Vector Calculus

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Contents

1 Curves **2**
1.1 Differentiating the Curve 3

Lecture 1: Introduction

21 Jan. 10:00

Problem. Why do we do Vector Calculus?

1. Calculus is important, and we want to apply it to a wider range of functions.
2. It is a tool that is needed throughout quantitative sciences.

Lecture notes are online.

We will learn to differentiate and integrate function (or maps) of the form

$$f : \mathbb{R}^m \to \mathbb{R}^n.$$

An element of $\mathbb{R}^m$ or $\mathbb{R}^n$ is a vector, so the subject is called vector calculus.

We present some examples of multivariable functions. In general, for a physicist, there are two types of functions, ones where the domain represents a physical space and the ones where the codomain represents a physical space.

1. A function $f : \mathbb{R} \to \mathbb{R}^n$ defines a *curve* in $\mathbb{R}^n$.

   In physics, we might think of $\mathbb{R}$ as time and $\mathbb{R}^n$ as space and write this as

   $$f : t \mapsto \mathbf{x}(t) \text{ with } \mathbf{x} \in \mathbb{R}^n.$$

   (Obviously we should take $n = 3$).

   Generalizing, a map
   $$f : \mathbb{R}^2 \to \mathbb{R}^n$$
   defines a *surface* in $\mathbb{R}^n$, and so on.

2. In other applications, the domain $\mathbb{R}^m$ might be viewed as physical space. For example, in physics a *scalar fielid* is a map

   $$f : \mathbb{R}^3 \to \mathbb{R}.$$

Example. The temperature $T(x)$ is a scalar field, as is the Higgs Field

A *vector field* is a map

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

where the domain is physical space and the codomain is something more abstract.

Example. The electric field $\mathbf{E}(\mathbf{x})$ and magnetic field $\mathbf{B}(\mathbf{x})$ are vector fields.

1 Curves

We consider maps of the form

$$f : \mathbb{R} \to \mathbb{R}^n.$$

We assign a coordinate t to $\mathbb{R}$ and the Cartesian coordinates on $\mathbb{R}^n$

$$\mathbf{x} = (x^1, \dots, x^n) = x^i \mathbf{e}_i$$

where $\mathbf{e}_i$ is orthonormal basis such that $\mathbf{e}_i \mathbf{e}_j = \delta_{ij}$. (For $\mathbb{R}^3$) we also use notation $\{\mathbf{e}_i\} = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$.

The image of the function f is a *parameterised curve* $\mathbf{x}(t)$, with t the parameter. We will call the curve C.

Example. Here we give some familiar examples of parameterised curves.

1. Consider the map $\mathbb{R} \to \mathbb{R}^3$ given by

   $$\mathbf{x}(t) = (at, bt^2, 0).$$

   The curve C is the parabola $ay = bx^2$ in the plane $z = 0$.

   Note. When plotting the curve, we lose information about the parameter t.

2. Consider $\mathbf{x}(t) = (\cos t, \sin t, t)$.

   The curve C is a helix. The choice of parameterisation is not unique. For example, the map $\mathbf{x}(t) = (\cos \lambda t, \sin \lambda t, \lambda t)$ gives exactly the same helix.

   Sometimes the choice of parameterisation matters.

   Example. If t is time and $\mathbf{x}(t)$ is position, the velocity is proportional to λ.

   But we will see that some questions are independent of the choice of parameterisation.

1.1 Differentiating the Curve

A vector function $\mathbf{x}(t)$ is *differentiable* at t if as $\delta t \to 0$, we have

$$\mathbf{x}(t+\delta t) - \mathbf{x}(t) = \dot{\mathbf{x}}(t)\delta t + O(\delta t^2).$$

Note. "Big-O" notation $O(\delta t^2)$ means terms are proportional to δt^2 or smaller.

In physics, the dot is usually used for time derivatives, and the prime is used for spacial derivatives. In math, these are used interchangeably.

We write

$$\delta \mathbf{x}(t) = \mathbf{x}(t+\delta t) - \mathbf{x}(t),$$

and the derivative is then

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} := \lim_{\delta t \to 0} \frac{\delta \mathbf{x}}{dt}.$$

Lecture 2

24 Jan. 10:00

If we're in Cartesian, then we just differentiate vector components

$$\mathbf{x}(t) = x^i(t)\mathbf{e}_i \implies \dot{\mathbf{x}}(t) = \dot{\mathbf{x}}^i\mathbf{e}_i.$$

We also have the identities

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\mathbf{g}) = \dot{\mathbf{f}}g + f\dot{\mathbf{g}}, \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\mathbf{g}) \qquad = \dot{\mathbf{f}}g + f\dot{\mathbf{g}}, \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{f}\mathbf{g}) = \dot{\mathbf{f}}g + f\dot{\mathbf{g}},$$

which can be proved by applying the product rule to the component.

Lecture 3

26 Jan. 10:00