Dynamics and Relativity

Jonathan Gai

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Contents

Lecture 1: Introduction

20 Jan. 10:00

1 Basic Concepts

Books

1. Classical Mechanics - Douglas (more examples)
2. Classical Mechanics - Tom Kibblet (more chatty)
3. Lecture Notes - David Tong

1.1 Newtonian Mechanics

A *particle* is an object of insignificant size. For now, its only attribute is its position.

For large objects, we take the center of mass to define the position and treat them like a particle.

To describe the position, we pick a *reference frame*: a choice of origin and 3 coordinate axes. With respect to this frame, a particle sweep out a *trajectory* $\mathbf{x}(t)$. (sometimes, we may write $\mathbf{r}(t)$).

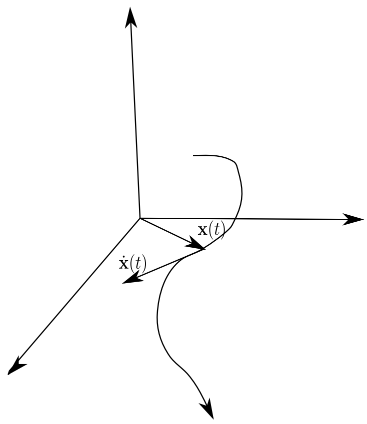

Figure 1: position and acceleration of a particle

Given two vector functions $\mathbf{f}(t)$ and $\mathbf{g}(t)$, $\frac{d}{dt}(\mathbf{f} \cdot \mathbf{g}) = \frac{d\mathbf{g}}{t} \cdot \mathbf{g} + \mathbf{f} \cdot \frac{d\mathbf{g}}{dt}$ and $\frac{d}{dt}(\mathbf{f} \times \mathbf{g}) = \frac{d\mathbf{g}}{t} \times \mathbf{g} + \mathbf{f} \times \frac{d\mathbf{g}}{dt}$.

1.2 Newtonian Laws of Motion

The framework of Newtonian mechanics rely on these axioms, known as *Newton's Laws*:

- **N1**: Left alone, a particle moves with constant velocity.

- **N2**: The rate of change of momentum is proportional to the force.
- **N3**: Every action has an equal and opposite reaction.

1.3 Inertial Frames and The First Law

For many reference frames, **N1** isn't true! It only holds for frames that are not themselves accelerating. Such frames are called *inertial frames*:

Definition 1.1: Inertial Frame

In an inertial frame, $\ddot{\mathbf{x}} = 0$ when left alone.

A better framing of the 1st law is (**N1′** inertial frames exist).

For most purposes, this room approximates an inertial frame.

1.4 Galilean Relativity

Inertial frames are not unique. Given an inertial frame S, in which a particle has coordinates $\mathbf{x}$, we can construct another inertial frame S' in which the coordinates of the particle are given by $\mathbf{x}'$.

1. Translations: $\mathbf{x}' = \mathbf{x} + \mathbf{a}$, where a is a constant.
2. Rotations: $\mathbf{x}' = R\mathbf{x}$, where R is a 3×3 matrix with $R^T R = I$.
3. (Galilean) Boost: $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$.

For each of these, if there is no force on a particle. $\ddot{\mathbf{x}} = 0 \implies \ddot{\mathbf{x}}' = 0$

The *Galilean principle of relativity* tells us that the laws of physics are the same

1. At every point in space.
2. No matter which direction you face.
3. No matter what constant velocity you move at.

4. At all moments in time.

The above are experimentally tested facts.

There is no such thing as "absolutely stationary", but notice that acceleration is absolute. You don't have to accelerate relative to something.

Lecture 2: Newton's Second Law

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1.5 Newton's Second Law

Here we present a different formulation of Newton's Second Law than A-levels.

Definition 1.2

The *equation of motion* of a particle subjected to a force $\mathbf{F}$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{\mathbf{x}}) = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}})$$

where $\mathbf{p} = m\dot{\mathbf{x}}$ is *momentum*, m is *inertial mass*, It is a measure of the reluctance of a particle to move.

When $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$ (so true in most situations), we have

Theorem 1.1: Newton's Second Law

$$m\ddot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}).$$

The equation above of motion is a second order differential equation, so we need to specify two initial conditions for each degree of freedom.

Example. When $\mathbf{x} \in \mathbb{R}^3$, we have three degrees of freedom, so we need six initial conditions.

There are two steps in any Newtonian mechanics problem:

- Write down the equations(s),

- Solve it.

2 Forces

2.1 Potentials in One Dimension

Consider a particle moving in a line with position $x(t)$. Suppose that $F = F(x)$, i.e. it depends on position, not on velocity. We define a *potential energy* $V(x)$ by

$$F = -\frac{\mathrm{d}V}{\mathrm{d}x} \quad \text{x} \quad V(x) = -\int_{x_0}^{x} \mathrm{d}x' F(x').$$

Note that the prime does not denote derivative here; x' is a dummy variable.

The equation of motion is

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}. \tag{1}$$

Proposition 2.1

The energy $E = \frac{1}{2}m\ddot{x} + V(x)$ is conserved (i.e. $\dot{E} = 0$) for any trajectory which obeys the equation of motion.

