FP: Questions and Comments

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Question and Comments

The following triplet satisfies Item 3.

- $\Big[G_1(b), G^w_{2|1}(y|b), G^l_{2|1}(y|b)\Big] = \Big[b, y^{2b}, y^b\Big]$ where the support is $(y, b) \in (0,1)^2$

Note that we have an open interval (0,1) to avoid $log(0) = -\infty$ happening. And, we have positive synergy case here. Each three subsection shows the proof.

1. *The RHS of (2) is increasing in* b^w_{2i}

For convenience, I let $y := b^w_{2i}$ and $b := b_{1i}$ and $M := I$. *RHS of (2)* is as follows.

$$y + \frac{H^w_2(y;b)}{h^w_2(y;b)} \tag{1}$$

We know that $H^w_2(y;b) = G^l_2(y|B_1 \le b)^{M-1}$. In our setting, we have the following.

빨간줄은 재컨펌한것이다 23년도 4월에.
그리고 이게 MC에 필요한 것들임

$$\begin{aligned} G^l_2(y|B_1 \le b) &= \frac{1}{G_1(b)} \int_0^b G^l_{2|1}(y|u)dG_1(u) \\ &= \frac{1}{G_1(b)} \int_0^b G^l_{2|1}(y|u)g_1(u)du \\ &= \frac{1}{b} \int_0^b y^u du \\ &= \frac{1}{b}\Big(\frac{y^b - 1}{\log(y)}\Big) \end{aligned} \tag{2}$$

Third equality comes out by using $G_1(b) = b$, $g_1(u) = 1$. As a result, we also have the following.

$$\begin{aligned} H^w_2(y;b) &= G^l_2(y|B_1 \le b)^{M-1} \\ &= \Big(\frac{y^b - 1}{b\log(y)}\Big)^{M-1} \end{aligned} \tag{3}$$

Then, we also have the following.

$$h_2^w(y;b) = \frac{dH_2^w(y;b)}{dy}$$
$$= \frac{d}{dy}\left(\frac{y^b-1}{b\log(y)}\right)^{M-1}$$
$$= \text{equivalent to the following} \tag{4}$$

두번다시 체크하지 말자. 맞아 $= \dfrac{b\left(\frac{y^b-1}{b\log(y)}\right)^{M-1}(M-1)\left(\frac{y^b}{y\log(y)} - \frac{y^b-1}{by\log(y)^2}\right)\log(y)}{y^b-1}$

As a result, we have the following.

$$y + \frac{H_2^w(y;b)}{h_2^w(y;b)} = \text{equivalent to the following} \tag{5}$$

$$y + \frac{\left(\frac{y^b-1}{b\log(y)}\right)^{1-M}\left(\frac{y^b-1}{b\log(y)}\right)^{M-1}\left(y^b-1\right)}{b(M-1)\left(\frac{y^b}{y\log(y)} - \frac{y^b-1}{by\log(y)^2}\right)\log(y)}$$

두번다시 체크 노노

Now, the derivative of Eq(5) w.r.t y **must be strictly positive for all** $(y,b) \in (0,1)^2$.

$$\frac{d}{dy}\left(y + \frac{H_2^w(y;b)}{h_2^w(y;b)}\right) = \text{equivalent to the following} \tag{6}$$

$$= \frac{y^7(M-1)\left(by^b\log(y) - y^b + 1\right)^3 + y^6\left(by^{b+1}\log(y) - y\left(y^b-1\right)\right)\left(by^b\log(y) - y^b + 1\right)^2 - \left(y^b-1\right)\left(by^{b+1}\log(y) + y\left(1-y^b\right)\right)\left(by^{b+6}(b-1)\log(y)^2 - 2by^{b+6}\log(y) + y^6\left(y^b-1\right)\log(y) + 2y^6\left(y^b-1\right)\right)}{y^7(M-1)\left(by^b\log(y) - y^b + 1\right)^3}$$

Is Eq(6) indeed strictly positive for all $(y,b) \in (0,1)^2$?

Yes it is.

And I didn't use any numerical approximation(e.g., trapezoid integration, Simpson integration) here.

