# FP: Questions and Comments

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## Contents



# 1 Comments

Each subsection introduces estimators equipped with a smooth CDF.

## 1.1 Eq(3) of OnEstimation1.3, $\hat{G}_{B_2^{\max}|B_1^{\max},Z,I}\left(b \mid b_1, z, I\right)$

Note the following notations and equations.

$$\overline{K}_{2\ell}(b) \equiv \int_{-\infty}^{\frac{b-B_{2\ell}^{max}}{h_2}} K(u)du = \int_{-\infty}^{b} \frac{1}{h_2} K\left(\frac{x - B_{2\ell}^{max}}{h_2}\right)dx \tag{1}$$

$$\hat{M}^w_2\left(b \mid b_1, z, I\right) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell \int_{-\infty}^{b} \frac{1}{h_2} K\left(\frac{x - B_{2\ell}^{max}}{h_2}\right)dx \tag{2}$$

$$\hat{m}^w_2(b|b_1, z, I) \equiv \frac{d}{db}\hat{M}^w_2\left(b \mid b_1, z, I\right) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell \frac{1}{h_2} K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right) \tag{3}$$

$$\hat{M}^l_2\left(b \mid b_1, z, I\right) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \lambda_\ell \int_{-\infty}^{b} \frac{1}{h_2} K\left(\frac{x - B_{2\ell}^{max}}{h_2}\right)dx \tag{4}$$

$$\hat{m}^l_2(b|b_1, z, I) \equiv \frac{d}{db}\hat{M}^l_2\left(b \mid b_1, z, I\right) = \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \lambda_\ell \frac{1}{h_2} K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right) \tag{5}$$

Then, we will have the following.

$$
\begin{aligned}
\hat{G}_{B_2^{\max}|B_1^{\max},Z,I}\left(b \mid b_1, z, I\right) &\equiv \hat{M}_2^w\left(b \mid b_1, z, I\right) + \hat{M}_2^l\left(b \mid b_1, z, I\right) \\
&= \sum_{\ell \in \mathcal{L}_I} \lambda_\ell \int_{-\infty}^{b} \frac{1}{h_2} K\left(\frac{x - B_{2\ell}^{max}}{h_2}\right) dx \\
&= \sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b) \qquad (6)
\end{aligned}
$$

where $\overline{K}_{2\ell}(b)$ is from Eq(1). **Eq(6) corresponds to Eq(3) of OnEstimation1.3.**

## 1.2 Eq(4), $\hat{G}_{2|1}^w\left(b_2 \mid b_1, z, I\right)$

Then $\hat{G}_{2|1}^w(b_2|b_1, z, I)$ which uses a smooth kernel is as follows.

$$
\begin{aligned}
\hat{G}_{2|1}^w\left(b_2 \mid b_1, z, I\right) &= \exp\left\{-\int_{-\infty}^{+\infty} \frac{\mathbb{1}\left[b_2 \leq b\right]}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} \left(\frac{d}{db} \hat{M}_2^w\left(b \mid b_1, z, I\right) db\right)\right\} \\
&= \exp\left\{-\int_{-\infty}^{+\infty} \frac{\mathbb{1}\left[b_2 \leq b\right]}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} \sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell \frac{1}{h_2} K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right) db\right\} \\
&= \exp\left\{-\frac{1}{h_2} \int_{b_2}^{\bar{b}} \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db\right\} \\
&= \prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2} \int_{b_2}^{\bar{b}} \frac{K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}} \overline{K}_{2\tilde{\ell}}(b)} db\right\} \qquad (7)
\end{aligned}
$$

where $\overline{K}_{2\tilde{\ell}}(b)$ above comes from Eq(1). **Eq(7) corresponds to Eq(4) of OnEstimation1.3.** Note that $\overline{K}_{2\tilde{\ell}}(b)$ typically has an explicit form, thus Eq(7) is straightforward to compute.

### 1.3 Eq(5), $\hat{G}_2^l(b_2|B_1 \le b_1, z, I)$

$\hat{G}_2^l(b_2|B_1 \le b_1, z, I)$ equipped with a smooth kernel will be as follows.

$$
\begin{aligned}
\hat{G}_2^l\left(b_2 \mid B_1 \le b_1, z, I\right) &= \exp\left\{-\frac{1}{I-1}\int_{-\infty}^{+\infty}\frac{\mathbb{1}\left[b_2 \le b\right]}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell \overline{K}_{2\ell}(b)}\left(\frac{d}{db}\hat{M}_2^l\left(b \mid b_1, z, I\right)db\right)\right\} \\
&= \exp\left\{-\frac{1}{I-1}\int_{-\infty}^{+\infty}\frac{\mathbb{1}\left[b_2 \le b\right]}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell \overline{K}_{2\ell}(b)}\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right)db\right\} \\
&= \exp\left\{-\frac{1}{h_2(I-1)}\int_{b_2}^{\bar{b}}\frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell \overline{K}_{2\ell}(b)}db\right\} \\
&= \prod_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}}\exp\left\{-\frac{\lambda_\ell}{h_2(I-1)}\int_{b_2}^{\bar{b}}\frac{K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}
\end{aligned}
\tag{8}
$$

Eq(8) is analogous to Eq(7), and it is straightforward to compute for the same reason. **Eq(8) corresponds to Eq(5) of OnEstimation1.3.**

### 1.4 Eqs(6)-(8), $\hat{m}_2^w(b|b_1, z, I), \hat{m}_2^l\left(b \mid b_1, z, I\right), \hat{g}_{B_2^{max}|B_1^{max}, Z, I}(b|b_1, z, I)$

$$
\hat{m}_2^w(b|b_1, z, I) = \sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}= W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right) \tag{9}
$$

$$
\hat{m}_2^l\left(b \mid b_1, z, I\right) = \sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right) \tag{10}
$$

$$
\hat{g}_{B_2^{max}|B_1^{max}, Z, I}(b|b_1, z, I) \equiv \sum_{\ell\in\mathcal{L}_I}\lambda_\ell\frac{1}{h_2}K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right) = \hat{m}_2^w(b|b_1, z, I) + \hat{m}_2^l(b|b_1, z, I) \tag{11}
$$

Each equation above comes from Eq(3) and Eq(5). **These three equations correspond to Eqs(6)-(8) of OnEstimation1.3.**

### 1.5 Eqs(9)-(10), $\hat{g}_{2|1}^w\left(b_2 \mid b_1, z, I\right), \hat{g}_2^l\left(b_2 \mid B_1 \le b_1, z, I\right)$

$$
\hat{g}_{2|1}^w\left(b_2 \mid b_1, z, I\right) = \frac{\hat{m}_2^w\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}, Z, I}\left(b_2 \mid b_1, z, I\right)}\hat{G}_{2|1}^w\left(b_2 \mid b_1, z, I\right) \tag{12}
$$

$$
\hat{g}_2^l\left(b_2 \mid B_1 \le b_1, z, I\right) = \frac{1}{I-1}\frac{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}, Z, I}\left(b_2 \mid b_1, z, I\right)}\hat{G}_2^l\left(b_2 \mid B_1 \le b_1, z, I\right) \tag{13}
$$

I copy and pasted Eqs(9)-(10) of OnEstimation1.3. We already know all the values.

