Geometry

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Contents

Lecture 1: Introduction

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1 Surfaces

1.1 Topological Surfaces

We start with some definitions.

Definition 1.1. A *topological surface* is a topological space Σ such that

1. **T1:** $\forall p \in \Sigma$ there is an open neighborhood $p \in U \subseteq \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subseteq \mathbb{R}^2$ with its usual Euclidean topology.
2. **T2:** Σ is Hausdorff and second countable.

Remark. We have the following remarks.

1. $\mathbb{R} \cong D(0,1)$, so homeomorphic to a disc is enough as stated in the definition.
2. A space X is *Hausdorff* if for $p \neq q \in X$, there exists disjoint open sets $p \in U$ and $q \in V$ in X.
3. A space X is *second countable* if it has a countable base i.e. $\exists \{u_i\}_{i \in \mathbb{N}}$ open sets s.t. every open set is a union of some u.
4. **T1** is the point and **T2** is for technical honesty.
5. If X is Hausdorff/ second countable, so are subspaces of X. In particular, Euclidean space has these properties. (For second countable, consider open balls with rational center and rational radius).

Example. Here we present some examples of topological surfaces.

1. $\mathbb{R}^2$, the plane.

2. Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \setminus Z$ where Z is closed:
   - $Z = \{0\}$,
   - $Z = \{(0,0)\} \cup \{(0, \frac{1}{n} \mid n = 1, 2, 3, \dots)\}$.
3. Graphs:

   Let $f : \mathbb{R}^2 \to R$ be a continuous function. The graph $\Gamma_f = \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\} \subseteq \mathbb{R}^2$ (subspace topology).

   Recall that if X, Y are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with U open and V open.

   It has the feature that $f : Z \to X \times Y$ is continuous if and open if the two projective maps are continuous. Commutative diagram

   Application: $\Gamma_f \subseteq X \times Y$, if $f : X \to Y$ is continuous, if homeomorphic to X.

   So $\Gamma_f \cong \mathbb{R}^2$ for any $f : \mathbb{R}^2 \to \mathbb{R}$ that is continuous, so Γ_f is a topological surface.

   Note. As a topological surface, Γ_f is independent of f, but later on as a geometric object, it will reflect features of f.
4. The sphere (subspace topology):

   $$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

   Stereographic projection Stereographic projection graph

   $$\pi_+ : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$$
   $$(x, y, z) \mapsto (\frac{x}{1-z}, \frac{y}{1-z})$$

   Note. The map is continuous and has an inverse, π_+ is a continuous bijection with continuous inverse, and hence a homeomorphism. Explicit formula for inverse

   Stereographic projection from the South Pole is also a homeomorphism from $S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2$.

   So S^2 is a topological surface:

   $\forall p \in S^2$, either p lies in the domain of π_+ or of π_- (or both) and so it lies in an open set homeomorphic to $\mathbb{R}^2$. (And Hausdorff and second countable from $\mathbb{R}^2$).

   Remark. S^2 has a global property as it is compact as a topological space, since it is a closed bounded set in $\mathbb{R}^3$.
5. The real projective place:

   The group $\mathbb{Z}/2$ acts on S^2 by homeomorphism via the *antipodal map* $a : S^2 \to S^2$.

   $$a(x, y, z) = (-x, -y, -t).$$

   i.e. There exists a homomorphism $\mathbb{Z}/2\mathbb{Z} \to \text{Homeo}(S^2)$, such that it maps the non-identity element to the antipodal map.

Definition 1.2. The *real projective plane* is the quotient space of S^2 given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/\mathbb{Z}/2\mathbb{Z}.$$

Lemma 1.1. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any straight line that goes through the origin meets the sphere exactly twice, and any such pair determines a straight line. ■

Graph of the sphere

Lemma 1.2. $\mathbb{RP}^2$ is a topological surface.

Proof. We check that it is Hausdorff:

Recall if X is a space and $q : X \to Y$ is a quotient map, $V \subseteq Y$ is open $\iff q^{-1}V \subseteq X$ open.

More balls

If $[p], [q] \in \mathbb{RP}^2$, then $\pm p, \pm q \in S^2$ are distinct antipodal pairs. Take small open discs around p, q and their antipodal images, as in the picture.

We can then take small balls $B_{\pm p}(\delta)$, $B_{\pm 1}(\delta)$, which intersects S^2 with open sets around $\pm q$ and $\pm p$.

$\mathbb{RP}^2$ is also second countable.

Let $\mathcal{U}$ be a countable base for the topology on S^2, such that for all $u \in \mathcal{U}$, the antipodal image is in $\mathcal{U}$.

Let $\bar{\mathcal{U}}$ be the family of open sets in $\mathbb{RP}^2$ of the form $q(u) \cup q(-u)$, $u \in \mathcal{U}$.

Now, if $v \in \mathbb{RP}^2$ is open, by definition $q^{-1}v$ is open in S^2, so $q^{-1}v$ contains some $u \in \mathcal{U}$, and hence contains $u \cup (-u)$. So $\bar{\mathcal{U}}$ is a countable base for the quotient topology on $\mathbb{RP}^2$ consider all such u that covers $q^{-1}v$.

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\bar{D}$ be a small (contained in an open hemisphere) closed disc neighborhood of $p \in S^2$.

If we consider q restricted to $\bar{D}$, it is a continuous map from a compact space to a Hausdorff space.

Also, on $\bar{D}$, the map q is injective. Recall "Topological inverse function theorem": A continuous bijection from a compact space to a Hausdorff space is a homeomorphism. So q restricted to the disk is a homeomorphism.

It then induces another homeomorphism of q restricted to D, and open disk contained in $\bar{D}$. So $[q] \in q(D)$ has an open neighborhood in $\mathbb{RP}^2$ that is homeomorphic to an open disk, and we are done. ■

Lecture 2: More Examples

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6. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

The *torus* $S^1 \times S^1$, with the subspace topology from $\mathbb{C}^2$ which is the product topology.

Lemma 1.3. The torus is a topological surface.

Proof. We consider the map

$$\begin{aligned} \mathbb{R}^2 &\to S^1 \times S^1 \subseteq \mathbb{C} \times \mathbb{C}, \\ (s,t) &\mapsto (e^{2\pi i s}, e^{2\pi i t}). \end{aligned}$$

Note that this induces map

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"] & S^1 \times S^1 \\ \mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, "\hat{e}"] & \end{tikzcd} .$$

That is, on the equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$, e is constant on equivalence classes, so it induces a map of sets $\mathbb{R}/\mathbb{Z}^2$. We can think of it as a quotient space equipped with the quotient topology.

$\mathbb{R}/\mathbb{Z}^2$ is compact. A continuous map from a compact space to a Hausdorff space that is a bijection is a homeomorphism.

Note we already know that $S^1 \times S^1$ is compact and Hausdorff. (closed and bounded in $\mathbb{R}^4$).

