Geometry

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Contents

Lecture 1: Introduction

21 Jan. 11:00

1 Surfaces

1.1 Topological Surfaces

We start with some definitions.

Definition 1.1. A *topological surface* is a topological space Σ such that

1. **T1:** $\forall p \in \Sigma$ there is an open neighborhood $p \in U \subseteq \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subseteq \mathbb{R}^2$ with its usual Euclidean topology.
2. **T2:** Σ is Hausdorff and second countable.

Remark. We have the following remarks.

1. $\mathbb{R} \cong D(0,1)$, so homeomorphic to a disc is enough as stated in the definition.
2. A space X is *Hausdorff* if for $p \neq q \in X$, there exists disjoint open sets $p \in U$ and $q \in V$ in X.
3. A space X is *second countable* if it has a countable base i.e. $\exists \{u_i\}_{i \in \mathbb{N}}$ open sets s.t. every open set is a union of some u.
4. **T1** is the point and **T2** is for technical honesty.
5. If X is Hausdorff/ second countable, so are subspaces of X. In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

Example. Here we present some examples of topological surfaces.

1. $\mathbb{R}^2$, the plane.

2. Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where Z is closed:
   - $Z = \{0\}$,
   - $Z = \{(0,0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \dots)\}$.
3. Graphs:

   Let $f : \mathbb{R}^2 \to R$ be a continuous function. The graph $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$ (subspace topology).

   Recall that if X, Y are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with U open and V open.

   It has the feature that $f : Z \to X \times Y$ is continuous if and open if the two projective maps are continuous.

   Commutative diagram

   Application: $\Gamma_f \subseteq X \times Y$, if $f : X \to Y$ is continuous, if homeomorphic to X.

   So $\Gamma_f \cong \mathbb{R}^2$ for any $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous, so Γ_f is a topological surface.

   Note. As a topological surface, Γ_f is independent of f, but later on as a geometric object, it will reflect features of f.
4. The sphere (subspace topology):
   $$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$
   Stereographic projection

   Stereographic projection graph
   $$\begin{aligned} \pi_+ : S^2 \setminus \{(0,0,1)\} &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \end{aligned}$$
   Note. The map is continuous and has an inverse, π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism.

   Explicit formula for inverse

   Stereographic projection from the South Pole is also a homeomorphism from $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$.

   So S^2 is a topological surface:

   $\forall p \in S^2$, either p lies in the domain of π_+ or of π_- (or both) and so it lies in an open set homeomorphic to $\mathbb{R}^2$. (And Hausdorff and second countable from $\mathbb{R}^2$).

   Remark. S^2 has a global property as it is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.
5. The real projective place:

   The group $\mathbb{Z}/2$ acts on S^2 by homeomorphism via the *antipodal map* $a : S^2 \to S^2$.
   $$a(x, y, z) = (-x, -y, -t).$$
   i.e. There exists a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \text{Homeo}(S^2)$, such that it maps the non-identity element to the antipodal map.

Definition 1.2. The *real projective plane* is the quotient space of S^2 given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}/2\mathbb{Z}.$$

Lemma 1.1. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ∎

Graph of the sphere

Lemma 1.2. $\mathbb{RP}^2$ is a topological surface.

Proof. We check that it is Hausdorff:

Recall if X is a space and $q : X \to Y$ is a quotient map, $V \subseteq Y$ is open $\iff q^{-1}V \subseteq X$ open.

More balls

If $[p], [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in S^2$ are distinct antipodal pairs. Take small open discs around p, q and their antipodal images, as in the picture.

We can then take small balls $B_{\pm p}(\delta)$, $B_{\pm 1}(\delta)$, which intersects S^2 with open sets around $\pm q$ and $\pm p$.

$\mathbb{RP}^2$ is also second countable.

Let $\mathcal{U}$ be a countable base for the topology on S^2, such that for all $u \in \mathcal{U}$, the antipodal image is in $\mathcal{U}$.