### 1.6 Eqs(12)-(13), $\hat{G}_1(\cdot \mid z, I), \hat{G}_1(\cdot \mid z, I)^{I-1}$

Note the following new notation.

$$\overline{K}_{1\ell}(\cdot) \equiv \int_{-\infty}^{\frac{\cdot - B_{1\ell}^{max}}{h_1}} K(u)du = \int_{-\infty}^{\cdot} \frac{1}{h_1} K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right)dx \tag{14}$$

This new notation is comparable to Eq(1), but we use a bandwidth $h_1$. Then we have the following.

$$\hat{G}_1(\cdot \mid z, I) = \left(\frac{\sum_{\ell \in \mathcal{L}_I} \overline{K}_{1\ell}(\cdot) K\left(\frac{z - Z_\ell}{h_z}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{z - Z_\ell}{h_z}\right)}\right)^{1/I} = \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(\cdot)\right)^{1/I} \tag{15}$$

$\overline{K}_{1\ell}(\cdot)$ typically has an explicit form. **Eq(15) corresponds to Eq(12) of OnEstimation1.3.**

$$\hat{G}_1(\cdot \mid z, I)^{I-1} = \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(\cdot)\right)^{(I-1)/I} \tag{16}$$

**Eq(16) corresponds to Eq(13) of OnEstimation1.3.** For future use, we will need the following equation.

$$\begin{aligned}
\frac{d}{dx}\hat{G}_1(x|z, I)^{I-1} &= \frac{d}{dx}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(x)\right)^{(I-1)/I} \\
&= \frac{d}{dx}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \int_{-\infty}^{x} \frac{1}{h_1} K\left(\frac{y - B_{1\ell}^{max}}{h_1}\right)dy\right)^{(I-1)/I} \\
&= \frac{I-1}{I}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \int_{-\infty}^{x} \frac{1}{h_1} K\left(\frac{y - B_{1\ell}^{max}}{h_1}\right)dy\right)^{-1/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell \frac{1}{h_1} K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right) \\
&= \frac{I-1}{h_1 I}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(x)\right)^{-1/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right) \qquad (17)
\end{aligned}$$

### 1.7 Eq(11), (14), $\hat{H}_2^w(\cdot; b_1, z, I), \hat{H}_2^l(\cdot; b_1, z, I)$.

I copy and paste Eq(11) of OnEstimation1.3 as follows.

$$\hat{H}_2^w(\cdot; b_1, z, I) = \hat{G}_2^l(\cdot \mid B_1 \le b_1, z, I)^{I-1} \tag{18}$$

$$\hat{H}_2^l(\cdot; b_1, z, I) = \underbrace{\frac{1}{1 - \hat{G}_1(b_1 \mid z, I)^{I-1}}}_{=:2} \underbrace{\int_{b_1}^{\bar{b}} \hat{G}_2^l(\cdot \mid B_1 \le x, z, I)^{I-2} \hat{G}_{2|1}^w(\cdot \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1}}_{=:1} \tag{19}$$

The RHS of Eq(18) uses Eq(8) which we can calculate.

\lambda_{\ell} here

The following is my derivation of Eq(19) where we first focus on the part ①.

$$\int_{b_1}^{\bar{b}} \hat{G}_2^l\left(\cdot \mid B_1 \leq x, z, I\right)^{I-2} \hat{G}_{2|1}^w(\cdot \mid x, z, I)\left(\frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1} dx\right)$$

$$= \int_{b_1}^{\bar{b}} \underbrace{\prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell (I-2)}{h_2(I-1)} \int_{\cdot}^{\bar{b}} \frac{K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}} \overline{K}_{2\tilde{\ell}}(b)} db\right\}}_{= \hat{G}_2^l(\cdot \mid B_1 \leq x, z, I)^{I-2}}$$

$$\underbrace{\prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2} \int_{\cdot}^{\bar{b}} \frac{K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}} \overline{K}_{2\tilde{\ell}}(b)} db\right\}}_{= \hat{G}_{2|1}^w(\cdot \mid x, z, I)}$$

$$\underbrace{\frac{I-1}{h_1 I}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(x)\right)^{-1/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right)}_{= \frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1}} dx \tag{20}$$

where each $\lambda_\ell$, $\lambda_{\tilde{\ell}}$, $\omega_\ell$ in Eq(20) is now as follows.

$$\lambda_\ell = K\left(\frac{x - B_{1\ell}^{\max}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right) / \left[\sum_{\ell \in \mathcal{L}_I} K\left(\frac{x - B_{1\ell}^{\max}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right)\right] \tag{21}$$

$$\lambda_{\tilde{\ell}} = K\left(\frac{x - B_{1\tilde{\ell}}^{\max}}{h_1}\right) K\left(\frac{z - Z_{\tilde{\ell}}}{h_z}\right) / \left[\sum_{\ell \in \mathcal{L}_I} K\left(\frac{x - B_{1\ell}^{\max}}{h_1}\right) K\left(\frac{z - Z_\ell}{h_z}\right)\right] \tag{22}$$

$$\omega_\ell = K\left(\frac{z - Z_\ell}{h_z}\right) / \sum_{\ell \in \mathcal{L}_I} K\left(\frac{z - Z_\ell}{h_z}\right) \tag{23}$$

Keep in mind that $\lambda_\ell$, $\lambda_{\tilde{\ell}}$ depend on $x$, which is the first auction bid in Eq(20) above. Eq(20) is a univariate integral on a compact support, where the integrand has an explicit expression. Thus, it can be computed efficiently through several univariate integration computer routines for each $(\cdot, b_1, Z, I)$

$\therefore$ **From here starts new update**

② in Eq(19) can be calculated from Eq(16). Combining both ① and ②, we can calculate Eq(19), where Eq(19) is Eq(14) in the OnEstimation1.3.

