

# Probability

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## Lecture 1: Probability Space

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**Example.** If we have a die with outcomes  $1, 2, \dots, 6$ .

1.  $\mathbb{P}(2) = \frac{1}{6}$
2.  $\mathbb{P}(\text{multiple of } 3) = \frac{2}{6} = \frac{1}{3}$
3.  $\mathbb{P}(\text{pair or a multiple of } 3) = \frac{4}{6} = \frac{2}{3}$

## 1 Formal Setup

We try to define a probability space rigorously in this section.

**Definition 1.1 (Probability Space).** We have the following,

1. Sample space  $\Omega$ , a set of outcomes.
2.  $\mathcal{F}$ , a collection of subsets of  $\Omega$  (called events).
3.  $\mathcal{F}$  is a  $\sigma$ -algebra if
  - (a) **F1:**  $\Omega \in \mathcal{F}$
  - (b) **F2:** if  $A \in \mathcal{F}$  then  $A^c \in \mathcal{F}$
  - (c) **F3:** For all countable collections  $\{A_n\}$  in  $\mathcal{F}$ ,  $\cup_n A_n \in \mathcal{F}$ .

Given  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$ , function  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$  is a probability measure if

1. **P1:** The probability function is nonnegative.
2. **P2:**  $\mathbb{P}(\Omega) = 1$
3. **P3:** For all countable collection  $\{A_n\}$  of disjoint events in  $\mathcal{F}$ , we have
$$\mathbb{P}(\cup_n A_n) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Then  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space.

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**Problem.** Why  $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ , not  $\mathbb{P} : \Omega \rightarrow [0, 1]$ ?

**Remark.** When  $\Omega$  is finite or countable,

1. In general:  $\mathcal{F} = P(\Omega)$ .
2.  $\mathbb{P}(2)$  is shorthand for  $\mathbb{P}(\{2\})$ .
3.  $\mathbb{P}$  is determined by  $\mathbb{P}(\{w\}), \forall w \in \Omega$ .

**Remark.** When  $\Omega$  is uncountable, a probability space behaves differently, as shown in the following example.

**Example.** If  $\Omega = [0, 1]$ , and we want to choose a real number, all equally likely.

If  $\mathbb{P}\{0\} = \alpha > 0$ , then  $\mathbb{P}(\{0, 1, \frac{1}{2}, \dots, \frac{1}{n}\}) = n\alpha$ . This cannot happen if  $n$  large, because we would have  $\mathbb{P} > 1$ . So  $\mathbb{P}(\{0\}) = 0$  or undefined.

**Property.** From the axioms, we want to prove the following properties of a probability space.

1.  $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ .

*Proof.*  $A, A^c$  disjoint.  $A \cup A^c = \Omega$ . So  $\mathbb{P}(A) + \mathbb{P}(A^c) = \mathbb{P}(\Omega) = 1$  ■

2.  $\mathbb{P}(\emptyset) = 0$
3. If  $A \subseteq B$ , then  $\mathbb{P}(A) \leq \mathbb{P}(B)$ .
4.  $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

## 1.1 Examples of Probability Spaces

**Example.** Here we list some concrete examples of probability spaces.

1.  $\Omega$  finite,  $\Omega = \{w_1, \dots, w_n\}$ ,  $\mathcal{F}$  = all subsets under uniform choice.

$\mathbb{P} : \mathcal{F} \rightarrow [0, 1], \mathbb{P}(A) = \frac{|A|}{|\Omega|}$ . In particular:  $\mathbb{P}(\{w\}) = \frac{1}{|\Omega|} \forall w \in \Omega$ .

2. If we are choosing without replacement  $n$  indistinguishable marbles that are labelled  $\{1, \dots, n\}$ . Pick  $k \leq n$  marbles uniformly at random.

Here we have  $\Omega = \{A \subseteq \{1, \dots, n\}, |A| = k\}$ ,  $|\Omega| = \binom{n}{k}$ .

3. If we have a well-shuffled deck of cards, and we uniformly chose permutation of 52 cards.

$\Omega = \{\text{all permutations of 52 cards}\}$ .  $|\Omega| = 52!$ .

Then we have

$$\mathbb{P}(\text{first three cards have the same suit}) = \frac{52 \cdot 12 \cdot 11 \cdot 49!}{52!} = \frac{22}{425}.$$