As for $S^2 \to \mathbb{RP}^2$, pick $[p] = q(p), p \in \mathbb{R}$ and a small closed disk $\overline{D}(p) \in \mathbb{R}^2$ such that for all $(n,m) \in \mathbb{Z} \setminus \{(0,0)\}$, we have $\overline{D}(p) \cap (\overline{D}(p) + (n,m)) = \varnothing$. Then e and q restricted to the small closed disk is injective. They are bijective continuous maps from compact spaces to Hausdorff spaces, so they are homeomorphisms. Restricting it further to a smaller open disk, and we have a neighborhood of $[p]$ that is homeomorphic to a disk. Since $[p]$ is arbitrary, and $S^1 \times S^1$ is a topological surface. ∎

Let P be a planar Euclidean polygon. Assume the edges are *oriented* and paired, and for simplicity assume the Euclidean length for $e, \hat{e}$ are equal if they are paired.

If $\{e, \hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respecting their orientations, say $f_{e\hat{e}} : e \to \hat{e}$.

There maps generate an equivalence relation on P where we identify $x \in P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4. $P/\sim$ (with the quotient topology) is a topological surface.

Example. The torus as $[0,1]^2/\sim$. We consider three different kinds of points.

If p is in the interior. We can find a small enough neighborhood that is injective, and again by topological inverse function theorem, that small enough disk is homeomorphic to an open disk.

If p is on the edge. Say $p = (0, y) \sim (1, y)$ and $\delta > 0$ is small enough such that a half disk of radius δ does not touch vertices. Define a map from the union of the half-disks to $B(0, \delta) \subseteq \mathbb{R}^2$ by $(x, y) \mapsto x, y - y_0$ and $(x, y) \mapsto (x-1, y-y_0)$ on each part of the half-disk. Recall if $X = A \cup B$ is a union of closed subspaces, and we have continuous maps $f : A \to Y, g : B \to Y$, and $f|_{A\cap B} = g|_{A\cap B}$, they define a continuous map from X to Y.

Explicitly: f_u, f_v are continuous on $u, v \in [0, 1]^2$ on each of the two half-disks, so they induce a continuous map on $qU, qV \subseteq T^2$.

In T^2, the two maps overlap but agree, so by the recalled fact, we can define a map from the torus to a disk in $\mathbb{R}^2$.

Finally, we use the usual argument (pass a closed disk, use T.I.F.T, pass back its interior), then it has an open neighborhood homeomorphic to a disc.

Analogously at the vertex of $[0, 1]^2$, we split it into 4 maps.

This shows that ${}^{[0,1]^2}/_{\sim}$ is a topological surface.

Proof. For a general planar polygon, We can consider the suitable disc for interior points, and points on the edge as well.

Our equivalence relation induces an equivalence relation on the vertices in the obvious fashion. If $v \in \mathrm{Vert}(P)$ has r vertices in its equivalence class. There are r sectors in P with a total angle of α_v. Any sector can be identified with a standard sector with angle $\frac{2\pi}{r}$. Combining the sectors, and we would get a disc as required.

If $r = 1$, we just glue the two neighboring edges together, and we get a cone. If we look from above, we get an open disc centered around the vertex.

These open neighborhoods of points in ${}^{P}/_{\sim}$ show that ${}^{P}/_{\sim}$ is locally homeomorphic to a disc. We can also see ${}^{P}/_{\sim}$ is Hausdorff and second countable:

It's Hausdorff because for any non-equivalent points, we can find discs with small enough radius that lie in different equivalence classes. They are open disjoint sets in the quotient space as well. So ${}^{P}/_{\sim}$ is Hausdorff.

For second countability, I can consider disks in the interior of P with rational centers and radii, and for e in the edge of P, there is an isometry from e to an interval. And the points on the edge with correspond to rational centered and rational radius discs. And at vertices allow rational radius sectors. This gives me a countable base. ■

Remark. This might look less rigorous, but it conveys the same information as providing explicit homeomorphisms.

Lecture 3

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7. Given topological surfaces Σ_1, Σ_2, I can remove an open disc from each and glue the resulting circles.

   Explicitly, I take $\Sigma_1 \setminus D_1 \coprod \Sigma_2 \setminus D_2$ and impose a quotient relation.

   $$\theta \in \partial D_1 \sim \theta \in \partial D_2$$

   where θ parametrizes $S^1 = \partial D_i$.

   The result $\Sigma_1 \# \Sigma_2$ is called the *connect sum* of Σ_1 and Σ_2. (In principle this depends on any choices, suppressed from the notation).

   Lemma 1.5. The connect sum $\Sigma_1 \# \Sigma_2$ is a topological surface.

Lecture 4

28 Jan. 2022

Definition 1.3. A *subdivision* of a compact topological surface Σ comprises

1. a finite set $V \in \Sigma$ of vertices;
2. a finite collection $E = \{e_i : [0,1] \to \Sigma\}_{i \in E}$ of *edges* such that
   - for all i, e_i is a continuous injection on its interior and $e_i^{-1} V = \{0, 1\}$,
   - e_i and e_j have disjoint image except perhaps at their endpoints in V;
3. such that each connected component of $\Sigma \setminus (\cup e_i[0,1] \cup V)$ is homeomorphic to an open disc called a *face*. (So the closure of a face has boundary $\overline{F} \setminus F$ lying in $E \cup V$).

A subdivision is a *triangulation* if each *closed* face (closure of a face) contains exactly 3 edges, and two closed faces are disjoint or meet in exactly one edge (or possible just one vertex).

Example.

- A cube displays a subdivision of S^2.
- A tetrahedron displays a triangulation of S^2.
- We can describe subdivisions using planar polygons. For example, the normal depiction of T^2 has 1 vertex, 2 edges and 1 face.
- By our definition, we can have degenerate subdivisions like a subdivision of S^2 with 1 vertex, 0 edge and 1 face.

Definition 1.4. The *Euler characteristic* of a subdivision is the number $\#V - \#E + \#F$.

Theorem 1.1.

1. Every compact topological surface admits subdivision, and indeed triangulation.
2. The Euler characteristic denoted $\chi(\Sigma)$ does not depend on the choice of subdivision and defines a topological invariant of the surface. (depends only on the homeomorphic type of Σ)

Example.

1. $\chi(S^2) = 2$;
2. $\chi(T^2) = 0$;
3. If Σ_1 and Σ_2 are compact topological surfaces, we can form $\Sigma_1\#\Sigma_2$ by removing an open disc $D_i \subseteq \Sigma_i$ which is a face of a triangulation, and gluing the boundary circles ∂D_i by a homeomorphism taking edges to edges.