## 1.8 Eqs(15)-(17), $\hat{h}_2^w\left(\cdot; b_1, z, I\right), \hat{h}_2^l\left(\cdot; b_1, z, I\right), \hat{\Psi}(\cdot; x, z, I)$

$$\hat{h}_2^w\left(\cdot; b_1, z, I\right) = (I-1)\hat{g}_2^l\left(\cdot \mid B_1 \leq b_1, z, I\right) \hat{G}_2^l\left(\cdot \mid B_1 \leq b_1, z, I\right)^{I-2} \tag{24}$$

**Eq(24) is the same equation as Eq(15) of OnEstimation1.3.** Each term inside the equation uses

Eq(13) and Eq(8), which implies we can calculate it.

**I copy and paste Eqs(16)-(17) from OnEstimation1.3 as follows.**

$$\hat{h}_2^l(\cdot; b_1, z, I) = \underbrace{\frac{1}{1 - \hat{G}_1(b_1 \mid z, I)^{I-1}}}_{②} \times \int_{b_1}^{\bar{b}} \hat{\Psi}(\cdot; x, z, I) \hat{G}_2^l(\cdot \mid B_1 \leq x, z, I)^{I-2} \hat{G}_{2|1}^w(\cdot \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1} \tag{25}$$

where

$$\hat{\Psi}(\cdot; x, z, I) \equiv (I-2) \frac{\hat{g}_2^l(\cdot \mid B_1 \leq x, z, I)}{\hat{G}_2^l(\cdot \mid B_1 \leq x, z, I)} + \frac{\hat{g}_{2|1}^w(\cdot \mid x, z, I)}{\hat{G}_{2|1}^w(\cdot \mid x, z, I)} \tag{26}$$

Note that no matter which CDF estimator we use (e.g., smooth or discrete), Eqs(25)-(26) hold. ② in Eq(25) is easily caclulated from Eq(16). Eq(26) is as follows.

$$\begin{aligned}
\hat{\Psi}(\cdot; x, z, I) &= \frac{I-2}{I-1} \frac{\hat{m}_2^l(\cdot \mid x, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}, Z, I}(\cdot \mid x, z, I)} + \frac{\hat{m}_2^w(\cdot \mid x, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}, Z, I}(\cdot \mid x, z, I)} \\
&= \frac{I-2}{I-1} \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \lambda_\ell \frac{1}{h_2} K\left(\frac{\cdot - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(\cdot)} + \frac{\sum_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \lambda_\ell \frac{1}{h_2} K\left(\frac{\cdot - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(\cdot)} \\
&= \frac{1}{h_2} \frac{1}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(\cdot)} \left( \sum_{\ell \in \mathcal{L}_I} \lambda_\ell^* K\left(\frac{\cdot - B_{2\ell}^{max}}{h_2}\right) \right)
\end{aligned} \tag{27}$$

First equality of Eq(27) holds by applying Eqs(12),(13) to Eq(26). Last equality holds if we define $\lambda_\ell^*$ as follows.

$$\lambda_\ell^* := \begin{cases} \lambda_\ell & \text{if } \ell \in \{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\} \\ \lambda_\ell \frac{I-2}{I-1} & \text{if } \ell \in \{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\} \end{cases} \tag{28}$$

Then, the integral part of Eq(25) is as follows.

$$
\begin{aligned}
&\int_{b_1}^{\bar{b}} \hat{\Psi}(\cdot; x, z, I) \hat{G}_2^l \left(\cdot \mid B_1 \leq x, z, I\right)^{I-2} \hat{G}_{2|1}^w(\cdot \mid x, z, I) \left(\frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1} dx\right) \\
&= \int_{b_1}^{\bar{b}} \underbrace{\frac{1}{h_2} \frac{1}{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(\cdot)} \left( \sum_{\ell \in \mathcal{L}_I} \lambda_\ell^* K\left(\frac{\cdot - B_{2\ell}^{max}}{h_2}\right)\right)}_{=\hat{\Psi}(\cdot;x,z,I)} \\
&\quad \underbrace{\prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}} \exp\left\{ -\frac{\lambda_\ell (I-2)}{h_2(I-1)} \int_{\cdot}^{\bar{b}} \frac{K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}} \overline{K}_{2\tilde{\ell}}(b)} db \right\}}_{=\hat{G}_2^l(\cdot \mid B_1 \leq x, z, I)^{I-2}} \\
&\quad \underbrace{\prod_{\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}} \exp\left\{ -\frac{\lambda_\ell}{h_2} \int_{\cdot}^{\bar{b}} \frac{K\left(\frac{b - B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell} \in \mathcal{L}_I} \lambda_{\tilde{\ell}} \overline{K}_{2\tilde{\ell}}(b)} db \right\}}_{=\hat{G}_{2|1}^w(\cdot \mid x, z, I)} \\
&\quad \underbrace{\frac{I-1}{h_1 I} \left( \sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(x) \right)^{-1/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell K\left(\frac{x - B_{1\ell}^{max}}{h_1}\right)}_{=\frac{d}{dx}\hat{G}_1(x \mid z, I)^{I-1}} \, dx
\end{aligned} \tag{29}
$$

Here, each $\lambda_\ell, \lambda_{\tilde{\ell}}, \omega_\ell$ corresponds to Eqs(21)-(23) and $\lambda_\ell^*$ corresponds to Eq(28). Similarly as in Eq(20), this Eq(29) is a univariate integral on a compact support, where the integrand has an explicit expression. Thus, it can be computed efficiently through several univariate integration computer routines for each $(\cdot, b_1, Z, I)$.

## 1.9 Eq(18), $\hat{\tilde{D}}\left(\cdot \mid b_1, z, I\right)$

Plug-in estimator for $\hat{\tilde{D}}\left(\cdot \mid b_1, z, I\right)$ will be as follows.

$$
\hat{\tilde{D}}\left(\cdot \mid b_1, z, I\right) = \int \mathbb{1}\left( x + \frac{\hat{H}_2^w\left(x; b_1, z, I\right)}{\hat{h}_2^w\left(x; b_1, z, I\right)} \leq \cdot \right) d\hat{G}_{2|1}^w\left(x \mid b_1, z, I\right) \tag{30}
$$

