Complex Analysis

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Lecture 1: Introduction

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Textbook

1. *Complex Analysis* -Ahlfors
2. *Real and Complex Analysis* -Rudin

1 Analytic Functions

Notation. We have the following notations.

- $\mathbb{C}$: complex plane.
- $\bar{z}$: complex conjugate of $z \in \mathbb{C}$.
- $|z|$: modulus of $z \in \mathbb{C}$.

  Note that $d(z, w) = |z - w|$ defines a metric on $\mathbb{C}$ (the usual or standard metric).
- $D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ is the open disk with centre a and radius r.

Definition 1.1. A subset $U \in \mathbb{C}$ is *open* if it is open with respect to the above metric, i.e. if for every $a \in U$ there exists $r > 0$

All topological notions we will use (limits, continuity of function, compactness)

This course is about complex valued functions of a single complex variables. i.e. function

$$f : A \to \mathbb{C}, \quad \text{where } A \subset \mathbb{C}.$$

Remark. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, we.

Almost exclusively we'll focus on differentiable function f. Let's start by recalling continuity.

Definition 1.2. The function f is *continuous* at a point $w \in A$ if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$z \in A, |z - w| < \delta \implies |f(z) - f(w) < \epsilon|.$$

This is the same as saying that $\lim_{z \to w} f(z) = f(w)$.

Remark. f is continuous if and only if u, v is continuous with respect to the metric on $\mathbb{R}^2$.

1.1 Complex Differentiation

We extend the definition of derivatives in this section.

Definition 1.3. We have some definitions of differentiation in $\mathbb{C}$.

1. f is *differentiable* at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}.$$

exists as a complex numbers. $f'(w)$ is called the derivative of f at w.

2. f is *holomorphic at* $w \in U$ if there is $\epsilon > 0$ such that $D(w, \epsilon) \subset U$ and f is differentiable at every point in $D(w, \epsilon)$. (Locally differentiable everywhere).

3. f is *holomorphic in* U if f is holomorphic at every point in U, or equivalently, f is differentiable at every point in U.

Remark. Sometimes we use "analytic" to mean holomorphic. But precisely "analytic" means that the function has a convergent Taylor series at the point. It means the function is very nice behaving. However, in $\mathbb{C}$, holomorphic if and only if analytic which we will prove later on.

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (as can easily be checked).

The chain rule for composite functions also holds:

$$f : U \to \mathbb{C},\ g : V \to \mathbb{C},\ f(U) \subset V,\ h = g \circ f : U \to \mathbb{C}.$$

If f is differentiable at $w \in U$ g is differentiable at $f(w)$, then h is differentiable at w and $h'(w) = g'(f(w))f'(w)$.

Problem. Write $f(z) = u(x, y) + iv(x, y), z = x + iy$. Is differentiability of f at a point $w = c + id \in U$ is the same as differentiability of u and v at (c, d)?

Recall from Analysis & Topology that $u : U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a "good affine approximation of u at (c, d)." i.e. if there is a linear transformation

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - (u(c, d) + L(x - c, y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} = 0.$$

The answer for the above question though is no. The theorem below characterizes differentiability of f in terms of u and v.

Theorem 1.1 (Cauchy-Riemann equations). The function $f = u+iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U \iff u, v : U \to \mathbb{R}$ are differentiable at $(c, d) \in U$ and u, v satisfy the Cauchy-Riemann equations at (c, d), i.e.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \quad \text{at} \quad (c, d). \end{aligned}$$

If f is differentiable at $w = c + id$, then $f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d)$ (and three other expressions following from the Cauchy-Riemann equations).

Proof. f is differentiable at w with derivative $f'(w) = p + iq$

$$\begin{aligned} &\iff \lim_{z\to w} \frac{f(z) - f(w)}{z - w} = p + iq \\ &\iff \lim_{z\to w} \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0 \end{aligned}$$

Writing $f = u + iv$ and separating real and imaginary parts, we have

$$\begin{aligned} &\iff \lim_{(x,y)\to(c,d)} \frac{u(x, y) - u(c, d) - p(x - c) + q(y - d)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0, \\ &\text{and} \lim_{(x,y))\to(c,d)} \frac{v(x, y) - v(c, d) - q(y - d) - p(x - c)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0. \end{aligned}$$

$\iff$ u is differentiable at (c, d) with $Du(c, d)(x, y) = px - qy$, and v is differentiable at (c, d) with $Dv(c, d)(x, y) = qx + py$.

