Geometry

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Contents

Lecture 1: Introduction

21 Jan. 11:00

1 Surfaces

1.1 Topological Surfaces

We start with some definitions.

Definition 1.1. A *topological surface* is a topological space Σ such that

1. **T1:** $\forall p \in \Sigma$ there is an open neighborhood $p \in U \subseteq \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subseteq \mathbb{R}^2$ with its usual Euclidean topology.
2. **T2:** Σ is Hausdorff and second countable.

Remark. We have the following remarks.

1. $\mathbb{R} \cong D(0,1)$, so homeomorphic to a disc is enough as stated in the definition.
2. A space X is *Hausdorff* if for $p \neq q \in X$, there exists disjoint open sets $p \in U$ and $q \in V$ in X.
3. A space X is *second countable* if it has a countable base i.e. $\exists \{u_i\}_{i \in \mathbb{N}}$ open sets s.t. every open set is a union of some u.
4. **T1** is the point and **T2** is for technical honesty.
5. If X is Hausdorff/ second countable, so are subspaces of X. In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

Example. Here we present some examples of topological surfaces.

1. $\mathbb{R}^2$, the plane.

2. Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where Z is closed:
   - $Z = \{0\}$,
   - $Z = \{(0,0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \ldots)\}$.
3. Graphs:

   Let $f : \mathbb{R}^2 \to R$ be a continuous function. The graph $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$ (subspace topology).

   Recall that if X, Y are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with U open and V open.

   It has the feature that $f : Z \to X \times Y$ is continuous if and open if the two projective maps are continuous.

   Commutative diagram

   Application: $\Gamma_f \subseteq X \times Y$, if $f : X \to Y$ is continuous, if homeomorphic to X.

   So $\Gamma_f \cong \mathbb{R}^2$ for any $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous, so Γ_f is a topological surface.

   Note. As a topological surface, Γ_f is independent of f, but later on as a geometric object, it will reflect features of f.
4. The sphere (subspace topology):

   $$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

   Stereographic projection

   Stereographic projection graph

   $$\pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$$
   $$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

   Note. The map is continuous and has an inverse, π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

   Explicit formula for inverse

   Stereographic projection from the South Pole is also a homeomorphism from $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$.

   So S^2 is a topological surface:

   $\forall p \in S^2$, either p lies in the domain of π_+ or of π_- (or both) and so it lies in an open set homeomorphic to $\mathbb{R}^2$. (And Hausdorff and second countable from $\mathbb{R}^2$).

   Remark. S^2 has a global property as it is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.
5. The real projective place:

   The group $\mathbb{Z}/2$ acts on S^2 by homeomorphism via the *antipodal map* $a : S^2 \to S^2$.

   $$a(x, y, z) = (-x, -y, -t).$$

   i.e. There exists a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathrm{Homeo}(S^2)$, such that it maps the non-identity element to the antipodal map.

Definition 1.2. The *real projective plane* is the quotient space of S^2 given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = {}^{S^2}/_{\mathbb{Z}/2\mathbb{Z}}.$$

Lemma 1.1. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ∎ Graph of the sphere

Lemma 1.2. $\mathbb{RP}^2$ is a topological surface.

Proof. We check that it is Hausdorff:

Recall if X is a space and $q : X \to Y$ is a quotient map, $V \subseteq Y$ is open $\iff q^{-1}V \subseteq X$ open. More balls

If $[p], [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in S^2$ are distinct antipodal pairs. Take small open discs around p, q and their antipodal images, as in the picture.

We can then take small balls $B_{\pm p}(\delta)$, $B_{\pm 1}(\delta)$, which intersects S^2 with open sets around $\pm q$ and $\pm p$.

$\mathbb{RP}^2$ is also second countable.

Let $\mathcal{U}$ be a countable base for the topology on S^2, such that for all $u \in \mathcal{U}$, the antipodal image is in $\mathcal{U}$.

Let $\bar{\mathcal{U}}$ be the family of open sets in $\mathbb{RP}^2$ of the form $q(u) \cup q(-u)$, $u \in \mathcal{U}$.

