# Analysis

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# Contents

### Lecture 1: Limits

Books:

- *A First Course in Mathematical Analysis* -Burkill

- *Calculus* -Spivak

- *Analysis I* -Tao

# 1 Limits and Convergence

## 1.1 Review from Numbers and Sets

**Notation.** We denote sequences by $a_n$ or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

**Definition 1.1.** We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists $N$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.

**Note.** $N = N(\epsilon)$ which is dependent on $\epsilon$. That is, if you want to go closer to $a$, sometimes you need to go higher in $N$.

**Definition 1.2 (limit of a sequence).** We say that a sequence is a

$$\left.\begin{array}{l} \textit{increasing sequence if } a_n \leq a_{n+1}, \\ \textit{decreasing sequence if } a_n \geq a_{n+1}, \end{array}\right\} \textit{monotone sequence}$$

$$\left.\begin{array}{l} \textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ \textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{array}\right\} \textit{strictly monotone sequence}$$

We also have

**Theorem 1.1 (Fundamental Axiom of the Real Numbers).** If $a_n \in \mathbb{R}$ and $a_n$ is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges*.

**Remark.** It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.

- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

**Definition 1.3 (supremum).** For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S$,

2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

**Note.** Supremum is unique, and there is a similar notion of infimum.

**Lemma 1.1 (Properties of Limits).**

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.

2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).

3. If $a_n = c$ for all $n$ then $a_n \to c$ as $n \to \infty$.

4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.

5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.

6. If $a_n \to a$ , then $\frac{1}{a_n} \to \frac{1}{a}$.

7. If $a_n < A$ for all $n$ and $a_n \to a$, then $a \le A$.

*Proof.*

1. Given $\epsilon > 0$, there exists $N_1$ such that $|a_n - a| < \epsilon, \forall n \ge N_1$, and there exists $N_2$ such that $|a_n - b| < \epsilon, \forall n \ge N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \ge N$,

   $$|a - b| \le |a_n - a| + |a_n - b| < 2\epsilon.$$

   If $a \ne b$, take $\epsilon = \frac{|a-b|}{3}$, we have

   $$|a - b| < \frac{2}{3}|a - b|. \notz$$

2. Given $\epsilon > 0$, there exists $N$ such that $|a_n - a| < \epsilon, \forall n \ge N$, Since $n_j \ge j$, we know

   $$\left|a_{n_j} - a\right| < \epsilon, \forall j \ge N.$$

   That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have

   $$|a_n b_n - ab| \le |a_n b_n - a_n b| + |a_n b - ab|$$
   $$= |a_n||b_n - b| + |b||a_n - a|.$$

   Given $\epsilon > 0$, there exists $N_1$ such that $|a_n - a| < \epsilon, \forall n \ge N_1$, and there exists $N_2$ such that $|b_n - b| < \epsilon, \forall n \ge N_2$.

   If $n \ge N_1(1)$, $|a_n - a| < 1$, so $|a_n| \le |a| + 1$.

   We have

   $$|a_n b_n - ab| \le \epsilon(|a| + 1 + |b|), \forall n \ge N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

   $\blacksquare$

> **Lemma 1.2.**
> $$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

*Proof.* $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit $a$.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma (1.1)}.$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma (1.1) $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma (1.1), we have $a = \frac{a}{2} \implies a = 0$. $\blacksquare$

**Remark.** The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

> **Definition 1.4.** We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists $N$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma (1.1) are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.

## Lecture 2: Bolzano–Weierstrass theorem

> **Theorem 1.2 (Bolzano-Weierstrass Theorem).** If $x_n \in R$ and there exists $K$ such that $|x_n| \leq K$ for all $n$, then we can find $n_1 < n_2 < n_3 < \ldots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. In other words, every bounded sequence has a convergent subsequence.

**Remark.** We say nothing about the uniqueness of the limit $x$.

For example, $x_n = (-1)^n$ has two subsequences tending to $-1$ and $1$ respectively.

*Proof.* Set $[a_1, b_1] = [-K, K]$. Let $c$ be the mid-point of $a_1, b_1$, consider the following alternatives,

1. $x_n \in [a_1, c]$ for infinitely many $n$.

2. $x_n \in [c, a_2]$ for infinitely many $n$.

Note that (1) and (2) can hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold, and we set $a_2 = c$ and $b_2 = b_1$.

We proceed as above to construct sequences $a_n, b_n$ such that $x_m \in [a_n, b_n]$ for infinitely many values of $m$. They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

$a_n$ is an increasing sequence and bounded, and $b_n$ is a decreasing sequence and bounded. By Fundamental Axiom, $a_n \to a \in [a_1, b_1], b_n \to b \in [a_1, b_1]$. Using Lemma (1.1), $b - a = \frac{b-a}{2} \implies a = b$.

Since $x_m \in [a_n, b_n]$ for infinitely many values of $m$, having chosen $n_j$ such that $x_{n_j} \in [a_j, b_j]$, that is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$. In other words, there is unlimited supply.

Hence, $a_j \leq x_{n_j} \leq b_j$, so $x_{n_j} \to a$. ∎

## 1.2 Cauchy Sequences

**Definition 1.5 (Cauchy Sequence).** $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\epsilon > 0 \; \exists N > 0$ such that $|a_n - a_m| < \epsilon \; \forall n, m > N$.

**Note.** $N$ is dependent on $\epsilon$.

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

**Lemma 1.3.** A convergent sequence is a Cauchy sequence.

*Proof.* If $a_n \to a$, given $\epsilon > 0$, exists $N$ such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Take $m, n \geq N$,
$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$
∎

**Lemma 1.4.** Every Cauchy sequence is convergent.

*Proof.* First we note that if $a_n$ is Cauchy, then it is bounded.

Take $\epsilon = 1$, $N = N(1)$ in the Cauchy property, then
$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$
We have
$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$
Let $K = \max\{1 + |a_N|, |a_n| \; n = 1, 2 \ldots, N - 1\}$.

Then $|a_n| \leq K$ for all n. By the Bolzano–Weierstrass theorem, $a_{n_j} \to a$. We must have $a_n \to a$.

Given $\epsilon > 0$, there exists $j_0$ such that for all $j \geq j_0$, $|a_{n_j} - a| < \epsilon$.

Also, there exists $N(\epsilon)$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N(\epsilon)$.

Take $j$ such that $n_j \geq \max\{N(\epsilon), n_{j_0}\}$. Then if $n \geq N(\epsilon)$,
$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < 2\epsilon.$$

Thus, on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

# 2 Series

**Definition 2.1.** If $a_n \in \mathbb{R}, \mathbb{C}$ We say that $\sum\limits_{j=1}^{\infty} a_j$ converges to $s$ if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to S$$

as $N \to \infty$. We write $\sum\limits_{j=1}^{\infty} a_j = s$. If $S_N$ does not converge, we say that $\sum\limits_{j=1}^{\infty} a_j$ *diverges.*

**Remark.** Any problem on series is really a problem about the sequence of partial sums.

**Lemma 2.1.**

1. If $\sum\limits_{j=1}^{\infty} a_j$ and $\sum\limits_{j=1}^{\infty} a_j$ converges, then so does $\sum\limits_{j=1}^{\infty} \lambda a_j + \mu b_j$, when $\lambda, \mu \in \mathbb{C}$;

2. Suppose there exists $N$ such that $a_i = b_i$ for all $i \geq N$. Then either $\sum\limits_{i=1}^{\infty} a_i$ and $\sum\limits_{i=1}^{\infty} b_i$ both converge or they both diverge. (initial terms do not matter for convergence)

*Proof.*    1. Exercise.

