

# ANALYSIS

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## Lecture 1: Limits

21 Jan. 11:00

Books:

- *A First Course in Mathematical Analysis* -Burkill
- *Calculus* -Spivak
- *Analysis I* -Tao

# 1 Limits and Convergence

## 1.1 Review from Numbers and Sets

**Notation.** We denote sequences by  $a_n$  or  $(a_n)_{n=1}^{\infty}$ , with  $a_n \in \mathbb{R}$ .

### Definition 1.1

We say that  $a_n \rightarrow a$  as  $n \rightarrow \infty$  if given  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon$  for all  $n \geq N$ .

**Note.**  $N = N(\epsilon)$  which is dependent on  $\epsilon$ . That is, if you want to go closer to  $a$ , sometimes you need to go higher in  $N$ .

### Definition 1.2: limit of a sequence

We say that a sequence is a

$$\begin{aligned} &\left. \begin{array}{l} \text{increasing sequence if } a_n \leq a_{n+1}, \\ \text{decreasing sequence if } a_n \geq a_{n+1}, \end{array} \right\} \text{monotone sequence} \\ &\left. \begin{array}{l} \text{strictly increasing sequence if } a_n < a_{n+1}, \\ \text{strictly decreasing sequence if } a_n > a_{n+1}. \end{array} \right\} \text{strictly monotone sequence} \end{aligned}$$

We also have

**Theorem 1.1: Fundamental Axiom of the Real Numbers**

If  $a_n \in \mathbb{R}$  and  $a_n$  is increasing and bounded above by  $A \in \mathbb{R}$ , then there exists  $a \in \mathbb{R}$  such that  $a_n \rightarrow a$  as  $n \rightarrow \infty$ .

That is, an increasing sequence of real numbers bounded above *converges*.

**Remark.** It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.
- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

**Definition 1.3: supremum**

For  $S \subseteq \mathbb{R}, S \neq \emptyset$ . We say that  $\sup S = k$  if

1.  $x \leq k, \quad \forall x \in S,$
2. given  $\epsilon > 0$ , there exists  $x \in S$  such that  $x > k - \epsilon$ .

**Note.** Supremum is unique, and there is a similar notion of infimum.

## Lemma 1.1: Properties of Limits

1. The limit is unique. That is, if  $a_n \rightarrow a$ , and  $a_n \rightarrow b$ , then  $a = b$ .
2. If  $a_n \rightarrow a$  as  $n \rightarrow \infty$  and  $n_1 < n_2 < n_3 \dots$ , then  $a_{n_j} \rightarrow a$  as  $j \rightarrow \infty$  (subsequences converge to the same limit).
3. If  $a_n = c$  for all  $n$  then  $a_n \rightarrow c$  as  $n \rightarrow \infty$ .
4. If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n + b_n \rightarrow a + b$ .
5. If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n b_n \rightarrow ab$ .
6. If  $a_n \rightarrow a$ , then  $\frac{1}{a_n} \rightarrow \frac{1}{a}$ .
7. If  $a_n < A$  for all  $n$  and  $a_n \rightarrow a$ , then  $a \leq A$ .

*Proof.*

1. Given  $\epsilon > 0$ , there exists  $N_1$  such that  $|a_n - a| < \epsilon, \forall n \geq N_1$ , and there exists  $N_2$  such that  $|a_n - b| < \epsilon, \forall n \geq N_2$ .

Take  $N = \max\{n_1, n_2\}$ , then if  $n \geq N$ ,

$$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$

If  $a \neq b$ , take  $\epsilon = \frac{|a-b|}{3}$ , we have

$$|a - b| < \frac{2}{3}|a - b|. \quad \text{✗}$$

2. Given  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon, \forall n \geq N$ , Since  $n_j \geq j$ , we know

$$|a_{n_j} - a| < \epsilon, \forall j \geq N.$$

That is,  $a_{n_j} \rightarrow a$  as  $j \rightarrow \infty$ .

5. We have

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n| |b_n - b| + |b| |a_n - a|. \end{aligned}$$

Given  $\epsilon > 0$ , there exists  $N_1$  such that  $|a_n - a| < \epsilon, \forall n \geq N_1$ , and there exists  $N_2$  such that  $|b_n - b| < \epsilon, \forall n \geq N_2$ .

If  $n \geq N_1(1)$ ,  $|a_n - a| < 1$ , so  $|a_n| \leq |a| + 1$ .

We have

$$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

■

#### Lemma 1.2

$$\frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*Proof.*  $\frac{1}{n}$  is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit  $a$ .

We claim that  $a = 0$ . We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \rightarrow \frac{a}{2} \text{ by Lemma 1.1.}$$

But  $\frac{1}{2n}$  is a subsequence, so by Lemma 1.1  $\frac{1}{2n} \rightarrow a$ . By uniqueness of limits proved again in Lemma 1.1, we have  $a = \frac{a}{2} \implies a = 0$ . ■

**Remark.** The definition of limit of a sequence makes perfect sense for  $a_n \in \mathbb{C}$  by replacing the absolute value with modulus.

#### Definition 1.4

We say that  $a_n \rightarrow a$  as  $n \rightarrow \infty$  if given  $\epsilon > 0$ , there exists  $N$  such that  $|a_n - a| < \epsilon$  for all  $n \geq N$ .

And the first six parts of Lemma 1.1 are the same over  $\mathbb{C}$ . The last one does not make sense over  $\mathbb{C}$  since it uses the order of  $\mathbb{R}$ .

## Lecture 2: Bolzano–Weierstrass theorem

24 Jan. 11:00

## Theorem 1.2: Bolzano-Weierstrass Theorem

If  $x_n \in \mathbb{R}$  and there exists  $K$  such that  $|x_n| \leq K$  for all  $n$ , then we can find  $n_1 < n_2 < n_3 < \dots$  and  $x \in \mathbb{R}$  such that  $x_{n_j} \rightarrow x$  as  $j \rightarrow \infty$ . In other words, every bounded sequence has a convergent subsequence.

**Remark.** We say nothing about the uniqueness of the limit  $x$ .

For example,  $x_n = (-1)^n$  has two subsequences tending to  $-1$  and  $1$  respectively.

*Proof.* Set  $[a_1, b_1] = [-K, K]$ . Let  $c$  be the mid-point of  $a_1, b_1$ , consider the following alternatives,

1.  $x_n \in [a_1, c]$  for infinitely many  $n$ .
2.  $x_n \in [c, b_1]$  for infinitely many  $n$ .

Note that (1) and (2) can hold at the same time. But if (1) holds, we set  $a_2 = a_1$  and  $b_2 = c$ . If (1) fails, we have that (2) must hold, and we set  $a_2 = c$  and  $b_2 = b_1$ .

We proceed as above to construct sequences  $a_n, b_n$  such that  $x_m \in [a_n, b_n]$  for infinitely many values of  $m$ . They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

$a_n$  is an increasing sequence and bounded, and  $b_n$  is a decreasing sequence and bounded. By Fundamental Axiom,  $a_n \rightarrow a \in [a_1, b_1]$ ,  $b_n \rightarrow b \in [a_1, b_1]$ . Using Lemma 1.1,  $b - a = \frac{b-a}{2} \implies a = b$ .

Since  $x_m \in [a_n, b_n]$  for infinitely many values of  $m$ , having chosen  $n_j$  such that  $x_{n_j} \in [a_j, b_j]$ , that is  $n_{j+1} > n_j$  such that  $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$ . In other words, there is unlimited supply.

Hence,  $a_j \leq x_{n_j} \leq b_j$ , so  $x_{n_j} \rightarrow a$ . ■

## 1.2 Cauchy Sequences

### Definition 1.5: Cauchy Sequence

A sequence  $a_n \in \mathbb{R}$  is called a *Cauchy sequence* if given  $\epsilon > 0 \exists N > 0$  such that  $|a_n - a_m| < \epsilon \forall n, m > N$ .

**Note.**  $N$  is dependent on  $\epsilon$ .

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

### Lemma 1.3

A convergent sequence is a Cauchy sequence.

*Proof.* If  $a_n \rightarrow a$ , given  $\epsilon > 0$ , exists  $N$  such that for all  $n \geq N$ ,  $|a_n - a| < \epsilon$ .

Take  $m, n \geq N$ ,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$

■

### Lemma 1.4

Every Cauchy sequence is convergent.

*Proof.* First we note that if  $a_n$  is Cauchy, then it is bounded.

Take  $\epsilon = 1$ ,  $N = N(1)$  in the Cauchy property, then

$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$

We have

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$

Let  $K = \max\{1 + |a_N|, |a_n| \mid n = 1, 2, \dots, N-1\}$ .

Then  $|a_n| \leq K$  for all  $n$ . By the Bolzano–Weierstrass theorem,  $a_{n_j} \rightarrow a$ . We must have  $a_n \rightarrow a$ .

Given  $\epsilon > 0$ , there exists  $j_0$  such that for all  $j \geq j_0$ ,  $|a_{n_j} - a| < \epsilon$ .

Also, there exists  $N(\epsilon)$  such that  $|a_m - a_n| < \epsilon$  for all  $m, n \geq N(\epsilon)$ .

Take  $j$  such that  $n_j \geq \max\{N(\epsilon), n_{j_0}\}$ . Then if  $n \geq N(\epsilon)$ ,

$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < 2\epsilon.$$

■

Thus, on  $\mathbb{R}$ , a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

## 2 Series

### Definition 2.1

If  $a_n \in \mathbb{R}, \mathbb{C}$  We say that  $\sum_{j=1}^{\infty} a_j$  converges to  $s$  if the sequence of partial sums

$$S_N = \sum_{j=1}^N a_j \rightarrow S$$

as  $N \rightarrow \infty$ . We write  $\sum_{j=1}^{\infty} a_j = s$ . If  $S_N$  does not converge, we say that  $\sum_{j=1}^{\infty} a_j$  *diverges*.

**Remark.** Any problem on series is really a problem about the sequence of partial sums.



## Lemma 2.1

1. If  $\sum_{j=1}^{\infty} a_j$  and  $\sum_{j=1}^{\infty} b_j$  converges, then so does  $\sum_{j=1}^{\infty} \lambda a_j + \mu b_j$ , when  $\lambda, \mu \in \mathbb{C}$ ;
2. Suppose there exists  $N$  such that  $a_i = b_i$  for all  $i \geq N$ . Then either  $\sum_{i=1}^{\infty} a_i$  and  $\sum_{i=1}^{\infty} b_i$  both converge or they both diverge. (initial terms do not matter for convergence)

*Proof.* 1. Exercise.

2. If we have  $n \geq N$ ,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^n a_i$$

$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^n b_i$$

So  $S_n - d_n = \sum_{i=1}^{N-1} a_i - b_i$  which is a constant. So  $S_n$  converges if and only if  $d_n$  does.

■

## Lecture 3

26 Jan. 11:00

We have the following important example,

**Example** (Geometric Series).  $x \in \mathbb{R}$ , set  $a_n = x^{n-1}$  with  $n \geq 1$ . So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \frac{1-x^n}{1-x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have  $S_n(1 - x) = 1 - x^n$ .

If  $|x| < 1$ ,  $x^n \rightarrow 0$  and  $S_n \rightarrow \frac{1}{1-x}$ .

If  $x > 1$ ,  $x^n \rightarrow \infty$  and  $S_n \rightarrow \infty$ .

If  $x < -1$ ,  $S_n$  does not converge (oscillates).

If  $x = -1$ ,  $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$ .

Thus, the geometric series converges if and only if  $|x| < 1$ .

To see for example that  $x^n \rightarrow 0$  if  $|x| < 1$ , consider first the case  $0 < x < 1$ . Write  $\frac{1}{x} = 1 + \delta$ ,  $\delta > 0$ , so  $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \rightarrow 0$  because  $(1 + \delta)^n \geq 1 + n\delta$  from binomial expansion.

#### Definition 2.2

$S_n \rightarrow \infty$  if given  $A$ , there exists an  $N$  such that  $S_n > A$  for all  $n > N$ .

$S_n \rightarrow -\infty$  if given  $A$ , there exists an  $N$  such that  $S_n < -A$  for all  $n > N$ .

#### Lemma 2.2

If  $\sum_{i=1}^{\infty} a_n$  converges, then  $\lim_{i \rightarrow \infty} a_i = 0$ .

*Proof.* Let  $S_n = \sum_{i=1}^n a_i$ , note that  $a_n = S_n - S_{n-1}$ . If  $S_n \rightarrow a$ , we have  $a_n \rightarrow 0$  because  $S_{n-1} \rightarrow a$  also. ■

**Remark.** The converse of the preceding lemma is false. One example is  $\sum \frac{1}{n}$ , the

harmonic series. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty} \frac{1}{i}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since  $\frac{1}{n+k} \geq \frac{1}{2n}$  for  $k = 1, 2, \dots, n$ .

So if  $S_n \rightarrow a$ , then  $S_{2n} \rightarrow a$ , also we have  $a \geq a + \frac{1}{2}$ . Contradiction.

## 2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

### Theorem 2.1: The Comparison Test

Suppose  $0 \leq b_n \leq a_n$  for all  $n$ . Then if  $\sum_{n=1}^{\infty} a_n$  converges, so does  $\sum_{n=1}^{\infty} b_n$ .

*Proof.* Let  $s_N = \sum_{n=1}^N a_n$ ,  $d_N = \sum_{n=1}^N b_n$ . Because  $b_n \leq a_n$ , we know  $d_N \leq s_N$ . But  $s_N \rightarrow s$ , then  $d_N \leq s_N \leq s$  for all  $n$ , and  $d_N$  is an increasing sequence bounded above. So  $d_N$  converges. ■

**Example.** We consider  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^N a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that  $\sum_{n=1}^{\infty} a_n$  converges, so  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges.

In fact, we get  $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2$ .

For the rest of the lecture, we establish two more tests.

**Theorem 2.2: Root test/ Cauchy's Test for Convergence**

Assume  $a_n \geq 0$  and  $a_n^{1/n} \rightarrow a$  as  $n \rightarrow \infty$ . Then if  $a < 1$ ,  $\sum_{n=1}^{\infty} a_n$  converges; if  $a > 1$ ,  $\sum_{n=1}^{\infty} a_n$  diverges.