In our context, we know the following holds,

$$
\begin{aligned}
x+\frac{\hat{H}_2^w\left(x;b_1,z,I\right)}{\hat{h}_2^w\left(x;b_1,z,I\right)} &= x+\frac{\hat{G}_2^l\left(x\mid B_1\leq b_1,z,I\right)^{I-1}}{(I-1)\hat{g}_2^l\left(x\mid B_1\leq b_1,z,I\right)\hat{G}_2^l\left(x\mid B_1\leq b_1,z,I\right)^{I-2}}\\
&= x+\frac{\hat{G}_2^l\left(x\mid B_1\leq b_1,z,I\right)}{(I-1)\hat{g}_2^l\left(x\mid B_1\leq b_1,z,I\right)}\\
&\boxed{= x+\frac{\hat{G}_{B_2^{\max}\mid B_1^{\max},Z,I}\left(x\mid b_1,z,I\right)}{\hat{m}_2^l\left(x\mid b_1,z,I\right)}}\\
&= x+\frac{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(x)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{x-B_{2\ell}^{max}}{h_2}\right)}.
\end{aligned}
\tag{31}
$$

The third equality holds because we know the following equality from Eq(13),

$$
\hat{g}_2^l\left(x\mid B_1\leq b_1,z,I\right)=\frac{1}{I-1}\frac{\hat{m}_2^l\left(x\mid b_1,z,I\right)}{\hat{G}_{B_2^{\max}\mid B_1^{\max},Z,I}\left(x\mid b_1,z,I\right)}\hat{G}_2^l\left(x\mid B_1\leq b_1,z,I\right).
$$

Given Eq(31), recall that $x+\frac{\hat{H}_2^w(x;b_1,z,I)}{\hat{h}_2^w(x;b_1,z,I)}$ should be strictly increasing in $x$ by assumption. Also, recall $\mathbb{1}\left(x+\frac{\hat{H}_2^w(x;b_1,z,I)}{\hat{h}_2^w(x;b_1,z,I)}\leq\cdot\right)$ inside Eq(30). Thus, in ideal situation, there exists a unique $\hat{b}^{w,*}(\cdot)\in[\underline{b},\overline{b}]$ such that it satisfies the following,[1]

$$
\hat{b}^{w,*}(\cdot)+\frac{\hat{H}_2^w\left(\hat{b}^{w,*}(\cdot);b_1,z,I\right)}{\hat{h}_2^w\left(\hat{b}^{w,*}(\cdot);b_1,z,I\right)}=\hat{b}^{w,*}(\cdot)+\frac{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(\hat{b}^{w,*}(\cdot))}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{\hat{b}^{w,*}(\cdot)-B_{2\ell}^{max}}{h_2}\right)}=\cdot
\tag{32}
$$

But, since $x+\frac{\hat{H}_2^w(x;b_1,z,I)}{\hat{h}_2^w(x;b_1,z,I)}$ is the empirical analog of $x+\frac{H_2^w(x;b_1,z,I)}{h_2^w(x;b_1,z,I)}$, the empirical analog(or estimate) may not necessarily be strictly increasing and so is its $\hat{b}^{w,*}(\cdot)$ may not be unique. Thus, I define $\hat{b}^{w,*}(\cdot)$ as *the smallest* solution of the following.

$$
\min_x\left(x+\frac{\hat{H}_2^w\left(x;b_1,z,I\right)}{\hat{h}_2^w\left(x;b_1,z,I\right)}-\cdot\right)^2
$$

Then, $\hat{b}^{w,*}(\cdot)$ will always be a unique number, and Eq(30) is equivalent to $\underline{\hat{G}_{2|1}^w(\hat{b}^{w,*}(\cdot)|b_1,z,I)}$, which we can calculate.

## 1.10 Eqs(19)-(21), $\hat{G}_{2|1}^l\left(\cdot\mid b_1,z,I\right)$, $\frac{\hat{G}_1(b_1|z,I)}{\hat{g}_1(b_1|z,I)}$, $\frac{\partial\hat{G}_2^l(\cdot|B_1\leq b_1,z,I)}{\partial b_1}$

$$
\hat{\hat{F}}_{2|1}\left(\cdot\mid b_1,z,I\right)=\int\mathbb{1}\left(x+\frac{\hat{H}_2^l\left(x;b_1,z,I\right)}{\hat{h}_2^l\left(x;b_1,z,I\right)}\leq\cdot\right)d\hat{G}_{2|1}^l\left(x\mid b_1,z,I\right)
\tag{33}
$$

Eq(33) requires $\hat{G}_{2|1}^l(x|b_1,z,I)$, which is equivalent to as follows. (Equation below is brought from Eq(19)

[1] Each $b^{w,*}(\cdot)-\varepsilon$ and $b^{w,*}(\cdot)+\varepsilon$ will cause "$<\cdot$", "$>\cdot$" respectively in Eq(32).

# Pesky part

of OnEstimation1.3.)

$$\hat{G}^l_{2|1}\left(\cdot \mid b_1, z, I\right) = \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right) + \frac{\hat{G}_1\left(b_1 \mid z, I\right)}{\hat{g}_1\left(b_1 \mid z, I\right)} \frac{\partial \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right)}{\partial b_1} \tag{34}$$

We know $\hat{G}^l_2(\cdot|B_1 \leq b_1, z, I)$ from Eq(8). Also, note the following holds by Eq(15).

$$\begin{aligned}\frac{\hat{G}_1\left(b_1 \mid z, I\right)}{\hat{g}_1\left(b_1 \mid z, I\right)} &= \frac{\left(\sum_{\ell\in\mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}\left(b_1\right)\right)^{1/I}}{\frac{1}{I}\left(\sum_{\ell\in\mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}\left(b_1\right)\right)^{(1-I)/I}\left(\sum_{\ell\in\mathcal{L}_I} \omega_\ell \frac{1}{h_1} K\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right)\right)} \\ &= I \frac{\sum_{\ell\in\mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}\left(b_1\right)}{\sum_{\ell\in\mathcal{L}_I} \omega_\ell \frac{1}{h_1} K\left(\frac{b_1 - B^{max}_{1\ell}}{h_1}\right)}\end{aligned} \tag{35}$$

where $\omega_\ell$ comes from Eq(23) — Eq(35) can easily be computed. **Each Eq(34) and Eq(35) corresponds to Eqs(19)-(20) of OnEstimation1.3, respectively.**

Regarding $\frac{\partial \hat{G}^l_2(\cdot|B_1 \leq b_1, z, I)}{\partial b_1}$ inside Eq(34), it is as follows.

$$\begin{aligned}&\frac{\partial \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right)}{\partial b_1} \\ &= \frac{\partial}{\partial b_1} \exp\left\{-\frac{1}{h_2(I-1)} \int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db\right\} \\ &= \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right) \frac{\partial}{\partial b_1}\left(-\frac{1}{h_2(I-1)} \int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db\right) \\ &= -\frac{1}{h_2(I-1)} \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right)\left(\int_\cdot^{\bar{b}} \frac{\partial}{\partial b_1} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db\right)\end{aligned}$$