2. If we have $n \geq N$,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^{n} a_i$$

$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^{n} b_i$$

So $S_n - d_n = \sum\limits_{i=1}^{N-1} a_i - b_i$ which is a constant. So $S_n$ converges if and only if $d_n$ does.

■

We have the following important example,

**Example (Geometric Series).** $x \in \mathbb{R}$, set $a_n = x^{n-1}$ with $n \geq 1$. So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \dfrac{1 - x^n}{1 - x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have $S_n(1 - x) = 1 - x^n$.

If $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$.

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$.

If $x < -1$, $S_n$ does not converge (oscillates).

If $x = -1$, $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$.

Thus, the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta, \delta > 0$, so $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \to 0$ because $(1 + \delta)^n \geq 1 + n\delta$ from binomial expansion.

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**Definition 2.2.** $S_n \to \infty$ if given $A$, there exists an $N$ such that $S_n > A$ for all $n > N$.

$S_n \to -\infty$ if given $A$, there exists an $N$ such that $S_n < -A$ for all $n > N$.

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**Lemma 2.2.** If $\sum_{i=1}^{\infty} a_n$ converges, then $\lim_{i \to \infty} a_i = 0$.

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*Proof.* Let $S_n = \sum_{i=1}^{\infty} a_i$, note that $a_n = S_n - s_{n-1}$. If $S_n \to a$, we have $a_n \to 0$ because $S_{n-1} \to a$ also. ∎

**Remark.** The converse of the preceding lemma is false. One example is $\sum \frac{1}{n}$,

the *harmonic series*. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$.

So if $S_n \to a$, then $S_{2n} \to a$, also we have $a \geq a + \frac{1}{2}$. Contradiction.

## 2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

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**Theorem 2.1 (The Comparison Test).** Suppose $0 \leq b_n \leq a_n$ for all $n$. Then if $\sum\limits_{n=1}^{\infty} a_n$ converges, so does $\sum\limits_{n=1}^{\infty} b_n$.

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*Proof.* Let $s_N = \sum\limits_{n=1}^{N} a_n$, $d_N = \sum\limits_{n=1}^{N} b_n$. Because $b_n \leq a_n$, we know $d_N \leq s_N$. But $s_N \to s$, then $d_n \leq s_n \leq 2$ for all n, and $d_N$ is a increasing sequence bounded above. So $d_N$ converges. ∎

**Example.** We consider $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that $\sum\limits_{n=1}^{\infty} a_n$ converges, so $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, we get $\sum\limits_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2$.

For the rest of the lecture, we establish two more tests.

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**Theorem 2.2 (Root test/ Cauchy's Test for Convergence).** Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges; if $a > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

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**Remark.** Nothing can be said if $a = 1$.

. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists $N$ such that $\forall n \geq N$,
$$a_n^{1/n} < r \implies a_n < r^n.$$
But since $r < 1$, the geometric series converges, and by comparison test, the series $\sum a_n$ converges as well.

To prove the second part of the theorem, if $a > 1$, for $n \geq N$,
$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus, $\sum\limits_{n=1}^{\infty} a_n$ diverges, since $a_n$ does not tend to zero. ∎

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**Theorem 2.3 (Ratio Test/ D'Alembert's Test).** Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges. If $\ell > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

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**Remark.** As before, nothing can be said for $\ell = 1$.

*Proof.* Supposed $\ell < 1$ and choose $r$ with $\ell < r < 1$. Then $\exists N$ such that $\forall n \geq N$,
$$\frac{a_{n+1}}{a_n} < r.$$
Therefore,
$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \ n > N.$$

So, $a_n < k r^n$ with $k$ independent of $n$. Since $\sum\limits_{n=1}^{\infty} r^n$ converges, so does $\sum\limits_{n=1}^{\infty} a_n$ by Comparison Test.

If $\ell > 1$, choose $1 < r < \ell$. Then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before
$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \ n > N.$$

So the series diverges. ∎

## Lecture 4

**Example.** To determine the convergence of $\sum\limits_{n=1}^{\infty} a_n = \frac{n}{2^n}$.

By ratio test,
$$\frac{n+1}{2^n} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1.$$
So we have convergence by ratio test.

However, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges, and ratio test gives limit 1, and $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, and ratio test gives limit 1. So ratio test is inconclusive if the limit is 1.

Since $n^{\frac{1}{n}} \to 1$ as $n \to \infty$, so root test is also inconclusive when the limit is 1.

To see this limit, write
$$n^{\frac{1}{n}} = 1 + \delta_n, \ \delta_n > 0.$$
So
$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2}\delta_n^2.$$
And $\delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$.

**Remark.** Use the root test when there is a nth power in the series.

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**Theorem 2.4 (Cauchy's Condensation Test).** Let $a_n$ be a decreasing sequence of positive terms. Then $\sum\limits_{n=1}^{\infty} a_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges.

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*Proof.* First we observe that if $a_n$ is decreasing
$$a_{2^k} \le a_{2^{k-1}+i} \le a_{2^{k-1}}$$
for all $k \ge 1$ and $1 \le i \le 2^{k-1}$.

Assume that $\sum\limits_{n=1}^{\infty} a_n$ converges with sum $A$. Then

$$2^{n-1}a_{2^n} = \underbrace{a_{2^n} + \cdots a_{2^n}}_{2^{n-1} \text{ times}}$$
$$\le a_{2^{n-1}+1} + \cdots + a_{2^n}$$
$$= \sum_{m=2^{n-1}+1}^{2^n} a_m.$$

Thus, $\sum\limits_{n=1}^{N} 2^{n-1}a_{2^n} \le \sum\limits_{n=1}^{N} \sum\limits_{m=2^{n-1}+1}^{2^n} a_m = \sum\limits_{m=2}^{2^N} a_m$. So

$$\sum_{n=1}^{N} 2^n a_{2^n} \le 2 \sum_{m=2}^{2^N} a_m \le 2(A - a_1).$$

Thus, $\sum\limits_{n=1}^{N} 2^n a_{2^n}$ being increasing and bounded above, converges.