**Remark.** Nothing can be said if  $a = 1$ .

. If  $a < 1$ , choose  $a < r < 1$ . By definition of limit and hypothesis, there exists  $N$  such that  $\forall n \geq N$ ,

$$a_n^{1/n} < r \implies a_n < r^n.$$

But since  $r < 1$ , the geometric series converges, and by comparison test, the series  $\sum a_n$  converges as well.

To prove the second part of the theorem, if  $a > 1$ , for  $n \geq N$ ,

$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus,  $\sum_{n=1}^{\infty} a_n$  diverges, since  $a_n$  does not tend to zero. ■

**Theorem 2.3: Ratio Test/ D'Alembert's Test**

Suppose  $a_n > 0$  and  $\frac{a_{n+1}}{a_n} \rightarrow \ell$ . If  $\ell < 1$ ,  $\sum_{n=1}^{\infty} a_n$  converges. If  $\ell > 1$ ,  $\sum_{n=1}^{\infty} a_n$  diverges.

**Remark.** As before, nothing can be said for  $\ell = 1$ .

*Proof.* Supposed  $\ell < 1$  and choose  $r$  with  $\ell < r < 1$ . Then  $\exists N$  such that  $\forall n \geq N$ ,

$$\frac{a_{n+1}}{a_n} < r.$$

Therefore,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \quad n > N.$$

So,  $a_n < kr^n$  with  $k$  independent of  $n$ . Since  $\sum_{n=1}^{\infty} r^n$  converges, so does  $\sum_{n=1}^{\infty} a_n$  by Comparison Test.

If  $\ell > 1$ , choose  $1 < r < \ell$ . Then  $\frac{a_{n+1}}{a_n} > r$  for all  $n \geq N$ , and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \quad n > N.$$

So the series diverges. ■

## Lecture 4

28 Jan. 2022

**Example.** To determine the convergence of  $\sum_{n=1}^{\infty} a_n = \frac{n}{2^n}$ .

By ratio test,

$$\frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{2n} \rightarrow \frac{1}{2} < 1.$$

So we have convergence by ratio test.

However,  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, and ratio test gives limit 1, and  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges, and ratio test gives limit 1. So ratio test is inconclusive if the limit is 1.

Since  $n^{\frac{1}{n}} \rightarrow 1$  as  $n \rightarrow \infty$ , so root test is also inconclusive when the limit is 1.

To see this limit, write

$$n^{\frac{1}{n}} = 1 + \delta_n, \quad \delta_n > 0.$$

So

$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2} \delta_n^2.$$

And  $\delta_n^2 < \frac{2}{n-1} \implies \delta_n \rightarrow 0$ .

**Remark.** Use the root test when there is a  $n$ th power in the series.

### Theorem 2.4: Cauchy's Condensation Test

Let  $a_n$  be a decreasing sequence of positive terms. Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{n=1}^{\infty} 2^n a_{2^n}$  converges.

*Proof.* First we observe that if  $a_n$  is decreasing

$$a_{2^k} \leq a_{2^{k-1}+i} \leq a_{2^{k-1}}$$

for all  $k \geq 1$  and  $1 \leq i \leq 2^{k-1}$ .

Assume that  $\sum_{n=1}^{\infty} a_n$  converges with sum  $A$ . Then

$$\begin{aligned} 2^{n-1}a_{2^n} &= \underbrace{a_{2^n} + \cdots + a_{2^n}}_{2^{n-1} \text{ times}} \\ &\leq a_{2^{n-1}+1} + \cdots + a_{2^n} \\ &= \sum_{m=2^{n-1}+1}^{2^n} a_m. \end{aligned}$$

Thus,  $\sum_{n=1}^N 2^{n-1}a_{2^n} \leq \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m = \sum_{m=2}^{2^N} a_m$ . So

$$\sum_{n=1}^N 2^n a_{2^n} \leq 2 \sum_{m=2}^{2^N} a_m \leq 2(A - a_1).$$

Thus,  $\sum_{n=1}^N 2^n a_{2^n}$  being increasing and bounded above, converges.

Conversely, assume  $\sum_{n=1}^{\infty} 2^n a_{2^n}$  converges to  $B$ , then

$$\begin{aligned} \sum_{m=2^{n-1}+1}^{2^n} a_m &= a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} \\ &\leq \underbrace{a_{2^{n-1}} + \cdots + a_{2^{n-1}}}_{2^{n-1} \text{ times}} = 2^{n-1}a_{2^{n-1}}. \end{aligned}$$

Similarly, we have

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^N 2^{n-1}a_{2^{n-1}} \leq B.$$

Therefore,  $\sum_{m=1}^N a_m$  is a bounded increasing sequence and thus it converges. ■

**Example.**  $\sum_{n=1}^{\infty} \frac{1}{n^k}$  for  $k > 0$  converges if and only if  $k > 1$ . First we note that  $\frac{1}{n^k}$  is a decreasing sequence of positive terms.

$$\frac{1}{(n+1)^k} < \frac{1}{n^k} \iff \left(\frac{n}{n+1}\right)^k < 1 \iff \frac{n}{n+1} < 1.$$

We use Cauchy condensation test, and we have

$$\begin{aligned} 2^n a_{2^n} &= 2^n \left( \frac{1}{2^n} \right)^k \\ &= 2^{n-nk} = (2^{1-k})^n. \end{aligned}$$

Which is a geometric series with the ratio  $2^{1-k}$ . So  $\sum_{n=1}^{\infty} \frac{1}{n^k}$  converges if and only if  $2^{1-k} < 1 \iff k > 1$ .

## 2.2 Alternating Series

### Theorem 2.5: Alternating Series Test

If  $a_n$  decreases and tends to 0 as  $n \rightarrow \infty$ , then the series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

**Example.**  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges.

*Proof.* The partial sum is

$$\begin{aligned} S_n &= a_1 - a_2 + \cdots + (-1)^{n+1} a_n \\ S_{2n} &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq S_{2n-1} \\ S_{2n} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1 \end{aligned}$$

So  $S_{2n}$  is increasing and bounded above, implying that  $S_{2n} \rightarrow S$ . The odd terms satisfy

$$S_{2n+1} = S_{2n} + a_{2n+1} \rightarrow S + 0 = S.$$

This implies that  $S_n$  converges to  $S$  as well. Given  $\epsilon$ , there exists  $N_1$  such that  $\forall n \geq N_1$ ,  $|S_{2n} - S| < \epsilon$ . We also know that there exists  $N_2$  such that  $\forall n \geq N_2$ ,  $|S_{2n+1} - S| < \epsilon$ . Take  $N = 2 \max\{N_1, N_2\} + 1$ , then if  $n \geq N$ ,  $|S_n - S| < \epsilon$ . So  $S_n \rightarrow S$ . ■

## Lecture 5

31 Jan. 2022

## 2.3 Absolute Convergence

## Definition 2.3

Take  $a_n \in \mathbb{C}$ . If  $\sum_{n=1}^{\infty} |a_n|$  is convergent, then the series is called *absolutely convergent*.

**Note.** Since  $|a_n| \geq 0$ . We can use the previous tests to check absolute convergence; this is particularly useful for  $a_n \in \mathbb{C}$ .

## Theorem 2.6

If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, then it is convergent.

*Proof.* Suppose first  $a_n \in \mathbb{R}$ . Let

$$v_n = \begin{cases} a_n, & \text{if } a_n \geq 0 \\ 0, & \text{if } a_n < 0 \end{cases}$$

and

$$w_n = \begin{cases} 0, & \text{if } a_n \geq 0 \\ -a_n, & \text{if } a_n < 0 \end{cases}.$$

We have  $v_n = \frac{|a_n| + a_n}{2}$ ,  $w_n = \frac{|a_n| - a_n}{2}$ . Clearly,  $v_n, w_n \geq 0$ . We also have  $|a_n| = v_n + w_n \geq v_n, w_n$ .

So by comparison test, if  $\sum_{n=1}^{\infty} |a_n|$  converges,  $\sum_{n=1}^{\infty} v_n, \sum_{n=1}^{\infty} w_n$  also converges.

If  $a_n \in \mathbb{C}$ , write  $a_n = x_n + iy_n$ . We have  $|x_n|, |y_n| \leq |a_n|$ . So  $\sum_{n=1}^{\infty} x_n$  and  $\sum_{n=1}^{\infty} y_n$  are absolutely convergent, so they are convergent. And  $\sum_{n=1}^{\infty} a_n$  converges as well. ■

**Example.**



1.  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges but not absolutely convergent.
2.  $\sum_{n=1}^{\infty} \frac{z^n}{2^n}$  for  $z \in \mathbb{C}$ . We check for absolute convergence first,  $\sum_{n=1}^{\infty} \left(\frac{|z|}{2}\right)^n$ . So if  $|z| < 2$ , the series is convergent by absolute convergence.

Otherwise, if  $|z| \geq 2$ ,  $\left|\frac{z}{2}\right| \geq 1$ .  $a_n$  does not tend to zero, hence the series diverge.

**Notation.** If  $\sum_{n=1}^{\infty} a_n$  converges but not absolutely convergent, it is sometimes called *conditional convergent*.

It is called conditional because the sum to which the series converges is conditional on the order in which elements of the sequence are taken.

**Example** (Example Sheet 1, Q7).  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$  and  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$  are two series with different sums. Let  $s_n$  be the partial sum of the first series, and  $t_n$  be the partial sum of the second series, then  $s_n \rightarrow s$  and  $t_n \rightarrow \frac{3s}{2}$ .

#### Definition 2.4

Let  $\sigma$  be a bijection of the positive integers,  $a'_n = a_{\sigma(n)}$  is a *rearrangement*.

#### Theorem 2.7

If  $\sum_{n=1}^{\infty} a_n$  is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

*Proof.* Again we do the proof first for  $a_n \in \mathbb{R}$ . Let  $\sum_{n=1}^{\infty} a'_n$  be a rearrangement of  $\sum_{n=1}^{\infty} a_n$ .

Let  $s_n = \sum_{i=1}^n a_i$  and  $t_n = \sum_{i=1}^n a'_i$ ,  $S = \sum_{n=1}^{\infty} a_n$ . Suppose first that  $a_n \geq 0$ . Given  $n$ , we can find  $q$  such that  $s_q$  contains every term of  $t_n$ . Because  $a_n \geq 0$ , we have

$$t_n \leq s_n \leq S.$$

So  $t_n$  is an increasing sequence bounded above so  $t_n \rightarrow t$ , and from the inequality above,  $t \leq s$ . By symmetry, we have  $s \leq t \implies s = t$ . If  $a_n$  has any negative term, consider  $v_n$  and  $w_n$  from Theorem 2.6. Consider  $\sum_{n=1}^{\infty} a'_n$ ,  $\sum_{n=1}^{\infty} v'_n$ ,  $\sum_{n=1}^{\infty} w'_n$ . Since  $\sum_{n=1}^{\infty} |a_n|$  converges,

both  $\sum_{n=1}^{\infty} v_n$  and  $\sum_{n=1}^{\infty} w_n$  converge. Using the fact that  $v_n, w_n \geq 0$ , we case above, we have  $\sum_{n=1}^{\infty} v'_n = \sum_{n=1}^{\infty} v_n$  and  $\sum_{n=1}^{\infty} w_n = \sum_{n=1}^{\infty} w'_n$ . But  $a_n = v_n - w_n$  so  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a'_n$ .

For the case  $a_n \in \mathbb{C}$ , we write  $a_n = x_n + iy_n$ . Since  $|x_i|, |y_i| \leq |a_n|$ ,  $\sum_{n=1}^{\infty} x_n$  and  $\sum_{n=1}^{\infty} y_n$  are absolutely convergent. By the previous case  $\sum_{n=1}^{\infty} x'_n = \sum_{n=1}^{\infty} x_n$ ,  $\sum_{n=1}^{\infty} y'_n = \sum_{n=1}^{\infty} y_n$ . Since  $a'_n = x'_n + iy'_n$  so  $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a'_n$ . ■

## Lecture 6

2 Feb. 2022

# 3 Functions

## 3.1 Continuity

Suppose  $E \subseteq \mathbb{C}$  is a non-empty subset, and we have a function  $f : E \rightarrow \mathbb{C}$  and a point  $a \in E$ . (this includes the case in which  $f$  is real-valued and  $E$  is a subset of  $\mathbb{R}$ )

### Definition 3.1

$f$  is *continuous* at  $a \in E$  if for every sequence  $z_n \in E$  with  $z_n \rightarrow a$ , we have  $f(z_n) \rightarrow f(a)$ .

### Definition 3.2: $\epsilon$ - $\delta$ Definition

$f$  is *continuous* at  $a \in E$ , if given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that if  $|z - a| < \delta, z \in E$ , then  $|f(z) - f(a)| < \epsilon$ .

We prove right away that the two definitions are equivalent.

### Theorem 3.1

The two definitions of continuity are equivalent.

*Proof.* We first prove the second definition implies the first definition. We know that

given  $\epsilon > 0, \exists \delta > 0$  such that  $|z - a| < \delta, z \in E$ , then  $|f(z) - f(a)| < \epsilon$ . Let  $z_n \rightarrow a$ , then  $\exists n_0$  such that  $\forall n \geq n_0$ , we have  $|z_n - a| < \delta$ . This implies, by the assumption,  $|f(z_n) - f(a)| < \epsilon$ . That is,  $f(z_n) \rightarrow f(a)$ .

Next, we prove the other direction. Assume  $f(z_n) \rightarrow f(a)$  whenever  $z_n \rightarrow a, z_n \in E$ . Suppose  $f$  is not continuous at  $a$  according to Definition 2.

$\exists \epsilon > 0$ , s.t.  $\forall \delta > 0$ , there exists  $z \in E$  s.t.  $|z - a| < \delta$  and  $|f(z) - f(a)| \geq \epsilon$ .

Let  $\delta = \frac{1}{n}$  from non-continuity defined above, we get  $z_n$  such that  $|z_n - a| < \frac{1}{n}$  and  $|f(z_n) - f(a)| \geq \epsilon$ . Clearly  $z_n \rightarrow a$ , but  $f(z_n)$  does not tend to  $f(a)$  because  $|f(z_n) - f(a)| \geq \epsilon$ . Contradiction. ■

### Proposition 3.1

$a \in E$ , and  $g, f : E \rightarrow \mathbb{C}$  are both continuous at  $a$ . So are the functions  $f(z) + g(z)$ ,  $f(z)g(z)$  and  $\lambda f(z)$  for any constant  $\lambda$ . In addition, if  $f(z) \neq 0 \forall z \in E$ , then  $\frac{1}{f(z)}$  is continuous at  $a$ .