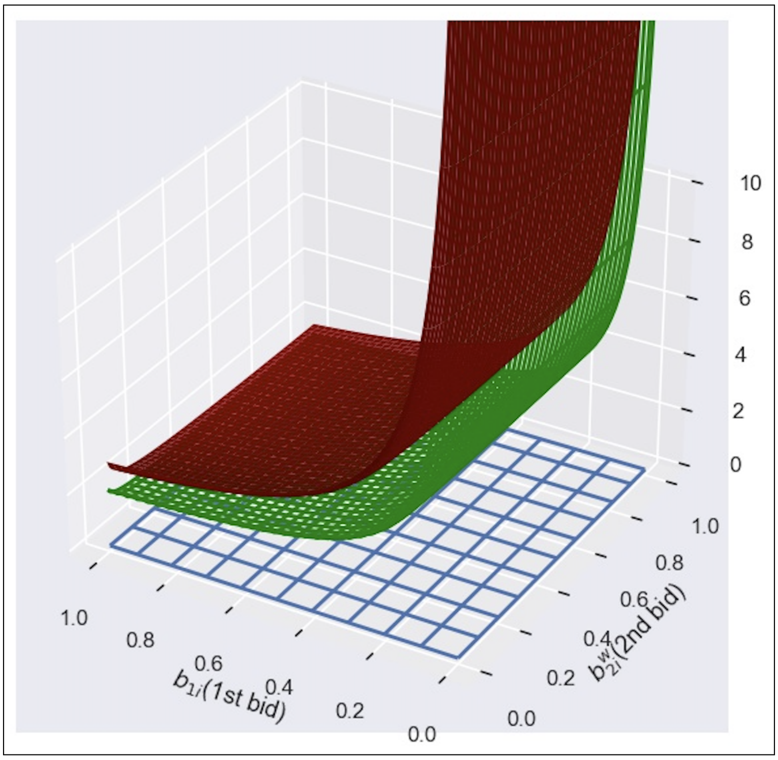

Figure 1: Value of Eq(6) on the support $(0,1)^2$

Red surface is the case when $M = 3$, and green surface is the case when $M = 5$. Notice that we have axes, $b^w_{2i} \equiv y$ and $b_{1i} \equiv b$. As you increase the number of bidders, M, the surface goes down, which is why the green surface is below the red surface. Then if I increase M to infinity, would it ever be below 0? Probably No. I tested with $M = 1,000$, $M = 10,000$, and the surface is always strictly above 1 (not 0, but 1).

For $M = 3$, argmin is $(b^w_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$ and the minimum surface value is 1.844.

For $M = 5$, argmin is $(b^w_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$ and the minimum surface value is 1.422.

Why is argmin not (1,1)? it is because I assumed we have an open interval plane $(0,1)^2$ Why is argmin not, say, $(b^w_{2i}, b_{1i}) = (\frac{9999}{10000}, \frac{9999}{10000})$? Yes, actually $(b^w_{2i}, b_{1i}) = (\frac{9999}{10000}, \frac{9999}{10000})$ outputs smaller surface value than $(b^w_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$. Then why did I say $(b^w_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$ is the argmin? It is because this plot's support is discrete 99×99 matrix where the largest element is $(\frac{99}{100}, \frac{99}{100})$ and the smallest element is $(\frac{1}{100}, \frac{1}{100})$.

As you go closer and closer to $(1,1)$, you achieve minimum surface value: $(b^w_{2i}, b_{1i}) = (\frac{99999}{100000}, \frac{99999}{100000})$ has smaller surface value than $(b^w_{2i}, b_{1i}) = (\frac{9999}{10000}, \frac{9999}{10000})$ for both $M = 3$, $M = 5$.