$\iff$ u, v are differentiable at (c, d) and $u_x(c, d) = p = v_y(c, d)$, and $u_y(c, d) = -q = -v_x(c, d)$ i.e. Cauchy-Riemann equations hold at (c, d).

We also get from the above that if f is differentiable at w, then $f'(w) = p+iq = u_x(c, d) + iv_x(c, d)$. ■

Remark. u, v satisfying the Cauchy-Riemann equations at a point does not guarantee differentiability of f. (See ex. sheet 1).

Remark. If we just want to show one direction:

Differentiability of f at $w = c + id$ implies the partial derivatives u_x, u_y, v_x, v_y exist and satisfy the Cauchy-Riemann equations.

The proof can be much simpler. We differentiate along $h = t$, and $h = it$ instead.

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Example. Here we give a non-example of Cauchy-Riemann equations. When $f(z) = \overline{z} = x - iy$. For this, $u(x, y) = x, v(x, y) = -y$, so $u_x = 1, v_y = 0, v_x = 0, v_y = -1$. So Cauchy-Riemann equations do not hold anywhere, and f is not differentiable at any point.

Corollary 1.1. Let $f = u + iv : U \to \mathbb{C}$. If u, v have continuous partial derivatives at a point $(c, d) \in U$ and satisfy the Cauchy-Riemann equations at (c, d), then f is differentiable at $w = c + id$.

In particular, if u, v are C^1 functions on U satisfying the Cauchy-Riemann equations in U, then f is holomorphic in U.

Proof. Continuity of partial derivatives of u implies that u is differentiable, and similarly for v from Analysis & Topology. So the corollary follows from Theorem 1.1. ■

Note. Quite remarkably, we can relax the requirement of continuity of partial derivatives of u, v in U to just continuity of u, v in U. Thus, if $f = u + iv$ is defined on an open set U and is continuous in U, and if u, v satisfy the Cauchy-Riemann equations in U, then f is holomorphic in U. This is called the **Looman-Menchoff** theorem. It is quite non-trivial to prove.

Remark. Complex differentiability is much more restrictive than real differentiability of real and imaginary parts (because of the additional requirement that C-R equations must hold). This leads to some surprising theorems compared to the real case. For instance:

1. if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, then f is constant! (Liouville's Theorem)
2. if $f : U \to \mathbb{C}$ is holomorphic, then f is automatically infinitely differentiable on U.

Note. (2) implies partial derivatives of u, v of all orders exists. So we can differentiate C-R equations to get

$$\begin{aligned} (u_x)_x = (v_y)_x &\implies u_{xx} = v_{yx} \\ (u_y)_y = (-v_x)_y &\implies u_{yy} = -v_{xy}. \end{aligned}$$

Since $v_{yx} = v_{xy}$ this gives, for $(x, y) \in U$,

$$\Delta u = u_{xx} + u_{yy} = 1.$$

Similarly, $\Delta u = 0$ in U.

That is, real and imaginary parts of a holomorphic function are harmonic.

For the next corollary, we need the following,

Definition 1.4.

- A *curve* is a continuous map $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subseteq \mathbb{R}$ is a closed interval. We say that γ is a C^1 *curve* if γ' exists and is continuous on $[a,]$. (If $\gamma(t) = x(t) + y(t)$ then $\gamma'(t) = x'(t) + iy'(t)$; at the end points, γ' is the one-sided derivative.)
- An open set $U \subseteq \mathbb{C}$ is *path connected* if for any two points $z, w \in U$, there is a curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$.
- A *domain* is a non-empty, open, path connected subset of C.

Corollary 1.2. If $U \subseteq C$ is a domain, $f : U \to \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant.

Proof. Write $f = u + iv$. By the Cauchy-Riemann equations, $f' = 0 \implies Du = 0, Dv = 0$ in U. Since U is a domain, this means (by a theorem from Analysis and Topology) that u, v are constants; that is, f is constant. ■

So far we've only seen very few explicit holomorphic functions (namely, polynomials on $\mathbb{C}$ and rational functions on their domains). We'd like to generate more. We do this by looking at power series.

1.2 Power Series

Recall (from IA Analysis)

Theorem 1.2 (Radius of Convergence). If (c_n) is a sequence of complex numbers then there is a unique number $R \in [0, \infty]$ such that the power series

$$\sum_{i=1}^{\infty} c_n (z - a)^n, z, a \in \mathbb{C}$$

converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly (with respect to the variable z) on the compact disk $\overline{D(a, r)} = \{z \in \mathbb{C} \mid |z - a| \leq r\}$.