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$$\begin{aligned}&= -\frac{1}{h_2(I-1)} \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right) \\ &\left(\int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \frac{\partial \lambda_\ell}{\partial b_1} K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)\right) - \sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I} \frac{\partial \lambda_\ell}{\partial b_1} \overline{K}_{2\ell}(b)\right)}{\left(\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)\right)^2} db\right) \\ &= -\frac{1}{h_2(I-1)} \hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right) \\ &\left(\int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \frac{\partial \lambda_\ell}{\partial b_1} K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db - \int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I} \frac{\partial \lambda_\ell}{\partial b_1} \overline{K}_{2\ell}(b)\right)}{\left(\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)\right)^2} db\right) \\ &= \frac{\hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right)}{h_2(I-1)} \int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I} \frac{\partial \lambda_\ell}{\partial b_1} \overline{K}_{2\ell}(b)\right)}{\left(\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)\right)^2} db \\ &\quad - \frac{\hat{G}^l_2\left(\cdot \mid B_1 \leq b_1, z, I\right)}{h_2(I-1)} \int_\cdot^{\bar{b}} \frac{\sum_{\{\ell\in\mathcal{L}_I : W_{1\ell}\neq W_{2\ell}\}} \frac{\partial \lambda_\ell}{\partial b_1} K\left(\frac{b - B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I} \lambda_\ell \overline{K}_{2\ell}(b)} db\end{aligned} \tag{36}$$

where the partial derivative above, $\frac{\partial \lambda_\ell}{\partial b_1}$, is as follows — I use two new notations, $\tilde{b}_{1\ell} = \frac{b_1 - B_{1\ell}^{max}}{h_1}$, $\tilde{z}_\ell = \frac{z - Z_\ell}{h_1}$ below.

$$
\begin{aligned}
\frac{\partial}{\partial b_1}\lambda_\ell &= \frac{\partial}{\partial b_1}\left(\frac{K\left(\frac{b_1 - B_{1\ell}^{\max}}{h_1}\right)\cancel{K\left(\frac{z - Z_\ell}{h_z}\right)}}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{\max}}{h_1}\right)\cancel{K\left(\frac{z - Z_\ell}{h_z}\right)}}\right) = \frac{\partial}{\partial b_1}\left(\frac{K(\tilde{b}_{1\ell})\cancel{K(\tilde{z}_\ell)}}{\sum_{\ell \in \mathcal{L}_I} K(\tilde{b}_{1\ell})\cancel{K(\tilde{z}_\ell)}}\right) \\
&= \left(\frac{1}{\sum_{\ell \in \mathcal{L}_I} K(\tilde{b}_{1\ell})K\left(\tilde{z}_\ell\right)}\right)^2 \frac{1}{h_1}\left(k(\tilde{b}_{1\ell})K(\tilde{z}_\ell)\sum_{\ell \in \mathcal{L}_I} K(\tilde{b}_{1\ell})K(\tilde{z}_\ell) - K(\tilde{b}_{1\ell})K(\tilde{z}_\ell)\sum_{\ell \in \mathcal{L}_I} k(\tilde{b}_{1\ell})K(\tilde{z}_\ell))\right) \\
&= \lambda_\ell \frac{1}{h_1}\left(\frac{k(\tilde{b}_{1\ell})}{K(\tilde{b}_{1\ell})} - \frac{\sum_{\ell \in \mathcal{L}_I} k(\tilde{b}_{1\ell})K(\tilde{z}_\ell)}{\sum_{\ell \in \mathcal{L}_I} K(\tilde{b}_{1\ell})K(\tilde{z}_\ell)}\right) \\
&\equiv \lambda_\ell \frac{1}{h_1}\left(\frac{k(\frac{b_1 - B_{1\ell}^{max}}{h_1})}{K(\frac{b_1 - B_{1\ell}^{max}}{h_1})} - \frac{\sum_{\ell \in \mathcal{L}_I} k(\frac{b_1 - B_{1\ell}^{max}}{h_1})K(\frac{z - Z_\ell}{h_1})}{\sum_{\ell \in \mathcal{L}_I} K(\frac{b_1 - B_{1\ell}^{max}}{h_1})K(\frac{z - Z_\ell}{h_1})}\right) \qquad (37)
\end{aligned}
$$

Note the last part of Eq(36) — $\frac{\partial \lambda_\ell}{\partial b_1}$ inside the integral does not depend on the index $b$. **I think Eq(36) can easily be computed, and this equation corresponds to Eq(21) of OnEstimation1.3.**

## 1.11 Eqs(22)-(23), Not Needed, $\hat{\hat{F}}_{2|1}\left(\cdot \mid b_1, z, I\right)$

Eq(22) of OnEstimation1.3 is not needed in our case.

For Eq(23) of OnEstimation1.3, I copy and paste Eq(33) below.

$$
\begin{aligned}
\hat{\hat{F}}_{2|1}\left(\cdot \mid b_1, z, I\right) &= \int \mathbb{1}\left(x + \frac{\hat{H}_2^l\left(x; b_1, z, I\right)}{\hat{h}_2^l\left(x; b_1, z, I\right)} \le \cdot\right) d\hat{G}_{2|1}^l\left(x \mid b_1, z, I\right) \\
&= \int_{\underline{b}}^{\hat{b}^{l,*}(\cdot)} d\hat{G}_{2|1}(x|b_1, z, I) \\
&= \hat{G}_{2|1}^l(\hat{b}^{l,*}(\cdot)|b_1, z, I) \qquad (38)
\end{aligned}
$$

where $\hat{b}^{l,*}(\cdot)$ satisfies the following.

$$
\begin{aligned}
&\hat{b}^{l,*}(\cdot) + \frac{\hat{H}_2^l\left(\hat{b}^{l,*}(\cdot); b_1, z, I\right)}{\hat{h}_2^l\left(\hat{b}^{l,*}(\cdot); b_1, z, I\right)} \\
&= \hat{b}^{l,*}(\cdot) + \frac{\int_{b_1}^{\bar{b}} \hat{G}_2^l\left(\hat{b}^{l,*}(\cdot) \mid B_1 \le x, z, I\right)^{I-2} \hat{G}_{2|1}^w(\hat{b}^{l,*}(\cdot) \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1}}{\int_{b_1}^{\bar{b}} \hat{\Psi}(\hat{b}^{l,*}(\cdot); x, z, I)\hat{G}_2^l\left(\hat{b}^{l,*}(\cdot) \mid B_1 \le x, z, I\right)^{I-2} \hat{G}_{2|1}^w(\hat{b}^{l,*}(\cdot) \mid x, z, I) d\hat{G}_1(x \mid z, I)^{I-1}} \\
&= \cdot \qquad (39)
\end{aligned}
$$

First equality of Eq(39) holds by Eqs(19), (25) — we already showed in previous subsections that we can compute both numerator and denominator. For the last equality, each $b^{l,*}(\cdot) - \varepsilon$ and $b^{l,*}(\cdot) + \varepsilon$ will cause "$< \cdot$", "$> \cdot$" respectively in Eq(39).