   The resulting surface $\Sigma_1\#\Sigma_2$ inherits a subdivision, and we have

   $$\chi(\Sigma_1\#\Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2.$$

   In particular, if we take a surface with g holes, which is $\Sigma_g = \#_{i=1}^{g} T^2$, then $\chi(\Sigma_g) = 2 - 2g$; g is called the *genus* of Σ.

Remark.

1. Part 1 is hard to prove.
2. You should believe Part 2 since I can turn a subdivision into a triangulation, and I can relate triangulations by local moves. It is easy to check both subdividing and switch diagonal preserves χ.

   But it is hard to rigorize this result, and you learn essentially nothing from the combinatorial proof. A much cleaner approach is developed in Part II algebraic topology.

Recalled if Σ is a topological surface, every $p \in \Sigma$ lies in an open neighborhood $p \in u \subseteq \Sigma$ with u homeomorphic to an open disc (or equivalently to $\mathbb{R}^2$).

Definition 1.5. A pair (u, ϕ) where u is an open set in Σ and $\phi : u \to V$ an open set in $\mathbb{R}^2$ which is a homeomorphism is called a *chart* for Σ. (If $p \in u$ we might say "a chart for Σ at p")

A collection $\{(u_i, \phi_1)_{i\in I}\}$ of charts such that $\cup_{i\in I} = \Sigma$ is called an *atlas* for Σ.

The inverse $\sigma = \phi^{-1} : v \to u \in \Sigma$ is called a *local parametrization* for Σ.

Example.

1. If $Z \subseteq \mathbb{R}^2$ is a closed set, $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas with one chart that is $(\mathbb{R}^2, \mathbf{id})$.

2. For S^2, we have an atlas with 2 charts, the 2 stereographic projections.

Definition 1.6. Let (u_i, ϕ_i) be charts containing p, the map

$$\phi_2 \circ \phi_1^{-1}\big|_{\phi_1(u_1 \cap u_2)}$$

is called the *transition map* between the charts. This is a homeomorphism of open sets in $\mathbb{R}^2$

Recall that if $V \subseteq \mathbb{R}^2$ and $V' \subseteq \mathbb{R}^m$ open subsets, then a map $f : V \to V'$ is called *smooth* if it is infinitely differentiable; that is, it has partial derivatives of all orders of all variables.

If $n = m$, a homeomorphism $f : V \to V'$ is called a *diffeomorphism* if it is smooth and has smooth inverse.

Definition 1.7. An *abstract smooth surface* Σ is a topological surface with an atlas of charts $\{(u_i, \phi_i)\}_{i \in I}$ such that all transition maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(u_i \cap u_j) \to \phi_i(u_i \cap u_j)$$

are diffeomorphisms of open sets in $\mathbb{R}^2$.

Note. It would not make sense to ask for the ϕ_i themselves to be smooth, as Σ is just a topological space.

Example (Example Sheet 2). The atlas of 2 charts with stereographic projections gives S^2 the structure of an abstract smooth surface.

Example. The torus $T^2 = {}^{\mathbb{R}^2}\!/\!_{\mathbb{Z}^2}$ is also an abstract smooth surface. Recall we obtained charts from the inverse of the projection restricted to small discs in $\mathbb{R}^2$, the ones that are disjoint from translation by $\mathbb{Z}^2 \setminus \{(0,0)\}$.

The transition maps are the translations, so T^2 inherits the structure of an abstract smooth surface.

Lecture 5

31 Jan. 2022

Explicitly, for

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"'] & S^1 \times S^1 \\ {}^{\mathbb{R}^2}\!/\!_{\mathbb{Z}^2} \arrow[ur, "\hat{e}"] & \end{tikzcd}.$$

Consider the atlas $\{(e(D_\epsilon(x,y)), e^{-1}$ on this image$\}$ where $\epsilon < \frac{1}{3}$, so the discs are disjoint from their nontrivial translations.

These are charts on T^2, and the transition maps are (restricted to the appropriate domain of) translation in $\mathbb{R}^2$. So T^2 has the structure (via this atlas) of an abstract smooth surface.

Remark (Philosophical). Being a topological surface is **structure**. (One can ask if a topological surface X is a topological surface)

Being an abstract smooth surface is **data**. (I have to give you an atlas of charts with smooth transition maps with smooth inverses. There could be many choices)

Definition 1.8. Let Σ be an abstract smooth surface, and $f : \Sigma \to \mathbb{R}^n$ a continuous map.

We say f is *smooth at a* $p \in \Sigma$ if whenever (u, ϕ) is a chart at p belonging to the smooth atlas for Σ, the map

$$f \circ \phi^{-1} : \phi(u) \longrightarrow \mathbb{R}^2$$

is smooth at $\phi(p) \in \mathbb{R}^2$.

Note. Smoothness of f at p is independent of choice of chart (u, ϕ) at p in the smooth atlas, because the transition map between two such charts is diffeomorphic.

Definition 1.9. If Σ_1, Σ_2 are abstract smooth surfaces, a map $f : \Sigma_1 \to \Sigma_2$ is *smooth* if it is smooth in the local charts. That is give (u, ϕ) at p and (u', ψ) at $f(p)$ (in our chosen smooth atlases), we want $\psi \circ f \circ \phi^{-1}$ smooth at $\phi^{-1}(p)$.

Note. Again, smoothness of f does not depend on their choices of charts at $p, f(p)$ provided I take charts from our smooth atlas.

Definition 1.10. Abstract smooth surfaces Σ_1, Σ_2 are *diffeomorphic* if there exists a homeomorphism

$$f : \Sigma_1 \to \Sigma_2$$

which is smooth and has smooth inverse.

Remark. We often pass from a given smooth atlas for an abstract smooth surface Σ to the *maximal* "compatible" such atlas. That is, I add to my atlas $\{(u_i, \phi_2)_{i \in I}\}$ for Σ all charts (V, ϕ) with the property that the transition maps are still all diffeomorphisms. (Technically, we use Zorn's Lemma)

If V, V' open sets in $\mathbb{R}^2$, then $f : V \to V'$ is smooth if it is infinitely differentiable.

Definition 1.11. If $Z \subseteq \mathbb{R}^2$ is an arbitrary subset, we say $f : Z \to \mathbb{R}^m$ continuous is smooth at $p \in Z$ if there exists open ball $p \in B \subseteq \mathbb{R}^n$ and a smooth map $F : B \to \mathbb{R}^m$ such that

$$F|_{B \cap Z} = f|_{B \cap Z}.$$

That is, f is locally the restriction of a smooth map defined on an open set.

Definition 1.12. If $X \subseteq \mathbb{R}^2$ and $Y \subseteq \mathbb{R}^2$ are subsets, we say X and Y are *diffeomorphic* if there exists continuous map $f : X \to Y$ such that f is a smooth homeomorphism with smooth inverse.

