The category Grp

6th June 2022

Todo list

Problem 1

Let $\varphi : G \to H$ be a morphism in a category C with products. Explain why there is a unique morphism

$$(\varphi \times \varphi) : G \times G \to H \times H.$$

(This morphism is defined explicitly for $C = \mathsf{Set}$ in §3.1.)

Solution. By definition of product, we know that $G \times G$ has natural projections to G

$$\begin{tikzcd}
 & G \arrow[r, "\varphi"] & H \\
G \times G \arrow[ur, "\pi_G"] \arrow[urr, dashed, "\varphi\circ\pi_G"'] \arrow[dr, "\pi'_G"'] \arrow[drr, dashed, "\varphi\circ\pi'_G"] & & \\
 & G \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

By universal property of product, we know that there exists a unique morphism $\varphi \times \varphi : G \times G \to H \times H$ such that the following diagram commutes.

$$\begin{tikzcd}
 & & H \\
G \times G \arrow[urr, bend left, "\varphi\circ\pi_G"] \arrow[r, dashed, "\varphi\times\varphi"'] \arrow[drr, bend right, "\varphi\circ\pi'_G"'] & H \times H \arrow[ur, "\pi_H"'] \arrow[dr, "\pi'_H"] & \\
 & & H
\end{tikzcd}$$

Putting it together, we know that $\varphi \times \varphi : G \times G \to H \times H$ is the unique morphism such that the following diagram commutes.

$$\begin{tikzcd}
G \arrow[d, "\varphi"] & G \times G \arrow[l, "\pi_G"'] \arrow[r, "\pi'_G"'] \arrow[d, dashed, "\varphi\times\varphi"] & G \arrow[d, "\varphi"'] \\
H & H \times H \arrow[l, "\pi_H"'] \arrow[r, "\pi'_H"] & H
\end{tikzcd}$$

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Problem 4

Let G, H be groups, and assume that $G \cong H \times G$. Can you conclude that H is trivial? (Hint: No. Can you construct a counterexample?)

Solution. When G is finite, we must have H trivial by considering G and H as sets.

But it doesn't have to be the case when G is infinite. Consider the case when $H = \mathbb{Z}/2\mathbb{Z}$ and $G = \prod_{i \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. Clearly, we must have $G \cong H \times G$. ■