2. *The RHS of (6) is increasing in* b^l_{2i}

For convenience, I let $y := b^l_{2i}$ and $b := b_{1i}$ and $M := I$. *RHS of (6)* is as follows.

$$y + \frac{H^l_2(y;b)}{h^l_2(y;b)} \tag{7}$$

We know that $H^l_2(y;b) = \frac{1}{1-G_1(b)^{M-1}}\int_b^1 G^l_2(y|B_1 \le u)^{M-2} G^w_{2|1}(y|u)dG_1(u)^{M-1}$. In our setting, we have the following.

$$\begin{aligned}
H^l_2(y;b) &= \frac{1}{1-G_1(b)^{M-1}}\int_b^1 G^l_2(y|B_1 \le u)^{M-2} G^w_{2|1}(y|u)dG_1(u)^{M-1} \\
&= \frac{1}{1-b^{M-1}}\int_b^1 \left(\frac{y^u-1}{u\log(y)}\right)^{M-2} y^{2u}(M-1)u^{M-2}du \\
&= (M-1)\frac{1}{1-b^{M-1}}\left(\frac{1}{\log(y)}\right)^{M-2}\int_b^1 \left(y^u-1\right)^{M-2} y^{2u}du \\
&= \frac{M-1}{1-b^{M-1}}\log(y)^{2-M}\int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j (y^u)^j(-1)^{M-2-j}y^{2u}du \\
&= \frac{M-1}{1-b^{M-1}}\underbrace{\log(y)^{2-M}}_{①}\underbrace{\int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j y^{u(2+j)}(-1)^{M-2-j}du}_{②} \\
&= \frac{M-1}{1-b^{M-1}}①②
\end{aligned} \tag{8}$$

Second equality comes from using Eq(2), $G^w_{2|1}(y|u) = y^{2u}$, $dG_1(u)^{M-1} = (M-1)G_1(u)^{M-2}g_1(u)du = (M-1)u^{M-2}du$. Rearranging yields the third equality. Fourth equality comes from using binomial theorem. Rearranging yields last equality. Then, $h^l_2(y;b)$ will be as follows, using Eq(8).

$$\begin{aligned}
\frac{d}{dy}H^l_2(y;b) &= \frac{d}{dy}\left(\frac{M-1}{1-b^{M-1}}\underbrace{\log(y)^{2-M}}_{①}\underbrace{\int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j y^{u(2+j)}(-1)^{M-2-j}du}_{②}\right) \\
&= \frac{M-1}{1-b^{M-1}}\left(①^{'}② + ①②^{'}\right)
\end{aligned} \tag{9}$$

Now, we know that taking derivative of Eq(7) w.r.t y **must be strictly positive for all** $(y,b) \in (0,1)^2$. The derivative of Eq(7) w.r.t y is,

$$\frac{d}{dy}\left(y + \frac{H^l_2(y;b)}{h^l_2(y;b)}\right) = 1 + \frac{hh - Hh^{'}}{h^2} = 2 - \frac{Hh^{'}}{h^2} \tag{10}$$

And we know the following results from Eq(8) and Eq(9).

$$H \equiv H_2^l(y;b) = \frac{M-1}{1-b^{M-1}} ①② \tag{11}$$

$$h \equiv h_2^l(y;b) = \frac{M-1}{1-b^{M-1}} \left(①^{'}② + ①②^{'} \right) \tag{12}$$

$$\begin{aligned} h^{'} \equiv \frac{d}{dy} h_2^l(y;b) &= \frac{M-1}{1-b^{M-1}} \left(①^{''}② + ①^{'}②^{'} + ①^{'}②^{'} + ①②^{''} \right) \\ &= \frac{M-1}{1-b^{M-1}} \left(①^{''}② + 2①^{'}②^{'} + ①②^{''} \right) \end{aligned} \tag{13}$$

Now, apply Eq(11), Eq(12), Eq(13) to Eq(10).

$$2 - \frac{Hh^{'}}{h^2} = 2 - \frac{①② \left(①^{''}② + 2①^{'}②^{'} + ①②^{''} \right)}{\left(①^{'}② + ①②^{'} \right)^2} \tag{14}$$

Each ①, $①^{'}$, $①^{''}$ is as follows.