Same as what we described in the subsection 1.9, $x + \frac{\hat{H}_2^l(x; b_1, z, I)}{\hat{h}_2^l(x; b_1, z, I)}$ is an empirical analog, so $\hat{b}^{l,*}(\cdot)$ may not be unique. Thus, we circumvent this problem the same way as we did in subsection 1.9 — $\hat{b}^{l,*}(\cdot)$ is *the smallest* solution of $\min_x \left(x + \frac{\hat{H}_2^l(x;b_1,z,I)}{\hat{h}_2^l(x;b_1,z,I)} - \cdot\right)^2$. Going back to Eq(38), we already know how to compute $\hat{G}_{2|1}^l(\cdot|b_1,z,I)$ so $\hat{\hat{F}}_{2|1}(\cdot|b_1,z,I)$ can be computed.

Same. I replaced it with a square

## 1.12 New Part. Estimation of $\hat{\hat{\delta}}(b_1,\cdot)$

This part follows exactly the same step as in the section (iv) of OnIdentification1.4. But, note that we will have the following — assume for now that the maximum number of possible bidders is $N$.

$$\hat{\hat{\delta}}(b_1, v_2; z) = \sum_{I=2}^{N} \frac{L_I}{\sum_{\tilde{I}=2}^{N} L_{\tilde{I}}} \hat{\hat{D}}^{-1}\left(\hat{\hat{F}}_{2|1}(v_2|b_1,z,I) \,\middle|\, b_1, z, I\right).$$

The equation above does the weighted average with respect to $L_I$. As a result, (pseudo) synergy function inherently depends on the condition $z$.

I revised this part too.

## 1.13 New Part. Estimation of $\hat{F}_1(\cdot|z,I)$, $\hat{F}_{2|1}(\cdot|b_1,z,I)$ and $\hat{\delta}(\cdot,\cdot)$

I copy and paste below Eq(10) of the 2FP note — plug-in estimator is as follows,

$$\begin{aligned} v_{1i} =& b_{1i} + \frac{1}{I-1}\frac{\hat{G}_1(b_{1i})}{\hat{g}_1(b_{1i})} - \underbrace{\int_{\underline{b}}^{\bar{b}} \left[\frac{\hat{H}_2^w(B_{2i}^w;b_{1i})}{\hat{h}_2^w(B_{2i}^w;b_{1i})}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \le b_{1i}\right)^{I-2}\hat{G}_{2|1}^l\left(B_{2i}^w \mid b_{1i}\right)\right] d\hat{G}_{2|1}^w(B_{2i}^w|b_{1i})}_{①} \\ &+ \underbrace{\int_{\underline{b}}^{\bar{b}} \left[\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l\left(B_{2i}^l \mid B_1 \le b_{1i}\right)^{I-2}\hat{G}_{2|1}^w\left(B_{2i}^l \mid b_{1i}\right)\right] d\hat{G}_{2|1}^l(B_{2i}^l|b_{1i})}_{②} \equiv \xi_1(b_{1i}). \end{aligned}$$

I will not write $(z,I)$ to save some space; throughout this subsection, assume there is always condition $(z,I)$. We already know $\frac{\hat{G}_1(b_{1i})}{\hat{g}_1(b_{1i})}$ from Eq(35), so what matters here is ① and ②,

$$① = \int_{\underline{b}}^{\bar{b}} \left[\frac{\hat{H}_2^w(B_{2i}^w;b_{1i})}{\hat{h}_2^w(B_{2i}^w;b_{1i})}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \le b_{1i}\right)^{I-2}\hat{G}_{2|1}^l\left(B_{2i}^w \mid b_{1i}\right)\hat{g}_{2|1}^w(B_{2i}^w|b_{1i})\right] dB_{2i}^w, \tag{40}$$

$$② = \int_{\underline{b}}^{\bar{b}} \left[\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l\left(B_{2i}^l \mid B_1 \le b_{1i}\right)^{I-2}\hat{G}_{2|1}^w\left(B_{2i}^l \mid b_{1i}\right)\hat{g}_{2|1}^l(B_{2i}^l|b_{1i})\right] dB_{2i}^l. \tag{41}$$

Integrand in Eq(40) is as follows — I use all the previous derivations.

$$\underbrace{\frac{\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-1}}{(I-1)\hat{g}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-2}}}_{\frac{\hat{H}_2^w}{\hat{h}_2^w}} \hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-2} \times$$
$$\underbrace{\left(\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_1\right) + \frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial \hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{\partial b_{1i}}\right)}_{\hat{G}_{2|1}^l} \times$$
$$\underbrace{\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right)}_{\hat{g}_{2|1}^w}.$$

Recalling that $\hat{g}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)$ inside $\frac{\hat{H}_2^w}{\hat{h}_2^w}$ is equivalent to $\frac{1}{I-1}\frac{\hat{m}_2^l\left(B_{2i}^w|b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^w|b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)$ from Eq(13), the integrand above becomes the following,

$$\frac{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-2} \times \underbrace{\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right)}_{\hat{g}_{2|1}^w}$$
$$\underbrace{\left(\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_1\right) + \frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial \hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{\partial b_{1i}}\right)}_{\hat{G}_{2|1}^l} =$$
$$\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-2}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right) \times$$
$$\left(\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_1\right) + \frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial \hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{\partial b_{1i}}\right) =$$
$$\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-1}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right) +$$
$$\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-2}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right)\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial \hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{\partial b_{1i}}.$$

Recall that $\frac{\partial \hat{G}_2^l\left(B_{2i}^w|B_1 \leq b_{1i}\right)}{\partial b_{1i}}$ comes from Eq(36) — I will express Eq(36) as follows,

$$\frac{\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^w}^{\bar{b}} f(x)dx - \frac{\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^w}^{\bar{b}} g(x)dx.$$