*Proof.* Using Definition 1 of continuity, this is obvious, using the analogous results for sequences. (Lemma 1.1)

For example,

$$z_n \rightarrow a \implies f(z_n) \rightarrow f(a), g(z_n) \rightarrow g(a) \implies f(z_n) + g(z_n) \rightarrow f(a) + g(a).$$

■

The function  $f(z) = z$  is continuous, so by using the proposition, we get that every polynomial is continuous at every point in  $\mathbb{C}$ .

**Note.** We say that  $f$  is *continuous on*  $E$  if it is continuous at every  $a \in E$ .

**Remark.** Still it is instructive to prove Proposition 3.1 directly from the  $\epsilon$ - $\delta$  definition.

Next we look at compositions.

**Theorem 3.2**

Let  $f : A \rightarrow \mathbb{C}$  and  $g : B \rightarrow \mathbb{C}$  be two functions such that  $f(A) \subseteq B$ . Suppose  $f$  is continuous at  $a \in A$  and  $g$  is continuous at  $f(a)$ , then  $g \circ f : A \rightarrow \mathbb{C}$  is continuous at  $a$ .

*Proof.* Take any sequence  $z_n \rightarrow a$ , by assumption we know  $f(z_n) \rightarrow f(a)$ . Set  $w_n = f(z_n) \in B$ . By continuity of  $g$ , we have  $g(w_n) \rightarrow g(f(a))$ , and we are done. ■

**Example.**

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases},$$

assuming that  $\sin x$  is continuous. (to be proved later) If  $x \neq 0$ , propositions proved above imply that  $f(x)$  is continuous at any  $x \neq 0$ .

However, it is discontinuous at 0. Consider the sequence satisfying

$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi.$$

We have  $f(x_n) \rightarrow 1, x_n \rightarrow 0$ , but  $f(0) = 0$ .

2. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

It's continuous at  $x \neq 0$  as above, and  $f$  is continuous at 0. Take  $x_n \rightarrow 0$ , then  $|f(x_n)| \leq |x_n|$  because  $\sin \frac{1}{x} \leq 1$ , so  $f(x_n) \rightarrow 0 = f(0)$ .

3. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}.$$

It is discontinuous at every point. If  $x \in \mathbb{Q}$ , take a sequence  $x_n \rightarrow x$  with  $x_n \notin \mathbb{Q}$ , then  $f(x_n) = 0 \not\rightarrow f(x) = 1$ . Similarly, if  $x \notin \mathbb{Q}$ , take  $x_n \rightarrow x$  with  $x_n \in \mathbb{Q}$ , we have  $f(x_n) = 1 \not\rightarrow f(x) = 0$ .

## Lecture 7

4 Jan. 2022

## 3.2 Limit of a function

$f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$ . We wish to define what is meant by  $\lim_{z \rightarrow a} f(z)$ , even when  $a$  might not be in  $E$ .

**Example.** The limit of  $\frac{\sin z}{z}$  as  $z \rightarrow 0$  with  $E = \mathbb{C} \setminus \{0\}$ .

Also, if  $E = \{0\} \cup [1, 2]$ , it does not make sense to speak about points  $z \in E, z \neq 0, z \rightarrow 0$ .

## Definition 3.3

If  $E \subseteq \mathbb{C}, a \in \mathbb{C}$ , we say that  $a$  is a *limit point* of  $E$  if for any  $\delta > 0, \exists z \in E$  such that  $0 < |z - a| < \delta$ .

**Remark.**  $a$  is a limit point if and only if there exists a sequence  $z_n \in E$  such that  $z_n \rightarrow a$  and  $z_n \neq a$  for all  $n$ .

## Definition 3.4

If  $f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$  and let  $a \in \mathbb{C}$  be a limit point of  $E$ . We say that  $\lim_{z \rightarrow a} f(z) = l$  (" $f$  tends to  $l$  as  $z$  tends to  $a$ ") if given  $\epsilon > 0, \exists \delta > 0$  such that whenever  $0 < |z - a| < \delta$  and  $z \in E$ , then  $|f(z) - l| < \epsilon$ .

Equivalently,  $f(z_n) \rightarrow l$  for every sequence  $z_n \in E, z_n \neq a$  and  $z_n \rightarrow a$ .

**Remark.** Straight from the definitions, we have that if  $a \in E$  is limit point, then  $\lim_{z \rightarrow a} f(z) = f(a)$  if and only if  $f$  is continuous at  $a$ .

If  $a \in E$  is *isolated* (i.e.  $a \in E$  is not a limit point), continuity of  $f$  at  $a$  always holds. The limit of functions has very similar properties to limit of sequences.

1. It is unique,  $f(z) \rightarrow A$  and  $f(z) \rightarrow B$  as  $z \rightarrow a$ , then

$$|A - B| \leq |A - f(z)| + |f(z) - B|.$$

If  $z \in E$  is such that  $0 < |z - a| < \min\{\delta_1, \delta_2\}$ , then  $|A - B| < 2\epsilon$ . So  $A = B$ . The existence of such  $z$  is a consequence of the condition that  $a$  is a limit point of  $E$ .

2.  $f(z) + g(z) \rightarrow A + B$ ;
3.  $f(z)g(z) \rightarrow AB$ ;
4. if  $B \neq 0$ ,  $\frac{f(z)}{g(z)} \rightarrow \frac{A}{B}$ . All proved in the same way as before.

### 3.3 The Intermediate Value Theorem

#### Theorem 3.3: Intermediate Value Theorem

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous and  $f(a) \neq f(b)$ , then  $f$  takes every value which lies between  $f(a)$  and  $f(b)$ .

*Proof.* Without loss of generality, suppose  $f(a) < f(b)$ . Take  $f(a) < \eta < f(b)$ . Let  $S = \{x \in [a, b] \mid f(x) < \eta\}$ . We note that  $a \in S$ , so  $S \neq \emptyset$ . Clearly  $S$  is bounded above by  $b$ . Then there is a supremum  $C$  where  $C \leq b$ . By definition of supremum, given  $n$ , there exists  $x_n \in S$  such that  $C - \frac{1}{n} < x_n \leq C$ . So  $x_n \rightarrow C$ . Since  $x_n \in A$ ,  $f(x_n) < \eta$ . By continuity of  $f$ ,  $f(x_n) \rightarrow f(C)$ . So  $f(C) \leq \eta$ .

Now observe that  $c \neq b$  because  $f(b) > \eta$ . Then for  $n$  large,  $C + \frac{1}{n} \in [a, b]$  and  $C + \frac{1}{n} \rightarrow C$ . Again by continuity  $f(C + \frac{1}{n}) \rightarrow f(C)$ . But since  $C + \frac{1}{2} > C$ ,  $f(C + \frac{1}{n}) \geq \epsilon$ . So  $f(C) \geq \eta \implies f(C) = \eta$ . ■

**Remark.** The theorem is very useful for finding zeroes or fixed points.

**Example.** Existence of the  $N$ -th root of a positive real number. Suppose

$$f(x) = x^N, \quad x \geq 0.$$

Let  $y$  be a positive real number.  $f$  is continuous on  $[0, 1 + y]$ , so

$$0 = f(0) < y < (1 + y)^N = f(1 + y).$$

By the IVT,  $C \in (0, 1 + y)$  such that  $f(C) = y$ , i.e.  $C^N = y$ .  $C$  is a positive  $N$ -th root of  $y$ .

We also have uniqueness. Exercise.

## Lecture 8

7 Feb. 2022

## 3.4 Bounds of a Continuous Function

## Theorem 3.4

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. Then there exists  $K$  such that  $|f(x)| \leq K$  for all  $x \in [a, b]$ .

*Proof.* We argue by contradiction. Suppose the statement is false. Then given any integer  $n \geq 1$ , there exists  $x_n \in [a, b]$  such that  $|f(x_n)| > n$ . By Bolzano-Weierstrass,  $x_n$  has a convergent subsequence  $x_{n_j} \rightarrow x$ . Since  $a \leq x_{n_j} \leq b$ , we must have  $x \in [a, b]$ . By the continuity of  $f$ ,  $f(x_{n_j}) \rightarrow f(x)$ . But  $|f(x_{n_j})| > n_j \rightarrow \infty$  as  $j \rightarrow \infty$ . Contradiction. ■

## Theorem 3.5: Extreme Value Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then there exists  $x_1, x_2 \in [a, b]$  such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all  $x \in [a, b]$ .

"A continuous function on a closed bounded interval is bounded and attains its bounds."

*Proof.* Let  $A = \{f(x) \mid x \in [a, b]\} = f([a, b])$ . By Theorem 3.4,  $A$  is bounded since it is clearly non-empty, it has a supremum  $M$ . By definition of supremum, given an integer  $n \geq 1$ , there exists  $x_n \in [a, b]$  such that  $M - \frac{1}{n} < f(x_n) \leq M$ . From Bolzano-Weierstrass, there exists  $x_{n_j} \rightarrow x \in [a, b]$ . Since  $f(x_{n_j}) \rightarrow M$ , by continuity of  $f$ , we get that  $f(x) = M$ . So  $x_2 := x$ .

We can prove similarly for the minimum. ■

*Proof 2.*  $A = f([a, b])$ ,  $M = \sup A$  as before. Suppose  $\nexists x_2$  such that  $f(x_2) = M$ . Let

$$g(x) = \frac{1}{M - f(x)}, x \in [a, b]$$

is defined and continuous on  $[a, b]$ . By Theorem 3.4 applied to  $g$ ,  $\exists k > 0$  such that  $g(x) < K$  for all  $x \in [a, b]$ . This means that  $f(x) \leq M - \frac{1}{k}$  for all  $x \in [a, b]$ . This is absurd because it contradicts that  $M$  is the supremum. ■

**Note.** Theorems 3.4 and 3.5 are false if the interval is not closed and bounded. For example,

$$f : (0, 1] \rightarrow \mathbb{R}, x \mapsto \frac{1}{x}.$$

### 3.5 Inverse Functions

#### Definition 3.5

$f$  is *increasing* for  $x \in [a, b]$  if  $f(x_1) \leq f(x_2)$  for all  $x_1, x_2$  such that  $a \leq x_1 < x_2 \leq b$ .

If  $f(x_1) < f(x_2)$ , we say that  $f$  is *strictly increasing*.

There are similar definitions for *decreasing* and *strictly decreasing*.

#### Theorem 3.6

$f : [a, b] \rightarrow \mathbb{R}$  is continuous and strictly increasing for  $x \in [a, b]$ . Let  $c = f(a)$  and  $d = f(b)$ . Then  $f : [a, b] \rightarrow [c, d]$  is bijective and the inverse  $g := f^{-1} : [c, d] \rightarrow [a, b]$  is also continuous and strictly increasing.

**Remark.** There is a similar statement for strictly decreasing function. Take  $c < k < d$ , from the IVT,  $\exists h$  such that  $f(h) = k$ . Since  $f$  is strictly increasing,  $h$  is unique. Define  $g(k) := h$  and this gives an inverse  $g : [c, d] \rightarrow [a, b]$  for  $f$ .

We first prove that  $g$  is strictly increasing. Take  $y_1 < y_2$  such that  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ . If  $x_2 \leq x_1$ , since  $f$  is increasing,  $f(x_2) \leq f(x_1) \implies y_2 \leq y_1$ . Absurd.

Next we prove continuity. Let  $\epsilon > 0$  be given, let  $k_1 = f(h - \epsilon)$  and  $k_2 = f(h + \epsilon)$ . Because  $f$  is strictly increasing, we have  $k_1 < k < k_2$ . If  $k_1 < y < k_2$ , we have  $h - \epsilon < g(y) < h + \epsilon$ . So we can just take  $\delta = \min\{k_2 - k, k - k_1\}$ . So  $g$  is continuous at  $k$ . Here we took  $k \in (c, d)$ . A very similar argument establishes continuity at the end points.



## Lecture 9

9 Feb. 2022

## 4 Differentiability

Let  $f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$ , most of the time  $E = \text{interval} \subseteq \mathbb{R}$ .

### Definition 4.1

Let  $x \in E$  be a point such that  $\exists x_n \in E$  with  $x_n \neq x$  and  $x_n \rightarrow x$  (i.e. a limit point),  $f$  is said to be *differentiable* at  $x$  with derivative  $f'(x)$  if

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = f'(x).$$

If  $f$  is differentiable at each  $x \in E$ , we say  $f$  is differentiable on  $E$ .

(Think of  $E$  as an interval or a disc in the case of  $\mathbb{C}$ .)

### Remark.

1. Other common notations include  $\frac{dy}{dx}$ ,  $\frac{df}{dx}$ .
2.  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ . ( $y = x + h$ )
3. Another look at the definition is the following.

Let  $\epsilon(h) := f(x+h) - f(x) - hf'(x)$ , then  $\lim_{h \rightarrow 0} \frac{\epsilon(h)}{h} = 0$ . We have also

$$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear in } h} + \epsilon(h).$$

Alternative definition of differentiability is  $f$  is differentiable at  $x$  if  $\exists A, E$  such that  $f(x+h) = f(x) + hA + \epsilon(h)$  where  $\lim_{h \rightarrow 0} \frac{\epsilon}{h} = 0$ . If such an  $A$  exists, then it is unique, since  $A = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ .

4. If  $f$  is differentiable at  $x$ , then  $f$  is continuous. Since  $\epsilon(h) \rightarrow 0$ , then  $f(x+h) \rightarrow f(x)$  as  $h \rightarrow 0$ .

5. Alternative ways of writing things:

$$f(x+h) = f(x) + hf'(x) + h\epsilon_f(h) \text{ with } \epsilon_f(h) \rightarrow 0 \text{ as } h \rightarrow 0.$$

Or,

$$f(x) = f(a) + (x-a)f'(a) + (x-a)\epsilon_f(x) \text{ with } \epsilon_f(x) \rightarrow 0 \text{ as } x \rightarrow a.$$

**Example.** If we have  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(x) = |x|$ . Clearly, we have  $f'(x) = 1$  if  $x > 0$  and  $f'(x) = -1$  if  $x < 0$ . Take  $h_n \downarrow 0$  at  $x = 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \rightarrow \infty} \frac{h_n}{h_n} = 1.$$

And take  $h_n \uparrow 0$  at  $x = 0$ , we have

$$\lim_{n \rightarrow \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \rightarrow \infty} \frac{-h_n}{h_n} = -1.$$

So  $f$  is not differentiable at  $x = 0$ .