$$① = \log(y)^{2-M} \tag{15}$$

$$①^{'} = (2-M)\log(y)^{1-M} \frac{1}{y} \tag{16}$$

$$①^{''} = (2-M)(1-M)\log(y)^{-M} \frac{1}{y^2} - (2-M)\log(y)^{1-M} \frac{1}{y^2} \tag{17}$$

Each ②, $②^{'}$, $②^{''}$ is as follows.

$$② = \int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j y^{u(2+j)} (-1)^{M-2-j} du \tag{18}$$

$$②^{'} = \int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j u(2+j) y^{u(2+j)-1} (-1)^{M-2-j} du \tag{19}$$

$$②^{''} = \int_b^1 \sum_{j=0}^{M-2} {}_{M-2}C_j u(2+j)(u(2+j)-1) y^{u(2+j)-2} (-1)^{M-2-j} du \tag{20}$$

Using the integration's linearity, Eq(18), Eq(19), Eq(20) becomes as follows.

$$② = \sum_{j=0}^{M-2} \left(\int_b^1 {}_{M-2}C_j y^{u(2+j)} (-1)^{M-2-j} du \right) \tag{21}$$

$$②^{'} = \sum_{j=0}^{M-2} \left(\int_b^1 {}_{M-2}C_j u(2+j) y^{u(2+j)-1} (-1)^{M-2-j} du \right) \tag{22}$$

$$②^{''} = \sum_{j=0}^{M-2} \left(\int_b^1 {}_{M-2}C_j u(2+j)(u(2+j)-1) y^{u(2+j)-2} (-1)^{M-2-j} du \right) \tag{23}$$

For Eq(21), Eq(22), Eq(23), all the terms inside the integral is integrable. For example, when $j = 0, M = 3$ we have the following.

$$\int_b^1 {}_{M-2}C_j y^{u(2+j)}(-1)^{M-2-j}du = -\int_b^1 y^{2u}du = -\Big(\frac{y^2-y^{2b}}{2\log(y)}\Big) \tag{24}$$

$$\int_b^1 {}_{M-2}C_j u(2+j)y^{u(2+j)-1}(-1)^{M-2-j}du = -\int_b^1 2uy^{2u-1}du \tag{25}$$

$$\int_b^1 {}_{M-2}C_j u(2+j)(u(2+j)-1)y^{u(2+j)-2}(-1)^{M-2-j}du = -\int_b^1 2u(2u-1)y^{2u-2}du \tag{26}$$

For Eq(25), Eq(26), they all have analytic expression, but I didn't write it here. Then, by Eq(15)-(17) and Eq(21)-(23), we have analytic expression for Eq(14).

Is Eq(14) indeed strictly positive for all $(y, b) \in (0, 1)^2$?

Yes it is.

And I didn't use any numerical approximation(e.g., trapezoid integration, Simpson integration) here.