Later I will express $f(x), g(x)$ as it is; integrand becomes as follows,

$$\begin{aligned}
&\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-1}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right) + \\
&\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-1}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right)\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{1}{h_2(I-1)}\int_{B_{2i}^w}^{\bar{b}} f(x)dx - \\
&\frac{\hat{m}_2^w\left(B_{2i}^w \mid b_{1i}\right)}{\hat{m}_2^l\left(B_{2i}^w \mid b_{1i}\right)}\hat{G}_2^l\left(B_{2i}^w \mid B_1 \leq b_{1i}\right)^{I-1}\hat{G}_{2|1}^w\left(B_{2i}^w \mid b_{1i}\right)\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{1}{h_2(I-1)}\int_{B_{2i}^w}^{\bar{b}} g(x)dx
\end{aligned} \qquad (42)$$

Eq(42) is equivalent to the integrand in ①. Thus, ① is as follows — I merely plugged in every previous derivation to Eq(42) to get the result below.

$$\int_{\underline{b}}^{\bar{b}} \Bigg[ \underbrace{\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}}_{\frac{\hat{m}^w_2}{\hat{m}^l_2}} \underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{(\hat{G}^l_2)^{I-1}} \times$$

$$\underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{\hat{G}^w_{2|1}} \Bigg] dB^w_{2i}+$$

$$\frac{\hat{G}_1(b_{1i})}{h_2(I-1)\hat{g}_1(b_{1i})}\int_{\underline{b}}^{\bar{b}} \Bigg[ \underbrace{\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}}_{\frac{\hat{m}^w_2}{\hat{m}^l_2}} \underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{(\hat{G}^l_2)^{I-1}} \times$$

$$\underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{\hat{G}^w_{2|1}} \int_{B^w_{2i}}^{\bar{b}} f(x)dx \Bigg] dB^w_{2i}-$$

$$\frac{\hat{G}_1(b_{1i})}{h_2(I-1)\hat{g}_1(b_{1i})}\int_{\underline{b}}^{\bar{b}} \Bigg[ \underbrace{\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}}_{\frac{\hat{m}^w_2}{\hat{m}^l_2}} \underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{(\hat{G}^l_2)^{I-1}} \times$$

$$\underbrace{\prod_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{\hat{G}^w_{2|1}} \int_{B^w_{2i}}^{\bar{b}} g(x)dx \Bigg] dB^w_{2i}.$$

The equation above is equivalent to as follows,

$$\int_{\underline{b}}^{\bar{b}} \left[ \frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)} \prod_{\{\ell\in\mathcal{L}_I\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}\right] dB^w_{2i}+$$
$$\frac{\hat{G}_1(b_{1i})}{h_2(I-1)\hat{g}_1(b_{1i})}\Bigg($$
$$\int_{\underline{b}}^{\bar{b}} \left[ \frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)} \prod_{\{\ell\in\mathcal{L}_I\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\} \int_{B^w_{2i}}^{\bar{b}} f(x)dx\right] dB^w_{2i}-$$
$$\int_{\underline{b}}^{\bar{b}} \left[ \frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}=W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}} \lambda_\ell K\left(\frac{B^w_{2i}-B^{max}_{2\ell}}{h_2}\right)} \prod_{\{\ell\in\mathcal{L}_I\}} \exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B^w_{2i}}^{\bar{b}} \frac{K\left(\frac{b-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\} \int_{B^w_{2i}}^{\bar{b}} g(x)dx\right] dB^w_{2i}\Bigg), \tag{43}$$

where $f(x)=\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell K\left(\frac{x-B^{max}_{2\ell}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell}{\partial b_{1i}}\overline{K}_{2\ell}(x)\right)}{(\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(x))^2}$ and $g(x)=\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\frac{\partial\lambda_\ell}{\partial b_{1i}}K\left(\frac{x-B^{max}_{2\ell}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(x)}$. Thus, Eq(43) is equivalent to ① in Eq(40). I am not sure whether Eq(43) could be computed. Indeed, every three integrals is a univariate integral with compact support, but all three integrands include $\int_{B^w_{2i}}^{\bar{b}}$, which makes me doubt the feasibility of computation. Now, we move to ②.

Integrand in Eq(41) is as follows — I use all the previous derivations.

$$\underbrace{\frac{\int_{b_{1i}}^{\bar{b}}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq x\right)^{I-2}\hat{G}^w_{2|1}(B^l_{2i}\mid x)d\hat{G}_1(x)^{I-1}}{\int_{b_{1i}}^{\bar{b}}\hat{\Psi}(B^l_{2i};x)\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq x\right)^{I-2}\hat{G}^w_{2|1}(B^l_{2i}\mid x)d\hat{G}_1(x)^{I-1}}}_{\frac{\hat{H}^l_2}{\hat{h}^l_2}}\hat{G}^l_2(B^l_{2i}|B_1\leq b_{1i})^{I-2}\hat{G}^w_{2|1}(B^l_{2i}|b_{1i})\times$$

$$\underbrace{\frac{\partial G^l_{2|1}(B^l_{2i}|b_{1i})}{\partial B^l_{2i}}}_{g^l_{2|1}(B^l_{2i}|b_{1i})},\tag{44}$$

where $\hat{G}^l_{2|1}\left(B^l_{2i}\mid b_{1i}\right)=\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)+\frac{\hat{G}_1(b_{1i})}{\hat{g}_1(b_{1i})}\frac{\partial\hat{G}^l_2\left(B^l_{2i}|B_1\leq b_{1i}\right)}{\partial b_{1i}}$ holds. Thus, $g^l_{2|1}(B^l_{2i}|b_{1i})$ will be,

$$\begin{aligned}
&g^l_{2|1}(B^l_{2i}|b_{1i})=\\
&\frac{\partial}{\partial B^l_{2i}}\left(\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)+\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)}{\partial b_{1i}}\right)=\\
&\hat{g}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)+\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial\hat{g}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)}{\partial b_{1i}}=\\
&\underbrace{\frac{1}{I-1}\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)}_{\hat{g}^l_2}\\
&+\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial}{\partial b_{1i}}\left(\underbrace{\frac{1}{I-1}\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)}_{\hat{g}^l_2}\right)=\\
&\frac{1}{I-1}\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)\\
&+\frac{1}{I-1}\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial}{\partial b_{1i}}\left(\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)\right)=\\
&\frac{1}{I-1}\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)\\
&+\frac{1}{I-1}\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial}{\partial b_{1i}}\left(\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\right)\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)\\
&+\frac{1}{I-1}\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\hat{m}^l_2\left(B^l_{2i}\mid b_{1i}\right)}{\hat{G}_{B^{\max}_2|B^{\max}_1}\left(B^l_{2i}\mid b_{1i}\right)}\frac{\partial\hat{G}^l_2\left(B^l_{2i}\mid B_1\leq b_{1i}\right)}{\partial b_{1i}}.
\end{aligned}$$