## 4.1 Differentiation of Sums, Products, etc

**Property.**

1. If  $f(x) = c$  for all  $x \in E$ , then  $f$  is differentiable with  $f'(x) = 0$ .

2.  $f, g$  are differentiable at  $x$ , then so is  $f + g$  and

$$(f + g)'(x) = f'(x) + g'(x).$$

3.  $f, g$  are differentiable at  $x$ , then so is  $fg$  and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

4.  $f$  differentiable at  $x$  and  $f(x) \neq 0$  for all  $x \in E$ , then  $\frac{1}{f}$  is differentiable at  $x$  and

$$\left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{[f(x)]^2}.$$

*Proof.*

$$1. \lim_{h \rightarrow 0} \frac{c-c}{h} = 0.$$

$$\begin{aligned}
 2. \quad & \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \quad \text{using properties of limits.} \\
 &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= f'(x) + g'(x)
 \end{aligned}$$

3. Let  $\phi(x) = f(x)g(x)$ , then we have

$$\begin{aligned}
 \frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
 &= f(x+h) \left[ \frac{g(x+h) - g(x)}{h} \right] + g(x) \left[ \frac{f(x+h) - f(x)}{h} \right].
 \end{aligned}$$

So we have  $\lim_{h \rightarrow 0} \frac{\phi(x+h) - \phi(x)}{h} = f(x)g'(x) + f'(x)g(x)$  using standard properties of limits and the fact that  $f$  is continuous at  $x$ .

4. Define again  $\phi(x) = \frac{1}{f(x)}$ , then

$$\begin{aligned}
 \frac{\phi(x+h) - \phi(x)}{h} &= \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} \\
 &= \frac{f(x) - f(x+h)}{hf(x)f(x+h)}.
 \end{aligned}$$

So we have  $\lim_{h \rightarrow 0} \frac{\phi(x+h) - \phi(x)}{h} = \frac{-f'(x)}{[f(x)]^2}$ .

■

**Remark.** From (3) and (4), we get

$$\left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

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**Example.** Consider  $f(x) = x^n$  with  $n \in \mathbb{Z}, n > 0$ . When  $n = 1$ , clearly we have  $f(x) = x$  and  $f'(x) = 1$ .

We claim that  $f'(x) = nx^{n-1}$ , and we prove it by induction,  $f(x) = xx^n = x^{n+1}$ . By product rule and inductive hypothesis,

$$f'(x) = x^n + x(nx^{n-1}) = (n+1)x^n.$$

Next, we consider  $f(x) = x^{-n}$  with  $n \in \mathbb{Z}, n > 0$ . If  $x \neq 0$ , use property 4.1, we have

$$f'(x) = \frac{-(x^n)'}{x^{2n}} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}.$$

So we know how to find derivatives of polynomials and rational functions.

We have the following useful result to differentiate a larger class of functions.

**Theorem 4.1: Chain Rule**

If  $f : U \rightarrow \mathbb{C}$  is such that  $f(x) \in V$  for  $x \in U$ . If  $f$  is differentiable at  $a \in U$  and  $g : V \rightarrow \mathbb{C}$  is differentiable at  $f(a)$ , then  $g \circ f$  is differentiable at  $a$  with

$$(g \circ f)'(a) = f'(a)g'(f(a)).$$

*Proof.* We know

$$f(x) = f(a) + (x - a)f'(a) + \epsilon_f(x)(x - a)$$

such that  $\lim_{x \rightarrow a} \epsilon_f(x) = 0$ , and

$$g(y) = g(b) + (y - b)g'(b) + \epsilon_g(y)(y - b)$$

with  $\lim_{y \rightarrow b} \epsilon_g(y) = 0$ . Let  $b = f(a)$ , and set  $\epsilon_f(a) = 0$  and  $\epsilon_g(b) = 0$  to make them continuous at  $x = a$  and  $f = b$ . Now  $y = f(x)$  gives

$$\begin{aligned} g(f(x)) &= g(b) + (f(x) - b)g'(b) + \epsilon_g(f(x))(f(x) - b) \\ &= g(f(a)) + [(x - a)f'(a) + \epsilon_f(x)(x - a)][g'(b) + \epsilon_g(f(x))] \\ &= g(f(a)) + (x - a)f'(a)g'(b) + \\ &\quad (x - a)[\epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))] \\ &= g(f(a)) + (x - a)f'(a)g'(b) + (x - a)\sigma(x). \end{aligned}$$

So it suffices to show  $\sigma(x) = \epsilon_f(x)g'(b) + \epsilon_g(f(x))(f'(a) + \epsilon_f(x))$  tends to 0 as  $x$  tends to  $a$ . We have clearly  $\epsilon_f(x)g'(b) \rightarrow 0$ ,  $\epsilon_g(f(x)) \rightarrow 0$  and  $f'(a) + \epsilon_f(x) \rightarrow f'(a)$ , so  $\lim_{x \rightarrow a} \sigma(x) = 0$ . ■

**Example.**

1. Consider  $f(x) = \sin(x^2)$ , and we have

$$f'(x) = 2x \cos(x^2).$$

2. Consider  $f(x) = \begin{cases} x \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ . From previous lectures, we know that  $f$  is continuous, and it is differentiable at every  $x \neq 0$  by the previous theorems. At  $x = 0$ , take  $t \neq 0$  and we have

$$\frac{f(t) - f(0)}{t - 0} = \sin\left(\frac{1}{t}\right).$$

Again from previous lecture, we know  $\lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t - 0}$  does not exist, so  $f$  is not differentiable at  $x = 0$ .

**4.2 The Mean Value Theorem****Theorem 4.2: Rolle's Theorem**

Let  $f : [a, b] \rightarrow \mathbb{R}$  continuous on  $[a, b]$  and differentiable on  $(a, b)$ . If  $f(a) = f(b)$ , then there exists  $c \in (a, b)$  such that  $f'(c) = 0$ .

*Proof.* Let  $M = \max_{x \in [a, b]} f(x)$ , and  $m = \min_{x \in [a, b]} f(x)$ . Theorem 3.5 says that these values are achieved. Let  $k = f(a) = f(b)$ . If  $M = m = k$ , then  $f$  is constant and  $f'(c) = 0$  for all  $c \in (a, b)$ .

If  $f$  not constant, then  $M > k$  or  $m < k$ . Suppose  $M > k$ . By Theorem 3.5, exist  $c \in (a, b)$  such that  $f(c) = M$ .

If  $f'(c) > 0$ , then there are values to right of  $c$  for which  $f(x) > f(c)$  because

$$f(h + c) - f(c) = h(f'(c) + \epsilon_f(h)).$$

Since  $\epsilon_f(h) \rightarrow 0$  as  $h \rightarrow 0$ ,  $f'(c) + \epsilon_f(h) > 0$  for  $h$  small. This contradicts that  $M$  is the maximum. Similarly, if  $f'(c) < 0$ , there exists  $x$  to the left of  $c$  for which  $f(x) > f(c)$ .

So we must have  $f'(c) = 0$ . ■

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## Theorem 4.3: Mean Value Theorem

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function which is differentiable on  $(a, b)$ , then  $\exists c \in (a, b)$  such that

$$f(b) - f(a) = f'(c)(b - a).$$

*Proof.* Write  $\phi(x) = f(x) - kx$ , and choose  $k$  such that  $\phi(a) = \phi(b)$ . So

$$f(b) - bk = f(a) - ak \implies k = \frac{f(b) - f(a)}{b - a}.$$

By Rolle's Theorem applied to  $\phi$ ,  $\exists c \in (a, b)$  such that  $\phi'(c) = 0$ . That is,  $f'(c) = k$ . ■

**Remark.** We will often write

$$f(a + h) = f(a) + hf'(a + \theta h)$$

with  $\theta \in (0, 1)$ . We need to be careful, and consider  $\theta = \theta(h)$ .

## Corollary 4.1

$f : [a, b] \rightarrow \mathbb{R}$  continuous and differentiable on  $(a, b)$ .

1. If  $f'(x) > 0$  for all  $x \in (a, b)$ , then  $f$  is strictly increasing. (i.e. if  $b \geq y > x \geq a$ , then  $f(y) > f(x)$ )
2. If  $f'(x) \geq 0$  for all  $x \in (a, b)$ , then  $f$  is increasing. (i.e. if  $b \geq y > x \geq a$ , then  $f(y) \geq f(x)$ )
3. If  $f'(x) = 0$  for all  $x \in (a, b)$ , then  $f$  is constant on  $[a, b]$ .

*Proof.*

1. MVT implies  $f(y) - f(x) = f'(c)(y - x)$ . And  $f'(c) > 0 \implies f(y) > f(x)$ .
2. MVT implies  $f(y) - f(x) = f'(c)(y - x)$ . And  $f'(c) \geq 0 \implies f(y) \geq f(x)$ .
3. Take  $x \in [a, b]$ . Then use the MVT in  $[a, x]$  to get  $c \in (a, x)$  such that  $f(x) - f(a) = f'(c)(x - a) = 0$ . So  $f(x) = f(a)$  and  $f$  is constant.


**Theorem 4.4: Inverse Function Theorem**

If  $f : [a, b] \rightarrow \mathbb{R}$  continuous and differentiable on  $(a, b)$  with  $f'(x) > 0$  for all  $x \in (a, b)$ . Let  $f(a) = c$  and  $f(b) = d$ , then the function  $f : [a, b] \rightarrow [c, d]$  is bijective and  $f^{-1}$  is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

*Proof.* By Corollary 4.1,  $f$  is strictly increasing in  $[a, b]$ . By Theorem 3.6,  $\exists g : [c, d] \rightarrow [a, b]$  which is a continuous strictly increasing inverse of  $f$ . We want to show that  $g$  is differentiable and  $g'(y) = \frac{1}{f'(x)}$  where  $y = f(x)$  and  $x \in (a, b)$ .

If  $k \neq 0$  is given, let  $h$  be given by  $y + k = f(x + h)$ . That is,  $g(y + k) = x + h$  for  $h \neq 0$ . Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)} = \frac{h}{f(x + h) - f(x)}.$$

Let  $k \rightarrow 0$ , then  $h \rightarrow 0$  because  $g$  is continuous. So we have

$$g'(y) = \lim_{k \rightarrow 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}.$$



**Example.** We take  $g(x) = x^{\frac{1}{q}}$  with  $x > 0$  and  $q$  positive integer. So  $f(x) = x^q$ , with  $f'(x) = qx^{q-1}$ .  $g$  is differentiable and so is  $f$ , and by Theorem 4.4,

$$g'(x) = \frac{1}{q(x^{\frac{1}{q}})^{q-1}} = \frac{1}{q}x^{\frac{1}{q}-1}.$$

**Remark.** If  $g(x) = x^r$  with  $r \in \mathbb{Q}$ , then  $g'(x) = rx^{r-1}$ .

Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  continuous and differentiable on  $(a, b)$  and  $g(a) \neq g(b)$ . Then the MVT gives us  $s, t \in (a, b)$  such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b - a)f'(s)}{(b - a)g'(t)} = \frac{f'(s)}{g'(t)}.$$

Cauchy showed that we can take  $s = t$ .

**Theorem 4.5: Cauchy's Mean Value Theorem**

Let  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous and differentiable on  $(a, b)$ . Then  $\exists t \in (a, b)$  such that

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a)).$$

**Remark.** We recover the MVT if we take  $g(x) = x$ .

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*Proof.* Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}.$$

We have  $\phi$  continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Also,  $\phi(a) = \phi(b) = 0$ . By Rolle's Theorem, there exists  $t \in (a, b)$  such that  $\phi'(t) = 0$ , and

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)]. \end{aligned}$$

So  $\phi'(t) = 0$  gives the result. ■

**Note.** Good choice of auxiliary function and Rolle's theorem proves the theorem.

**Example (L' Hopital's Rule).** If we want to find  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x}$ , we have

$$\frac{e^x - 1}{\sin x} = \frac{e^x - x^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

for some  $t \in (0, x)$  by Cauchy's Mean Value Theorem. So

$$\frac{e^x - 1}{\sin x} \rightarrow 1$$

as  $x \rightarrow 0$ .

Goal: We want to extend the MVT to include higher order derivatives.



**Theorem 4.6: Taylor's Theorem with Lagrange's reminder**

Suppose  $f$  and its derivatives up to order  $n - 1$  are continuous in  $[a, a + h]$  and  $f^{(n)}$  exists for  $x \in (a, a + h)$ , then

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n f^{(n)}(a + \theta h)}{n!}$$

where  $\theta \in (0, 1)$ .

**Note.**

1. For  $n = 1$ , we get back the MVT, so this is an " $n$ -th order MVT".
2.  $R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h)$  is known as Lagrange's form of the remainder.

*Proof.* Define for  $0 \leq t \leq h$

$$\phi(t) = f(a + t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}b$$

where we choose  $b$  such that  $\phi(h) = 0$ , and we clearly have  $\phi(0) = 0$ . (Recall that in the proof of the MVT, we used  $f(x) - kx$  and picked  $k$  that we can use Rolle's Theorem. We also have that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0.$$

We use Rolle's Theorem  $n$  times. Since  $\phi(0) = \phi(h) = 0$ ,  $\phi'(h_1) = 0$  for some  $0 < h_1 < h$ . And since  $\phi'(0) = \phi'(h_1) = 0$ , we have  $\phi''(h_2) = 0$  for some  $0 < h_2 < h_1$ . Finally,  $\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0$ . So  $\phi^{(n)}(h_n) = 0$  with  $0 < h_n < h_{n-1} < \cdots < h$ . So  $h_n = \theta h$  for  $\theta \in (0, 1)$ , now

$$\phi^{(n)}(t) = f^{(n)}(a + t) - b \implies b = f^{(n)}(a + \theta h).$$

Set  $t = h$ ,  $\phi(h) = 0$  and put this value of  $b$  to the second line in the proof. ■

**Theorem 4.7: Taylor's Theorem with Cauchy's reminder**

Suppose  $f$  and its derivatives up to order  $n - 1$  are continuous in  $[a, a + h]$  and  $f^{(n)}$  exists for  $x \in (a, a + h)$ , and if  $a = 0$  for simplification, then we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{(1 - \theta)^{n-1} f^{(n)}(\theta h) h^n}{(n-1)!}$$

with  $\theta \in (0, 1)$ .