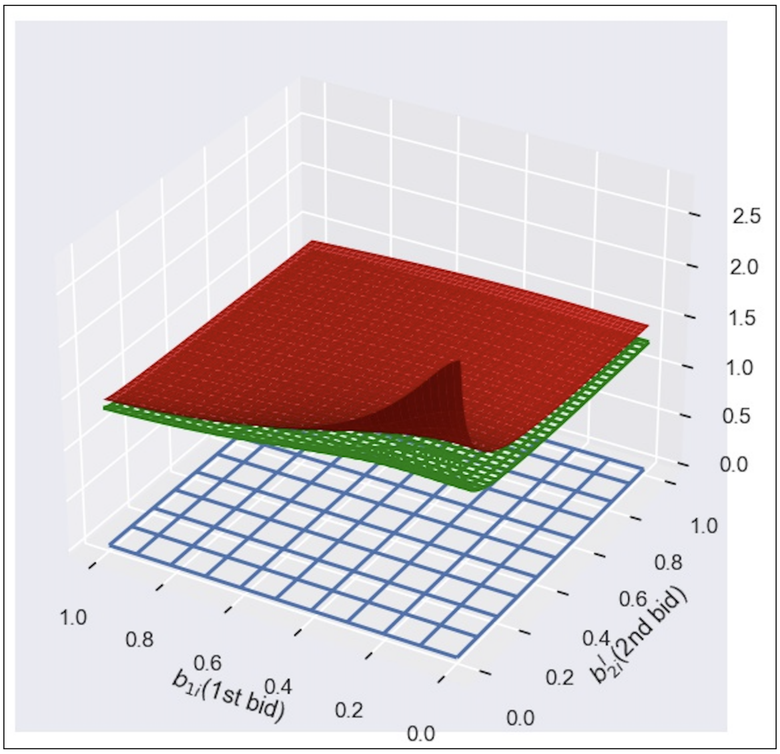

Figure 2: Value of Eq(14) on the support $(0, 1)^2$

Red surface is the case when $M = 3$, and green surface is the case when $M = 5$. Notice that we have axes, $b^l_{2i} \equiv y$ and $b_{1i} \equiv b$. As you increase the number of bidders, M, the surface goes down, which is why the green surface is below the red surface.

Then, if I increase M to infinity, would it ever be below 0? Here, I am not sure; as I increase M too much to such as 500, then still the surface is strictly above 0. But I feel(without any good reasoning) that outputs when M is extremely high are not so much credible; it is because increasing M increases the number of terms inside the integral in Eq(24)-(26). Thus, the number of terms in the denominator of Eq(14) grows exponentially.

For $M = 3$, argmin is $(b^l_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$ and the minimum surface value is 1.388.

For $M = 5$, argmin is $(b^l_{2i}, b_{1i}) = (\frac{99}{100}, \frac{99}{100})$ and the minimum surface value is 1.266.

Same as in the case of Figure 1, As I get closer and closer to $(b^l_{2i}, b_{1i}) = (1, 1)$, the surface value decreases for both $M = 3, M = 5$.

3. *the RHS of (10) is increasing in* b_{1i}

Let y be the second auction bid, and $b := b_{1i}$. Then, the RHS of *Equation* (10) on the FP note is the same as follows.

$$b + \frac{1}{M-1}\frac{G_1(b)}{g_1(b)} - \underbrace{\left(\int_0^1 \frac{H_2^w(y;b)}{h_2^w(y;b)} G_2^l(y|B_1 \le b)^{M-2} G^l_{2|1}(y|b)\, dG^w_{2|1}(y|b)\right)}_{③} \tag{27}$$
$$+ \underbrace{\left(\int_0^1 \frac{H_2^l(y;b)}{h_2^l(y;b)} G_2^l(y|B_1 \le b)^{M-2} G^w_{2|1}(y|b)\, dG^l_{2|1}(y|b)\right)}_{④}$$

where the ③, ④ are as follows.

$$\begin{aligned} ③ &= \int_0^1 \frac{H_2^w}{h_2^w} G_2^l(y|B_1 \le b)^{M-2} G^l_{2|1}(y|b)\, dG^w_{2|1}(y|b) \\ &= \int_0^1 \frac{H_2^w}{h_2^w} G_2^l(y|B_1 \le b)^{M-2} G^l_{2|1}(y|b)\, g^w_{2|1}(y|b) dy \\ &= \int_0^1 \frac{H_2^w}{h_2^w} \Big(\frac{y^b - 1}{b\log(y)}\Big)^{M-2} y^b\, 2by^{2b-1}\, dy \\ &= \int_0^1 2by^{3b-1} \Big(\frac{y^b - 1}{b\log(y)}\Big)^{M-2} \frac{H_2^w}{h_2^w}\, dy \end{aligned} \tag{28}$$

Second equality comes from $dG^w_{2|1}(y|b) = g^w_{2|1}(y|b)dy$, third equality comes from Eq(2), $g^w_{2|1}(y|b) = 2b\,y^{2b-1}$, and last equality comes from rearrangement.

$$\begin{aligned}
④ &= \int_0^1 \frac{H_2^l}{h_2^l} G_2^l(y|B_1 \le b)^{M-2} G^w_{2|1}(y|b)\, dG^l_{2|1}(y|b) \\
&= \int_0^1 \frac{H_2^l}{h_2^l} \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} y^{2b}\, dG^l_{2|1}(y|b) \\
&= \int_0^1 \frac{H_2^l}{h_2^l} \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} y^{2b}\, g^l_{2|1}(y|b)dy \\
&= \int_0^1 \frac{H_2^l}{h_2^l} \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} y^{2b}\, by^{b-1}dy \\
&= \int_0^1 \frac{H_2^l}{h_2^l} by^{3b-1} \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} dy
\end{aligned} \tag{29}$$