Proof. We have the following,

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} \\ &= \dot{x}(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}) \\ &= 0 \quad \text{by (1)}\end{aligned}$$

∎

Note, however, if $F = F(x, \dot{x})$, there is no conserved quantity.

Example (Harmonic Oscillater)**.** When $V = \frac{1}{2}kx^2$, (1) becomes $m\ddot{x}] = -kx$. And the general solution is

$$x(t) = A\cos(wt) + B\sin(wt)$$

where $w := \sqrt{\frac{k}{m}}$ is the *angular frequency*. A and B are integration constants.

It's simple to show that $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is constant.

The time taken to complete a cycle is the *period* $T = \frac{2\pi}{\omega}$.

For a general potential $V(x)$, the conserved quantity allows us to solve any one-dimensional problem.

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + V(x) \\ \implies \frac{\mathrm{d}x}{\mathrm{d}t} &= \pm\sqrt{\frac{2}{m}(E - V(x))} \\ \implies t - t_0 &= \pm\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}} \end{aligned}$$

which is the solution, and we just need to do the integral.

Motion in a Potential

Sometimes even if you can't do the integral, it's simple to get a qualitative picture of the solution.

Example. If the potential is $V(x) = m(x^3 - 3x)$. If we drop the particle at $x = x_0$, the

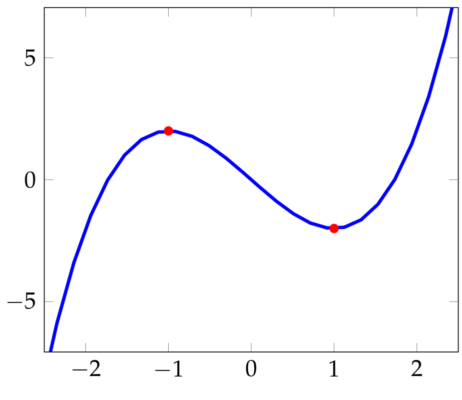

Figure 2: Graph of the potential function

total energy $E = V(x_0)$ which is all potential. We have the following several cases,

1. $x_0 = \pm 1 \implies$ particle stays there at all time because (1) says there must be no force.

2. $x_0 \in (-1, 2) \implies$ particle oscillates back and force in dip.

   The initial energy is all potential, and it turns this into kinetic energy and falls down the dip.

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3. $x_0 > 2$ or $x_0 < -1 \implies$ particle keeps on going.

4. $x_0 = 2 \implies$ is a special case. It reaches $x = -1$, but how long does it take?

   Write $x = -1 + \epsilon$ and as $\epsilon \to 0$,

$$V(x) \approx 2m - 3m\epsilon^2.$$

   So we have

$$\begin{aligned} t \setminus t_0 0 &= -\int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}} \\ &= -\int_{\epsilon_0}^{\epsilon} \frac{d\epsilon'}{\sqrt{6}\epsilon'} \\ &= -\frac{1}{\sqrt{6}} \log(\frac{\epsilon}{\epsilon_0}). \end{aligned}$$

   So $t \to \infty$ as $\epsilon \to 0$ that is it takes infinite time.

Definition 2.1

A particle placed at an *equilibrium point* x_0 will stay there for all time.
$\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$ so equilibrium points obey $\left.\frac{\mathrm{d}V}{\mathrm{d}x}\right|_{x_0} = 0$ that is critical points of V

We can look at motion near equilibrium point. Taylor expanding,

$$V(x) \approx V(x_0) + \frac{1}{2}(x - x_0)^2 V''(x) + \dots.$$

There are several cases,

1. $V''(x_0) > 0 \implies$ minimum of V, similar to potential of harmonic oscillator
$$m\ddot{x} \approx -V''(x_0)(x - x_0).$$
This point is *stable*. Particle oscillates with frequency $\omega = \sqrt{\frac{V''(x_0)}{m}}$.

2. $V''(x_0) < 0 \implies$ minimum of V, the point is *unstable*.
$$x - x_0 \approx Ae^{\alpha t} + Be^{-\alpha t}$$
with $\alpha = \sqrt{\frac{-V''(x_0)}{m}}$. $A \neq 0$, and the particle moves quickly away from x_0.

3. $V''(x_0) = 0 \implies$ more work needed. We need to expand out higher terms of the Taylor expansion.

Example (the pendulum)**.** The equation of motion is
$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$
The energy is
$$E = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta.$$
When

1. $E > mgl \implies \dot{\theta} \neq 0$ for all t.

2. $E < mgl \implies \dot{\theta} = 0$ for some point θ_0.

   The pendulum oscillates back and forth and
$$E = -mgl\cos\theta_0.$$

   Using the general solution for 1-dimensional system,
$$\begin{aligned} T &= 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + (\frac{2g}{l})\cos\theta}} \\ &= 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}}. \end{aligned}$$
   The integral is a bit tricky. But for small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, so
$$T \approx 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\theta_0^2 - \theta^2}}.$$
   Note the independence of θ_0.

This is the result for the harmonic oscillator, of course.