*Proof.* Define

$$F(t) = f(h) - f(t) - (h - t)f'(t) - \cdots - \frac{(h - t)^{n-1} f^{(n-1)}(t)}{(n-1)!}$$

with  $t \in [0, h]$ . Note that we have

$$\begin{aligned} F'(t) &= -f'(t) + f'(t) - (h - t)f''(t) + (h - t)f''(t) - \cdots - \frac{(h - t)^{n-1}}{(n-1)!}f^{(n)}(t) \\ &= -\frac{(h - t)^{n-1}}{(n-1)!}f^{(n)}(t). \end{aligned}$$

Set

$$\phi(t) = F(t) - \left(\frac{h-t}{h}\right)^p F(0)$$

with  $p \in \mathbb{Z}, 1 \leq p \leq n$ . Then  $\phi(0) = \phi(h) = 0$ , and by Rolle's,  $\exists \theta \in (0, 1)$  such that  $\phi'(\theta h) = 0$ . But,

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1 - \theta)^{p-1}}{h} F(0) = 0.$$

So

$$0 = -\frac{h^{n-1}(1 - \theta)^{n-1}}{(n-1)!}f^{(n)}(\theta h) + \frac{p(1 - \theta)^{p-1}}{h}[f(h) - \cdots - \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)].$$

Rearranging the two sides, and we get

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{h^n(1 - \theta)^{n-1}}{(n-1)!p(1 - \theta)^{p-1}}f^{(n)}(\theta h).$$

Taking  $p = n$ , we get Lagrange's reminder, and taking  $p = 1$  gives Cauchy's reminder.

■

## Lecture 13

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To get a Taylor series for  $f$ , one needs to show that  $R_n \rightarrow 0$  as  $n \rightarrow \infty$ . This requires “estimates” and “effort”.

**Remark.** Theorems 4.6 and 4.7 work equally well in an interval  $[a + h, a]$  with  $h < 0$ .

**Example.** The binomial series

$$f(x) = (1 + x)^r, r \in \mathbb{Q}.$$

We claim that  $|x| < 1$ , then

$$(1 + x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where

$$\binom{r}{n} = \frac{r(r-1) \cdots (r-n+1)}{n!}.$$

*Proof.* Clearly,

$$f^{(n)}(x) = r(r-1) \cdots (r-n+1)(1+x)^{r-n}.$$

If  $r \in \mathbb{Z}_{\geq 0}$ , then  $f^{(n+1)} = 0$ , we have a polynomial of degree  $r$ .

In general, by Lagrange’s reminder, we have

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \binom{r}{n} \frac{x^n}{(1 + \theta x)^{n-r}}.$$

Note that  $\theta$  depends on both  $x$  and  $n$ .

For  $0 < x < 1$ ,  $(1 + \theta x)^{n-r} > 1$  for  $n > r$ . Now observe that the series  $\sum \binom{r}{n}x^n$  is absolutely convergent for  $|x| < 1$ . Indeed, by the ratio test,

$$\begin{aligned} a_n &= \binom{r}{n} x^n \\ \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{r(r-1)(r-n+1) \cdots (r-n)x^{n+1}}{(n+1)!} \right| \left| \frac{n!}{r(r-1) \cdots (r-n+1)x^n} \right| \\ &= \left| \frac{(r-n)x}{n+1} \right| \end{aligned}$$

which tends to a value less than 1. In particular,  $a_n \rightarrow 0$  and  $\binom{r}{n}x^n \rightarrow 0$ .

Hence, for  $n > r$ , and  $0 < x < 1$ , we have that  $|R_n| \leq \left| \binom{r}{n} x^n \right| = |a_n| \rightarrow 0$  as  $n \rightarrow \infty$ .

So the claim is proved in the range  $0 \leq x < 1$ . If  $-1 < x < 0$ , the argument above breaks, but Cauchy's form for  $R_n$  works.

$$\begin{aligned} R_n &= \frac{(1-\theta)^{n-1} r(r-1) \cdots (r-n+1) (1+\theta h)^{r-n} x^n}{(n-1)!} \\ &= \frac{r(r-1) \cdots (r-n+1)}{(n-1)!} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^n \\ &= r \binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \left( \frac{1-\theta}{1+\theta x} \right)^{n-1}. \end{aligned}$$

So  $|R_n| \leq \left| r \binom{r-1}{n-1} x^n \right| (1+\theta x)^{r-1}$ . Check that  $(1+\theta x)^{r-1} \leq \max\{1, (1+x)^{r-1}\}$ . Let  $K_r = |r| \max\{1, (1+x)^{r-1}\}$  is independent of  $n$ . So we have

$$|R_n| \leq |K_r| \left| \binom{r-1}{n-1} x^n \right| \rightarrow 0.$$

So  $R_n \rightarrow 0$  as  $n \rightarrow \infty$ . ■

### 4.3 Remarks on Complex Differentiation

Formally, for function  $f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$ , we have properties for sums, products, chain rule etc. But it is much more restrictive than differentiability on the real line.

**Example.**  $f : \mathbb{C} \rightarrow \mathbb{C}$ , with  $z \mapsto \bar{z}$ . We consider the sequence  $z_n = z + \frac{1}{n} \rightarrow z$ .

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1.$$

If we approach it vertically instead, taking  $z_n = z + \frac{i}{n} \rightarrow z$ , we have

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1.$$

So  $\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z}$  does not exist.  $f$  is nowhere  $\mathbb{C}$ -differentiable.

If we consider it as a function on  $\mathbb{R}^2$ ,  $f(x, y) = (x, -y)$ . It is real differentiable.

In fact, if a function is complex differentiable, it is infinitely complex differentiable. It is discussed in more detail in IB Complex Analysis.

## Lecture 14

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## 5 Power Series

We want to look at

$$\sum_{n=0}^{\infty} a_n z^n,$$

with  $z \in \mathbb{C}, a_n \in \mathbb{C}$ . The case  $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ , with  $z$  fixed can be reduced to power series around 0 by translation.

### Lemma 5.1

If  $\sum_{n=0}^{\infty} a_n z_1^n$  converges and  $|z| < |z_1|$ , then  $\sum_{n=0}^{\infty} a_n z^n$  converges absolutely.

*Proof.* Since  $\sum_{n=0}^{\infty} a_n z_1^n$  converges,  $a_n z_1^n \rightarrow 0$ . Thus, there exists  $K > 0$  such that  $|a_n z_1^n| \leq K$  for all  $n$ .

Then,

$$|a_n z^n| = |a_n z_1^n| \frac{|z_1^n|}{|z_1^n|} \leq K \left| \frac{z}{z_1} \right|^n.$$

Since the geometric series  $\sum_{n=0}^{\infty} \left| \frac{z}{z_1} \right|^n$  converges, the lemma follows by comparison. ■

Using this lemma, we will prove that every power series has a *radius of convergence*.

### Theorem 5.1

A power series either

1. converges absolutely for all  $z$ , or
2. converges absolutely for all  $z$  inside a circle  $|z| = R$  and diverges for all  $z$  outside it, or
3. converges for  $z = 0$  only.

**Definition 5.1**

The circle  $|z| = R$  is called the *circle of convergence* and  $R$  the *radius of convergence*.

In (1) of Theorem 5.1, we agree that  $R = \infty$ , and in (3)  $R = 0$ , so  $R \in [0, \infty]$ .

*Proof.* Let  $S = \{x \in \mathbb{R} \mid x \geq 0, \sum a_n x^n \text{ converges}\}$ . Clearly  $0 \in S$ . By (5.1), if  $x_1 \in S$ , then  $[0, x_1] \subseteq S$ . If  $S = [0, \infty)$ , we have case 1.

Otherwise, there exists a finite supremum for  $S$ .  $R = \sup S < \infty, R \geq 0$ . If  $R > 0$ , we'll prove that if  $|z_1| < R$ , then  $\sum a_n z_1^n$  converges absolutely. Pick  $R_0$  such that  $|z_1| < R_0 < R$ , then  $R_0 \in S$  and the series converges for  $z = R_0$ . By (5.1),  $\sum |a_n z_1^n|$  converges.

Finally, we show that if  $|z_2| > R$ , then the series does not converge for  $z_2$ . Pick  $R < R_0 < |z_2|$ . If  $\sum a_n z_2^n$  converges, by (5.1),  $\sum a_n R_0^n$  would be convergent, which contradicts that  $R = \sup S$ . ■

The following lemma is useful for computing  $R$ .

**Lemma 5.2**

If  $\left| \frac{a_{n+1}}{a_n} \right| \rightarrow \ell$ , as  $n \rightarrow \infty$ , then  $R = \frac{1}{\ell}$ .

*Proof.* By the ratio test, we have absolute convergence if

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| = \ell |z| < 1.$$

So if  $|z| < \frac{1}{\ell}$ , we have absolute convergence. If  $|z| > \frac{1}{\ell}$ , the series diverges, again by the ratio test. ■

**Remark.** One can also use the Root Test to get that if  $|a_n|^{\frac{1}{n}} \rightarrow \ell$ , then  $R = \frac{1}{\ell}$ .

**Example.**

$$1. \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \rightarrow 0 = \ell.$$

So  $R = \infty$ .

2. Geometric Series,  $\sum_{n=0}^{\infty} z^n$ .

By ratio test, we have  $R = 1$ . Note that at  $|z| = 1$ , we have divergence.

3.  $\sum_{n=0}^{\infty} n!z^n$ .

We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!}{n!} = n+1 \rightarrow \infty.$$

So  $R = 0$ .

4.  $\sum_{n=1}^{\infty} \frac{z^n}{n}$ , and we have  $R = 1$ . For  $z = 1$ , it diverges. What happens for  $|z| = 1$  and  $z \neq 1$ ?

Consider  $\sum_{n=1}^{\infty} \frac{z^n}{n} (1 - z)$ , we have the partial sum

$$\begin{aligned} S_N &= \sum_{n=1}^N \left( \frac{z^n - z^{n+1}}{n} \right) \\ &= \sum_{n=1}^N \frac{z^n}{n} - \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} \\ &= \sum_{n=1}^N \frac{z^n}{n} - \sum_{n=2}^{N+1} \frac{z^n}{n-1} \\ &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^N z^n \left( \frac{1}{n} - \frac{1}{n-1} \right) \\ &= z - \frac{z^{N+1}}{N} + \sum_{n=2}^N -\frac{z^n}{n(n-1)}. \end{aligned}$$

If  $|z| = 1$ ,  $\frac{z^{N+1}}{N} \rightarrow 0$  as  $N \rightarrow \infty$  and  $\sum \frac{1}{n(n-1)}$  converges, so  $S_N$  converges.

5.  $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ , and we have  $R = 1$ . But it converges for all  $z$  with  $|z| = 1$ .

Conclusion is that, in principle, nothing can be said about  $|z| = R$  and each case has to be discussed separately.

Within the radius of convergence, “life is great”. Power series behave as if “they were polynomials”.

## Lecture 15

23 Feb. 2022

### Theorem 5.2

$f(z) = \sum_{n=0}^{\infty} a_n z^n$  has radius convergence  $R$ . Then  $f$  is differentiable at all points with  $|z| < R$  with

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

*Proof (non examinable).*

Watch the lecture and finish the proof.



## 5.1 The Standard Functions

In this section, we will discuss exponential, logarithmic, trigonometric, etc.

We have already seen that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has  $R = \infty$ . Define  $e : \mathbb{C} \rightarrow \mathbb{C}$ ,  $z \mapsto \sum_{n=0}^{\infty} \frac{z^n}{n!}$ . From Theorem 5.2,  $e$  is differentiable, and  $e'(z) = e(z)$ .

If  $F : \mathbb{C} \rightarrow \mathbb{C}$  has  $F'(z) = 0$  for all  $z \in \mathbb{C}$ , then  $F$  is constant.

*Proof.* Consider  $g(t) = F(tz)$ , and chain rule gives  $g'(t) = F'(tz)z = 0$ . If  $g(t) = u(t) + iv(t)$ . It is immediate that  $g'(t) = u'(t) + iv'(t)$ . So  $u'(t) = v'(t) = 0$ . By



previously proved corollary, we have  $u(t), v(t)$  constant. Thus,  $F(z)$  is constant. ■

Now let  $a, b \in \mathbb{C}$ . Consider

$$F(z) = e(a + b - z)e(z).$$

We have  $F'(z) = 0$ , so  $F$  is constant.

$$e(a + b - z)e(z) = F(0) = e(a + b).$$

Setting  $z = b$  gives

$$e(a)e(b) = e(a + b).$$

## Lecture 16

25 Feb. 2022

Now we restrict to  $\mathbb{R}$ ,  $e : \mathbb{R} \rightarrow \mathbb{R}$ .

### Theorem 5.3

1.  $e : \mathbb{R} \rightarrow \mathbb{R}$  is everywhere differentiable and  $e'(x) = e(x)$ .
2.  $e(x + y) = e(x)e(y)$ .
3.  $e(x) > 0$  for all  $x \in \mathbb{R}$ .
4.  $e$  is strictly increasing.
5.  $e(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , and  $e(x) \rightarrow 0$  as  $x \rightarrow -\infty$ .
6.  $e : \mathbb{R} \rightarrow (0, \infty)$  is a bijection.

*Proof.* We proved (1) and (2) above.

To prove (3), clearly we have  $e(x) > 0$  for  $x \geq 0$  and  $e(0) = 1$ . Also,  $e(0) = e(x - x) = e(x)e(-x) = 1$ . So  $e(-x) > 0$  for  $x > 0$ .

To prove (4), we have  $e'(x) = e(x) > 0$  so  $e$  is strictly increasing.

To prove (5),  $e(x) > 1 + x$  for  $x > 0$ . So if  $x \rightarrow \infty$ , clearly  $e(x) \rightarrow \infty$ . For  $x > 0$ , since  $e(-x) = \frac{1}{e(x)}$ , then  $e(x) \rightarrow 0$  as  $x \rightarrow -\infty$ .