Let $\bar{\mathcal{U}}$ be the family of open sets in $\mathbb{RP}^2$ of the form $q(u) \cup q(-u)$, $u \in \mathcal{U}$.

Now, if $v \in \mathbb{RP}^2$ is open, by definition $q^{-1}v$ is open in S^2, so $q^{-1}v$ contains some $u \in \mathcal{U}$, and hence contains $u \cup (-u)$. So $\bar{\mathcal{U}}$ is a countable base for the quotient topology on $\mathbb{RP}^2$ consider all such u that covers $q^{-1}v$.

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\bar{D}$ be a small (contained in an open hemisphere) closed disc neighborhood of $p \in S^2$.

If we consider q restricted to $\bar{D}$, it is a continuous map from a compact space to a Hausdorff space.

Also, on $\bar{D}$, the map q is injective. Recall "Topological inverse function theorem": A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. So q restricted to the disk is a homeomorphism.

It then induces another homeomorphism of q restricted to D, and open disk contained in $\bar{D}$. So $[q] \in q(D)$ has an open neighborhood in $\mathbb{RP}^2$ that is homeomorphic to an open disk, and we are done. ∎

Lecture 2: More Examples

24 Jan. 11:00

6. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

The *torus* $S^1 \times S^1$, with the subspace topology from $\mathbb{C}^2$ which is the product topology.

Lemma 1.3. The torus is a topological surface.

Proof. We consider the map

$$\begin{aligned} \mathbb{R}^2 &\to S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}, \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

Note that this induces map

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, "\hat{e}"] & \end{tikzcd}.$$

That is, on the equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$, e is constant on equivalence classes, so it induces a map of sets $\mathbb{R}/\mathbb{Z}^2$. We can think of it as a quotient space equipped with the quotient topology.

$\mathbb{R}/\mathbb{Z}^2$ is compact. A continuous map from a compact space to a Hausdorff space that is a bijection is a homeomorphism.

Note we already know that $S^1 \times S^1$ is compact and Hausdorff. (closed and bounded in $\mathbb{R}^4$).

As for $S^2 \to \mathbb{RP}^2$, pick $[p] = q(p), p \in \mathbb{R}$ and a small closed disk $\overline{D}(p) \in \mathbb{R}^2$ such that for all $(n,m) \in \mathbb{Z}^2$, we have $\overline{D}(p) \cap (\overline{D}(p) + (n,m)) = \varnothing$. Then e and q restricted to the small closed disk is injective. They are bijective continuous maps from compact spaces to Hausdorff spaces, so they are homeomorphisms. Restricting it further to a smaller open disk, and we have a neighborhood of $[p]$ that is homeomorphic to a disk. Since $[p]$ is arbitrary, and $S^1 \times S^1$ is a topological surface. ■

7. Let P be a planar Euclidean polygon. Assume the edges are *oriented* and paired, and for simplicity assume the Euclidean length for $e, \hat{e}$ are equal if they are paired.

   If $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say $f_{e\hat{e}} : e \to \hat{e}$.

   There maps generate an equivalence relation on P where we identify $x \in P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. $P/\sim$ (with the quotient topology) is a topological surface.

Example. The torus as $[0,1]^2/\sim$. We consider three different kinds of points.

If p is in the interior. We can find a small enough neighborhood that is injective, and again by topological inverse function theorem, that small enough disk is homeomorphic to an open disk.

If p is on the edge. Say $p = (0, y) \sim (1, y)$ and $\delta > 0$ is small enough such that a half disk of radius δ does not touch vertices. Define a map from the union of the half-disks to $B(0, \delta) \subseteq \mathbb{R}^2$ by $(x, y) \mapsto x, y - y_0$ and $(x, y) \mapsto (x-1, y-y_0)$ on each part of the half-disk. Recall if $X = A \cup B$ is a union of closed subspaces, and we have continuous maps $f : A \to Y, g : B \to Y$, and $f|_{A \cap B} = g|_{A \cap B}$, they define a continuous map from X to Y.