Definition 1.13. A *smooth surface in* R^3 is a subspace $\Sigma \subseteq \mathbb{R}^3$ such that for all $p \in \Sigma$, $\exists$ and open set $p \in U \subseteq \Sigma$ such that U is diffeomorphic to an open set in $\mathbb{R}^2$.

That is, for all $p \in \Sigma$, there exists open ball $p \in B \subseteq \mathbb{R}^3$ such that if $U = B \cap \Sigma$ and a smooth map to an open set $f : B \to V \subseteq \mathbb{R}^2$ such that $f|_U : U \to V$ is a homeomorphism, and the inverse map $V \to U \subseteq \Sigma \subseteq \mathbb{R}^3$ is also smooth.

Theorem 1.2. For a subset $\Sigma \in \mathbb{R}^3$, the following are equivalent:

1. Σ is a smooth surface in $\mathbb{R}^3$.
2. Σ is locally the graph of a smooth function over one of the co-ordinate planes. That is, $\forall p \in \Sigma$, exists open $p \in B \subseteq \mathbb{R}^3$ and open $V \subseteq \mathbb{R}^3$ such that

$$\Sigma \cap B = \{(x, y, g(x, y)) \mid g : V \to \mathbb{R} \text{ smooth}\},$$

   or a graph over the $x - z$ or $y - z$ plane, locally.
3. Σ is locally cut out by a smooth function with nonzero derivative. That is, $\forall p \in \Sigma$, there exists open $p \in B \subseteq \mathbb{R}^3$ and $f : B \to \mathbb{R}$ smooth such that

$$\Sigma \cap B = f^{-1}(0) \;\&\; Df|_x \neq 0 \;\forall x \in B.$$

4. Σ is locally the image of an *allowable* parametrization. That is, if $p \in \Sigma$, there exists open $p \in U \subseteq \Sigma$ and smooth chart map $\sigma : V \subseteq \mathbb{R}^2 \to U \subseteq \mathbb{R}^3$ such that σ is homeomorphic and $D\sigma|_x$ has rank 2 for all $x \in V$.

Remark. Part 2 says that if Σ is a smooth surface in $\mathbb{R}^3$, each $p \in \Sigma$ belongs to a chart (U, ϕ) where ϕ is one of (the restriction of) $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$. (The co-ordinate plane projections)

For example, the transition map

$$(x, y) \mapsto (x, y, g(x, y)) \mapsto (y, g(x, y))$$

has inverse

$$(y, z) \mapsto (h(y, z), y, z) \mapsto (h(y, z), y).$$

That is, all the transition maps between such charts contains projection maps and the smooth maps involved in defining Σ as a graph. This gives Σ the structure of an abstract smooth surface.

Our next goal is to prove the theorem. The non-trivial work comes from the inverse function theorem.

Theorem 1.3 (Inverse Function Theorem). Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be continuously differentiable. Let $p \in U$ and $f(p) = q$ and suppose $Df|_p$ is invertible.

Then there is an open neighborhood V of q and a differentiable map $g : V \to \mathbb{R}^n$ and $g(q) = p$ with image an open neighborhood $U' \subseteq U$ of p such that $f \circ g = \mathbf{id}_V$.

If f is smooth, so is g.

Remark. We also have $Dg|_q = (Df|_p)^{-1}$ by chain rule.

Lecture 6

2 Feb. 2022

Inverse Function Theorem is about maps $f : \mathbb{R}^n \to \mathbb{R}^n$ with $Df|_p$ an isomorphism. If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, can ask about what to conclude if $Df|_p$ is onto. That is, $Df|_p = (\frac{\partial f_i}{\partial x_j})_{n\times m}$ has full rank. After permuting the co-ordinates, we can assume last m columns are independent. Then we have the following.

Theorem 1.4 (Implicit Function Theorem). Let $p = (x_0, y_0) \in U \subseteq \mathbb{R}^k \times \mathbb{R}^l = \{(x, y) \mid x \in \mathbb{R}^k, y \in \mathbb{R}^l\}$ with U open and a map $f : U \to \mathbb{R}^l$ that sends $p \mapsto 0$ with $(\frac{\partial f_i}{\partial y_i})_{l\times l}$ an isomorphism at p. Then there is an open neighborhood of $x_0 \in V \subseteq \mathbb{R}^k$ and a continuously differentiable map $g : V \to \mathbb{R}^l$ that maps $x_0 \mapsto y_0$ such that if $(x, y) \in U \cap (V \times \mathbb{R}^l)$, then

$$f(x, y) = 0 \iff y = g(x).$$

If f is smooth, so is g.

Proof. Introduce

$$\begin{aligned} F\colon U &\longrightarrow \mathbb{R}^k \times \mathbb{R}^l \\ (x, y) &\longmapsto (x, f(x, y)) \end{aligned}$$

Then $DF = \begin{pmatrix} I & \frac{\partial f_i}{\partial x_j} \\ 0 & \frac{\partial f_i}{\partial y_j} \end{pmatrix}$, so $DF|_{(x_0,y_0)}$ is an isomorphism.

So inverse function theorem says that F is locally invertible near $F(x_0, y_0)$. That is the point $(x_0, f(x_0, y_0)) = (x_0, 0)$. Take a product open neighborhood

$$(x_0, 0) \in V \times V' \quad V \subseteq \mathbb{R}^k, V' \subseteq \mathbb{R}^l$$

and the continuously differentiable inverse

$$G : V \times V' \to U' \subseteq U \subseteq \mathbb{R}^k \times \mathbb{R}^l$$

has the property $F \circ G = \mathbf{id}_{V \times V'}$. Write $G(x, y) = (\phi(x, y), \psi(x, y))$, then

$$F \circ G(x, y) = (\phi(x, y), f(\phi(x, y), \psi(x, y))) = (x, y).$$

So $\phi(x, y) = x$. So G has the form

$$(x, y) \mapsto (x, \phi(x, y))$$

and $f(x, \psi(x, y)) = y$ when $(x, y) \in V \times V'$, so $f(x, y) = 0 \iff y = \psi(x, 0)$.

Define $g : V \to \mathbb{R}^l \quad x \mapsto \psi(x, 0)$ and the map does indeed send $x_0 \mapsto y_0$, and this is what we want. ■

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth, $f(x_0, y_0) = 0$, and suppose $\frac{\partial f}{\partial y}\Big|_{(x_0, y_0)}$. Then there exists a smooth $g : (x_0 - \epsilon, x_0 + \epsilon) \to \mathbb{R}$, $g(x_0) = y_0$ such that

$$f(x, y) = 0 \iff y = g(x)$$

for (x, y) in some open neighborhood of (x_0, y_0).

Since $f(x, g(x)) = 0$, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} g'(x) = 0 \implies g'(x) = \frac{f_x}{f_y}$$

noting that $f_y \neq 0$ near (x_0, y_0).

The level set $f(x, y) = 0$ is "implicitly" described via g, a function that we have an integral expression.