Conversely, assume $\sum\limits_{n=1}^{\infty} 2^n a_{2^n}$ converges to $B$, then

$$\sum_{m=2^{n-1}+1}^{2^n} a_m = a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n}$$
$$\le \underbrace{a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} \text{times}} = 2^{n-1}a_{2^{n-1}}.$$

Similarly, we have

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^{N} \sum_{m=2^{n-1}+1}^{2^n} a_m \le \sum_{n=1}^{N} 2^{n-1} a_{2^{n-1}} \le B.$$

Therefore, $\sum_{m=1}^{N} a_m$ is a bounded increasing sequence and thus it converges. ∎

**Example.** $\sum_{n=1}^{\infty} \frac{1}{n^k}$ for $k > 0$ converges if and only if $k > 1$. First we note that $\frac{1}{n^k}$ is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)k} < \frac{1}{n^k} \iff \left(\frac{n}{n+1}\right)^k < 1 \iff \frac{n}{n+1} < 1.$$

We use Cauchy condensation test, and we have

$$\begin{aligned} 2^n a_{2^n} &= 2^n \left(\frac{1}{2^n}\right)^k \\ &= 2^{n-nk} = (2^{1-k})^n. \end{aligned}$$

Which is a geometric series with the ratio $2^{1-k}$. So $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges if and only if $2^{1-k} < 1 \iff k > 1$.

## 2.2 Alternating Series

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**Theorem 2.5 (Alternating Series Test).** If $a_n$ decreases and tends to 0 as $n \to \infty$, then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

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**Example.** $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

*Proof.* The partial sum is

$$\begin{aligned} S_n &= a_1 - a_2 + \cdots + (-1)^{n+1} a_n \\ S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \ge S_{2n-1} \\ S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \le a_1 \end{aligned}$$

So $S_{2n}$ is increasing and bounded above, implying that $S_{2n} \to S$. The odd terms satisfy

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S + 0 = S.$$

This implies that $S_n$ converges to $S$ as well. Given $\epsilon$, there exists $N_1$ such that $\forall n \ge N_1, |S_{2n} - S| < \epsilon$. We also know that there exists $N_2$ such that $\forall n \ge N_2, |S_{2n+1} - S| < \epsilon$. Take $N = 2\max\{N_1, N_2\} + 1$, then if $n \ge N, |S_k - S| < \epsilon$. So $S_k \to S$. ∎

## 2.3 Absolute Convergence

> **Definition 2.3.** Take $a_n \in \mathbb{C}$. If $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent, then the series is called *absolutely convergent*.

**Note.** Since $|a_n| \geq 0$. We can use the previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

> **Theorem 2.6.** If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

*Proof.* Suppose first $a_n \in \mathbb{R}$. Let

$$v_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$w_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

We have $v_n = \frac{|a_n|+a_n}{2}, w_n = \frac{|a_n|-a_n}{2}$. Clearly, $v_n, w_n \geq 0$. We also have $|a_n| = v_n + w_n \geq v_n, w_n$.

So by comparison test, if $\sum\limits_{n=1}^{\infty} |a_n|$ converges, $\sum\limits_{-=1}^{\infty} v_n, \sum\limits_{n=1}^{\infty} w_n$ also converges.

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$. We have $|x_n|, |y_n| \leq |a_n|$. So $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are absolutely convergent, so they are convergent. And $\sum\limits_{n=1}^{\infty} a_n$ converges as well. ∎

**Example.**

1. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but not absolutely convergent.

2. $\sum\limits_{n=1}^{\infty} \frac{z^n}{2^n}$ for $z \in \mathbb{C}$. We check for absolute convergence first, $\sum\limits_{n=1}^{\infty} \left(\frac{|z|}{2}\right)^n$. So if $|z| < 2$, the series is convergent by absolute convergence.

   Otherwise, if $|z| \geq 2$, $\left|\frac{z}{2}\right| \geq 1$. $a_n$ does not tend to zero, hence the series diverge.

**Notation.** If $\sum\limits_{n=1}^{\infty} a_n$ converges but not absolutely convergent, it is sometimes called *conditional convergent*.

It is called conditional because the sum to which the series converges is conditional on the order in which elements of the sequence are taken.

**Example (Example Sheet 1, Q7).** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ and $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$ are two series with different sums. Let $s_n$ be the partial sum of the first series, and $t_n$ be the partial sum of the second series, then $s_n \to s$ and $t_n \to \frac{3s}{2}$.

---

**Definition 2.4.** Let $\sigma$ be a bijection of the positive integers, $a_n' = a_{\sigma(n)}$ is a *rearrangement*.

---

**Theorem 2.7.** If $\sum\limits_{n=1}^{\infty} a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

---

*Proof.* Again we do the proof first for $a_n \in \mathbb{R}$. Let $\sum\limits_{n=1}^{\infty} a_n'$ be a rearrangement of $\sum\limits_{n=1}^{\infty} a_n$. Let $s_n = \sum\limits_{i=1}^{n} a_i$ and $t_n = \sum\limits_{i=1}^{n} a_i'$, $S = \sum\limits_{n=1}^{\infty} a_n$. Suppose first that $a_n \geq 0$. Given $n$, we can find $q$ such that $s_q$ contains every term of $t_n$. Because $a_n \geq 0$, we have
$$t_n \leq s_n \leq S.$$

So $t_n$ is an increasing sequence bounded above so $t_n \to t$, and from the inequality above, $t \leq s$. By symmetry, we have $s \leq t \implies s = t$. If $a_n$ has any negative term, consider $v_n$ and $w_n$ from Theorem (2.6). Consider $\sum\limits_{n=1}^{\infty} a_n'$, $\sum\limits_{n=1}^{\infty} v_n'$, $\sum\limits_{n=1}^{\infty} w_n'$. Since $\sum\limits_{n=1}^{\infty} |a_n|$ converges, both $\sum\limits_{n=1}^{\infty} v_n$ and $\sum\limits_{n=1}^{\infty} w_n$ converge. Using the fact that $v_n, w_n \geq 0$, we case above, we have $\sum\limits_{n=1}^{\infty} v_n' = \sum\limits_{n=1}^{\infty} v_n$ and $\sum\limits_{n=1}^{\infty} w_n = \sum\limits_{n=1}^{\infty} w_n'$. But $a_n = v_n - w_n$ so $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_n'$.

For the case $a_n \in \mathbb{C}$, we write $a_n = x_n + iy_n$. Since $|x_i|, |y_i| \leq |a_n|$, $\sum\limits_{n=1}^{\infty} x_n$ and $\sum\limits_{n=1}^{\infty} y_n$ are absolutely convergent. By the previous case $\sum\limits_{n=1}^{\infty} x_n' = \sum\limits_{n=1}^{\infty} x_n$, $\sum\limits_{n=1}^{\infty} y_n' = \sum\limits_{n=1}^{\infty} y_n$. Since $a_n' = x_n' + iy_n'$ so $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} a_n'$. ∎

## Lecture 6
2 Feb. 2022

# 3 Functions

## 3.1 Continuity

Suppose $E \subseteq \mathbb{C}$ is a non-empty subset, and we have a function $f : E \to \mathbb{C}$ and a point $a \in E$. (this includes the case in which $f$ is real-valued and $E$ is a subset of $\mathbb{R}$)

**Definition 3.1.** $f$ is *continuous* at $a \in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$.

**Definition 3.2 ($\epsilon$-$\delta$ Definition).** $f$ is *continuous* at $a \in E$, if given $\epsilon > 0$, $\exists \delta > 0$ such that if $|z - a| < \delta, z \in E$, then $|f(z) - f(a)| < \epsilon$.

