& g(f(a)) + (x-a)f'(a)g'(b) + (x-a)\sigma(x).
\end{aligned}$$

So it suffices to show $\sigma(x) = \epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))$ tends to 0 as x tends to a. We have clearly $\epsilon_f(x)g'(b) \to 0$, $\epsilon_g(f(x)) \to 0$ and $f'(a) + \epsilon_f(x) \to f'(a)$, so $\lim_{x\to a} \sigma(x) = 0$. ■

Example.

1. Consider $f(x) = \sin(x^2)$, and we have

$$f'(x) = 2x\cos(x^2).$$

2. Consider $f(x) = \begin{cases} x\sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. From previous lectures, we know that f is continuous, and it is differentiable at every $x \neq 0$ by the previous theorems. At $x = 0$, take $t \neq 0$ and we have
$$\frac{f(t) - f(0)}{t - 0} = \sin(\frac{1}{t}).$$
Again from previous lecture, we know $\lim\limits_{t \to 0} \frac{f(t)-f(0)}{t-0}$ does not exist, so f is not differentiable at $x = 0$.

4.2 The Mean Value Theorem

Theorem 4.2 (Rolle's Theorem). Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Let $M = \max_{x\in[a,b]} f(x)$, and $m = \min_{x\in[a,b]} f(x)$. Theorem (3.5) says that these values are achieved. Let $k = f(a) = f(b)$. If $M = m = k$, then f is constant and $f'(c) = 0$ for all $c \in (a, b)$.

If f not constant, then $M > k$ or $m < k$. Suppose $M > k$. By Theorem (3.5), exist $c \in (a, b)$ such that $f(c) = M$.

If $f'(c) > 0$, then there are values to right of c for which $f(x) > f(c)$ because
$$f(h + c) - f(c) = h(f'(c) + \epsilon_f(h)).$$
Since $\epsilon_f(h) \to 0$ as $h \to 0$, $f'(c) + \epsilon_f(h) > 0$ for h small. This contradicts that M is the maximum. Similarly, if $f'(c) < 0$, there exists x to the left of c for which $f(x) > f(c)$.

So we must have $f'(c) = 0$. ■

Lecture 11

14 Feb. 2022

Theorem 4.3 (Mean Value Theorem). Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b), then $\exists c \in (a, b)$ such that
$$f(b) - f(a) = f'(c)(b - a).$$

Proof. Write $\phi(x) = f(x) - kx$, and choose k such that $\phi(a) = \phi(b)$. So
$$f(b) - bk = f(a) - ak \implies k = \frac{f(b) - f(a)}{b - a}.$$
By Rolle's Theorem applied to ϕ, $\exists c \in (a, b)$ such that $\phi'(c) = 0$. That is, $f'(c) = k$. ■

Remark. We will often write

$$f(a+h) = f(a) + hf'(a+\theta h)$$

with $\theta \in (0,1)$. We need to be careful, and consider $\theta = \theta(h)$.

Corollary 4.1. $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b).

1. If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)
2. If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)
3. If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.

Proof.

1. MVT implies $f(y) - f(x) = f'(c)(y-x)$. And $f'(c) > 0 \implies f(y) > f(x)$.
2. MVT implies $f(y) - f(x) = f'(c)(y-x)$. And $f'(c) \geq 0 \implies f(y) \geq f(x)$.
3. Take $x \in [a,b]$. Then use the MVT in $[a,x]$ to get $c \in (a,x)$ such that $f(x) - f(a) = f'(c)(x-a) = 0$. So $f(x) = f(a)$ and f is constant.

■

Theorem 4.4 (Inverse Function Theorem). If $f : [a,b] \to \mathbb{R}$ continuous and differentiable on (a,b) with $f'(x) > 0$ for all $x \in (a,b)$. Let $f(a) = c$ and $f(b) = d$, then the function $f : [a,b] \to [c,d]$ is bijective and f^{-1} is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Proof. By Corollary (4.1), f is strictly increasing in $[a,b]$. By theorem (3.6), $\exists g : [c,d] \to [a,b]$ which is a continuous strictly increasing inverse of f. We want to show that g is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$ and $x \in (a,b)$.