For (6), injectivity follows right away from being strictly increasing. To prove surjectivity, take any  $y \in (0, \infty)$ . Since  $e(x) \rightarrow \infty$  as  $x \rightarrow \infty$  and  $e(x) \rightarrow 0$  as  $x \rightarrow -\infty$ . We can find  $a, b$  such that  $e(a) < y < e(b)$ . By the IVT, there exists  $x \in \mathbb{R}$  such that  $e(x) = y$ . ■

**Remark.**  $e : (\mathbb{R}, +) \rightarrow ((0, \infty), \cdot)$  is a group isomorphism.

Since  $e$  is a bijection, we have an inverse  $\ell : (0, \infty) \rightarrow \mathbb{R}$ .

#### Theorem 5.4

1.  $\ell : (0, \infty) \rightarrow \mathbb{R}$  is a bijection and  $\ell(e(x)) = x$  for all  $x \in \mathbb{R}$  and  $e(\ell(t)) = t$  for all  $t \in (0, \infty)$ .
2.  $\ell$  is differentiable and  $\ell'(t) = \frac{1}{t}$ .
3.  $\ell(xy) = \ell(x) + \ell(y)$  for all  $x, y \in (0, \infty)$ .

*Proof.*

1. Obvious from the definition of  $e$ .
2. Inverse rule gives that  $\ell$  is differentiable and

$$\ell'(t) = \frac{1}{e'(\ell(t))} = \frac{1}{t}.$$

3. From IA Groups, if  $e$  is an isomorphism, so is its inverse.

■

Now define for  $\alpha \in \mathbb{R}$  and  $x > 0$ ,

$$r_\alpha(x) = e(\alpha \ell(x)).$$

**Theorem 5.5**

Suppose  $x, y > 0$ , and  $\alpha, \beta \in \mathbb{R}$ , then

1.  $r_\alpha(xy) = r_\alpha(x)r_\alpha(y)$ ;
2.  $r_{\alpha+\beta} = r_\alpha(x)r_\beta(x)$ ;
3.  $r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$ ;
4.  $r_1(x) = x$  and  $r_0(x) = 1$ .

*Proof.*

1.  $r_\alpha(xy) = e(\alpha\ell(xy))$   
 $= e(\alpha\ell(x) + \alpha\ell(y))$   
 $= e(\alpha\ell(x))e(\alpha\ell(y))$   
 $= r_\alpha(x)r_\alpha(y).$
2.  $r_{\alpha+\beta}(x) = e((\alpha + \beta)\ell(x))$   
 $= e(\alpha\ell(x))e(\beta\ell(x))$   
 $= r_\alpha(x)r_\beta(x).$
3.  $r_\alpha(r_\beta(x)) = r_\alpha(e(\beta\ell(x)))$   
 $= e(\alpha\ell \circ e(\beta\ell(x)))$   
 $= e(\alpha\beta\ell(x))$   
 $= r_{\alpha\beta}(x).$
4.  $r_1(x) = e(\ell(x)) = x$  and  $r_0(x) = e(0\ell(x)) = e(0) = 1$

■

For some  $n \in \mathbb{Z}_{\geq 1}$ , then

$$r_n(x) = r_{\underbrace{1 + \dots + 1}_n}(x) = \underbrace{x \cdots x}_n = x^n.$$

We also have

$$r_1(x)r_{-1}(x) = r_0(x) = 1 \implies r_{-1}(x) = \frac{1}{x}.$$

So  $r_{-n}(x) = \frac{1}{x^n}$ . Next we consider for  $q \in \mathbb{Z}_{\geq 1}$ ,

$$(r_{\frac{1}{q}})^q = r_1(x) = x \implies r_{\frac{1}{q}}(x) = x^{\frac{1}{q}}.$$

And thus we also have  $r_{\frac{p}{q}}(x) = (r_{\frac{1}{q}}(x))^p = x^{\frac{p}{q}}$ . Thus,  $r_\alpha(x)$  agrees with  $x^\alpha$  when  $\alpha \in \mathbb{Q}$  as previously defined.

Now we give the above functions names:

1.  $\exp(x) = e(x)$  for  $x \in \mathbb{R}$ ;
2.  $\log x = \ell(x)$  for  $x \in (0, \infty)$ ;
3.  $x^\alpha = r_\alpha(x)$  for  $\alpha \in \mathbb{R}, x \in (0, \infty)$ .

If  $e(x) = e(x \log e)$  where  $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ , we have  $e(x) = r_x(e) = e^x$ .  $\exp(x)$  is also a power, which we may as well write as  $e^x$ .

Finally, we compute

$$\begin{aligned} (x^\alpha)' &= (e^{\alpha \log x})' \\ &= e^{\alpha \log x} \alpha \frac{1}{x} \\ &= \alpha x^{\alpha-1}. \end{aligned}$$

And we have

$$(a^x)' = (e^{x \log a})' = e^{x \log a} \log a = a^x \log a.$$

## Lecture 17: Trigonometric Functions

28 Feb. 2022

**Remark** ("Exponentials beat polynomials"). We have

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^k} = \infty. \quad (k > 0)$$

We use power series to prove it. We have

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} > \frac{x^n}{n!}. \quad (x > 0)$$

Now we pick  $n > k$ , and we have

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \rightarrow \infty \text{ as } x \rightarrow \infty.$$

## Definition 5.2: Trigonometric Functions

$$\begin{aligned}\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \\ \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\end{aligned}$$

Both power series have infinite radius of convergence and by Theorem 5.2, we get

$$(\sin z)' = \cos z, \quad (\cos z)' = -\sin z.$$

Also note that we have

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \sum_{n=1}^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_{n=1}^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!} = \cos z + i \sin z.$$

Similarly,

$$e^{-iz} = \cos z - i \sin z.$$

So we can write  $\cos z$  and  $\sin z$  as

$$\begin{aligned}\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\ \sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}).\end{aligned}$$

From this, we get many trigonometric identities. For example,

$$\cos z = \cos(-z), \quad \sin(-z) = -\sin z, \quad \cos(0) = 1, \quad \sin(0) = 0.$$

And the addition formulas

1.  $\sin(z + w) = \sin z \cos w + \cos z \sin w;$
2.  $\cos(z + w) = \cos z \cos w - \sin z \sin w.$

They essentially follow from  $e^{a+b} = e^a e^b$ . To prove (2), write

$$\cos(z + w) = \frac{1}{2}(e^{i(z+w)} + e^{-i(z+w)}),$$

and expand  $\cos z \cos w - \sin z \sin w$  similarly to get the result. We can also get

$$\sin^2 z + \cos^2 z = 1 \quad \forall z \in \mathbb{C} \tag{1}$$

by direct computation.

Now if  $x \in \mathbb{R}$ , then  $\sin x, \cos x \in \mathbb{R}$ , and eq. (1) gives  $|\sin x|, |\cos x| \leq 1$ . We should be careful that they don't have to be bounded when  $z$  is not real. For example

$$\cos(iy) = \frac{1}{e^{-y} + e^y}, \quad y \in \mathbb{R}$$

So  $\cos(iy) \rightarrow \infty$  as  $y \rightarrow \infty$ .

#### Proposition 5.1

There is a smallest positive number  $w$  (where  $\sqrt{2} < \frac{w}{2} < \sqrt{3}$ ) such that

$$\cos \frac{w}{2} = 0.$$

*Proof.* If  $0 < x < 2$ ,

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

since  $0 < x < 2 \implies \frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!}$ , and  $(\cos x)' = -\sin x < 0$  for  $0 < x < 2$ . So  $\cos x$  is strictly decreasing on  $(0, 2)$ . We will show that  $\cos \sqrt{2} > 0$  and  $\cos \sqrt{3} < 0$ . Then by the intermediate value theorem, the existence of  $w$  follows. Now we prove that.

$$\begin{aligned} \cos \sqrt{2} &= \left(\frac{(\sqrt{2})^2}{4!} - \frac{(\sqrt{2})^6}{6!}\right) + \cdots > 0 \\ \cos \sqrt{3} &= 1 - \frac{3}{2} + \frac{9}{4!} - \left(\frac{x^6}{6!} - \frac{x^8}{8!}\right) - \cdots = -\frac{1}{8} - \cdots < 0, \end{aligned}$$

and we are done. ■

#### Corollary 5.1

$$\sin \frac{w}{2} = 1.$$

*Proof.* We have  $\sin^2 \frac{w}{2} + \cos^2 \frac{w}{2} = 0$ , and we know that  $\sin$  is positive on  $(0, 2)$ . ■

Now we define  $\pi = w$ .

#### Theorem 5.6

1.  $\sin(z + \frac{\pi}{2}) = \cos z, \cos(z + \frac{\pi}{2}) = -\sin z;$
2.  $\sin(z + \pi) = -\sin z, \cos(z + \pi) = -\cos z;$
3.  $\sin(z + 2\pi) = \sin z, \cos(z + 2\pi) = \cos z.$

*Proof.* It is immediate from addition formulas and  $\cos \frac{\pi}{2} = 0, \sin \frac{\pi}{2} = 1$ . ■

This implies

$$\begin{aligned} e^{iz+2\pi i} &= \cos(z + 2\pi) + i \sin(z + 2\pi) \\ &= \cos z + i \sin z \\ &= e^{iz}. \end{aligned}$$

So  $e^z$  is periodic with period  $2\pi i$ .

**Remark** ("Relation with geometry"). Given two vectors  $x, y \in \mathbb{R}^2$ , define  $x \cdot y$  as in Part IA Vector and Matrices,

$$x \cdot y = x_1 y_1 + x_2 y_2.$$

Cauchy-Schwarz gives  $|x \cdot y| \leq \|x\| \|y\|$ . So, for  $x \neq 0, y \neq 0$ ,

$$-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1.$$

Define the angle between  $x$  and  $y$  as the unique  $\theta \in [0, \pi]$  with  $\cos \theta = \frac{x \cdot y}{\|x\| \|y\|}$ . And we recover the unit circle picture by defining angle this way.

## Lecture 18: Hyperbolic Functions

2 Mar. 2022

#### Definition 5.3

$$\begin{aligned} \cosh z &= \frac{1}{2}(e^z + e^{-z}) \\ \sinh z &= \frac{1}{2}(e^z - e^{-z}) \end{aligned}$$

We clearly have the relationships

$$\begin{aligned}\cosh z &= \cos(iz) \\ \sinh z &= -i \sin(iz) \\ (\cosh z)' &= \sinh z \\ (\sinh z)' &= \cosh z \\ \cosh^2 z - \sinh^2 z &= 1.\end{aligned}$$

The rest of the trigonometric functions (tan, cot, sec, csc) are defined in the usual way.

## 6 Integration

Suppose we have a function  $f : [a, b] \rightarrow \mathbb{R}$  bounded. That is  $\exists K$  such that  $|f(x)| \leq K$  for all  $x \in [a, b]$ .

### Definition 6.1

A *dissection* (or *partition*)  $\mathcal{D}$  of  $[a, b]$  is a finite subset of  $[a, b]$  containing the end points  $a$  and  $b$ . We write

$$\mathcal{D} = \{x_0, x_1, \dots, x_n\}$$

with  $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ .

A picture of the function

### Definition 6.2

We define the *upper sum* and *lower sum* associated with  $\mathcal{D}$  by

$$\begin{aligned}S(f, \mathcal{D}) &= \sum_{j=1}^n (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) && \text{(upper)} \\ \mathcal{S}(f, \mathcal{D}) &= \sum_{j=1}^n (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) && \text{(lower)}\end{aligned}$$

Clearly,  $\mathcal{S}(f, \mathcal{D}) \leq S(f, \mathcal{D})$  for any  $\mathcal{D}$ .



## Lemma 6.1

If  $\mathcal{D}, \mathcal{D}'$  are dissections with  $\mathcal{D}' \supseteq \mathcal{D}$ , then

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D}') \geq \mathcal{S}(f, \mathcal{D}') \geq \mathcal{S}(f, \mathcal{D}).$$

*Proof.*  $S(f, \mathcal{D}') \geq \mathcal{S}(f, \mathcal{D}')$  is obvious.

Suppose  $\mathcal{D}'$  contains an extra point than  $\mathcal{D}$ , let's say  $y \in (x_{r-1}, x_r)$ . Clearly,

$$\sup_{x \in [x_{r-1}, y]} f(x), \sup_{x \in [y, x_r]} f(x) \leq \sup_{x \in [x_{r-1}, x_r]} f(x).$$

So we have

$$(x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x) \geq (y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} f(x).$$

And the same argument goes for  $\mathcal{S}$  and the same if  $\mathcal{D}'$  has more extra points than  $\mathcal{D}$ . ■

## Lemma 6.2

Suppose  $\mathcal{D}_1, \mathcal{D}_2$  are two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathcal{S}(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq \mathcal{S}(f, \mathcal{D}_2).$$

So  $S(f, \mathcal{D}_1) \geq \mathcal{S}(f, \mathcal{D}_2)$ .

*Proof.* Take  $\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$  and apply the previous lemma. ■

## Definition 6.3

The *upper integral* of  $f$  is  $I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$ .

The *lower integral* of  $f$  is  $I_*(f) = \sup_{\mathcal{D}} \mathcal{S}(f, \mathcal{D})$ .

The supremum, infimum always exists by Lemma 6.2 and picking an arbitrary  $\mathcal{D}_1, \mathcal{D}_2$  respectively.

And by Lemma 6.2,

$$I^*(f) \geq I_*(f)$$

because

$$\begin{aligned} S(f, \mathcal{D}_1) &\geq \mathcal{S}(f, \mathcal{D}_2) \\ I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_1) &\geq \mathcal{S}(f, \mathcal{D}_2) \\ I^*(f) &\geq \sup_{\mathcal{D}_2} \mathcal{S}(f, \mathcal{D}_2) = I_*(f). \end{aligned}$$

#### Definition 6.4

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is said to be *Riemann integrable* (or just *integrable*) if  $I^*(f) = I_*(f)$ .