We prove right away that the two definitions are equivalent.

**Theorem 3.1.** The two definitions of continuity are equivalent.

*Proof.* We first prove the second definition implies the first definition. We know that given $\epsilon > 0, \exists \delta > 0$ such that $|z - a| < \delta, z \in E$, then $|f(z) - f(z)| < \epsilon$. Let $z_n \to a$, then $\exists n_0$ such that $\forall n \geq n_0$, we have $|z_n - a| < \delta$. This implies, by the assumption, $|f(z_n) - f(a)| < \epsilon$. That is, $f(z_n) \to f(a)$.

Next, we prove the other direction. Assume $f(z_n) \to f(a)$ whenever $z_n \to a, z_n \in E$. Suppose $f$ is not continuous at $a$ according to Definition 2.

$\exists \epsilon > 0$, s.t. $\forall \delta > 0$, there exists $z \in E$ s.t. $|z - a| < \delta$ and $|f(z) - f(a)| \geq \epsilon$.

Let $\delta = \frac{1}{n}$ from non-continuity defined above, we get $z_n$ such that $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \epsilon$. Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \epsilon$. Contradiction. ∎

**Proposition 3.1.** $a \in E$, and $g, f : E \to \mathbb{C}$ are both continuous at $a$. So are the functions $f(z) + g(z)$, $f(z)g(z)$ and $\lambda f(z)$ for any constant $\lambda$. In addition, if $f(z) \neq 0 \ \forall z \in E$, then $\frac{1}{f(z)}$ is continuous at $a$.

*Proof.* Using Definition 1 of continuity, this is obvious, using the analogous results for sequences. (Lemma (1.1))

For example,

$$z_n \to a \implies f(z_n) \to f(a), g(z_n) \to g(a) \implies f(z_n) + g(z_n) \to f(a) + g(a).$$

∎

The function $f(z) = z$ is continuous, so by using the proposition, we get that every polynomial is continuous at every point in $\mathbb{C}$.

**Note.** We say that $f$ is *continuous on E* if it is continuous at every $a \in E$.

**Remark.** Still it is instructive to prove Proposition (3.1) directly from the $\epsilon$-$\delta$ definition.

Next we look at compositions.

> **Theorem 3.2.** Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions such that $f(A) \subseteq B$. Suppose $f$ is continuous at $a \in A$ and $g$ is continuous at $f(a)$, then $g \circ f : A \to \mathbb{C}$ is continuous at $a$.

*Proof.* Take any sequence $z_n \to a$, by assumption we know $f(z_n) \to f(a)$. Set $w_n = f(z_n) \in B$. By continuity of $g$, we have $g(w_n) \to g(f(a))$, and we are done. ∎

**Example.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$
f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},
$$

   assuming that $\sin x$ is continuous. (to be proved later) If $x \neq 0$, propositions proved above imply that $f(x)$ is continuous at any $x \neq 0$.

   However, it is discontinuous at 0. Consider the sequence satisfying

$$
\frac{1}{x_n} = (2n + \frac{1}{2})\pi.
$$

   We have $f(x_n) \to 1, x_n \to 0$, but $f(0) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$
f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.
$$

   It's continuous at $x \neq 0$ as above, and $f$ is continuous at 0. Take $x_n \to 0$, then $|f(x_n)| \leq |x_n|$ because $\sin\frac{1}{x} \leq 1$, so $f(x_n) \to 0 = f(0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be

$$
f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.
$$

   It is discontinuous at every point. If $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then $f(x_n) = 0 \nrightarrow f(x) = 1$. Similarly, if $x \notin \mathbb{Q}$, take $x_n \to x$ with $x_n \in \mathbb{Q}$, we have $f(x_n) = 1 \nrightarrow f(x) = 0$.

# Lecture 7

## 3.2   Limit of a function

$f : E \subseteq \mathbb{C} \to \mathbb{C}$. We wish to define what is meant by $\lim\limits_{z \to a} f(z)$, even when $a$ might not be in $E$.

**Example.** The limit of $\frac{\sin z}{z}$ as $z \to 0$ with $E = \mathbb{C}\{0\}$.

Also, if $E = \{0\} \cup [1, 2]$, it does not make sense to speak about points $z \in E, z \neq 0, z \to 0$.

**Definition 3.3.** If $E \subseteq \mathbb{C}, a \in \mathbb{C}$, we say that $a$ is a *limit point* of $E$ if for any $\delta > 0, \exists z \in E$ such that $0 < |z - a| < \delta$.

**Remark.** $a$ is a limit point if and only if there exists a sequence $z_n \in E$ such that $z_n \to a$ and $z_n \neq a$ for all $n$.

**Definition 3.4.** If $f : E \subseteq \mathbb{C} \to \mathbb{C}$ and let $a \in \mathbb{C}$ be a limit point of $E$. We say that $\lim_{z \to a} f(z) = l$ ("$f$ tends to $l$ as $z$ tends to $a$") if given $\epsilon > 0, \exists \delta > 0$ such that whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \epsilon$.

Equivalently, $f(z_n) \to l$ for every sequence $z_n \in E, z_n \neq a$ and $z_n \to a$.

**Remark.** Straight from the definitions, we have that if $a \in E$ is limit point, then $\lim_{z \to a} f(z) = f(a)$ if and only if $f$ is continuous at $a$.

If $a \in E$ is *isolated* (i.e. $a \in E$ is not a limit point), continuity of $f$ at $a$ always holds. The limit of functions has very similar properties to limit of sequences.

1. It is unique, $f(z) \to A$ and $f(z) \to B$ as $z \to a$, then

$$|A - B| \leq |A - f(z)| + |f(z) - B|.$$

   If $z \in E$ is such that $0 < |z - a| < \min\{\delta_1, \delta_2\}$, then $|A - B| < 2\epsilon$. So $A = B$. The existence of such $z$ is a consequence of the condition that $a$ is a limit point of $E$.

2. $f(z) + g(z) \to A + B$;

3. $f(z)g(z) \to AB$;

4. if $B \neq 0$, $\frac{f(z)}{g(z)} \to \frac{A}{B}$. All proved in the same way as before.

## 3.3 The Intermediate Value Theorem

**Theorem 3.3 (Intermediate Value Theorem).** If $f : [a, b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$, then $f$ takes every value which lies between $f(a)$ and $f(b)$.

*Proof.* Without loss of generality, suppose $f(a) < f(b)$. Take $f(a) < \eta < f(b)$. Let $S = \{x \in [a, b] \mid f(x) < \eta\}$. We note that $a \in S$, so $S \neq \varnothing$. Clearly $S$ is bounded above by $b$. Then there is a supremum $C$ where $C \leq b$. By definition of supremum, given $n$, there exists $x_n \in S$ such that $C - \frac{1}{n} < x_n \leq C$. So $x_n \to C$. Since $x_n \in A$, $f(x_n) < \eta$. By continuity of $f$, $f(x_n) \to f(C)$. So $f(c) \leq \eta$.