If $k \neq 0$ is given, let h be given by $y + k = f(x+h)$. That is, $g(y+k) = x+h$ for $h \neq 0$. Then

$$\frac{g(y+k) - g(y)}{k} = \frac{x+h-x}{f(x+h) - f(x)} = \frac{h}{f(x+h) - f(x)}.$$

Let $k \to 0$, then $h \to 0$ because g is continuous. So we have

$$g'(y) = \lim_{k \to 0} \frac{g(y+k) - g(y)}{k} = \frac{1}{f'(x)}.$$

■

Example. We take $g(x) = x^{\frac{1}{q}}$ with $x > 0$ and q positive integer. So $f(x) = x^q$, with $f'(x) = qx^{q-1}$. g is differentiable and so is g, and by Theorem (4.4),

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q}x^{\frac{1}{q}-1}.$$

Remark. If $g(x) = x^r$ with $r \in \mathbb{Q}$, then $g'(x) = rx^{r-1}$.

Suppose $f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b-a)f'(s)}{(b-a)g'(t)} = \frac{f'(s)}{g'(t)}.$$

Cauchy showed that we can take $s = t$.

Theorem 4.5 (Cauchy's Mean Value Theorem). Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then $\exists t \in (a, b)$ such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a)).$$

Remark. We recover the MVT if we take $g(x) = x$.

Lecture 12

16 Feb. 2022

Proof. Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}.$$

We have ϕ continuous on $[a, b]$ and differentiable on (a, b). Also, $\phi(a) = \phi(b) = 0$. By Rolle's Theorem, there exists $t \in (a, b)$ such that $\phi'(t) = 0$, and

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)]. \end{aligned}$$

So $\phi'(t) = 0$ gives the result. ∎

Note. Good choice of auxiliary function and Rolle's theorem proves the theorem.

Example (L' Hopital's Rule). If we want to find $\lim\limits_{x \to 0} \frac{e^x - 1}{\sin x}$, we have

$$\frac{e^x - 1}{\sin x} = \frac{e^x - x^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

for some $t \in (0, x)$ by Cauchy's Mean Value Theorem. So

$$\frac{e^x - 1}{\sin x} \to 1$$

as $x \to 0$.

Goal: We want to extend the MVT to include higher order derivatives.

Theorem 4.6 (Taylor's Theorem with Lagrange's reminder). Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!f''(a)} + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n f^{(n)}(a+\theta h)}{n!}$$

where $\theta \in (0,1)$.

Note.

1. For $n = 1$, we get back the MVT, so this is an "n-th order MVT".
2. $R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$ is known as Lagrange's form of the remainder.

Proof. Define for $0 \le t \le h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{t^n}{n!} b$$

where we choose b such that $\phi(h) = 0$, and we clearly have $\phi(0) = 0$. (Recall that in the proof of the MVT, we used $f(x) - kx$ and picked k that we can use Rolle's Theorem. We also have that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0.$$

We use Rolle's Theorem n times. Since $\phi(0) = \phi(h) = 0$, $\phi'(h_1) = 0$ for some $0 < h_1 < h$. And since $\phi'(0) = \phi'(h_1) = 0$, we have $\phi''(h_2) = 0$ for some $0 < h_2 < h_1$. Finally, $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$. So $\phi^{(n)}(h_n) = 0$ with $0 < h_n < h_{n-1} < \cdots < h$. So $h_n = \theta h$ for $\theta \in (0,1)$, now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - b \implies b = f^{(n)}(a+\theta h).$$

Set $t = h$, $\phi(h) = 0$ and put this value of b to the second line in the proof. ■

Theorem 4.7 (Taylor's Theorem with Cauchy's reminder). Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, and if $a = 0$ for simplification, then we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1-\theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!}$$

with $\theta \in (0,1)$.

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1} f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$. Note that we have

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$
$$= -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t).$$

Set

$$\phi(t) = F(t) - (\frac{h-t}{h})^p F(0)$$

with $p \in \mathbb{Z}, 1 \leq p \leq n$. Then $\phi(0) = \phi(h) = 0$, and by Rolle's, $\exists \theta \in (0,1)$ such that $\phi'(\theta h) = 0$. But,

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h}F(0) = 0.$$

So

$$0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}[f(h) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)].$$

Rearranging the two sides, and we get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!p(1-\theta)^{p-1}}f^{(n)}(\theta h).$$

Taking $p = n$, we get Lagrange's reminder, and taking $p = 1$ gives Cauchy's reminder. ■