Analysis

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January 28, 2022

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Lecture 1: Limits

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Books:

- *A First Course in Mathematical Analysis* -Burkill
- *Calculus* -Spivak
- *Analysis I* -Tao

1 Limits and Convergence

1.1 Review from Numbers and Sets

Notation. We denote sequences by a_n or $(a_n)_{n=1}^{\infty}$, with $a_n \in \mathbb{R}$.

Definition 1.1. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

Note. $N = N(\epsilon)$ which is dependent on ϵ. That is, if you want to go closer to a, sometimes you need to go higher in N.

Definition 1.2 (limit of a sequence). We say that a sequence is a

$$\left.\begin{aligned} &\textit{increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{decreasing sequence if } a_n \geq a_{n+1}, \end{aligned}\right\} \textit{monotone sequence}$$

$$\left.\begin{aligned} &\textit{strictly increasing sequence if } a_n \leq a_{n+1}, \\ &\textit{strictly decreasing sequence if } a_n \geq a_{n+1}. \end{aligned}\right\} \textit{strictly monotone sequence}$$

We also have

Theorem 1.1 (Fundamental Axiom of the Real Numbers). If $a_n \in \mathbb{R}$ and a_n is increasing and bounded above by $A \in R$, then there exists $a \in \mathbb{R}$ such that $a_n \to n$ as $n \to \infty$.

That is, an increasing sequence of real numbers bounded above *converges.*

Remark. It is equivalent to the following,

- A decreasing sequence of real numbers bounded below converges.
- Every non-empty set of real numbers bounded above has a *supremum* (Least Upper Bound Axiom).

Definition 1.3 (supremum). For $S \subseteq \mathbb{R}, S \neq \varnothing$. We say that $\sup S = k$ if

1. $x \leq k, \quad \forall x \in S$,
2. given $\epsilon > 0$, there exists $x \in S$ such that $x > k - \epsilon$.

Note. Supremum is unique, and there is a similar notion of infimum.

Lemma 1.1 (Properties of Limits).

1. The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$.
2. If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 \ldots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit).
3. If $a_n = c$ for all n then $a_n \to c$ as $n \to \infty$.
4. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$.
5. If $a_n \to a$ and $b_n \to b$, then $a_n b_n \to ab$.
6. If $a_n \to a$, then $\frac{1}{a_n} \to \frac{1}{a}$.
7. If $a_n < A$ for all n and $a_n \to a$, then $a \leq A$.

Proof.

1. Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|a_n - b| < \epsilon, \forall n \geq N_2$.

   Take $N = \max\{n_1, n_2\}$, then if $n \geq N$,

   $$|a - b| \leq |a_n - a| + |a_n - b| < 2\epsilon.$$

   If $a \neq b$, take $\epsilon = \frac{|a-b|}{3}$, we have

   $$|a - b| < \frac{2}{3}|a - b|. \lightning$$

2. Given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon, \forall n \geq N$, Since $n_j \geq j$, we know

   $$\left|a_{n_j} - a\right| < \epsilon, \forall j \geq N.$$

   That is, $a_{n_j} \to a$ as $j \to \infty$.

5. We have

   $$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a|. \end{aligned}$$

   Given $\epsilon > 0$, there exists N_1 such that $|a_n - a| < \epsilon, \forall n \geq N_1$, and there exists N_2 such that $|b_n - b| < \epsilon, \forall n \geq N_2$.

   If $n \geq N_1(1)$, $|a_n - a| < 1$, so $|a_n| \leq |a| + 1$.

   We have

   $$|a_n b_n - ab| \leq \epsilon(|a| + 1 + |b|), \forall n \geq N_3(\epsilon) = \max\{N_1(1), N_1(\epsilon), N_2(\epsilon)\}.$$

∎

Lemma 1.2.

$$\frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Proof. $\frac{1}{n}$ is a decreasing sequence that is bounded below. By the Fundamental Axiom, it has a limit a.

We claim that $a = 0$. We have

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2} \text{ by Lemma (1.1).}$$

But $\frac{1}{2n}$ is a subsequence, so by Lemma (1.1) $\frac{1}{2n} \to a$. By uniqueness of limits proved again in Lemma (1.1), we have $a = \frac{a}{2} \implies a = 0$. ∎

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$ by replacing the absolute value with modulus.

Definition 1.4. We say that $a_n \to a$ as $n \to \infty$ if given $\epsilon > 0$, there exists N such that $|a_n - a| < \epsilon$ for all $n \geq N$.

And the first six parts of Lemma (1.1) are the same over $\mathbb{C}$. The last one does not make sense over $\mathbb{C}$ since it uses the order of $\mathbb{R}$.

Lecture 2: Bolzano–Weierstrass theorem

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Theorem 1.2 (Bolzano-Weierstrass Theorem). If $x_n \in R$ and there exists K such that $|x_n| \leq K$ for all n, then we can find $n_1 < n_2 < n_3 < \ldots$ and $x \in \mathbb{R}$ such that $x_{n_j} \to x$ as $j \to \infty$. In other words, every bounded sequence has a convergent subsequence.