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be smooth, and $f(x_0, y_0, z_0) = 0$. Let $\Sigma = f^{-1}(0)$, and assume that $Df|_{(x_0, y_0, z_0)} \neq 0$. Permuting co-ordinates if necessary, $\frac{\partial f}{\partial z}\Big|_{(x_0, y_0, z_0)} \neq 0$. Then there exists an open neighborhood $(x_0, y_0) \in V \subseteq \mathbb{R}^2$ and a smooth $g : V \to \mathbb{R}$ that maps $(x_0, y_0) \mapsto z_0$ such that in open set $(x_0, y_0, z_0) \in U$,

$$f^{-1}(0) \cap U = \Sigma \cap U = \mathrm{Graph}(g).$$

That is, the $f^{-1}(0) \cap U = \{(x, y, g(x, y)) \mid (x, y) \in V\}$.

Recall Theorem (1.2), and we present a proof.

Proof.

1. Part 2 implies all others.

- If Σ locally is $\{(x, y, g(x, y))\}$, then one gets a chart from the projection π_{xy} which is smooth and defined on an open neighborhood of points of Σ in its domain. (c.f. last lecture)
- If Σ is locally $\{(x, y, g(x, y))\}$, it's locally cut out by $f(x, y, t) = t - g(x, y)$. Clearly $\frac{\partial f}{\partial z} \neq 0$. So Part 2 implies Part 3.
- The parametrization $\sigma(x, y) := (x, y, g(x, y))$ is allowable as it is smooth and $\sigma_x = (1, 0, g_x)$ and $\sigma_y = (0, 1, g_y)$ are linearly independent and σ is injective.

2. Part 1 implies Part 4 is part of the definition of being a smooth surface in $\mathbb{R}^3$ and hence locally diffeomorphic to $\mathbb{R}^2$. (At $p \in \Sigma$, Σ locally diffeomorphic to $\mathbb{R}^2$ and the inverse of such a local diffeomorphism gives an allowable parametrization)
3. Part 3 to Part 2 was the second example of Implicit Function Theorem.
4. We will show Part 4 implying Part 2 and Part 1, and we are done.

   Let $p \in \Sigma$ and $\sigma : V \subseteq \mathbb{R}^2 \to \Sigma \subseteq \mathbb{R}^3$ satisfies $\sigma(0) = p \in U \subseteq \Sigma$, then if $\sigma = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v))$, and

$$D\sigma = \begin{pmatrix} \frac{\partial \sigma_1}{\partial u} & \frac{\partial \sigma_1}{\partial v} \\ \frac{\partial \sigma_2}{\partial u} & \frac{\partial \sigma_2}{\partial v} \\ \frac{\partial \sigma_3}{\partial u} & \frac{\partial \sigma_3}{\partial v} \end{pmatrix}$$

   has rank 2, and so there exists two rows defining an invertible matrix at 0. Suppose without loss of generality that the first 2 rows define an invertible matrix, and let $\mathrm{pr} := \pi_{xy}$ and consider the map $\mathrm{pr} \circ \sigma : V \to \mathbb{R}^2$.

   By Inverse Function Theorem (since the derivative is an isomorphism at 0) says that it is locally invertible. So Σ is a graph considering the inverse of the projection map, i.e. Part 2 holds.

   Moreover, if we let $\phi = \mathrm{pr} \circ \sigma$, then

$$B(p, \delta) \subseteq \mathbb{R}^3 \ni (x, y, z) \mapsto \phi^{-1}(x, y).$$

   Here $\phi^{-1} : W \subseteq \mathrm{pr}(B(p, \delta)) \to \Sigma$ which is locally defined, smooth on an open set in $\mathbb{R}^3$ which is a local inverse of σ. That is, $\sigma^{-1} = \phi^{-1} \circ \mathrm{pr}$.

   So Part 4 implies Part 1.

■

Example. The unit sphere $S^2 \subseteq \mathbb{R}^2$ is $f^{-1}(0)$ for

$$f : \mathbb{R}^3 \to \mathbb{R} \quad (x, y, z) \mapsto x^2 + y^2 + z^2 - 1.$$

If $p \in S^2$, $Df|_p \neq 0$, so S^2 is a smooth surface in $\mathbb{R}^3$.

Example (Surfaces of revolution). Let $\gamma : [a, b] \to \mathbb{R}^3$ be a smooth map with image in the $x - z$ plane:

$$\gamma(t) = (f(t), 0, g(t)).$$

We assume that γ is injective, $\gamma'(t) \neq 0$ for all t and $f > 0$.

The associated *surface of revolution* has allowable parametrizations

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $(u,v) \in (a,b) \times (\theta, \theta + 2\pi)$ for some initial fixed $\theta \in [0, 2\pi]$.

Note. We first think about the derivatives

$$\begin{aligned}\sigma_u &= (f_u(u)\cos v, f_u(u)\sin v, g_u(u)),\\ \sigma_v &= (-f(u)\sin v, f(u)\cos v, 0).\end{aligned}$$

And we have $\|\sigma_u \times \sigma_v\| = f^2((f')^2 + (g')^2) \neq 0$ by conditions on the original curve γ, so $D\sigma$ has rank 2 and σ is injective on given domain, so allowable.

Example. The orthogonal group $O(3)$ acts on S^2 by diffeomorphisms.

Proof. Any $A \in O(3)$ defines an invertible linear (thus smooth) map $\mathbb{R}^3 \to \mathbb{R}^3$ preserving S^2. So the induced map on S^2 is by homeomorphism which is smooth in our definition. (It is smooth globally, so smooth locally as well)

Compare it with the action of the Möbius group on $S^2 = \mathbb{C} \cup \{\infty\}$. ■

Lecture 7

4 Jan. 2020

If V, V' are open subsets of $\mathbb{R}^2$, and $f : V \to V'$ a diffeomorphism, then at $x \in V$, $Df|_x \in GL_2(\mathbb{R})$. (invertible as f is a diffeomorphism)

Let $GL_2^+(\mathbb{R} \leq GL_2(\mathbb{R}))$ be the subgroup of matrices of positive determinant. We say f is *orientation-preserving* if $Df|_x \in GL_2^+(\mathbb{R})$ for all $x \in V$.

Definition 1.14. An abstract smooth surface Σ is *orientable* if it admits an atlas $\{(u_i, \phi_i) \mid \cup u_i = \Sigma\}$ such that the transition maps are orientation-preserving diffeomorphisms of open subsets of $\mathbb{R}^2$.

A choice of such an atlas is an *orientation* of Σ, and we say Σ is *oriented*.

Remark. An oriented atlas (in this sense) belongs to a maximal compatible oriented smooth atlas.

Lemma 1.6. If Σ_1 and Σ_2 are abstract smooth surfaces, and they're diffeomorphic, then Σ_1 is orientable if and only if Σ_2 is orientable.