Now observe that $c \neq b$ because $f(b) > \eta$. Then for $n$ large, $C + \frac{1}{n} \in [a, b]$ and $C + \frac{1}{n} \to C$. Again by continuity $f(C + \frac{1}{n}) \to f(C)$. But since $C + \frac{1}{2} > C$, $f(C + \frac{1}{n}) \geq \epsilon$. So $f(c) \geq \eta \implies f(c) = \eta$. ∎

**Remark.** The theorem is very useful for finding zeroes or fixed points.

**Example.** Existence of the $N$-th root of a positive real number. Suppose

$$f(x) = x^N, \quad x \geq 0.$$

Let $y$ be a positive real number. $f$ is continuous on $[0, 1+y]$, so

$$0 = f(0) < y < (1+y)^N = f(1+y).$$

By the IVT, $C \in (0, 1+y)$ such that $f(c) = y$, i.e. $C^N = y$. $C$ is a positive $N$-th root of $y$.

We also have uniqueness. Exercise.

## Lecture 8

### 3.4  Bounds of a Continuous Function

> **Theorem 3.4.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $K$ such that $|f(x)| \leq K$ for all $x \in [a, b]$.

*Proof.* We argue by contradiction. Suppose the statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass, $x_n$ has a convergent subsequence $x_{n_j} \to x$. Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By the continuity of $f$, $f(x_{n_j}) \to f(x)$. But $|f(x_{n_j})| > nj \to \infty$ as $j \to \infty$. Contradiction. ∎

> **Theorem 3.5 (Extreme Value Theorem).** Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $x_1, x_2 \in [a, b]$ such that
>
> $$f(x_1) \leq f(x) \leq f(x_2)$$
>
> for all $x \in [a, b]$.
>
> "A continuous function on a closed bounded interval is bounded and attains its bounds."

*Proof.* Let $A = \{f(x) \mid x \in [a, b]\} = f([a, b])$. By Theorem (3.4), $A$ is bounded since it is clearly non-empty, it has a supremum $M$. By definition of supremum, given an integer $n \geq 1$, there exists $x_n \in [a, b]$ such that $M - \frac{1}{n} < f(x_n) \leq M$. From Bolzano-Weierstrass, there exists $x_{n_j} \to x \in [a, b]$. Since $f(x_{n_j}) \to M$, by continuity of $f$, we get that $f(x) = M$. So $x_2 := x$.

We can prove similarly for the minimum. ∎

*Proof 2.* $A = f([a, b]), M = \sup A$ as before. Suppose $\nexists x_2$ such that $f(x_2) = M$. Let

$$g(x) = \frac{1}{M - f(x)}, x \in [a, b]$$

is defined and continuous on $[a,b]$. By Theorem (3.4) applied to $g$, $\exists k > 0$ such that $g(x) < K$ for all $x \in [a,b]$. This means that $f(x) \leq M - \frac{1}{k}$ for all $x \in [a,b]$. This is absurd because it contradicts that $M$ is the supremum. ∎

**Note.** Theorems (3.4) and (3.5) are false if the interval is not closed and bounded. For example,

$$f : (0,1] \to \mathbb{R}, x \mapsto \frac{1}{x}.$$

## 3.5 Inverse Functions

**Definition 3.5.** $f$ is *increasing* for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all $x_1, x_2$ such that $a \leq x_1 < x_2 \leq b$.

If $f(x_1) < f(x_2)$, we say that $f$ is *strictly increasing*.

There are similar definitions for *decreasing* and *strictly decreasing*.

**Theorem 3.6.** $f : [a,b] \to \mathbb{R}$ is continuous and strictly increasing for $x \in [a,b]$. Let $c = f(a)$ and $d = f(b)$. Then $f : [a,b] \to [c,d]$ is bijective and the inverse $g \coloneqq f^{-1} : [c,d] \to [a,b]$ is also continuous and strictly increasing.

**Remark.** There is a similar statement for strictly decreasing function. Take $c < k < d$, from the IVT, $\exists h$ such that $f(h) = k$. Since $f$ is strictly increasing, $h$ is unique. Define $g(k) \coloneqq h$ and this gives an inverse $g : [c,d] \to [a,b]$ for $f$.

We first prove that $g$ is strictly increasing. Take $y_1 < y_2$ such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$. If $x_2 \leq x_1$, since $f$ is increasing, $f(x_2) \leq f(x_1) \implies y_2 \leq y_1$. Absurd.

Next we prove continuity. Let $\epsilon > 0$ be given, let $k_1 = f(h-\epsilon)$ and $k_2 = f(h+\epsilon)$. Because $f$ is strictly increasing, we have $k_1 < k < k_2$. If $k_1 < y < k_2$, we have $h - \epsilon < g(y) < h + \epsilon$. So we can just take $\delta = \min\{k_2 - k, k - k_1\}$. So $g$ is continuous at $k$. Here we took $k \in (c,d)$. A very similar argument establishes continuity at the end points.

## Lecture 9

# 4 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, most of the time $E = $ interval $\subseteq \mathbb{R}$.

**Definition 4.1.** Let $x \in E$ be a point such that $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point), $f$ is said to be *differentiable* at $x$ with derivative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x).$$

If $f$ is differentiable at each $x \in E$, we say $f$ is differentiable on $E$.

(Think of $E$ as an interval or a disc in the case of $\mathbb{C}$.)

**Remark.**

1. Other common notations include $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}f}{\mathrm{d}x}$.

2. $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. $(y = x + h)$

3. Another look at the definition is the following.

   Let $\epsilon(h) := f(x + h) - f(x) - hf'(x)$, then $\lim_{h \to 0} \frac{\epsilon(h)}{h} = 0$. We have also

   $$f(x + h) = f(x) + \underbrace{hf'(x)}_{\text{linear in } h} + \epsilon(h).$$

   Alternative definition of differentiability is $f$ is differentiable at $x$ if $\exists A, E$ such that $f(x + h) = f(x) + hA + \epsilon(h)$ where $\lim_{h \to 0} \frac{\epsilon}{h} = 0$. If such an $A$ exists, then it is unique, since $A = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

4. If $f$ is differentiable at $x$, then $f$ is continuous. Since $\epsilon(h) \to 0$, then $f(x + h) \to f(x)$ as $h \to 0$.

5. Alternative ways of writing things:

   $$f(x + h) = f(x) + hf'(x) + h\epsilon_f(h) \text{ with } \epsilon_f(h) \to 0 \text{ as } h \to 0.$$

   Or,

   $$f(x) = f(a) + (x - a)f'(a) + (x - a)\epsilon_f(x) \text{ with } \epsilon_f(x) \to 0 \text{ as } x \to a.$$

**Example.** If we have $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = |x|$. Clearly, we have $f'(x) = 1$ if $x > 0$ and $f'(x) = -1$ if $x < 0$. Take $h_n \downarrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{h_n}{h_n} = 1.$$

And take $h_n \uparrow 0$ at $x = 0$, we have

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \to \infty} \frac{-h_n}{h_n} = -1.$$

So $f$ is not differentiable at $x = 0$.