Remark. We say nothing about the uniqueness of the limit x.

For example, $x_n = (-1)^n$ has two subsequences tending to -1 and 1 respectively.

Proof. Set $[a_1, b_1] = [-K, K]$. Let c be the mid-point of a_1, b_1, consider the following alternatives,

1. $x_n \in [a_1, c]$ for infinitely many n.
2. $x_n \in [c, a_2]$ for infinitely many n.

Note that (1) and (2) can hold at the same time. But if (1) holds, we set $a_2 = a_1$ and $b_2 = c$. If (1) fails, we have that (2) must hold, and we set $a_2 = c$ and $b_2 = b_1$.

We proceed as above to construct sequences a_n, b_n such that $x_m \in [a_n, b_n]$ for infinitely many values of m. They also satisfy

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}, \quad b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}.$$

a_n is an increasing sequence and bounded, and b_n is a decreasing sequence and bounded. By Fundamental Axiom, $a_n \to a \in [a_1, b_1], b_n \to b \in [a_1, b_1]$. Using Lemma (1.1), $b - a = \frac{b-a}{2} \implies a = b$.

Since $x_m \in [a_n, b_n]$ for infinitely many values of m, having chosen n_j such that $x_{n_j} \in [a_j, b_j]$, that is $n_{j+1} > n_j$ such that $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$. In other words, there is unlimited supply.

Hence, $a_j \leq x_{n_j} \leq b_j$, so $x_{n_j} \to a$. ■

1.2 Cauchy Sequences

Definition 1.5 (Cauchy Sequence). $a_n \in \mathbb{R}$ is called a *Cauchy sequence* if given $\epsilon > 0$ $\exists N > 0$ such that $|a_n - a_m| < \epsilon$ $\forall n, m > N$.

Note. N is dependent on ϵ.

A function is Cauchy if after you wait long enough, any two elements in the sequence would be close enough.

Lemma 1.3. A convergent sequence is a Cauchy sequence.

Proof. If $a_n \to a$, given $\epsilon > 0$, exists N such that for all $n \geq N$, $|a_n - a| < \epsilon$.

Take $m, n \geq N$,
$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\epsilon.$$
∎

Lemma 1.4. Every Cauchy sequence is convergent.

Proof. First we note that if a_n is Cauchy, then it is bounded.

Take $\epsilon = 1$, $N = N(1)$ in the Cauchy property, then
$$|a_n - a_m| < 1, \quad n, m \geq N(1).$$

We have
$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \quad \forall m \geq N.$$

Let $K = \max\{1 + |a_N|, |a_n| \; n = 1, 2 \ldots, N-1\}$.

Then $|a_n| \leq K$ for all n. By the Bolzano–Weierstrass theorem, $a_{n_j} \to a$. We must have $a_n \to a$.

Given $\epsilon > 0$, there exists j_0 such that for all $j \geq j_0$, $\left|a_{n_j} - a\right| < \epsilon$.

Also, there exists $N(\epsilon)$ such that $|a_m - a_n| < \epsilon$ for all $m, n \geq N(\epsilon)$.

Take j such that $n_j \geq \max\{N(\epsilon), n_{j_0}\}$. Then if $n \geq N(\epsilon)$,
$$|a_n - a| \leq \left|a_n - a_{n_j}\right| + \left|a_{n_j} - a\right| < 2\epsilon.$$
∎

Thus, on $\mathbb{R}$, a sequence is convergent if and only if it is Cauchy.

The old fashion name of this is called the "general principle of convergence".

It is a useful property because we don't need what the limit actually is.

2 Series

Definition 2.1. If $a_n \in \mathbb{R}, \mathbb{C}$ We say that $\sum_{j=1}^{\infty} a_j$ converges to s if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to S$$

as $N \to \infty$. We write $\sum_{j=1}^{\infty} a_j = s$. If S_N does not converge, we say that $\sum_{j=1}^{\infty} a_j$ *diverges.*

Remark. Any problem on series is really a problem about the sequence of partial sums.

Lemma 2.1.

1. If $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} a_j$ converges, then so does $\sum_{j=1}^{\infty} \lambda a_j + \mu b_j$, when $\lambda, \mu \in \mathbb{C}$;
2. Suppose there exists N such that $a_i = b_i$ for all $i \geq N$. Then either $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge or they both diverge. (initial terms do not matter for convergence)

Proof. 1. Exercise.

2. If we have $n \geq N$,

$$S_n = \sum_{i=1}^{N-1} a_i + \sum_{i=N}^{n} a_i$$
$$d_n = \sum_{i=1}^{N-1} b_i + \sum_{i=N}^{n} b_i$$

So $S_n - d_n = \sum_{i=1}^{N-1} a_i - b_i$ which is a constant. So S_n converges if and only if d_n does.