Third and fourth equality come from $dG^l_{2|1}(y|b) = g^l_{2|1}(y|b)dy = b\,y^{b-1}dy$, and last equality comes from rearrangement. Using Eq(28) and Eq(29), Eq(27) becomes as follows.

$$\begin{aligned}
&b + \frac{1}{M-1}\frac{G_1(b)}{g_1(b)} + \int_0^1 \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} by^{3b-1}\left(-2\frac{H_2^w}{h_2^w} + \frac{H_2^l}{h_2^l}\right) dy \\
&= b + \frac{1}{M-1}b + \int_0^1 \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} by^{3b-1}\left(-2\frac{H_2^w}{h_2^w} + \frac{H_2^l}{h_2^l}\right) dy
\end{aligned} \tag{30}$$

Also, the following holds.

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$$-2\frac{H_2^w}{h_2^w} = -2\frac{y(y^b-1)\log(y)}{(M-1)(by^b\log(y) - y^b + 1)} \tag{31}$$

$$\frac{H_2^l}{h_2^l} = \frac{①②}{①'② + ①②'} \tag{32}$$

Eq(31) comes from the Eq(5); rearranging the term leads to Eq(31). Eq(32) comes from Eq(11) and Eq(12). So Eq(27)≡Eq(30) is equivalent to the following.

$$b + \frac{1}{M-1}b + \int_0^1 \left(\frac{y^b-1}{b\log(y)}\right)^{M-2} by^{3b-1}\left(-2\frac{y(y^b-1)\log(y)}{(M-1)(by^b\log(y)-y^b+1)} + \frac{①②}{①'② + ①②'}\right) dy \tag{33}$$

But unluckily, I could not derive analytic expression for Eq(33); that is, we had analytic expressions such as Eq(6) and Eq(14) before, but here I wasn't able to. I haven't tried any numerical integration(e.g., Simpson, Midpoint, Trapezoid) for Eq(33), but I thought it wouldn't work because Eq(33) is so complicated. So I plugged in $M = 3$ to find a simple expression. Then Eq(33) becomes as follows.

$$
\begin{aligned}
&b+\frac{1}{2}b+\int_0^1\Big(\frac{y^b-1}{b\log(y)}\Big)by^{3b-1}\Big(-\frac{y(y^b-1)\log(y)}{by^b\log(y)-y^b+1}+\frac{\textcircled{1}\textcircled{2}}{\textcircled{1}'\textcircled{2}+\textcircled{1}\textcircled{2}'}\Big)dy=\\
&=\frac{3}{2}b-\int_0^1\frac{(y^b-1)^2y^{3b}}{by^b\log(y)-y^b+1}\,dy+\int_0^1\Big(\frac{y^b-1}{b\log(y)}\Big)by^{3b-1}\Big(\frac{\textcircled{1}\textcircled{2}}{\textcircled{1}'\textcircled{2}+\textcircled{1}\textcircled{2}'}\Big)dy\\
&=\frac{3}{2}b-\int_0^1\frac{(y^b-1)^2y^{3b}}{by^b\log(y)-y^b+1}\,dy+\int_0^1\Big(\frac{y^b-1}{b\log(y)}\Big)by^{3b-1}\Big(\frac{1}{\frac{\textcircled{1}'}{\textcircled{1}}+\frac{\textcircled{2}'}{\textcircled{2}}}\Big)dy
\end{aligned}
\tag{34}
$$

All the equalities above come from rearrangement. In case of $M=3$, we have the following results.

