Probability

Jonathan Gai

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Contents

Lecture 1: Probability Space

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Example. If we have a die with outcomes 1, 2, . . . , 6.

1. $\mathbb{P}(2) = \frac{1}{6}$
2. $\mathbb{P}(\text{multiple of } 3) = \frac{2}{6} = \frac{1}{3}$
3. $\mathbb{P}(\text{pair or a multiple of } 3) = \frac{4}{6} = \frac{2}{3}$

1 Formal Setup

We try to define a probability space rigorously in this section.

Definition 1.1 (Probability Space). We have the following,

1. Sample space Ω, a set of outcomes.
2. $\mathcal{F}$, a collection of subsets of Ω (called events).
3. $\mathcal{F}$ is a σ-algebra if
   (a) **F1**: $\Omega \in \mathcal{F}$
   (b) **F2**: if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
   (c) **F3**: For all countable collections $\{A_n\}$ in $\mathcal{F}$, $\cup_n A_n \in \mathcal{F}$.

Given σ-algebra $\mathcal{F}$ on Ω, function $\mathbb{P} : \mathcal{F} \to [0,1]$ is a probability measure if

1. **P1**: The probability function is nonnegative.
2. **P2**: $\mathbb{P}(\Omega) = 1$
3. **P3**: For all countable collection $\{A_n\}$ of disjoint events in $\mathcal{F}$, we have $\mathbb{P}(\cup_n A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$.

Then $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Problem. Why $\mathbb{P} : \mathcal{F} \to [0,1]$, not $\mathbb{P} : \Omega \to [0,1]$?

We will justify the definition in the following examples.

Example. When Ω is finite or countable,

1. In general: $\mathcal{F} = \mathcal{P}(\Omega)$.
2. $\mathbb{P}(2)$ is shorthand for $\mathbb{P}(\{2\})$.
3. $\mathbb{P}$ is determined by $\mathbb{P}(\{w\}), \forall w \in \Omega$.

Remark. When Ω is uncountable, a probability space behaves differently, as shown in the following example.

Example. If $\Omega = [0,1]$, and we want to choose a real number, all equally likely.

If $\mathbb{P}\{0\} = \alpha > 0$, then $\mathbb{P}(\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}\} = n\alpha)$. This cannot happen if n large, because we would have $\mathbb{P} > 1$. So $\mathbb{P}(\{0\}) = 0$ or undefined.

Example. When Ω is infinitely countable (e.g., $\Omega = \mathbb{N}$ or $\Omega = \mathbb{Q} \cap [0,1]$), however, it is not possible to choose uniformly. Suppose it is possible, there are two possibilities

- If $\mathbb{P}(\{\omega\}) = \alpha \quad \forall \omega \in \Omega$,

  then $\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = \infty$. $\lightning$
- If $\mathbb{P}(\{\omega\}) = 0 \quad \forall \omega \in \Omega$,

  then $\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 0$. $\lightning$

So it is not possible to have one such uniform probability space. But that's fine as there exists many other interesting probability measures on a infinite countably set.

Property. From the axioms, we want to prove the following properties of a probability space.

1. $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

   Proof. A, A^c disjoint. $A \cup A^c = \Omega$. So $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$ ■

2. $\mathbb{P}(\varnothing) = 0$
3. If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
4. $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

1.1 Examples of Probability Spaces

Example. Here we list some concrete examples of probability spaces.

1. Ω finite, $\Omega = \{w_1, \dots, w_n\}$, $\mathcal{F}$ = all subsets under uniform choice.

   $\mathbb{P} : \mathcal{F} \to [0,1], \mathbb{P}(A) = \frac{|A|}{|\Omega|}$. In particular: $\mathbb{P}(\{w\}) = \frac{1}{|\Omega|} \forall w \in \Omega$.

2. If we are choosing without replacement n indistinguishable marbles that are labelled $\{1, \dots, n\}$. Pick $k \leq n$ marbles uniformly at random.

   Here we have $\Omega = \{A \subseteq \{1, \dots, n\}, |A| = k, |\Omega| = \binom{n}{k}$.

3. If we have a well-shuffled deck of cards, and we uniformly chose permutation of 52 cards.

   $\Omega = \{\text{all permutations of 52 cards}\}$. $|\Omega| = 52!$.

   Then we have

   $$\mathbb{P}(\text{first three cards have the same suit}) = \frac{52 \cdot 12 \cdot 11 \cdot 49!}{52!} = \frac{22}{425}.$$

Lecture 2: Finite Probability Space

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Example (Coincidental Birthday). There we have n people, what is the probability that at least two share a birthday? To be precise, we first make the following assumptions,

- No leap years; (365 days in a year)
- All birthdays are equally likely.