## 4.1 Differentiation of Sums, Products, etc

**Property.**

1. If $f(x) = c$ for all $x \in E$, then $f$ is differentiable with $f'(x) = 0$.

2. $f, g$ are differentiable at $x$, then so is $f + g$ and
$$(f + g)'(x) = f'(x) + g'(x).$$

3. $f, g$ are differentiable at $x$, then so is $fg$ and
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

4. $f$ differentiable at $x$ and $f(x) \neq 0$ for all $x \in E$, then $\frac{1}{f}$ is differentiable at $x$ and
$$(\frac{1}{f})'(x) = \frac{-f'(x)}{[f(x)]^2}.$$

*Proof.*

1. $\lim\limits_{h \to 0} \frac{c - c}{h} = 0$.

2. $\lim\limits_{h \to 0} \dfrac{f(x + h) + g(x + h) - f(x) - g(x)}{h}$     using properties of limits.
$$= \lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim\limits_{h \to 0} \frac{g(x + h) - g(x)}{h}$$
$$= f'(x) + g'(x)$$

3. Let $\phi(x) = f(x)g(x)$, then we have
$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$
$$= f(x + h)[\frac{g(x + h) - g(x)}{h}] + g(x)[\frac{f(x + h) - f(x)}{h}].$$

So we have $\lim\limits_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = f(x)g'(x) + f'(x)g(x)$ using standard properties of limits and the fact that $f$ is continuous at $x$.

4. Define again $\phi(x) = \frac{1}{f(x)}$, then
$$\frac{\phi(x + h) - \phi(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h}$$
$$= \frac{f(x) - f(x + h)}{h f(x) f(x + h)}.$$

So we have $\lim\limits_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h} = \frac{-f(x)}{[f(x)]^2}$.

$\blacksquare$

**Remark.** From (3) and (4), we get
$$\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

**Example.** Consider $f(x) = x^n$ with $n \in \mathbb{Z}, n > 0$. When $n = 1$, clearly we have $f(x) = x$ and $f'(x) = 1$.

We claim that $f'(x) = nx^{n-1}$, and we prove it by induction, $f(x) = xx^n = x^{n+1}$. By product rule and inductive hypothesis,

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Next, we consider $f(x) = x^{-n}$ with $n \in \mathbb{Z}, n > 0$. If $x \neq 0$, use Proposition (4.1), we have

$$f'(x) = \frac{-(x^n)'}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

So we know how to find derivatives of polynomials and rational functions.

We have the following useful result to differentiate a larger class of functions.

---

**Theorem 4.1 (Chain Rule).** If $f : U \to \mathbb{C}$ is such that $f(x) \in V$ for $x \in U$. If $f$ is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at $a$ with

$$(g \circ f)'(a) = f'(a)g'(f(a)).$$

---

*Proof.* We know

$$f(x) = f(a) + (x - a)f'(a) + \epsilon_f(x)(x - a)$$

such that $\lim_{x \to a} \epsilon_f(x) = 0$, and

$$g(y) = g(b) + (y - b)g'(b) + \epsilon_g(y)(y - b)$$

with $\lim_{y \to b} \epsilon_g(y) = 0$. Let $b = f(a)$, and set $\epsilon_f(a) = 0$ and $\epsilon_g(b) = 0$ to make them continuous at $x = a$ and $f = b$. Now $y = f(x)$ gives

$$\begin{aligned}
g(f(x)) =& g(b) + (f(x) - b)g'(b) + \epsilon_g(f(x))(f(x) - b) \\
=& g(f(a)) + [(x - a)f'(a) + \epsilon_f(x)(x - a)][g'(b) + \epsilon_g(f(x))] \\
=& g(f(a)) + (x - a)f'(a)g'(b) + \\
& (x - a)[\epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))] \\
=& g(f(a)) + (x - a)f'(a)g'(b) + (x - a)\sigma(x).
\end{aligned}$$

So it suffices to show $\sigma(x) = \epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))$ tends to 0 as $x$ tends to $a$. We have clearly $\epsilon_f(x)g'(b) \to 0$, $\epsilon_g(f(x)) \to 0$ and $f'(a) + \epsilon_f(x) \to f'(a)$, so $\lim_{x \to a} \sigma(x) = 0$. ∎

**Example.**

1. Consider $f(x) = \sin(x^2)$, and we have

$$f'(x) = 2x \cos(x^2).$$

2. Consider $f(x) = \begin{cases} x\sin(\dfrac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. From previous lectures, we know that $f$ is continuous, and it is differentiable at every $x \neq 0$ by the previous theorems. At $x = 0$, take $t \neq 0$ and we have

$$\frac{f(t) - f(0)}{t - 0} = \sin(\frac{1}{t}).$$

Again from previous lecture, we know $\lim\limits_{t \to 0} \frac{f(t) - f(0)}{t - 0}$ does not exist, so $f$ is not differentiable at $x = 0$.

## 4.2   The Mean Value Theorem

**Theorem 4.2 (Rolle's Theorem).** Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$ and differentiable on $(a, b)$. If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

*Proof.* Let $M = \max_{x \in [a,b]} f(x)$, and $m = \min_{x \in [a,b]} f(x)$. Theorem (3.5) says that these values are achieved. Let $k = f(a) = f(b)$. If $M = m = k$, then $f$ is constant and $f'(c) = 0$ for all $c \in (a, b)$.

If $f$ not constant, then $M > k$ or $m < k$. Suppose $M > k$. By Theorem (3.5), exist $c \in (a, b)$ such that $f(c) = M$.

If $f'(c) > 0$, then there are values to right of $c$ for which $f(x) > f(c)$ because

$$f(h + c) - f(c) = h(f'(c) + \epsilon_f(h)).$$

Since $\epsilon_f(h) \to 0$ as $h \to 0$, $f'(c) + \epsilon_f(h) > 0$ for $h$ small. This contradicts that $M$ is the maximum. Similarly, if $f'(c) < 0$, there exists $x$ to the left of $c$ for which $f(x) > f(c)$.

So we must have $f'(c) = 0$. ∎

## Lecture 11

**Theorem 4.3 (Mean Value Theorem).** Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on $(a, b)$, then $\exists c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

*Proof.* Write $\phi(x) = f(x) - kx$, and choose $k$ such that $\phi(a) = \phi(b)$. So

$$f(b) - bk = f(a) - ak \implies k = \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem applied to $\phi$, $\exists c \in (a, b)$ such that $\phi'(c) = 0$. That is, $f'(c) = k$. ∎

**Remark.** We will often write

$$f(a+h) = f(a) + hf'(a + \theta h)$$

with $\theta \in (0, 1)$. We need to be careful, and consider $\theta = \theta(h)$.

---

**Corollary 4.1.** $f : [a, b] \to \mathbb{R}$ continuous and differentiable on $(a, b)$.

1. If $f'(x) > 0$ for all $x \in (a, b)$, then $f$ is strictly increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

2. If $f'(x) \geq 0$ for all $x \in (a, b)$, then $f$ is increasing. (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)

3. If $f'(x) = 0$ for all $x \in (a, b)$, then $f$ is constant on $[a, b]$.

---

*Proof.*

1. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) > 0 \implies f(y) > f(x)$.