$$
\frac{\textcircled{1}'}{\textcircled{1}}=\frac{(2-M)\log(y)^{1-M}\frac{1}{y}}{\log(y)^{2-M}}=(2-M)\frac{1}{y\log(y)}=-\frac{1}{y\log(y)}
\tag{35}
$$

$$
\begin{aligned}
\frac{\textcircled{2}'}{\textcircled{2}}&=\frac{-\int_b^1{}_1C_0 2uy^{2u-1}du+\int_b^1{}_1C_1 3uy^{3u-1}du}{-\int_b^1{}_1C_0 y^{2u}du+\int_b^1{}_1C_1 y^{3u}du}=\\
&=\Bigg(-\Big(\frac{2\log(y)[y^2-by^{2b}]-y^2+y^{2b}}{2y\log^2(y)}\Big)+\Big(\frac{3\log(y)[y^3-by^{3b}]-y^3+y^{3b}}{3y\log^2(y)}\Big)\Bigg)\Bigg(-\frac{y^2-y^{2b}}{2\log(y)}+\frac{y^3-y^{3b}}{3\log(y)}\Bigg)^{-1}\\
&=\Bigg(\frac{-6\log(y)[y^2-by^{2b}]+3y^2-3y^{2b}+6\log(y)[y^3-by^{3b}]-2y^3+2y^{3b}}{6y\log^2(y)}\Bigg)\Bigg(\frac{-3y^2+3y^{2b}+2y^3-2y^{3b}}{6\log(y)}\Bigg)^{-1}\\
&=\frac{-6\log(y)[y^2-by^{2b}]+3y^2-3y^{2b}+6\log(y)[y^3-by^{3b}]-2y^3+2y^{3b}}{y\log(y)[-3y^2+3y^{2b}+2y^3-2y^{3b}]}
\end{aligned}
\tag{36}
$$

Eq(36) comes from Eq(21) and Eq(22). As a result, we also have the following.

$$
\begin{aligned}
\frac{\textcircled{1}'}{\textcircled{1}}+\frac{\textcircled{2}'}{\textcircled{2}}&=-\frac{1}{y\log(y)}+\frac{-6\log(y)[y^2-by^{2b}]+3y^2-3y^{2b}+6\log(y)[y^3-by^{3b}]-2y^3+2y^{3b}}{y\log(y)[-3y^2+3y^{2b}+2y^3-2y^{3b}]}\\
&=\frac{-6\log(y)[y^2-by^{2b}]+6y^2-6y^{2b}+6\log(y)[y^3-by^{3b}]-4y^3+4y^{3b}}{y\log(y)[-3y^2+3y^{2b}+2y^3-2y^{3b}]}
\end{aligned}
\tag{37}
$$

Plugging Eq(37) to Eq(34) yields the following.

$$
\begin{aligned}
&\frac{3}{2}b-\int_0^1\frac{(y^b-1)^2y^{3b}}{by^b\log(y)-y^b+1}\,dy+\int_0^1\Big(\frac{y^b-1}{b\log(y)}\Big)by^{3b-1}\Big(\frac{1}{\frac{\textcircled{1}'}{\textcircled{1}}+\frac{\textcircled{2}'}{\textcircled{2}}}\Big)dy\\
&=\frac{3}{2}b-\int_0^1\frac{(y^b-1)^2y^{3b}}{by^b\log(y)-y^b+1}\,dy\\
&+\int_0^1\Big(\frac{y^b-1}{b\log(y)}\Big)by^{3b-1}\Big(\frac{y\log(y)[-3y^2+3y^{2b}+2y^3-2y^{3b}]}{-6\log(y)[y^2-by^{2b}]+6y^2-6y^{2b}+6\log(y)[y^3-by^{3b}]-4y^3+4y^{3b}}\Big)dy\\
&=\frac{3}{2}b-\int_0^1\frac{(y^b-1)^2y^{3b}}{by^b\log(y)-y^b+1}\,dy\\
&+\int_0^1(y^{4b}-y^{3b})\frac{-3y^2+3y^{2b}+2y^3-2y^{3b}}{-6\log(y)[y^2-by^{2b}]+6y^2-6y^{2b}+6\log(y)[y^3-by^{3b}]-4y^3+4y^{3b}}dy
\end{aligned}
\tag{38}
$$

