Universal properties

11th June 2022

Problem 2

Prove that $\varnothing$ is the unique initial object in Set.

Solution. Assume otherwise $\varnothing$ and $\varnothing'$ are two initial objects in Set. They must be isomorphic since they are both initial objects. Thus, they are both empty. The statements $x \in \varnothing \implies x \in \varnothing'$ and $x \in \varnothing' \implies x \in \varnothing$ are both vacuously true. So $\varnothing \subseteq \varnothing'$ and $\varnothing' \subseteq \varnothing'$, implying $\varnothing = \varnothing'$. ■

Problem 3

Prove that final objects are unique up to isomorphism.

Solution. Let F_1, F_2 be two final objects in the category C. There is a unique morphism $F_1 \to F_1$ and $F_2 \to F_2$, so it must be the identity morphisms 1_{F_1} and 1_{F_2}. Because F_2 is a final object, there is a unique morphism $f : F_1 \to F_2$, and because F_1 is a final object, there is a unique morphism $g : F_2 \to F_1$. We want to show that f is an isomorphism. gf is a morphism from F_1 to F_1, so it must be the identity. That is

$$gf = 1_{F_1}.$$

Similarly, we have

$$fg = 1_{F_2}.$$

So f is an isomorphism, and $F_1 \cong F_2$.

Q3

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Problem 6

Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the *divisibility* relation. Thus, there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names?

Solution. If $a, b \in \mathbb{Z}^+$, and the product $a \times b$ satisfies the condition that $d \mid a$ and $d \mid b$ implies $d \mid a \times b$. We also want $a \times b \mid a$ and $a \times b \mid b$. In other words, we have $a \times b = \gcd(a, b)$.

Similarly, we have $a \amalg b = \mathrm{lcm}(a, b)$. ■