Proof. Supposed $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism, and Σ_2 is orientable and equipped with an oriented smooth atlas.

Let's consider the atlas on Σ_1 and charts of the form $(f^{-1}u, \phi \circ f|_{f^{-1}u})$ where (u, ϕ) is a chart at $f(p)$ in our atlas for Σ_2.

A transition map between 2 such is exactly a transition map in the Σ_2 atlas.

To prove it differently, in the maximal smooth atlas we already have for Σ_1, we'll allow charts $(\tilde{U}, \tilde{\psi})$ at p exact when for any (U, ψ) at $f(p)$ in the Σ_2 atlas, the map $\psi \circ f \circ \tilde{\psi}^{-1}$ preserves orientation.

(If atlas on Σ_2 was maximal as an oriented atlas, this recover previous set of charts.) ■

Remark.

1. There's no really sensible classification of all smooth or topological surfaces. For example, $\mathbb{R}^2 \setminus Z$ for Z closed in $\mathbb{R}^2$ already realizes uncountably many homeomorphism types.

   By contrast, compact smooth surfaces up to diffeomorphism are classified by the Euler characteristic and the orientability.

2. There is a definition of orientation preserving homeomorphism, which needs Algebraic Topology.

3. We can get other structures on an abstract smooth surface by asking for a smooth atlas such that if ϕ_1, ϕ_2^{-1} is one of our transition maps, then $D(\phi_1\phi_2^{-1})\big|_x \in G \leq GL_2(\mathbb{R})$ a subgroup of general linear group. For example, taking $\{e\}$ leads to *Euclidean Surfaces*. And $G = GL_1(\mathbb{C} \leq GL_2(\mathbb{R}))$ which leads to the theory of Riemann surfaces.

Example. We consider the Möbius band created by an open strip $(a, b) \times [c, d]$. It turns out that an abstract smooth surface is orientable if and only if it contains no subsurface homeomorphic to the Möbius band.

So we say a topological surface is orientable if and only if it contains no subsurface (open set) homeomorphic to a Möbius band, as an ad hoc definition.

Example.

1. For S^2 with the atlas of two stereographic projections, you computed the transition maps
$$(u, v) \mapsto (\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2})$$
   on $\mathbb{R}^2 \setminus \{0\}$ in example sheet 1, and check this is orientation preserving.

2. For T^2, we exhibited an atlas such that all the transition maps are translations of $\mathbb{R}^2$ (restricted to appropriate open discs). So T^2 is oriented (and even Euclidean).

We want to investigate orientability for surfaces in $\mathbb{R}^3$ next. Recall an *affine subspace* of a vector space is a translation of a linear subspace.

Definition 1.15. Let Σ be a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$. Fix an allowable parametrization

$$\sigma : V \to U \subseteq \Sigma \quad 0 \mapsto p.$$

Then the *tangent plane* $T_p\Sigma$ of Σ at p is $\operatorname{Im}(D\sigma|_0) \subseteq \mathbb{R}^3$, a 2d vector subspace of $\mathbb{R}^3$.

The *affine tangent plane* of Σ at p is $p + T_p\Sigma \subseteq \mathbb{R}^3$.

Lemma 1.7. $T_p\Sigma$ is well-defined, i.e., independent of the choice of allowable parametrization near p.

Proof. We will give two proofs.

1. If $\sigma : V \to U \subseteq \Sigma$ and $\tilde{\sigma} : \tilde{V} \to \tilde{U} \subseteq \Sigma$ are two allowable parametrizations near p. There is a transition map $\sigma^{-1} \circ \tilde{\sigma}$ which is a diffeomorphism of open sets in $\mathbb{R}^2$. That means I can write

   $$\tilde{\sigma} = \sigma \circ (\sigma^{-1} \circ \tilde{\sigma}).$$

   Because the transition map is an isomorphism near p, The images of σ and $\tilde{\sigma}$ are the same.

2. Let $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ be a smooth map such that γ has image inside Σ and $\gamma(0) = p$. Then I claim $\gamma'(0) \in T_p\Sigma$.

   If $\sigma : V \to U$ is the allowable parametrization around p, and ϵ small enough so $\operatorname{Im}(\gamma) \in U$, then I can write

   $$\gamma(t) = \sigma(u(t), v(t))$$

   for smooth functions $u, v : (-\epsilon, \epsilon) \to V$. Then

   $$\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t) \in \operatorname{Im} D\sigma.$$

   This exhibits that

   $$T_p\Sigma = \operatorname{span}\{\gamma'(0) \mid \gamma \text{ a smooth curve as above}\}.$$

■

Definition 1.16. If Σ is a smooth surface is $\mathbb{R}^3$ and $p \in \Sigma$, the *normal direction* to Σ at p is just $(T_p\Sigma)^\perp$ (the Euclidean orthogonal complement of $T_p\Sigma$ with respect to the Euclidean inner product).

So at each $p \in \Sigma$, there are two unit length normal vectors.

The affine tangent plane is the best linear approximation to Σ at p.

Definition 1.17. A smooth surface in $\mathbb{R}^3$ is *two-sided* if it admits a continuous global choice of unit normal vector.

Lecture 8

7 Feb. 2022

Lemma 1.8. A smooth surface in $\mathbb{R}^3$ is orientable with its abstract smooth surface structure if and only if it is two-sided.

Proof. Let $\sigma : V \to U \in \Sigma$ be an allowable parametrization for $U \in \Sigma$ and say $\sigma(0) = p$. Define the *positive* unit normal with respect to σ at p to be the normal vector $n_\sigma(p)$ such that

$$\{\sigma_u, \sigma_v, n_\sigma(p)\}, \{e_1, e_2, e_3\}$$

are related by a positive determinant change of basis matrix, where $\{e_1, e_2, e_3\}$ is the standard basis. Explicitly,

$$n_\sigma(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}.$$

If $\tilde{\sigma}$ is another allowable parametrization,

$$\begin{aligned}\tilde{\sigma} \colon \tilde{V} &\longrightarrow \tilde{U} \subseteq \Sigma \\ 0 &\longmapsto p,\end{aligned}$$

and suppose Σ is orientable as an abstract smooth surface and $\tilde{\sigma}$ belongs to the same oriented atlas. So

$$\sigma = \tilde{\sigma} \circ \phi$$

where ϕ is a transition map. If we write $D\phi|_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta. \end{pmatrix}$ Chain rule says

$$\begin{aligned}\sigma_u &= \alpha\tilde{\sigma}_u + \gamma\tilde{\sigma}_v \\ \sigma_v &= \beta\tilde{\sigma}_u + \delta\tilde{\sigma}_v\end{aligned}$$

and we have $\sigma_u \times \sigma_v = \det(D\phi|_0)\tilde{\sigma}_u \times \tilde{\sigma}_v$. The determinant is positive since $\sigma, \tilde{\sigma}$ belong to the same oriented atlas, so the positive unit normal at p was intrinsic; it depends on the orientation of Σ but no choice of allowable parametrization in the oriented atlas. And the expression $\frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ is continuous. i.e. Σ is 2-sided.