R is called the radius of convergence of the power series. Note that there is no claim about convergence when $|z - a| = R, R > 0$. There are various expressions for R. For example,

- $R = \sup\{r \geq 0 \mid |c_n| r^n \to 0 \text{ as } n \to \infty\}$,
- $\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$.

Theorem 1.3. Let $\sum_{n=0}^{\infty} c_n(z-a)^n$ be a power series with radius of convergence equal to $R > 0$. Fix $a \in \mathbb{C}$, and define $f : D(a,R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} (z-a)^n$. Then:

1. f is holomorphic on $D(a,R)$;
2. the derived series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has radius of convergence equal to R, and $f'(x) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ for all $z \in D(a,R)$;
3. f has derivatives of all orders on $D(a,R)$), and $c_n = \frac{f^{(n)}(a)}{n!}$;
4. if f vanishes on $D(a,\epsilon)$ for some $\epsilon > 0$, then $f = 0$ on $D(a,R)$.

Remark.
1. This theorem provides a way to generate a large class of holomorphic functions on a disk.
2. Later we will show that every holomorphic function is locally given by a power series (Taylor series theorem). Once we have that, part (iii) above gives that holomorphic functions are automatically infinitely differentiable in their domain. (regardless of what the domain looks like)

Proof. 1, 2. By considering $g(z) = f(z+a)$, we assume without loss of generality that $a = 0$. So we have $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in D(0,R)$, with radius of convergence $R > 0$.

The derived series $\sum_{n=1}^{\infty} nc_n z^{n-1}$ will have some radius of convergence $R_1 \in [0,\infty]$.

To see that $R_1 \geq R$, let $z \in D(0,R)$ be arbitrary, and choose ρ such that $|z| < \rho < R$. Then

$$n|c_n||z|^{n-1} = n|c_n|\left|\frac{z}{\rho}\right|^{n-1} \rho^{n-1} \leq |c_n|\rho^{n-1}$$

for sufficient large n (as $n\left|\frac{z}{\rho}\right|^{n-1} \to 0$ as $n \to \infty$). Since $\sum |c_n| p^{n-1}$ converges, it follows that $\sum_{n=1}^{\infty} n|c_n||z|^{n-1}$ converges. Thus, $D(0,R) \subseteq D(0,R_1)$; that is, $R_1 \geq R$.

Since

$$|c_n||z|^n \leq n|c_n||z|^n = |z|(n|c_n||z|^{n-1}),$$

if $\sum n|c_n||z|^{n-1}$ converges then so does $\sum_{n=0}^{\infty} |c_n||z|^n$, so $R_1 \leq R$. So $R_1 = R$.

To prove that f is differentiable with $f'(z) = \sum\limits_{n=1}^{\infty} nc_n z^{n-1}$, fix $z \in D(0, R)$. We prove that $g : D(0, R) \to \mathbb{C}$ is continuous instead.

$$g(w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z}, & \text{if } w \neq z \\ \displaystyle\sum_{n=1}^{\infty} nc_n z^{n-1}, & \text{if } w = z \end{cases}.$$

We have $g(w) = \sum\limits_{n=1}^{\infty} h_n(w)$ where

$$h_n(w) = \begin{cases} \dfrac{c_n(w^n - z^n)}{w - z}, & \text{if } w \neq z \\ nc_n z^{n-1}, & \text{if } w = z \end{cases}.$$

h_n is continuous on $D(0, R)$ (since $w \mapsto w^n$ is differentiable with derivative nw^{n-1}). Using $\frac{w^n - z^n}{w - z} = \sum\limits_{j=0}^{n-1} z^j w^{n-1-j}$, we get that for any r with $|z| < r < R$ and any $w \in D(0, r)$, $h_n(w) \leq n|c_n| r^{n-1} = M_n$. Since $\sum\limits_{n=1}^{\infty} M_1 < \infty$, we have by the Weierstrass M-test, $\sum h_n$ converges uniformly on $D(0, r)$. But a uniform limit of continuous functions is continuous, so g is continuous in $D(0, r)$ and in particular in z.

3. Repeated apply (2). The formula $c_n = \frac{f^(n)(a)}{n!}$ follows by differentiating the series n times and setting $z = a$.

4.

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