# Geometry

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## Contents



### Lecture 1: Introduction

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# 1 Surfaces

## 1.1 Topological Surfaces

We start with some definitions.

**Definition 1.1.** A *topological surface* is a topological space $\Sigma$ such that

1. **T1:** $\forall p \in \Sigma$ there is an open neighborhood $p \in U \subseteq \Sigma$ such that $U$ is homeomorphic to $\mathbb{R}^2$ , or a disc $D^2 \subseteq \mathbb{R}^2$ with its usual Euclidean topology.
2. **T2:** $\Sigma$ is Hausdorff and second countable.

**Remark.** We have the following remarks.

1. $\mathbb{R} \cong D(0,1)$.
2. A space $X$ is *Hausdorff* if for $p \neq q \in X$, there exists disjoint open sets $p \in U$ and $q \in V$ in $X$.
3. A space $X$ is *second countable* if it has a countable base i.e. $\exists \{u_i\}_{i\in\mathbb{N}}$ open sets s.t. every open set is a union of some $u$.
4. **T1** is the point and **T2** is for technical honesty.
5. If $X$ is Hausdorff/ second countable, so are subspaces of $X$. In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

**Example.** Here we present some examples of topological surfaces.

1. $\mathbb{R}^2$ the plane
2. Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where $Z$ is closed:

- $Z = \{0\}$,
- $Z = \{(0,0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \ldots)\}$.

3. Graphs:

   Let $f : \mathbb{R}^2 \to R$ be a continuous function. The graph $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$ (subspace topology).

   Recall that if $X, Y$ are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with $U$ open and $V$ open.

   It has the feature that $f : Z \to X \times Y$ is continuous if and open if the two projective maps are continuous.

   Commutative diagram

   Application: $\Gamma_f \subseteq X \times Y$, if $f : X \to Y$ is continuous, if homeomorphic to $X$.

   So $\Gamma_f \cong \mathbb{R}^2$ for any $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous, so $\Gamma_f$ is a topological surface.

   **Note.** As a topological surface, $\Gamma_f$ is independent of $f$, but later on as a geometric object, it will reflect features of $f$.

4. The sphere (subspace topology):

   $$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

   Stereographic projection

   Stereographic projection graph

   $$\pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$$
   $$(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$$

   **Note.** The map is continuous and has an inverse, $\pi_+$ is a continuous bijection with continuous inverse, and hence a homeomorphism.

   Explicit formula for inverse

   Stereographic projection from the South Pole is also a homeomorphism from $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$.

   So $S^2$ is a topological surface:

   $\forall p \in S^2$, either $p$ lies in the domain of $\pi_+$ or of $\pi_-$ (or both) and so it lies in an open set homeomorphic to $\mathbb{R}^2$. (And Hausdorff and second countable from $\mathbb{R}^2$).

   **Remark.** $S^2$ has a global property as it is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$ .

5. The real projective place:

   The group $\mathbb{Z}/2$ acts on $S^2$ by homeomorphism via the *antipodal map* $a : S^2 \to S^2$.

   $$a(x, y, z) = (-x, -y, -t).$$

   i.e. There exists a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \text{Homeo}(S^2)$, such that it maps the non-identity element to the antipodal map.

**Definition 1.2.** The *real projective plane* is the quotient space of $S^2$ given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}/2\mathbb{Z}.$$

**Lemma 1.1.** As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin.

*Proof.* Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ∎

Graph of the sphere

**Lemma 1.2.** $\mathbb{RP}^2$ is a topological surface.

*Proof.* We check that it is Hausdorff:

Recall if $X$ is a space and $q : X \to Y$ is a quotient map, $V \subseteq Y$ is open $\iff q^{-1}V \subseteq X$ open.

More balls

If $[p], [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in S^2$ are distinct antipodal pairs. Take small open discs around $p, q$ and their antipodal images, as in the picture. ∎