Now, if $v \in \mathbb{RP}^2$ is open, by definition $q^{-1}v$ is open in S^2, so $q^{-1}v$ contains some $u \in \mathcal{U}$, and hence contains $u \cup (-u)$. So $\bar{\mathcal{U}}$ is a countable base for the quotient topology on $\mathbb{RP}^2$ consider all such u that covers $q^{-1}v$.

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\bar{D}$ be a small (contained in an open hemisphere) closed disc neighborhood of $p \in S^2$.

If we consider q restricted to $\bar{D}$, it is a continuous map from a compact space to a Hausdorff space.

Also, on $\bar{D}$, the map q is injective. Recall "Topological inverse function theorem": A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. So q restricted to the disk is a homeomorphism.

It then induces another homeomorphism of q restricted to D, and open disk contained in $\bar{D}$. So $[q] \in q(D)$ has an open neighborhood in $\mathbb{RP}^2$ that is homeomorphic to an open disk, and we are done. ∎

Lecture 2: More Examples

24 Jan. 11:00

6. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

The *torus* $S^1 \times S^1$, with the subspace topology from $\mathbb{C}^2$ which is the product topology.

Lemma 1.3. The torus is a topological surface.

Proof. We consider the map

$$\begin{aligned} \mathbb{R}^2 &\to S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}, \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

Note that this induces map

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, "\hat{e}"] & \end{tikzcd}.$$

That is, on the equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$, e is constant on equivalence classes, so it induces a map of sets $\mathbb{R}/\mathbb{Z}^2$. We can think of it as a quotient space equipped with the quotient topology.

$\mathbb{R}/\mathbb{Z}^2$ is compact. A continuous map from a compact space to a Hausdorff space that is a bijection is a homeomorphism.

Note we already know that $S^1 \times S^1$ is compact and Hausdorff. (closed and bounded in $\mathbb{R}^4$).

As for $S^2 \to \mathbb{RP}^2$, pick $[p] = q(p), p \in \mathbb{R}$ and a small closed disk $\overline{D}(p) \in \mathbb{R}^2$ such that for all $(n,m) \in \mathbb{Z} \setminus \{(0,0)\}$, we have $\overline{D}(p) \cap (\overline{D}(p) + (n,m)) = \varnothing$. Then e and q restricted to the small closed disk is injective. They are bijective continuous maps from compact spaces to Hausdorff spaces, so they are homeomorphisms. Restricting it further to a smaller open disk, and we have a neighborhood of $[p]$ that is homeomorphic to a disk. Since $[p]$ is arbitrary, and $S^1 \times S^1$ is a topological surface. ∎

Let P be a planar Euclidean polygon. Assume the edges are *oriented* and paired, and for simplicity assume the Euclidean length for $e, \hat{e}$ are equal if they are paired.

If $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say $f_{e\hat{e}} : e \to \hat{e}$.

There maps generate an equivalence relation on P where we identify $x \in P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. $P/\sim$ (with the quotient topology) is a topological surface.

Example. The torus as $[0,1]^2/\sim$. We consider three different kinds of points.

If p is in the interior. We can find a small enough neighborhood that is injective, and again by topological inverse function theorem, that small enough disk is homeomorphic to an open disk.

If p is on the edge. Say $p = (0, y) \sim (1, y)$ and $\delta > 0$ is small enough such that a half disk of radius δ does not touch vertices. Define a map from the union of the half-disks to $B(0, \delta) \subseteq \mathbb{R}^2$ by $(x, y) \mapsto x, y - y_0$ and $(x, y) \mapsto (x-1, y-y_0)$ on each part of the half-disk. Recall if $X = A \cup B$ is a union of closed subspaces, and we have continuous maps $f : A \to Y, g : B \to Y$, and $f|_{A \cap B} = g|_{A \cap B}$, they define a continuous map from X to Y.