We have the probability space

$$\begin{aligned}
\Omega &= \{1, \dots, 365\}^n \\
\mathcal{F} &= \mathcal{P}(\Omega) \\
A &= \{\text{at least 2 people share birthday}\} \\
A^c &= \{\text{all } n \text{ birthdays are different}\}.
\end{aligned}$$

So we have the probability

$$\mathbb{P}(A^c) = \frac{365 \times 364 \times \ldots \times (365-n-1)}{365^n},$$
$$\mathbb{P}(A) = 1 - \frac{365 \times 364 \times \ldots \times (365-n-1)}{365^n}.$$

Remark.

- We note several special n values,

$$\begin{aligned} n = 22 \quad &: \quad \mathbb{P}(A) \approx 0.479 \\ n = 23 \quad &: \quad \mathbb{P}(A) \approx 0.507 \\ n \geq 366 \quad &: \quad \mathbb{P}(A) = 1 \end{aligned}$$

- The probability of birthday is not equal in real life though. It is more likely to be born about 9 months after christmas.
- Sometimes it would be easier to calculate the probability of the complement of an event.

1.2 Combinatorial Analysis

If Ω is a finite set such that $|\Omega| = n$,

Problem. How many ways to partition Ω into k disjoint subsets $\Omega_1, \ldots \Omega_k$ with $|\Omega_i| = n_i$ $(\sum_{i=1}^{k} n_i = n)$?

The total number of ways M is

$$\begin{aligned} M &= \binom{n}{n_i}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2\cdots-n_{k-1}}{n_k} \\ &= \binom{n}{n_i}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n_k}{n_k} \\ &= \frac{n!}{n!(n-n_1)!} \times \frac{(n-n_1)!}{n_2!(n-n_1-n_2)!} \times \cdots \times \frac{(n-n_1-n_2-\cdots-n_{k-1})!}{x_k!0!} \\ &= \frac{n!}{n_1!n_2!\cdots n_k!} \\ &= \binom{n}{n_1, n_2, \ldots, n_k} \end{aligned}$$

which is called the *multinomial coefficient*, and denoted by the last term in the equations.

Remark. The ordering of the subsets do matter in this setting.

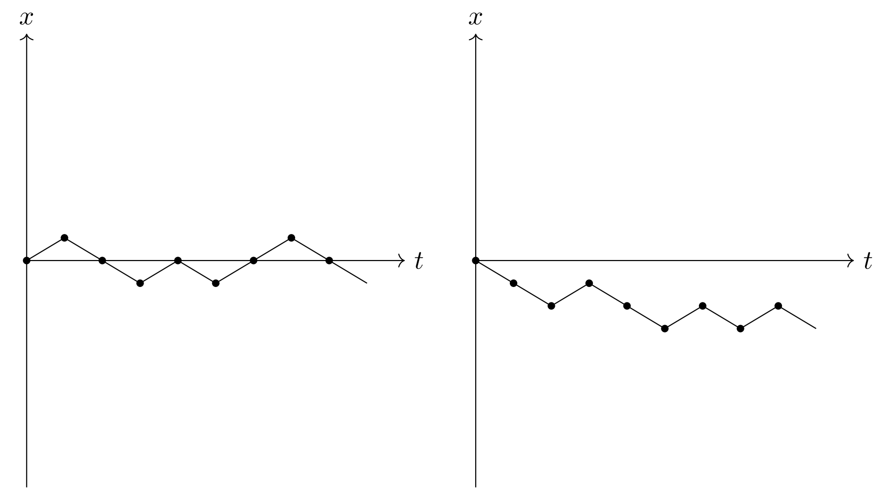

Figure 1: Random Walks

1.3 Random Walks

We have the following uniform probability space

$$\Omega = \{(x_0, x_1, \ldots, x_n) \mid x_0 = 0, |x_k - x_{k-1}| = 1, k = 1, \ldots, n\},$$
$$|\Omega| = 2^n.$$

Problem. What's $\mathbb{P}(x_n = 0)$ and $\mathbb{P}(x_n = n)$?

We have $\mathbb{P}(x_n = n) = \frac{1}{2^n}$.

When n is odd, $\mathbb{P}(x_n = 0) = 0$ because after every step the value changes parity. To find the probability when n is even, we need to choose $\frac{n}{2}$ ks for which $x_k = x_{k-1} + 1$, and the rest $x_k = x_{k-1} - 1$. So

$$\begin{aligned} \mathbb{P}(x_n = 0) &= 2^{-n}\binom{n}{n/2} \\ &= \frac{n!}{2^n[(\frac{n}{2})]!}. \end{aligned}$$

Problem. What happens when n is large?

We next present Stirling's Formula, and we adopt the following notation for the time being.

Notation. If (a_n), b_n are two sequences, we say $a_n \sim b_n$ as $n \to \infty$ if $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.

Theorem 1.1 (Stirling's Formula).

$$n! \sim \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \quad \text{as} \quad n \to \infty.$$

We also have the weaker version

$$\log(n!) \sim n \log n.$$