Probability

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Contents

Lecture 1: Probability Space

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Example. If we have a die with outcomes 1, 2, . . . , 6.

1. $\mathbb{P}(2) = \frac{1}{6}$
2. $\mathbb{P}(\text{multiple of } 3) = \frac{2}{6} = \frac{1}{3}$
3. $\mathbb{P}(\text{pair or a multiple of } 3) = \frac{4}{6} = \frac{2}{3}$

1 Formal Setup

We try to define a probability space rigorously in this section.

Definition 1.1 (Probability Space). We have the following,

1. Sample space Ω, a set of outcomes.
2. $\mathcal{F}$, a collection of subsets of Ω (called events).
3. $\mathcal{F}$ is a σ-algebra if
   (a) **F1**: $\Omega \in \mathcal{F}$
   (b) **F2**: if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
   (c) **F3**: For all countable collections $\{A_n\}$ in $\mathcal{F}$, $\cup_n A_n \in \mathcal{F}$.

Given σ-algebra $\mathcal{F}$ on Ω, function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure if

1. **P1**: The probability function is nonnegative.
2. **P2**: $\mathbb{P}(\Omega) = 1$
3. **P3**: For all countable collection $\{A_n\}$ of disjoint events in $\mathcal{F}$, we have $\mathbb{P}(\cup_n A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Problem. Why $\mathbb{P} : \mathcal{F} \to [0,1]$, not $\mathbb{P} : \Omega \to [0,1]$?

Remark. When Ω is finite or countable,

1. In general: $\mathcal{F} = P(\Omega)$.
2. $\mathbb{P}(2)$ is shorthand for $\mathbb{P}(\{2\})$.
3. $\mathbb{P}$ is determined by $\mathbb{P}(\{w\}), \forall w \in \Omega$.

Remark. When Ω is uncountable, a probability space behaves differently, as shown in the following example.

Example. If $\Omega = [0,1]$, and we want to choose a real number, all equally likely.

If $\mathbb{P}\{0\} = \alpha > 0$, then $\mathbb{P}(\{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}\} = n\alpha)$. This cannot happen if n large, because we would have $\mathbb{P} > 1$. So $\mathbb{P}(\{0\}) = 0$ or undefined.

Property. From the axioms, we want to prove the following properties of a probability space.

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

   Proof. A, A^c disjoint. $A \cup A^c = \Omega$. So $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$ ∎
2. $\mathbb{P}(\varnothing) = 0$
3. If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
4. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

1.1 Examples of Probability Spaces

Example. Here we list some concrete examples of probability spaces.

1. Ω finite, $\Omega = \{w_1, \ldots, w_n\}$, $\mathcal{F}$ = all subsets under uniform choice.

   $\mathbb{P} : \mathcal{F} \to [0,1], \mathbb{P}(A) = \frac{|A|}{|\Omega|}$. In particular: $\mathbb{P}(\{w\}) = \frac{1}{|\Omega|} \forall w \in \Omega$.
2. If we are choosing without replacement n indistinguishable marbles that are labelled $\{1, \ldots, n\}$. Pick $k \leq n$ marbles uniformly at random.

   Here we have $\Omega = \{A \subseteq \{1, \ldots, n\}$, $|A| = k$, $|\Omega| = \binom{n}{k}$.
3. If we have a well-shuffled deck of cards, and we uniformly chose permutation of 52 cards.

   $\Omega = \{$all permutations of 52 cards$\}$. $|\Omega| = 52!$.

   Then we have

   $$\mathbb{P}(\text{first three cards have the same suit}) = \frac{52 \cdot 12 \cdot 11 \cdot 49!}{52!} = \frac{22}{425}.$$