And we set

$$\int_a^b f(x) \, dx = I^*(f) = I_*(f) = \int_a^b f.$$

**Example.** The function  $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 0, & \text{if } x \notin \mathbb{Q} \cap [0, 1] \end{cases}$  is not Riemann integrable since

$$\begin{aligned} \sup_{[x_{j-1}, x_j]} f(x) &= 1, & \inf_{[x_{j-1}, x_j]} f(x) &= 0 \\ \implies S(f, \mathcal{D}) &= 1, & \mathcal{S}(f, \mathcal{D}) &= 0 \quad \forall \mathcal{D} \end{aligned}$$

So  $I^*(f) = 1$ , but  $I_*(f) = 0$ .

## Lecture 19

4 Mar. 2022

We first develop a useful criterion for integrability.

#### Theorem 6.1

A bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if and only if given  $\epsilon > 0$ ,  $\exists \mathcal{D}$  such that

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) < \epsilon.$$

*Proof.* For every dissection  $\mathcal{D}$ , we have

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}).$$

If the given condition holds, then for all  $\epsilon > 0$ , we can find  $\mathcal{D}$  such that

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) < \epsilon.$$

So  $I^*(f) = I_*(f)$ .

Conversely, if  $f$  is integrable, by definition of sup and inf, there are partitions  $\mathcal{D}_1$  and  $\mathcal{D}_2$  such that

$$\begin{aligned} \int_a^b f \, dx - \frac{\epsilon}{2} &= I_*(f) - \frac{\epsilon}{2} < \mathcal{S}(f, \mathcal{D}_1) \\ \int_a^b f \, dx + \frac{\epsilon}{2} &= I^*(f) + \frac{\epsilon}{2} > S(f, \mathcal{D}_2) \end{aligned}$$

By Lemma 6.2,

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) - \mathcal{S}(f, \mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f, \mathcal{D}_2) - \mathcal{S}(f, \mathcal{D}_1) < \epsilon.$$

■

We now use this criterion to show that monotone and continuous functions are integrable.

**Remark.** Monotone and continuous functions are bounded.

#### Theorem 6.2

Let  $f : [a, b] \rightarrow \mathbb{R}$  be monotone. Then  $f$  is Riemann integrable.

*Proof.* Suppose without loss of generality that  $f$  is increasing (same proof for  $f$  decreasing), then  $\sup_{x \in [x_{j-1}, x_j]} f(x) = f(x_j)$  and  $\inf_{x \in [x_{j-1}, x_j]} f(x) = f(x_{j-1})$ . Thus,

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) (f(x_j) - f(x_{j-1})).$$

Now choose

$$\mathcal{D} = \left\{ a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b \right\}.$$

In other words,  $x_j = a + \frac{(b-a)j}{n}$  for  $0 \leq j \leq n$ . So

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) = \frac{b-a}{n} (f(b) - f(a)).$$

Take  $n$  large enough such that  $\frac{b-a}{n} (f(b) - f(a)) < \epsilon$  and use Theorem 6.1. ■

To prove that of continuous functions, we need an auxiliary lemma.

### Lemma 6.3

For  $f : [a, b] \rightarrow \mathbb{R}$  continuous. Given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ . (*uniform continuity*)

The point is that  $\delta$  works for all  $x, y$  as long as  $|x - y| < \delta$ . (in the definition of continuity of  $f$  at  $x$ , the  $\delta$  depends on  $x$ )

*Proof.* Suppose the claim is false. Then  $\exists \epsilon > 0$  such that  $\forall \delta > 0$ , we can find  $x, y \in [a, b]$  such that  $|x - y| < \delta$ , but  $|f(x) - f(y)| \geq \epsilon$ .

Take  $\delta = \frac{1}{n}$  to get  $x_n, y_n \in [a, b]$  with  $|x_n - y_n| < \frac{1}{n}$ , but  $|f(x_n) - f(y_n)| \geq \epsilon$ . By Bolzano-Weierstrass theorem, we know there exists  $x_{n_k} \rightarrow c \in [a, b]$ , and

$$|y_{n_k} - c| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - c| \rightarrow 0.$$

So  $y_{n_k} \rightarrow c$ . But  $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$ . Let  $k \rightarrow \infty$ , by continuity of  $f$ ,

$$|f(c) - f(c)| \geq \epsilon.$$

Contradiction. ■

### Theorem 6.3

Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous, then  $f$  is Riemann integrable.

*Proof.* By Lemma 6.3, given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ . Let  $\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b\}$ . Choose  $n$  large enough such that  $\frac{b-a}{n} < \delta$ , then for  $x, y \in [x_{j-1}, x_j]$ ,  $|f(x) - f(y)| < \epsilon$ , since  $|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$ . Observe that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j). \quad p_j, q_j \in [x_{j-1}, x_j]$$

Minimum and maximum are achieved due to continuity. So

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1})(f(p_j) - f(q_j)) < \epsilon(b-a).$$

And we are done by Theorem 6.1. ■

## Lecture 20

7 Mar. 2022

Functions more complicated than monotone or continuous can be Riemann integrable.

**Example.** Consider  $f:[0,1] \rightarrow \mathbb{R}$ . Clearly,  

$$x \mapsto \begin{cases} 1/q, & \text{if } x = p/q \in (0,1] \text{ in its lowest term} \\ 0, & \text{otherwise} \end{cases}$$
 $\mathcal{S}(f, \mathcal{D}) = 0$  for all  $\mathcal{D}$ . We will show that given  $\epsilon > 0$ ,  $\exists \mathcal{D}$  such that  $S(f, \mathcal{D}) < \epsilon$ . This implies that  $f$  is integrable with  $\int_0^1 f \, dx = 0$ .

Consider the set

$$\{x \in [0,1] \mid f(x) \geq \frac{1}{N}\} = \left\{ \frac{p}{q} \mid 1 \leq q \leq N, 1 \leq p \leq q \right\}.$$

Take  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \frac{\epsilon}{2}$ . This is a finite set with

$$0 < t_1 < t_2 < \cdots < t_R = 1.$$

Consider a dissection  $\mathcal{D}$  of  $[a, b]$  such that

1. Each  $t_k$  with  $1 \leq k \leq R$  is in some  $(x_{j-1}, x_j)$ ;
2. For all  $k$ , the unique interval containing  $t_k$  has length at most  $\epsilon/2R$ .

Such dissection clearly exists. Note that  $f \leq 1$  everywhere, and

$$S(f, \mathcal{D}) \leq \frac{1}{N} + \frac{\epsilon}{2} < \epsilon.$$

The function is integrable but has countable many discontinuities.

## 6.1 Elementary Properties of the Integral

Let  $f, g$  be bounded and integrable functions on  $[a, b]$ .

1. If  $f \leq g$  on  $[a, b]$ , then

$$\int_a^b f \leq \int_a^b g.$$

2.  $f + g$  is integrable on  $[a, b]$  and

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

3. For any constant  $k$ ,  $kg$  is integrable and

$$\int_a^b kf = k \int_a^b f.$$

4.  $|f|$  is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

5. The product  $fg$  is integrable.

*Proof.* 1. If  $f \leq g$ , then

$$\begin{aligned} \int_a^b f &= I^*(f) \leq S(f, \mathcal{D}) \leq S(g, \mathcal{D}) \\ \implies \int_a^b f &= I^*(f) \leq I^*(g) = \int_a^b g. \end{aligned}$$

2. We have  $\sup_{[x_{j-1}, x_j]} (f + g) \leq \sup_{[x_{j-1}, x_j]} f + \sup_{[x_{j-1}, x_j]} g$ , so

$$S(f + g, \mathcal{D}) \leq S(f, \mathcal{D}) + S(g, \mathcal{D}).$$

Now for dissections  $\mathcal{D}_1$  and  $\mathcal{D}_2$ ,

$$\begin{aligned} I^*(f + g) &\leq S(f + g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\leq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) + S(g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\leq S(f, \mathcal{D}_1) + S(g, \mathcal{D}_2). \end{aligned}$$

Fix  $\mathcal{D}_1$  and take inf over  $\mathcal{D}_2$  to get

$$I^*(f + g) \leq S(f, \mathcal{D}_1) + I^*(g).$$

Again, take inf over all  $\mathcal{D}_1$ , we have

$$I^*(f + g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g.$$

Similarly,  $\int_a^b f + \int_a^b g \leq I_*(f + g)$ , so  $f + g$  is integrable with the integral equal to the sum of integrals.

3. Exercise.

4. Consider the function  $f_+(x) = \max(f(x), 0)$ , we have

$$\sup_{[x_{j-1}, x_j]} f_+ - \inf_{[x_{j-1}, x_j]} f_+ \leq \sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f.$$

We know that given  $\epsilon > 0$ , there exists  $\mathcal{D}$  such that

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) (\sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f) < \epsilon.$$

By inequality above, we have

$$S(f_+, \mathcal{D}) - \mathcal{S}(f_+, \mathcal{D}) \leq S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) < \epsilon.$$

Note  $|f| = 2f_+ - f$ , and by (2) and (3),  $|f|$  is integrable. Since  $-|f| \leq f \leq |f|$ , and by property (1),  $\left| \int_a^b f \right| \leq \int_a^b |f|$ .

5. Take  $f$  integrable and  $f \geq 0$ . Then

$$\begin{aligned} \sup_{[x_{j-1}, x_j]} f^2 &= (\sup_{[x_{j-1}, x_j]} f)^2 = (M_j)^2 \\ \inf_{[x_{j-1}, x_j]} f^2 &= (\inf_{[x_{j-1}, x_j]} f)^2 = (m_j)^2. \end{aligned}$$

Note  $M_j + m_j < 2K$  for some  $K$  since  $f$  is bounded. Thus,

$$\begin{aligned} S(f^2, \mathcal{D}) - \mathcal{S}(f^2, \mathcal{D}) &= \sum_{j=1}^n (x_j - x_{j-1}) (M_j^2 - m_j^2) \\ &= \sum_{j=1}^n (x_j - x_{j-1}) (M_j + m_j) (M_j - m_j) \\ &\leq 2K (S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D})) \end{aligned}$$

Using Theorem 6.1, we deduce that  $f^2$  is integrable. Now take any  $f$ , then  $|f| \geq 0$ . Since  $f^2 = |f|^2$ , we deduce that  $f^2$  is integrable for any  $f$ . Finally, for  $fg$ , note

$$4fg = (f + g)^2 - (f - g)^2.$$

And we are done since the right-hand side is integrable. ■

## Lecture 21

9 Mar. 2022

We have an additional property.

6. If  $f$  is integrable on  $[a, b]$ . If  $a < c < b$ , then  $f$  is integrable over  $[a, c]$  and  $c, b]$ , and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Conversely, if  $f$  is integrable over  $[a, c]$  and  $[c, b]$ , then  $f$  is integrable over  $[a, b]$ , and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

*Proof.* We first make two observations. If  $\mathcal{D}_1$  is a dissection of  $[a, c]$  and  $\mathcal{D}_2$  is a dissection of  $[c, b]$ , then  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$  is a dissection of  $[a, b]$  and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2).$$

Also, if  $\mathcal{D}$  is a dissection of  $[a, b]$ , then

$$S(f, \mathcal{D}) \geq S(f, \mathcal{D} \cup \{c\}) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2)$$

where  $\mathcal{D}_1$  dissects  $[a, c]$  and  $\mathcal{D}_2$  dissects  $[c, b]$ . From the first inequality, we have

$$I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

From the second inequality, we have

$$I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

So, we have

$$I^*(f) = I^*(f|_{[a,c]}) + I^*(f|_{[c,b]}).$$

Similarly, the lower integral

$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]}).$$

Thus,

$$0 \leq I^*(f) - I_*(f) = (I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})) + (I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})).$$

From this (6) follows right away. ■

We have the convention that if  $a > b$ , then

$$\int_a^b f = - \int_b^a f;$$

If  $a = b$ , we agree that its value is zero. With this convention, if  $|f| \leq K$ ,

$$\left| \int_a^b f \right| \leq K|b - a|.$$



## 6.2 The Fundamental Theorem of Calculus

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a bounded and integrable function, and we write

$$F(x) = \int_a^x f(t) \, dt. \quad x \in [a, b]$$

### Theorem 6.4

$F$  is continuous.

*Proof.* Consider the difference  $F(x+h) - F(x) = \int_x^{x+h} f(t) \, dt$ . We have

$$\begin{aligned} |F(x+h) - F(x)| &= \left| \int_x^{x+h} f(t) \, dt \right| \\ &\leq K|h| \end{aligned}$$

if  $|f(t)| \leq K$  for all  $t \in [a, b]$ . Now let  $h \rightarrow 0$  and we are done. (In fact,  $F$  is Lipschitz continuous) ■

### Theorem 6.5: FTC

If in addition  $f$  is continuous at  $x$ , then  $F$  is differentiable at  $x$ , and

$$F'(x) = f(x).$$

*Proof.* We consider

$$\begin{aligned} &\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \quad x+h \in [a, b], h \neq 0 \\ &= \frac{1}{|h|} \left| \int_x^{x+h} f(t) \, dt - hf(x) \right| \\ &= \frac{1}{|h|} \left| \int_x^{x+h} f(t) - f(x) \, dt \right|. \end{aligned}$$

Since  $f$  is continuous at  $x$ , given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $|t - x| < \delta \implies |f(t) - f(x)| < \epsilon$ . So if  $|h| < \delta$ , we have

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \leq \frac{1}{|h|} \epsilon |h| = \epsilon.$$

That is,

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

■

**Example.** Consider the function  $f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in (0, 1] \end{cases}$ . Taking the integral of  $f$ , and we have  $F(x) = -1 + |x|$ . It is differentiable everywhere except for  $x = 0$  where there is a discontinuity on  $f$ .

#### Corollary 6.1: Integration is the inverse of differentiation

If  $f = g'$  is continuous on  $[a, b]$ , then

$$\int_a^x f(t) \, dt = g(x) - g(a). \quad \forall x \in [a, b].$$

*Proof.* From Theorem 6.5,  $F - g$  has zero derivative in  $[a, b]$ . So  $F - g$  is constant, and since  $F(a) = 0$ , we have  $F(x) = g(x) - g(a)$ . ■

Every continuous function has an *indefinite integral* or *antiderivative* written  $\int f(x) \, dx$  which is determined up to a constant.