Explicitly: f_u, f_v are continuous on $u, v \in [0, 1]^2$ on each of the two half-disks, so they induce a continuous map on $qU, qV \subseteq T^2$.

In T^2, the two maps overlap but agree, so by the recalled fact, we can define a map from the torus to a disk in $\mathbb{R}^2$.

Finally, we use the usual argument (pass a closed disk, use T.I.F.T, pass back its interior), then it has an open neighborhood homeomorphic to a disc.

Analogously at the vertex of $[0, 1]^2$, we split it into 4 maps.

This shows that ${}^{[0,1]^2}/_{\sim}$ is a topological surface.

Proof. For a general planar polygon, We can consider the suitable disc for interior points, and points on the edge as well.

Our equivalence relation induces an equivalence relation on the vertices in the obvious fashion. If $v \in \mathrm{Vert}(P)$ has r vertices in its equivalence class. There are r sectors in P with a total angle of α_v. Any sector can be identified with a standard sector with angle $\frac{2\pi}{r}$. Combining the sectors, and we would get a disc as required.

If $r = 1$, we just glue the two neighboring edges together, and we get a cone. If we look from above, we get an open disc centered around the vertex.

These open neighborhoods of points in ${}^{P}/_{\sim}$ show that ${}^{P}/_{\sim}$ is locally homeomorphic to a disc. We can also see ${}^{P}/_{\sim}$ is Hausdorff and second countable:

It's Hausdorff because for any non-equivalent points, we can find discs with small enough radius that lie in different equivalence classes. They are open disjoint sets in the quotient space as well. So ${}^{P}/_{\sim}$ is Hausdorff.

For second countability, I can consider disks in the interior of P with rational centers and radii, and for e in the edge of P, there is an isometry from e to an interval. And the points on the edge with correspond to rational centered and rational radius discs. And at vertices allow rational radius sectors. This gives me a countable base. ■

Remark. This might look less rigorous, but it conveys the same information as providing explicit homeomorphisms.

Lecture 3

26 Jan. 11:00

7. Given topological surfaces Σ_1, Σ_2, I can remove an open disc from each and glue the resulting circles.

   Explicitly, I take $\Sigma_1 \setminus D_1 \coprod \Sigma_2 \setminus D_2$ and impose a quotient relation.

   $$\theta \in \partial D_1 \sim \theta \in \partial D_2$$

   where θ parametrizes $S^1 = \partial D_i$.

   The result $\Sigma_1 \# \Sigma_2$ is called the *connect sum* of Σ_1 and Σ_2. (In principle this depends on any choices, suppressed from the notation).

   Lemma 1.5. The connect sum $\Sigma_1 \# \Sigma_2$ is a topological surface.

Lecture 4

28 Jan. 2022

Definition 1.3. A *subdivision* of a compact topological surface Σ comprises

1. a finite set $V \in \Sigma$ of vertices;
2. a finite collection $E = \{e_i : [0,1] \to \Sigma\}_{i \in E}$ of *edges* such that
   - for all i, e_i is a continuous injection on its interior and $e_i^{-1}V = \{e_i(0), e_i(1)\}$,
   - e_i and e_j have disjoint image except perhaps at their endpoints in V;
3. such that each connected component of $\Sigma \setminus (\cup e_i[0,1] \cup V)$ is homeomorphic to an open disc called a *face*. (So the closure of a face has boundary $\overline{F} \setminus F$ lying in $E \cup V$).

A subdivision is a *triangulation* if each *closed* face (closure of a face) contains exactly 3 edges, and two closed faces are disjoint or meet in exactly one edge (or possible just one vertex).

what

Example.

- A cube displays a subdivision of S^2.
- A tetrahedron displays a triangulation of S^2.
- We can describe subdivisions using planar polygons. For example, the normal depiction of T^2 has 1 vertex, 2 edges and 1 face.
- By our definition, we can have degenerate subdivisions like a subdivision of S^2 with 1 vertex, 0 edge and 1 face.

Definition 1.4. The *Euler characteristic* of a subdivision is the number $\#V - \#E + \#F$.

