Variational Principles

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Contents

Lecture 1

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Motivation

0.1 The Brachistochrone Problem

Problem. Particle slides on a wire under influence of gravity between two fixed points A, B. Which shape of the wire gives the shortest travel time, starting from rest?

The travel time is $T = \int_A^B \frac{\mathrm{d}\ell}{v(x,y)}$, and by energy conservation, and by energy conservation

$$\frac{1}{2}mv^2 + mgy = 0 \implies v = \sqrt{-2gy}.$$

So

$$T[y] = \frac{1}{\sqrt{2g}} \int_0^{x_2} \frac{\sqrt{1 + (y')^2}}{\sqrt{-y}} \,\mathrm{d}x$$

subject to $y(0) = 0$, $y(x_2) = y_2$.

0.2 Geodesics

Problem. What is the shortest path γ between two points A, B on a surface.

Take $\Sigma = \mathbb{R}^2$. The distance along γ is

$$D[y] = \int_A^B \mathrm{d}\ell = \int_{x_1}^{x_2} \sqrt{1 + (y')^2}\,\mathrm{d}x,$$

and we want to minimize D by varying γ.

0.3 Introduction

In general, we want to minimize (maximize)

$$F[y] = \int_{x_1}^{x_2} f(x, y, y')\,\mathrm{d}x$$

among all functions s.t. $y(x_1) = y_1$, $y(x_2) = y_2$. The expression is a *functional*. (A function on a space of functions). Functions map numbers to numbers, and functionals map functions to numbers.

Area under the graph is when $f = y$ and the length of a curve is when $f = \sqrt{1 + (y')^2}$.

Calculus of variations finds extrema of functionals on space of functions.

Notation.

- $C(\mathbb{R})$ is space of continuous functions on $\mathbb{R}$.
- $C^k(\mathbb{R})$ is the space of functions with continuous kth derivatives.
- $C^k_{(\alpha,\beta)}$ is the space of functions with continuous kth derivatives and $f(\alpha) = f(\beta)$.

We need to specify the function space beforehand. It is a branch of functional analysis—Part III analysis on the space of functions, while Analysis I is analysis on the number line. Variational Principles follows principles in Nature, where the laws follow from extremizing functionals.

Example (Fermat's Principle)**.** Light between two pints travel along paths which require least time.

Example (Principle of Least Action)**.** Let T be the kinetic energy $\frac{m|\dot{\mathbf{x}}|^2}{2}$ and potential energy $V = V(\mathbf{x})$.

$$S[\gamma] = \int_{t_1}^{t_2} (T - V)\, \mathrm{d}t$$

is minimized along paths of motion.

Leibniz commented on this, saying that "we live in the best of all worlds".

Feynman's take on this: "This is wrong. In quantum theorem the motion takes place along all possible path with different possibilities". [See Part III QFT]

In this course, we discuss

1. necessary condition for extrema of the Euler Lagrange Equations;
2. lots of examples (geometry, physics, problems with constraints);
3. second variation (some sufficient condition of extrema).

The following books will be useful

1. Gelfand - Fomin "Calculus of Variations".
2. DAMTP notes (e.g. P. Townsend).
3. Lectures are self-contained.