**Remark.** We have solved the ODE

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}.$$

## Lecture 22

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#### Corollary 6.2: Integration by Parts

Suppose  $f'$  and  $g'$  exist and are continuous on  $[a, b]$ , then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'.$$

*Proof.* By the product rule,

$$(fg)' = f'g + fg'.$$

By Theorem 6.5,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'.$$

■

### Corollary 6.3: Integration by Substitution

Let  $g : [\alpha, \beta] \rightarrow [a, b]$  be a continuous function with  $g(\alpha) = a$  and  $g(\beta) = b$ , and  $g'$  exists and is continuous on  $[\alpha, \beta]$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then

$$\int_a^b f(x) \, dx = \int_\alpha^\beta f(g(t))g'(t) \, dt.$$

*Proof.* Set  $F(x) = \int_a^x f(t) \, dt$ , and let  $h(t) = F(g(t))$ . (defined as  $g$  takes values in  $[a, b]$ ) Then we have

$$\begin{aligned} \int_\alpha^\beta f(g(t))g'(t) \, dt &= \int_\alpha^\beta F'(g(t))g'(t) \, dt \\ &= \int_\alpha^\beta h'(t) \, dt \\ &= h(\beta) - h(\alpha) \\ &= F(b) - F(a) \\ &= \int_a^b f(x) \, dx. \end{aligned}$$

■

### Theorem 6.6: Taylor's theorem with remainder an integral

Let  $f^{(n)}(x)$  be continuous for  $x \in [0, h]$ . Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) \, dt.$$

*Proof.* Substitute  $u = th$ ,

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u) \, du.$$

Integrating by parts, we get the following

$$R_n = \frac{-h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u) \, du.$$

If we integrate by parts  $n-1$  times, we arrive at

$$\begin{aligned} R_n &= -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \dots - hf'(0) + \int_0^h f'(u) \, du \\ &= -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \dots - hf'(0) + f(h) - f(0). \end{aligned}$$

■

Now we can get the Cauchy and Lagrange form of the remainder. However, note that the proof above uses continuity of  $f^{(n)}$  not just the existence as we proved before.

#### Theorem 6.7

Suppose  $f, g : [a, b] \rightarrow \mathbb{R}$  be continuous functions with  $g(x) \neq 0$  for all  $x \in (a, b)$ . Then there exists  $c \in (a, b)$  such that

$$\int_a^b f(x)g(x) \, dx = f(c) \int_a^b g(x) \, dx.$$

**Note.** If we take  $g(x) = 1$ , we get

$$\int_a^b f(x) \, dx = f(c)(b-a).$$

*Proof.* We are going to use Cauchy's MVT. (Theorem 4.5) Let

$$F(x) = \int_a^x fg, \quad G(x) = \int_a^x g.$$

Theorem 4.5 proves the existence of  $c \in (a, b)$  such that

$$\begin{aligned} (F(b) - F(a))G'(c) &= F'(c)(G(b) - G(a)) \\ \left(\int_a^b fg\right)g(c) &= f(c)g(c) \int_a^b g \end{aligned}$$

Since  $g(c) \neq 0$ , we simplify and get the required form. ■

Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) dt.$$

First we use Theorem 6.7 with  $g(x) = 1$ , to get

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h)$$

with some  $\theta \in (0,1)$  which is the Cauchy's form of remainder. To get Lagrange's form, we use Theorem 6.7 with  $g(t) = (1-t)^{n-1}$  which is greater than 0 for  $t \in (0,1)$ . We know there exists  $\theta \in (0,1)$  such that

$$\begin{aligned} R_n &= \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \int_0^1 (1-t)^{n-1} dt \\ &= \frac{h^n}{n!} f^{(n)}(\theta h) \end{aligned}$$

which is the Lagrange's form of the remainder.

## Lecture 23

14 Mar. 2022

### 6.3 Improper Integrals

#### Definition 6.5

Suppose  $f : [a, \infty) \rightarrow \mathbb{R}$  is integrable (and bounded) on every interval  $[a, R]$  and that as  $R \rightarrow \infty$ ,

$$\int_a^R f(x) dx \rightarrow \ell.$$

Then we say that  $\int_a^\infty f(x) dx$  *exists* or *converges* and that its value is  $\ell$ . If  $\int_a^R f(x) dx$  does not tend to a limit, we say that  $\int_a^\infty f(x) dx$  *diverges*.

A similar definition applies to  $\int_{-\infty}^a f(x) dx$

If  $\int_a^\infty f = \ell_1$  and  $\int_{-\infty}^a f = \ell_2$ , we write  $\int_{-\infty}^\infty f = \ell_1 + \ell_2$ . (independent of the particular value of  $a$ )

**Remark.** The last part is not the same as saying that  $\lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$  exists. It is stronger since  $\int_{-R}^R x dx = 0$ .

**Example.** The integral  $\int_1^\infty \frac{dx}{x^k}$  converges if and only if  $k > 1$ .

Indeed, if  $k \neq 1$ ,

$$\int_1^R \frac{dx}{x^k} = \frac{x^{1-k}}{1-k} \Big|_1^R = \frac{R^{1-k} - 1}{1-k}$$

and as  $R \rightarrow \infty$ , this limit is finite if and only if  $k > 1$ . If  $k = 1$ ,

$$\int_1^R \frac{dx}{x} = \log R \rightarrow \infty.$$

**Remark.** 1.  $1/\sqrt{x}$  is continuous on  $[\delta, 1]$  for any  $\delta > 0$ , and

$$\int_\delta^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_\delta^1 = 2 - 2\sqrt{\delta} \rightarrow 2$$

as  $\delta \rightarrow 0$ . The function is unbounded on  $(0, 1]$ , but the limit of the integral exists.

However,

$$\int_0^1 \frac{dx}{x} = \lim_{\delta \rightarrow 0} \int_\delta^1 \frac{dx}{x} = \lim_{\delta \rightarrow 0} \log x \Big|_\delta^1 = \lim_{\delta \rightarrow 0} (\log 1 - \log \delta)$$

does not exist.

2. If  $f \geq 0$  and  $g \geq 0$  for  $x \geq a$ , and

$$f(x) \leq Kg(x), \quad \forall x \geq a$$

Then  $\int_a^\infty g$  converges implies  $\int_a^\infty f$  converges and  $\int_a^\infty f \leq K \int_a^\infty g$ .

Just note that  $\int_a^R f \leq K \int_a^R g$ . The function  $R \mapsto \int_a^R f$  is increasing because  $f \geq 0$ , and bounded above because  $\int_a^\infty g$  converges. Take  $\ell = \sup_{R \geq a} \int_a^R f < \infty$ . Now check that

$$\lim_{R \rightarrow \infty} \int_a^R f = \ell.$$

Given  $\epsilon > 0$ , there exists  $R_0$  such that

$$\int_a^{R_0} f \geq \ell - \epsilon.$$

Thus, if  $R \geq R_0$ ,

$$\int_a^R f \geq \int_a^{R_0} f \geq \ell - \epsilon.$$

And the limit follows.

**Example.** Consider the integral  $\int_0^\infty e^{-x^2/2} dx$ . We have

$$e^{-x^2/2} \leq e^{-x/2}, \quad x \geq 1$$

Since

$$\int_1^R e^{-x/2} dx = \frac{1}{2}(e^{-1/2} - e^{-R/2}) \rightarrow \frac{e^{-1/2}}{2},$$

the integral  $\int_0^\infty e^{-x^2/2} dx$  converges.

3. We know that if  $\sum a_n$  converges, then  $a_n \rightarrow 0$ . However,  $\int_a^\infty f$  converges may not imply that  $f \rightarrow 0$ .

For example, consider a function of smaller and smaller bumps to the highest point with value 1.

#### Theorem 6.8: Integral test

Let  $f(x)$  be a positive decreasing function for  $x \geq 1$ . Then

1. the integral  $\int_1^\infty f(x) dx$  and the series  $\sum_1^\infty f(n)$  both converge or diverge;
2. As  $n \rightarrow \infty$ ,  $\sum_{r=1}^n f(r) - \int_1^n f(x) dx$  tends to a limit  $\ell$  such that  $0 \leq \ell \leq f(1)$ .

*Proof.* Note that  $f$  decreasing implies that  $f$  is integrable on every bounded subinterval by Theorem 6.2.

If  $n-1 \leq x \leq n$ , then  $f(n-1) \geq f(x) \geq f(n)$ , so

$$f(n-1) \geq \int_{n-1}^n f(x) dx \geq f(n).$$

Adding the terms together, we have

$$\sum_1^{n-1} f(r) \geq \int_1^n f(x) dx \geq \sum_2^n f(r).$$

From this, claim (1) is clear. For the proof of (2), set  $\phi(n) = \sum_1^n f(r) - \int_1^n f(x) dx$ , then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x) dx \leq 0.$$

So  $\phi(n)$  is decreasing. We also have  $0 < \phi(n) \leq f(1)$ . Thus,  $\phi(n)$  tends to a limit  $\ell$  such that  $0 \leq \ell < f(1)$ . ■

## Lecture 24

16 Mar. 2022

**Example.** 1. The series  $\sum_1^\infty \frac{1}{n^k}$  converges if and only if  $k > 1$ .

We proved last lecture that  $\int_1^\infty \frac{1}{x^k}$  converges if and only if  $k > 1$ , and the convergence follows from integral test.

2. The series  $\sum_2^\infty \frac{1}{n \log n}$  is divergent by considering  $f(x) = \frac{1}{x \log x}$  for  $x \geq 2$ .

$$\int_2^R \frac{dx}{x \log x} = \log(\log x) \Big|_2^R = \log(\log R) - \log(\log 2) \rightarrow \infty$$

as  $R \rightarrow \infty$ . So divergence follows from integral test.

## Corollary 6.4: Euler's constant

As  $n \rightarrow \infty$ ,  $1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \rightarrow \gamma$  with  $0 \leq \gamma \leq 1$ .

*Proof.* Set  $f(x) = 1/x$ , and use Theorem 6.8. ■

**Remark.** It is still an open question whether  $\gamma$  is irrational. ( $\gamma \approx 0.577$ )

We have seen that monotone functions and continuous function are Riemann integrable. We can generalize this a bit and say that piece-wise continuous functions are integrable.

## Definition 6.6: A

unction  $f : [a, b] \rightarrow \mathbb{R}$  is said to be piece-wise continuous if there is a dissection  $\mathcal{D} = \{a = x_0, x_1, \dots, x_n = b\}$  such that

1.  $f$  is continuous on  $(x_{j-1}, x_j)$  for all  $j$ ;
2. the one-sided limits  $\lim_{x \rightarrow x_{j-1}^+} f(x), \lim_{x \rightarrow x_j^-} f(x)$  exist.

It is now an exercise to check that  $f$  is Riemann integrable; just check that  $f|_{[x_{j-1}, x_j]}$  is integrable for each  $j$  (the values of  $f$  at the end points won't really matter) and use additivity of domains.

**Problem.** How large can the discontinuity set of  $f$  be while  $f$  is still Riemann integrable?



Recall the example  $f(x) = \begin{cases} 1/q, & x = p/q \\ 0, & \text{otherwise} \end{cases}$  which has a countably infinite set of discontinuities.

—non-examinable—

The question is answered by Henri Lebesgue.

Bounded function  $f : [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if and only if the set of discontinuity points has *measure zero*.

#### Definition 6.7

Let  $\ell(I)$  be the length of an interval  $I$ . A subset  $A \subseteq \mathbb{R}$  is said to have *measure zero* if for each  $\epsilon > 0$ , there exists a countable collection of intervals  $I_j$  such that  $A \subseteq \bigcup_{j=1}^{\infty} I_j$  and  $\sum_{j=1}^{\infty} \ell(I_j) < \epsilon$ .

#### Lemma 6.4

1. Every countable set has measure zero.
2. If  $B$  has measure zero and  $A \subseteq B$ , then  $A$  has measure zero.
3. If  $A_k$  has measure zero for all  $k \in \mathbb{N}$ , then  $\bigcup_{k \in \mathbb{N}} A_k$  also has measure zero.

We use the *oscillation* of  $f$ , which is for  $I$  an interval

$$\omega_f(I) = \sup_I f - \inf_I f.$$

And the oscillation of  $f$  at a point is

$$\omega_f(x) = \lim_{\epsilon \rightarrow 0} \omega_f(x - \epsilon, x + \epsilon).$$

#### Lemma 6.5

$f$  is continuous at  $x$  if and only if  $\omega_f(x) = 0$ .

*Brief Sketch of proof of Lebesgue's Criterion.* We consider the set

$$D = \{x \in [a, b] \mid f \text{ discontinuous at } x\} = \{x \mid \omega_f(x) > 0\}.$$

---

Let  $N(\alpha) = \{x \mid \omega_f(x) \geq \alpha\}$ , then  $D = \cup_1^\infty N(1/k)$ . We want to show that  $D$  has measure zero.

Let  $\epsilon > 0$  be given, there exists  $\mathcal{D}$  such that

$$S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) = \sum_{j=1}^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\epsilon \alpha}{2}.$$

Consider  $F = \{j \mid (x_{j-1}, x_j) \cap N(\alpha) \neq \emptyset\}$ , then for each  $j \in F$ ,

$$\omega_f([x_{j-1}, x_j]) \geq \alpha.$$

So

$$\alpha \sum_{j \in F} (x_j - x_{j-1}) \leq S(f, \mathcal{D}) - \mathcal{S}(f, \mathcal{D}) < \frac{\epsilon \alpha}{2}.$$

And  $\sum_{j \in F} (x_j - x_{j-1}) < \epsilon/2$ . These cover  $N(\alpha)$  except perhaps for  $\mathcal{D}$ . But these can be covered by intervals of total length less than  $\epsilon/2$ . So  $N(\alpha)$  can be covered by intervals of total length less than  $\epsilon$ . That is,  $N(\alpha)$  has measure zero.

For the other direction, let  $\epsilon > 0$  be given, consider  $N(\epsilon) \subseteq D$ , so  $N(\epsilon)$  has measure zero.  $N(\epsilon)$  is closed and bounded, so it can be covered by finitely many open intervals of total length less than  $\epsilon$  because it has measure zero.  $N(\epsilon) = \cup_{i=1}^m U_i$ , and  $K = [a, b] \setminus \cup_{i=1}^m U_i$  compact. So it can be covered by finitely many intervals  $J_j$  such that  $\omega_f(J_j) < \epsilon$ . And we are done by considering the partition with these intervals. ■