So Eq(38) is the RHS of *Equation (10)* on the FP note, given that $M=3$. Then, **the derivative of Eq(38) w.r.t b should be strictly positive for all $b\in(0,1)$**. The derivative of Eq(38) w.r.t b is as follows.

$$\frac{3}{2} - \int_0^1 \mathrm{FIRST}\, dy + \int_0^1 \mathrm{SECOND}\, dy \tag{39}$$

$$\mathrm{FIRST} = \frac{y^{3b}\left(y^b - 1\right)\left(-by^b\left(y^b - 1\right)\log(y) + \left(5y^b - 3\right)\left(by^b\log(y) - y^b + 1\right)\right)\log(y)}{\left(by^b\log(y) - y^b + 1\right)^2}$$

$$\mathrm{SECOND} = \frac{\begin{array}{l} y^{3b}\Big(-3y^{2b}\left(y^b - 1\right)\left(2y^3 - 3y^2 - 2y^{3b} + 3y^{2b}\right)\left(2b\log(y) - y^b\left(3b\log(y) + 1\right) + 2y^b - 1\right) \\ + \left(6y^{2b}\left(y^b - 1\right)^2 - \left(4y^b - 3\right)\left(2y^3 - 3y^2 - 2y^{3b} + 3y^{2b}\right)\right)\left(2y^3 - 3y^2 - 2y^{3b} + 3y^{2b} - 3\left(by^{2b} - y^2\right)\log(y) + 3\left(by^{3b} - y^3\right)\log(y)\right)\Big)\log(y) \end{array}}{2\left(2y^3 - 3y^2 - 2y^{3b} + 3y^{2b} - 3\left(by^{2b} - y^2\right)\log(y) + 3\left(by^{3b} - y^3\right)\log(y)\right)^2}$$

So, is Eq(39) strictly positive for all $b \in (0, 1)$?

Yes it is.

But here, I had to use numerical integration, as you can see in Eq(39). I used the Simpson rule because it performs better than Midpoint or Trapezoid.

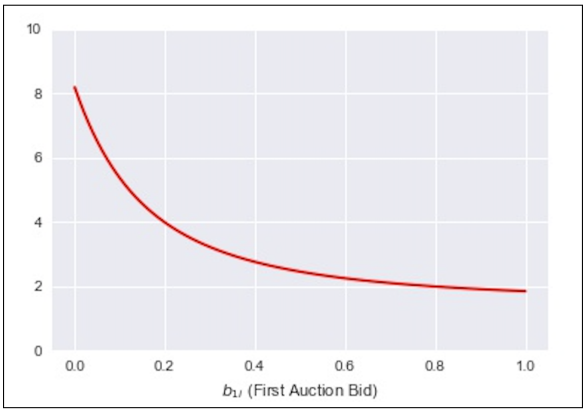

Figure 3: Value of Eq(39) on the support $(0, 1)$

As you see, the red curve is strictly above 0. Note that we are in $M = 3$ case, so there is only one curve. If we were to plot $M = 5$ curve, I guess it would slightly be below the red curve. The lowest curve value is achieved as you get closer and closer to 1. In both Figure 1 and Figure 2, we also had the same finding; as you get closer and closer to the end point of the support, your surface value decreases. Also, why is the support one dimension? It is because now, unlike in Figure 1 and Figure 2, our second auction bid y is integrated out as you can see in Eq(39).

4. Conclusion

- I showed that the triplet $\left[G_1(b), G^w_{2|1}(y|b), G^l_{2|1}(y|b)\right] = \left[b, y^{2b}, y^b\right]$, where the support is $(y, b) \in (0, 1)^2$, satisfies Item 3 on the FP note, in case we have $M = 3$.

- For $M = 5$, I would have to check Figure 3. But I have a strong feeling that the Triplet above will also satisfy Item 3 as well in $M = 5$
- As I wrote in on page 2 and page 6, I didn't resort to numerical integration for Figure 1 and Figure 2; it is because I got analytic expression. But, for Figure 3, I had to resort to numerical method(Simpson Method).