Variational Principles

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4th May 2022

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Lecture 1

29 Apr. 2022

Motivation

0.1 The Brachistochrone Problem

Problem. Particle slides on a wire under influence of gravity between two fixed points A, B. Which shape of the wire gives the shortest travel time, starting from rest?

The travel time is $T = \int_A^B \frac{\mathrm{d}\ell}{v(x,y)}$, and by energy conservation, and by energy conservation

$$\frac{1}{2}mv^2 + mgy = 0 \implies v = \sqrt{-2gy}.$$

So

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1+(y')^2}}{\sqrt{-y}}\,\mathrm{d}x$$

subject to $y(0) = 0$, $y(x_2) = y_2$.

0.2 Geodesics

Problem. What is the shortest path γ between two points A, B on a surface.

Take $\Sigma = \mathbb{R}^2$. The distance along γ is

$$D[y] = \int_A^B \mathrm{d}\ell = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\, \mathrm{d}x,$$

and we want to minimize D by varying γ.