

The category Grp

6th June 2022

Todo list

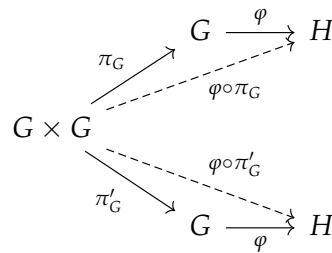
Problem 1

Let $\varphi : G \rightarrow H$ be a morphism in a category \mathcal{C} with products. Explain why there is a unique morphism

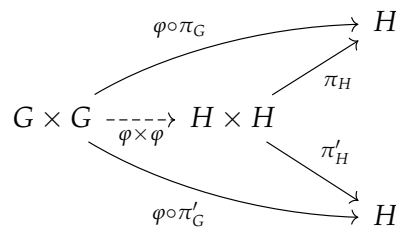
$$(\varphi \times \varphi) : G \times G \rightarrow H \times H.$$

(This morphism is defined explicitly for $\mathcal{C} = \text{Set}$ in §3.1.)

Solution. By definition of product, we know that $G \times G$ has natural projections to G



By universal property of product, we know that there exists a unique morphism $\varphi \times \varphi : G \times G \rightarrow H \times H$ such that the following diagram commutes.



Putting it together, we know that $\varphi \times \varphi : G \times G \rightarrow H \times H$ is the unique morphism such that the following diagram commutes.

$$\begin{array}{ccccc} G & \xleftarrow{\pi_G} & G \times G & \xrightarrow{\pi'_G} & G \\ \downarrow \varphi & & \downarrow \varphi \times \varphi & & \downarrow \varphi \\ H & \xleftarrow{\pi_H} & H \times H & \xrightarrow{\pi'_H} & H \end{array}$$

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Problem 4

Let G, H be groups, and assume that $G \cong H \times G$. Can you conclude that H is trivial? (Hint: No. Can you construct a counterexample?)

Solution. When G is finite, we must have H trivial by considering G and H as sets.

But it doesn't have to be the case when G is infinite. Consider the case when $H = \mathbb{Z}/2\mathbb{Z}$ and $G = \prod_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. Clearly, we must have $G \cong H \times G$. ■