Conversely, if Σ is 2-sided, and I have a global continuous choice of normal vector. I can consider the subatlas of the natural smooth atlas such that allow a chart (u, ϕ) if associated parametrization $\phi^{-1} = \sigma$ makes $\{\sigma_u, \sigma_v, n\}$ a positive basis for $\mathbb{R}^3$. Similarly, we know that the transition maps between such charts have positive determinant, and are orientation-preserving. So Σ is continuous. ■

Lemma 1.9. If Σ is a smooth surface in $\mathbb{R}^3$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth map that preserves Σ set-wise. Then $DA|_p : \mathbb{R}^3 \to \mathbb{R}^3$ sends $T_p\Sigma$ to $T_{A(p)}\Sigma$ whenever $p \in \Sigma$.

Proof. Suppose $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ is a smooth map such that $\text{Im}(\gamma) \subseteq \Sigma$ and $\gamma(0) = p$. Recall $T_p\Sigma$ is spanned by $\gamma'(0)$ for such γ.

Now $A \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^3$ also has image in Σ, and by chain rule,

$$T_{A(p)}\Sigma \ni D(A \circ \gamma)|_0 = DA|_{\gamma(0)} \cdot D\gamma|_0 = DA|_p \cdot (\gamma'(0)).$$

■

Example. If $S^2 \subseteq \mathbb{R}^3$, then the normal line $(T_p\Sigma)^\perp = (T_pS^2)^\perp = \text{span}(p)$ is the line through p. (since SO_3 acts transitively on S^2, check this at the one point suffices) So there is at each point an outwards-pointing normal vector $n(p)$. (such that $p \notin \mathbb{R}_{\geq 0}n(p) + p$)

So S^2 is 2-sided, and so orientable.

Example (A Möbius Band). Let

$$\sigma(t, \theta) = ((1 - t\sin\frac{\theta}{2})\cos\theta, (1 - t\sin\frac{\theta}{2})\sin\theta, t\cos\frac{\theta}{2}),$$

where

$$(t, \theta) \in V_1 = \{t \in (-\frac{1}{2}, \frac{1}{2}), \theta \in (0, 2\pi))\},$$

or

$$(t, \theta) \in V_2 = \{t \in (-\frac{1}{2}, \frac{1}{2}), \theta \in (-\pi, \pi))\}.$$

We start with the unit circle $x^2 + y^2 = 1$ in the xy-plane ($t = 0$ on the surface), and we take an open interval of length 1. And this line rotates as you move around the circle such that it has rotated by $\frac{\theta}{2}$ at point θ.

We can check if we define σ_i on V_i, then σ_i is allowable. (smooth, injective and $D\sigma_i$ injective)

By direct computation, we have

$$\sigma_t \times \sigma_\theta = (-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}) = n_\theta$$

which is already unit length.

Also note $\theta \to 0^+, n_\theta \to (-1, 0, 0)$ and $\theta \to 2\pi^-, n_\theta \to (1, 0, 0)$. So the surface is not 2-sided.

We're now starting a new chapter on geometry surfaces in $\mathbb{R}^3$: especially length, area and curvature.

Let $\gamma : (a, b) \to \mathbb{R}^3$ smooth. The *length* of γ is $L(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t$. If $s : (A, B) \to (a, b)$ is monotonically increasing, and let $\tau(t) = \gamma(s(t))$, then

$$L(\tau) = \int_A^B \|\tau'(t)\| \, \mathrm{d}x = \int_A^B \|\gamma(s(t))\| |s'(t)| \, \mathrm{d}x = L(\gamma)$$

by change of variables formula since $s'(t) \geq 0$.

Lemma 1.10. If $\gamma : (a,b) \to \mathbb{R}^3$ is continuously differentiable and $\gamma'(t) \neq 0\ \forall t$ then γ can be parametrized by arc-length. (i.e. in a parameter s such that $|\gamma'(s) = 1|\ \forall s$)

Proof. Exercise. ∎

Let Σ be a smooth surface in $\mathbb{R}^3$ and let $\sigma : V \to U \subseteq \Sigma$ allowable. If $\gamma : (a,b) \to \mathbb{R}^3$ is smooth and has image in U, then there exists $u(t), v(t) : (a,b) \to V$ such that $\gamma(t) = \sigma(u(t), v(t))$. And we have

$$\begin{aligned} \gamma'(t) &= \sigma_u u'(t) + \sigma_v v'(t), \\ \|\gamma'(t)\|^2 &= Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2 \end{aligned}$$

where $\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle = \|\sigma_u\|^2 \\ F &= \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle \\ G &= \langle \sigma_v, \sigma_v \rangle = \|\sigma_v\|^2 \end{aligned}$ are smooth functions on V, and $\langle , \rangle$ is the Euclidean inner product. Note that E, F, G depend only on σ, but not on γ.

Definition 1.18. The *First Fundamental Form* (FFF) of Σ in the parametrization is the expression

$$E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2.$$

The notation is designed to remind you that if $\gamma : (a,b) \to \mathbb{R}^3$ lands in $\sigma(V) = U \subseteq \Sigma$, then

$$L(\gamma) = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2}\,\mathrm{d}t$$

where $\gamma(t) = \sigma(u(t), v(t))$.

Remark. Really the Euclidean inner product $\langle , \rangle$ on $\mathbb{R}^3$ gives me an inner product on $T_p\Sigma \subseteq \mathbb{R}^3$. If I pick a parametrization σ,

$$T_p\Sigma = \mathrm{Im}(D\sigma|_0) = \mathrm{span}\{\sigma_u, \sigma_v\}$$

given $\sigma(0) = p$. FFF is a symmetric bilinear form on $T_p\Sigma$ (varying smoothly in p), expressed in a basis coming from the parametrization σ. So often helpful to consider $\left(\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right)$ which is the matrix of the bilinear form.

Example.

1. The plane $\mathbb{R}^2_{xy} \subseteq \mathbb{R}^3$ has parametrization $\sigma(u,v) = (u,v,0)$, so $\sigma_u = (1,0,0)$ and $\sigma_0, 1, 0$, so FFF is $\mathrm{d}u^2 + \mathrm{d}v^2$.
2. Or in polar co-ordinates, $\sigma(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ for $r \in (0,\infty), \theta \in (0, 2\pi)$. Now we have $\sigma_r = (\cos\theta, \cos\theta, 0)$ and $\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$ and FFF is $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$. So the first fundamental form is dependent on the parametrization.