■

Lecture 3

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We have the following important example,

Example (Geometric Series). $x \in \mathbb{R}$, set $a_n = x^{n-1}$ with $n \geq 1$. So the partial sums are

$$S_n = \sum_{i=1}^{\infty} a_i = 1 + x + x^2 + \cdots + x^{n-1}.$$

Then we have

$$S_n = \begin{cases} \dfrac{1-x^n}{1-x}, & \text{if } x \neq 1 \\ n, & \text{if } x = 1 \end{cases}.$$

You can derive this by the equation

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n,$$

and we have $S_n(1-x) = 1 - x^n$.

If $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$.

If $x > 1$, $x^n \to \infty$ and $S_n \to \infty$.

If $x < -1$, S_n does not converge (oscillates).

If $x = -1$, $S_n = \begin{cases} 1, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$.

Thus, the geometric series converges if and only if $|x| < 1$.

To see for example that $x^n \to 0$ if $|x| < 1$, consider first the case $0 < x < 1$. Write $\frac{1}{x} = 1 + \delta, \delta > 0$, so $x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+n\delta} \to 0$ because $(1+\delta)^n \geq 1 + n\delta$ from binomial expansion.

Definition 2.2. $S_n \to \infty$ if given A, there exists an N such that $S_n > A$ for all $n > N$.

$S_n \to -\infty$ if given A, there exists an N such that $S_n < -A$ for all $n > N$.

Lemma 2.2. If $\sum_{i=1}^{\infty} a_n$ converges, then $\lim_{i\to\infty} a_i = 0$.

Proof. Let $S_n = \sum_{i=1}^{\infty} a_i$, note that $a_n = S_n - s_{n-1}$. If $S_n \to a$, we have $a_n \to 0$ because $S_{n-1} \to a$ also. ∎

Remark. The converse of the preceding lemma is false. One example is $\sum \frac{1}{n}$, the *harmonic series*. We can see that it diverges because

$$S_n = \sum_{i=1}^{\infty}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > S_n + \frac{1}{2}$$

since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$.

So if $S_n \to a$, then $S_{2n} \to a$, also we have $a \geq a + \frac{1}{2}$. Contradiction.

2.1 Series of Non-negative Terms

We first consider sequences with positive terms, but it gives monotonicity of partial sums.

> **Theorem 2.1 (The Comparison Test).** Suppose $0 \leq b_n \leq a_n$ for all n. Then if $\sum\limits_{n=1}^{\infty} a_n$ converges, so does $\sum\limits_{n=1}^{\infty} b_n$.

Proof. Let $s_N = \sum\limits_{n=1}^{N} a_n$, $d_N = \sum\limits_{n=1}^{N} b_n$. Because $b_n \leq a_n$, we know $d_N \leq s_N$. But $s_N \to s$, then $d_n \leq s_n \leq 2$ for all n, and d_N is a increasing sequence bounded above. So d_N converges. ■

Example. We consider $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$. We have

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

So we have

$$\sum_{n=2}^{N} a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} = 1 - \frac{1}{N}.$$

It is clear that $\sum\limits_{n=1}^{\infty} a_n$ converges, so $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

In fact, we get $\sum\limits_{n=1}^{\frac{1}{n^2}} \leq 1 + 1 = 2$.

For the rest of the lecture, we establish two more tests.

> **Theorem 2.2 (Root test/ Cauchy's Test for Convergence).** Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges; if $a > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Remark. Nothing can be said if $a = 1$.

. If $a < 1$, choose $a < r < 1$. By definition of limit and hypothesis, there exists N such that $\forall n \geq N$,

$$a_n^{1/n} < r \implies a_n < r^n.$$

But since $r < 1$, the geometric series converges, and by comparison test, the series $\sum a_n$ converges as well.

To prove the second part of the theorem, if $a > 1$, for $n \geq N$,

$$a_n^{1/n} > 1 \implies a_n > 1.$$

Thus, $\sum\limits_{n=1}^{\infty} a_n$ diverges, since a_n does not tend to zero. ■

Theorem 2.3 (Ratio Test/ D'Alembert's Test). Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to \ell$. If $\ell < 1$, $\sum\limits_{n=1}^{\infty} a_n$ converges. If $\ell > 1$, $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Remark. As before, nothing can be said for $\ell = 1$.

Proof. Supposed $\ell < 1$ and choose r with $\ell < r < 1$. Then $\exists N$ such that $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r.$$

Therefore,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \ n > N.$$

So, $a_n < kr^n$ with k independent of n. Since $\sum\limits_{n=1}^{\infty} r^n$ converges, so does $\sum\limits_{n=1}^{\infty} a_n$ by Comparison Test.

If $\ell > 1$, choose $1 < r < \ell$. Then $\frac{a_{n+1}}{a_n} > r$ for all $n \geq N$, and as before

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \ n > N.$$

So the series diverges. ■