2. MVT implies $f(y) - f(x) = f'(c)(y - x)$. And $f'(c) \geq 0 \implies f(y) \geq f(x)$.

3. Take $x \in [a, b]$. Then use the MVT in $[a, x]$ to get $c \in (a, x)$ such that $f(x) - f(a) = f'(c)(x - a) = 0$. So $f(x) = f(a)$ and $f$ is constant.

∎

---

**Theorem 4.4 (Inverse Function Theorem).** If $f : [a, b] \to \mathbb{R}$ continuous and differentiable on $(a, b)$ with $f'(x) > 0$ for all $x \in (a, b)$. Let $f(a) = c$ and $f(b) = d$, then the function $f : [a, b] \to [c, d]$ is bijective and $f^{-1}$ is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

---

*Proof.* By Corollary (4.1), $f$ is strictly increasing in $[a, b]$. By theorem (3.6), $\exists g : [c, d] \to [a, b]$ which is a continuous strictly increasing inverse of $f$. We want to show that $g$ is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$ and $x \in (a, b)$.

If $k \neq 0$ is given, let $h$ be given by $y + k = f(x + h)$. That is, $g(y + k) = x + h$ for $h \neq 0$. Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)} = \frac{h}{f(x + h) - f(x)}.$$

Let $k \to 0$, then $h \to 0$ because $g$ is continuous. So we have

$$g'(y) = \lim_{k \to 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}.$$

∎

**Example.** We take $g(x) = x^{\frac{1}{q}}$ with $x > 0$ and $q$ positive integer. So $f(x) = x^q$, with $f'(x) = qx^{q-1}$. $g$ is differentiable and so is $g$, and by Theorem (4.4),

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q}x^{\frac{1}{q}-1}.$$

**Remark.** If $g(x) = x^r$ with $r \in \mathbb{Q}$, then $g'(x) = rx^{r-1}$.

Suppose $f, g : [a, b] \to \mathbb{R}$ continuous and differentiable on $(a, b)$ and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b-a)f'(s)}{(b-a)g'(t)} = \frac{f'(s)}{g'(t)}.$$

Cauchy showed that we can take $s = t$.

---

**Theorem 4.5 (Cauchy's Mean Value Theorem).** Let $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on $(a, b)$. Then $\exists t \in (a, b)$ such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a)).$$

---

**Remark.** We recover the MVT if we take $g(x) = x$.

# Lecture 12

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*Proof.* Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}.$$

We have $\phi$ continuous on $[a, b]$ and differentiable on $(a, b)$. Also, $\phi(a) = \phi(b) = 0$. By Rolle's Theorem, there exists $t \in (a, b)$ such that $\phi'(t) = 0$, and

$$\phi'(x) = f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x)$$
$$= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)].$$

So $\phi'(t) = 0$ gives the result. ∎

**Note.** Good choice of auxiliary function and Rolle's theorem proves the theorem.

**Example (L' Hopital's Rule).** If we want to find $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$, we have

$$\frac{e^x - 1}{\sin x} = \frac{e^x - x^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

for some $t \in (0, x)$ by Cauchy's Mean Value Theorem. So

$$\frac{e^x - 1}{\sin x} \to 1$$

as $x \to 0$.

Goal: We want to extend the MVT to include higher order derivatives.

4 DIFFERENTIABILITY

**Theorem 4.6 (Taylor's Theorem with Lagrange's reminder).** Suppose $f$ and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!f''(a)} + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n f^{(n)}(a+\theta h)}{n!}$$

where $\theta \in (0, 1)$.

**Note.**

1. For $n = 1$, we get back the MVT, so this is an "$n$-th order MVT".

2. $R_n = \frac{h^n}{n!} f^{(n)}(a + \theta h)$ is known as Lagrange's form of the remainder.

*Proof.* Define for $0 \leq t \leq h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}b$$

where we choose $b$ such that $\phi(h) = 0$, and we clearly have $\phi(0) = 0$. (Recall that in the proof of the MVT, we used $f(x) - kx$ and picked $k$ that we can use Rolle's Theorem. We also have that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0.$$

We use Rolle's Theorem $n$ times. Since $\phi(0) = \phi(h) = 0$, $\phi'(h_1) = 0$ for some $0 < h_1 < h$. And since $\phi'(0) = \phi'(h_1) = 0$, we have $\phi''(h_2) = 0$ for some $0 < h_2 < h_1$. Finally, $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$. So $\phi^{(n)}(h_n) = 0$ with $0 < h_n < h_{n-1} < \cdots < h$. So $h_n = \theta h$ for $\theta \in (0, 1)$, now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - b \implies b = f^{(n)}(a + \theta h).$$

Set $t = h$, $\phi(h) = 0$ and put this value of $b$ to the second line in the proof. ∎

**Theorem 4.7 (Taylor's Theorem with Cauchy's reminder).** Suppose $f$ and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exists for $x \in (a, a+h)$, and if $a = 0$ for simplification, then we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1-\theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!}$$

with $\theta \in (0, 1)$.

*Proof.* Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$. Note that we have

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$= -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t).$$

Set

$$\phi(t) = F(t) - (\frac{h-t}{h})^p F(0)$$

with $p \in \mathbb{Z}, 1 \le p \le n$. Then $\phi(0) = \phi(h) = 0$, and by Rolle's, $\exists \theta \in (0, 1)$ such that $\phi'(\theta h) = 0$. But,

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h}F(0) = 0.$$

So

$$0 = -\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h}[f(h) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)].$$

Rearranging the two sides, and we get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!p(1-\theta)^{p-1}}f^{(n)}(\theta h).$$

Taking $p = n$, we get Lagrange's reminder, and taking $p = 1$ gives Cauchy's reminder. ∎

## Lecture 13

To get a Taylor series for $f$, one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort".

**Remark.** Theorems (4.6) and (4.7) work equally well in an interval $[a + h, a]$ with $h < 0$.

**Example.** The binomial series

$$f(x) = (1+x)^r, r \in \mathbb{Q}.$$

We claim that $|x| < 1$, then

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}.$$

*Proof.* Clearly,

$$f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}.$$

If $r \in \mathbb{Z}_{\geq 0}$, then $f^{(n+1)} = 0$, we have a polynomial of degree $r$.

In general, by Lagrange's reminder, we have

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \binom{r}{n} \frac{x^n}{(1 + \theta x)^{n-r}}.$$

Note that $\theta$ depends on both $x$ and $n$.

For $0 < x < 1$, $(1 + \theta x)^{n-r} > 1$ for $n > r$. Now observe that the series $\sum \binom{r}{n} x^n$ is absolutely convergent for $|x| < 1$. Indeed, by the ratio test,

$$a_n = \binom{r}{n} x^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{r(r-1)(r-n+1)\cdots(r-n)x^{n+1}}{(n+1)!} \right| \left| \frac{n!}{r(r-1)\cdots(r-n+1)x^n} \right|$$

$$= \left| \frac{(r-n)x}{n+1} \right|$$

which tends to a value less than 1. In particular, $a_n \to 0$ and $\binom{r}{n} x^n \to 0$.