Lecture 9

9 Feb. 2022

Definition 1.19. Let Σ, Σ' be smooth surfaces in $\mathbb{R}$. We say Σ and Σ' are isometric is there is a diffeomorphism $f : \Sigma \to \Sigma'$ such that for every smooth curve $\gamma : (a, b) \to \Sigma$,

$$L_\Sigma(\gamma) = L_{\Sigma'}(f \circ \phi).$$

Example. If $\Sigma' = f(\Sigma)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a rigid motion; i.e.

$$f : v \longmapsto Av + b$$

where $A \in O(3)$ and $b \in \mathbb{R}^3$. (so f preserves $\langle , \rangle_{\mathrm{Eucl}}$ on $\mathbb{R}^3$), then $f : \Sigma \to \Sigma'$ is an isometry.

Note. In the definition, importantly f is only a priori defined on Σ, not all not $\mathbb{R}^3$.

Often we're really interested in a local statement.

We say Σ, Σ' are *locally isometric* (near point $p \in \Sigma$ and $q \in \Sigma'$) if there exists open neighborhoods $p \in U \subseteq \Sigma$ and $q \in U' \subseteq \Sigma'$ which are isometric.

Lemma 1.11. Smooth surfaces Σ, Σ' in $\mathbb{R}^3$ are locally isometric near $p \in \Sigma$ and $q \in \Sigma'$ if and only if there exist allowable parametrizations

$$\begin{aligned} \sigma &: V \to U \subseteq \Sigma \\ \sigma' &: V \to U' \subseteq \Sigma' \end{aligned}$$

for which the FFFs are equivalent (equal as functions on V).

Proof. We know (by definition) that the FFF of σ determines the lengths of all curves on Σ inside $\sigma(V) = U$.

We will allow length of curves determine the FFF of the parametrization. Given $\sigma : V \to U \subseteq \Sigma$, w.l.o.g. $V = B(0, \delta)$ for some $\delta > 0$ with $\sigma(0) = p$, and consider

$$\begin{aligned} \gamma_\epsilon : [0, \epsilon] &\to U \subseteq \Sigma \quad \sigma < \delta \\ t &\mapsto \sigma(t, 0). \end{aligned}$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\epsilon} L(\gamma_\epsilon) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \int_0^\epsilon \sqrt{E(t, 0)} \,\mathrm{d}t \\ &= \sqrt{E(\epsilon, 0)}. \end{aligned}$$

So $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\big|_{\epsilon=0} L(\gamma_\epsilon) = \sqrt{E(0, 0)}$. So lengths of curves γ_ϵ determines E at p.

Analogously, $\chi_\epsilon : [0, \epsilon] \to \Sigma, t \mapsto \sigma(0, t)$, and their lengths determine $\sqrt{G(0, 0)}$.

Then $\lambda_\epsilon : [0,\epsilon] \to \Sigma, t \mapsto \sigma(t,t)$ determine $\sqrt{(E+2F+G)(0,0)}$, so we can determine F knowing E, G. ■

Example.

1. The sphere $\{x^2+y^2+z^2=a^2\} \subseteq \mathbb{R}^3$ has an open set with allowable parametrization

$$\sigma(u,v) = (a\cos u\cos v, a\cos u\sin u, a\sin u)$$

with latitude $u \in (-\pi,\pi)$ and longitude $v \in (0,2\pi)$. It parametrizes the complement of a half great circle. We compute the FFF as follows

$$\begin{aligned}\sigma_u &= (-a\sin u\cos v, -a\sin u\sin v, a\cos u)\\ \sigma_v &= (-a\cos u\sin v, a\cos u\cos v, 0).\end{aligned}$$

And we have

$$E = \sigma_u\cdot\sigma_u = a^2, \quad F = \sigma_u\cdot\sigma_v = 0, \quad G = \sigma_v\cdot\sigma_v = a^2\cos^2 u.$$

So the FFF is $a^2 \mathrm{d}u^2 + a^2\cos^2 u \mathrm{d}v^2$.

2. Surface of revolution: take

$$\eta(t) = (f(t), 0, g(t))$$

in xy-plane, and rotate about z axis; we have

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

We have

$$\begin{aligned}\sigma_u &= (f_u\cos v, f_u\sin v, g_u)\\ \sigma_u &= (f_u\cos v, f_u\sin v, g_u)\\ FFF &: (f_u^2+g_u^2)\mathrm{d}u^2 + f^2\mathrm{d}v^2.\end{aligned}$$

3. Cone: If we have a cone with angle $\tan^{-1}(a)$. For $u>0, v\in(0,2\pi)$,

$$\sigma(u,v) = (au\cos v, au\sin v, u)$$

parametrizes complement of one line on the cone. We have from above the FFF being $(1+a^2)\mathrm{d}u^2 + a^2u^2\mathrm{d}v^2$.

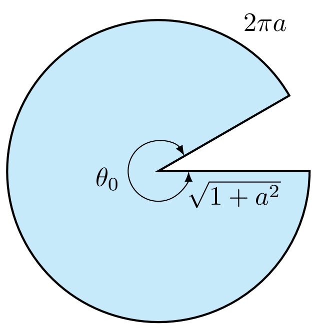

Figure 1: Cone Cut Open

If I cut open the cone and unfold it, I get a plane sector as shown in Figure 1. We have $\theta_0 = \frac{2\pi a}{\sqrt{1+a^2}}$. Parametrize this plane sector by

$$\sigma(r,\theta) = (\sqrt{1+a^2} r \cos\left(\frac{a\theta}{\sqrt{1+a^2}}\right), \sqrt{1+a^2} r \sin\left(\frac{a\theta}{\sqrt{1+a^2}}\right), 0)$$

with $r > 0, \theta \in (0, \theta_0)$.

We have the FFF $(1+a^2)\mathrm{d}r^2 + r^2 a^2 \mathrm{d}\theta^2$. So the cone is locally isometric to the plane.

Note. The cone and the plane cannot be globally isometric, since they are not homeomorphic.

The cone is homeomorphic to $S^1 \times \mathbb{R}$; in the plane $\mathbb{R}^2$, every compact set K lies inside a larger compact set $K' = \overline{B(0,N)}$ with $N > 0$ and such that $\mathbb{R}^2 \setminus K'$ is connected. But on $S^1 \times \mathbb{R}$, a circle that goes around the cone would have the property that for any $K' \supseteq K$, $(S^1 \times \mathbb{R}) \setminus K'$ is disconnected.

So they cannot be homeomorphic since homeomorphism preserves connectedness.

Let Σ be a smooth surface in $\mathbb{R}^3$, $q \in \Sigma$, and take two allowable parametrizations around p,

$$\begin{aligned} \sigma :& V \to U \subseteq \Sigma \quad \sigma(0) = p \\ \tilde{\sigma} :& \tilde{V} \to U \subseteq \Sigma \quad \tilde{\sigma}(0) = p \end{aligned}$$

We have a transition map F :

Lecture 10

11 Feb. 2022