Theorem 1.1.

1. Every compact topological surface admits subdivision, and indeed triangulation.
2. The Euler characteristic denoted $\chi(\Sigma)$ does not depend on the choice of subdivision and defines a topological invariant of the surface. (depends only on the homeomorphic type of Σ)

Example.

1. $\chi(S^2) = 2$;
2. $\chi(T^2) = 0$;
3. If Σ_1 and Σ_2 are compact topological surfaces, we can form $\Sigma_1 \# \Sigma_2$ by removing an open disc $D_i \subseteq \Sigma_i$ which is a face of a triangulation, and gluing the boundary circles ∂D_i by a homeomorphism taking edges to edges.

   The resulting surface $\Sigma_1 \# \Sigma_2$ inherits a subdivision, and we have

   $$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

   In particular, if we take a surface with g holes, which is $\Sigma_g = \#_{i=1}^{g} T^2$, then $\chi(\Sigma_g) = 2 - 2g$; g is called the *genus* of Σ.

Remark.

1. Part 1 is hard to prove.
2. You should believe Part 2 since I can turn a subdivision into a triangulation, and I can relate triangulations by local moves. It is easy to check both subdividing and switch diagonal preserves χ.

   But it is hard to rigorize this result, and you learn essentially nothing from the combinatorial proof. A much cleaner approach is developed in Part II algebraic topology.

Recalled if Σ is a topological surface, every $p \in \Sigma$ lies in an open neighborhood $p \in u \subseteq \Sigma$ with u homeomorphic to an open disc (or equivalently to $\mathbb{R}^2$).

Definition 1.5. A pair (u, ϕ) where u is an open set in Σ and $\phi : u \to V$ an open set in $\mathbb{R}^2$ which is a homeomorphism is called a *chart* for Σ. (If $p \in u$ we might say "a chart for Σ at p")

A collection $\{(u_i, \phi_1)_{i \in I}\}$ of charts such that $@cup_{i \in I} = \Sigma$ is called an *atlas* for Σ.

The inverse $\sigma = \phi^{-1} : v \to u \in \Sigma$ is called a *local parametrization* for Σ.

Example.

1. If $Z \subseteq \mathbb{R}^2$ is a closed set, $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas with one chart that is $(\mathbb{R}^2, \mathbf{id})$.

2. For S^2, we have an atlas with 2 charts, the 2 stereographic projections.

Definition 1.6. Let (u_i, ϕ_i) be charts containing p, the map

$$\phi_2 \circ \phi_1^{-1}\big|_{\phi_1(u_1 \cap u_2)}$$

is called the *transition map* between the charts. This is a homeomorphism of open sets in $\mathbb{R}^2$

Recall that if $V \subseteq \mathbb{R}^2$ and $V' \subseteq \mathbb{R}^m$ open subsets, then a map $f : V \to V'$ is called *smooth* if it is infinitely differentiable; that is, it has partial derivatives of all orders of all variables.

If $n = m$, a homeomorphism $f : V \to V'$ is called a *diffeomorphism* if it is smooth and has smooth inverse.

Definition 1.7. An *abstract smooth surface* Σ is a topological surface with an atlas of charts $\{(u_i, \phi_i)\}_{i \in I}$ such that all transition maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(u_i \cap u_j) \to \phi_i(u_i \cap u_j)$$

are diffeomorphisms of open sets in $\mathbb{R}^2$.

Note. It would not make sense to ask for the ϕ_i themselves to be smooth, as Σ is just a topological space.

Example (Example Sheet 2). The atlas of 2 charts with stereographic projections gives S^2 the structure of an abstract smooth surface.

Example. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is also an abstract smooth surface. Recall we obtained charts from the inverse of the projection restricted to small discs in $\mathbb{R}^2$, the ones that are disjoint from translation by $\mathbb{Z}^2 \setminus \{(0,0)\}$.

The transition maps are the translations, so T^2 inherits the structure of an abstract smooth surface.