Hence, for $n > r$, and $0 < x < 1$, we have that $|R_n| \leq \left| \binom{r}{n} x^n \right| = |a_n| \to 0$ as $n \to \infty$.

So the claim is proved in the range $0 \leq x < 1$. If $-1 < x < 0$, the argument above breaks, but Cauchy's form for $R_n$ works.

$$R_n = \frac{(1-\theta)^{n-1} r(r-1)\cdots(r-n+1)(1+\theta h)^{r-n} x^n}{(n-1)!}$$

$$= \frac{r(r-1)\cdots(r-n+1)}{(n-1)!} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^n$$

$$= r \binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \left( \frac{1-\theta}{1+\theta x} \right)^{n-1}.$$

So $|R_n| \leq \left| r \binom{r-1}{n-1} x^n \right| (1+\theta x)^{r-1}$. Check that $(1+\theta x)^{r-1} \leq \max\{1, (1+x)^{r-1}\}$. Let $K_r = |r| \max\{1, (1+x)^{r-1}\}$ is independent of $n$. So we have

$$|R_n| \leq |K_r| \left| \binom{r-1}{n-1} x^n \right| \to 0.$$

So $R_n \to 0$ as $n \to \infty$. ∎

## 4.3   Remarks on Complex Differentiation

Formally, for function $f : E \subseteq \mathbb{C} \to \mathbb{C}$, we have properties for sums, products, chain rule etc. But it is much more restrictive than differentiability on the real line.

**Example.** $f : \mathbb{C} \to \mathbb{C}$, with $z \mapsto \overline{z}$. We consider the sequence $z_n = z + \frac{1}{n} \to z$.

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\overline{z} + \frac{1}{n} - \overline{z}}{z + \frac{1}{n} - z} = 1.$$

If we approach it vertically instead, taking $z_n = z + \frac{i}{n} \to z$, we have

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\overline{z} - \frac{i}{n} - \overline{z}}{z + \frac{i}{n} - z} = -1.$$

So $\lim\limits_{w \to z} \frac{f(w) - f(z)}{w - z}$ does not exist. $f$ is nowhere $\mathbb{C}$-differentiable.

If we consider it as a function on $\mathbb{R}^2$, $f(x, y) = (x, -y)$. It is real differentiable.

In fact, if a function is complex differentiable, it is infinitely complex differentiable. It is discussed in more detail in IB Complex Analysis.

## Lecture 14

## 5   Power Series

We want to look at

$$\sum_{n=0}^{\infty} a_n z^n,$$

with $z \in \mathbb{C}, a_n \in \mathbb{C}$. The case $\sum\limits_{n=0}^{\infty} a_n (z - z_0)^n$, with $z$ fixed can be reduced to power series around 0 by translation.

---

**Lemma 5.1.** If $\sum\limits_{n=0}^{\infty} a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum\limits_{n=0}^{\infty} a_n z^n$ converges absolutely.

---

*Proof.* Since $\sum\limits_{n=0}^{\infty} a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus, there exists $K > 0$ such that $|a_n z_1^n| \le K$ for all $n$.

Then,

$$|a_n z^n| = |a_n z^n| \frac{|z_1^n|}{|z_1^n|} \le K \left| \frac{z}{z_1} \right|^n.$$

Since the geometric series $\sum\limits_{n=0}^{\infty} \left| \frac{z}{z_1} \right|^n$ converges, the lemma follows by comparison. ∎

Using this lemma, we will prove that every power series has a *radius of convergence*.

---

**Theorem 5.1.** A power series either

1. converges absolutely for all $z$, or

2. converges absolutely for all $z$ inside a circle $|z| = R$ and diverges for all $z$ outsider it, or

3. converges for $z = 0$ only.

---

> **Definition 5.1.** The circle $|z| = R$ is called the *circle of convergence* and $R$ the *radius of convergence*.
>
> In (1) of Theorem (5.1), we agree that $R = \infty$, and in (3) $R = 0$, so $R \in [0, \infty]$.

*Proof.* Let $S = \{x \in \mathbb{R} \mid x \geq 0, \sum a_n x^n \text{ converges}\}$. Clearly $0 \in S$. By Lemma (5.1), if $x_1 \in S$, then $[0, x_1] \subseteq S$. If $S = [0, \infty)$, we have case 1.

Otherwise, there exists a finite supremum for $S$. $R = \sup S < \infty, R \geq 0$. If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely. Pick $R_0$ such that $|z_1| < R_0 < R$, then $R_0 \in S$ and the series converges for $z = R_0$. By Lemma (5.1), $\sum |a_n z_1^n|$ converges.

Finally, we show that if $|z_2| > R$, then the series does not converge for $z_2$. Pick $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges, by Lemma (5.1), $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. ∎

The following lemma is useful for computing $R$.

> **Lemma 5.2.** If $\left| \frac{a_{n+1}}{a_n} \right| \to \ell$, as $n \to \infty$, then $R = \frac{1}{\ell}$.

*Proof.* By the ratio test, we have absolute convergence if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| = \ell |z| < 1.$$

So if $|z| < \frac{1}{\ell}$, we have absolute convergence. If $|z| > \frac{1}{\ell}$, the series diverges, again by the ratio test. ∎

**Remark.** One can also use the Root Test to get that if $|a_n|^{\frac{1}{n}} \to \ell$, then $R = \frac{1}{\ell}$.

**Example.**    1. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

   We have
$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = \ell.$$
   So $R = \infty$.

   2. Geometric Series, $\sum_{n=0}^{\infty} z^n$.

   By ratio test, we have $R = 1$. Note that at $|z| = 1$, we have divergence.

   3. $\sum_{n=0}^{\infty} n! z^n$.

   We have
$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{n!} = n+1 \to \infty.$$
   So $R = 0$.

4. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$, and we have $R = 1$. For $z = 1$, it diverges. What happens for $|z| = 1$ and $z \neq 1$?

Consider $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}(1 - z)$, we have the partial sum

$$
\begin{aligned}
S_N &= \sum_{n=1}^{N} \left( \frac{z^n - z^{n+1}}{n} \right) \\
&= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} \\
&= \sum_{n=1}^{N} \frac{z^n}{n} - \sum_{n=2}^{N+1} \frac{z^n}{n-1} \\
&= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} z^n \left( \frac{1}{n} - \frac{1}{n-1} \right) \\
&= z - \frac{z^{N+1}}{N} + \sum_{n=2}^{N} -\frac{z^n}{n(n-1)}.
\end{aligned}
$$

If $|z| = 1$, $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum \frac{1}{n(n-1)}$ converges, so $S_N$ converges.

5. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$, and we have $R = 1$. But it converges for all $z$ with $|z| = 1$.

Conclusion is that, in principle, nothing can be said about $|z| = R$ and each case has to be discussed separately.

Within the radius of convergence, "life is great". Power series behave as if "they were polynomials".

## Lecture 15

23 Feb. 2022