Using this result, Eq(44) becomes as follows,

$$\begin{aligned}
&\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l(B_{2i}^l|B_1\le b_{1i})^{I-1}\hat{G}_{2|1}^w(B_{2i}^l|b_{1i})\frac{1}{I-1}\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\\
&+\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l(B_{2i}^l|B_1\le b_{1i})^{I-1}\hat{G}_{2|1}^w(B_{2i}^l|b_{1i})\frac{1}{I-1}\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\partial}{\partial b_{1i}}\left(\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\right)\\
&+\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l(B_{2i}^l|B_1\le b_{1i})^{I-2}\hat{G}_{2|1}^w(B_{2i}^l|b_{1i})\frac{1}{I-1}\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\frac{\partial\hat{G}_2^l\left(B_{2i}^l\mid B_1\le b_{1i}\right)}{\partial b_{1i}}.
\end{aligned} \tag{45}$$

Eq(45) uses the following,

$$\begin{aligned}
&\frac{\partial}{\partial b_{1i}}\left(\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\right)\\
&=\frac{\partial}{\partial b_{1i}}\left(\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}\right)\\
&=\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\frac{\partial\lambda_\ell}{\partial b_{1i}}\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}\\
&\quad-\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)\sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell}{\partial b_{1i}}\overline{K}_{2\ell}(B_{2i}^l)}{\left(\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)\right)^2}.
\end{aligned}$$

Also, express $\frac{\partial\hat{G}_2^l\left(B_{2i}^l|B_1\le b_{1i}\right)}{\partial b_{1i}}$ inside Eq(44) as follows — we used the same trick before,

$$\frac{\hat{G}_2^l\left(B_{2i}^l\mid B_1\le b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^l}^{\bar{b}}f(x)dx-\frac{\hat{G}_2^l\left(B_{2i}^l\mid B_1\le b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^l}^{\bar{b}}g(x)dx.$$

Then, ② is as follows — I temporarily use the new notation, $\beta(B_{2i}^l;b_{1i})\equiv\frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l(B_{2i}^l|B_1\le b_{1i})^{I-1}\hat{G}_{2|1}^w(B_{2i}^l|b_{1i})\frac{1}{I-1}$. The first line of Eq(45) will be as follows,

$$\int_{\underline{b}}^{\bar{b}}\beta(B_{2i}^l;b_{1i})\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}dB_{2i}^l \tag{46}$$

The second line of Eq(45) will be as follows,

$$\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\frac{\partial\lambda_\ell}{\partial b_{1i}}\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}dB_{2i}^l$$
$$-\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)\sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell}{\partial b_{1i}}\overline{K}_{2\ell}(B_{2i}^l)}{\left(\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)\right)^2}dB_{2i}^l. \tag{47}$$

The last line of Eq(45) will be as follows,

$$\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\frac{\beta(B_{2i}^l;b_{1i})}{\hat{G}_2^l(B_{2i}^l|B_1\leq b_{1i})}\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\frac{\hat{G}_2^l\left(B_{2i}^l\mid B_1\leq b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^l}^{\overline{b}}f(x)dx\ \ dB_{2i}^l$$
$$-\frac{\hat{G}_1\left(b_{1i}\right)}{\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\frac{\beta(B_{2i}^l;b_{1i})}{\hat{G}_2^l(B_{2i}^l|B_1\leq b_{1i})}\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\frac{\hat{G}_2^l\left(B_{2i}^l\mid B_1\leq b_{1i}\right)}{h_2(I-1)}\int_{B_{2i}^l}^{\overline{b}}g(x)dx\ \ dB_{2i}^l,$$

which can be rearranged as follows,

$$\frac{\hat{G}_1\left(b_{1i}\right)}{h_2(I-1)\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\int_{B_{2i}^l}^{\overline{b}}f(x)dx\ \ dB_{2i}^l$$
$$-\frac{\hat{G}_1\left(b_{1i}\right)}{h_2(I-1)\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\hat{m}_2^l\left(B_{2i}^l\mid b_{1i}\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(B_{2i}^l\mid b_{1i}\right)}\int_{B_{2i}^l}^{\overline{b}}g(x)dx\ \ dB_{2i}^l,$$

which is equivalent to as follows,

$$\frac{\hat{G}_1\left(b_{1i}\right)}{h_2(I-1)\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}\int_{B_{2i}^l}^{\overline{b}}f(x)dx\ \ dB_{2i}^l$$
$$-\frac{\hat{G}_1\left(b_{1i}\right)}{h_2(I-1)\hat{g}_1\left(b_{1i}\right)}\int_{\underline{b}}^{\overline{b}}\beta(B_{2i}^l;b_{1i})\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell\frac{1}{h_2}K\left(\frac{B_{2i}^l-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(B_{2i}^l)}\int_{B_{2i}^l}^{\overline{b}}g(x)dx\ \ dB_{2i}^l. \tag{48}$$

Thus, ② is as follows:

$$② = \text{Eq(46)+Eq(47)+Eq(48)}, \tag{49}$$

where Eq(49) uses each abbreviated notations:

$$\begin{aligned}\beta(B_{2i}^l;b_{1i}) &\equiv \frac{\hat{H}_2^l(B_{2i}^l;b_{1i})}{\hat{h}_2^l(B_{2i}^l;b_{1i})}\hat{G}_2^l(B_{2i}^l|B_1\leq b_{1i})^{I-1}\hat{G}_{2|1}^w(B_{2i}^l|b_{1i})\frac{1}{I-1}\\ &= \underbrace{\frac{\int_{b_{1i}}^{\bar{b}}\hat{G}_2^l\left(B_{2i}^l\mid B_1\leq x\right)^{I-2}\hat{G}_{2|1}^w(B_{2i}^l\mid x)d\hat{G}_1(x)^{I-1}}{\int_{b_{1i}}^{\bar{b}}\hat{\Psi}(B_{2i}^l;x)\hat{G}_2^l\left(B_{2i}^l\mid B_1\leq x\right)^{I-2}\hat{G}_{2|1}^w(B_{2i}^l\mid x)d\hat{G}_1(x)^{I-1}}}_{\frac{\hat{H}_2^l}{\hat{h}_2^l}}\times\\ &\underbrace{\prod_{\{\ell\in\mathcal{L}_I\}}\exp\left\{-\frac{\lambda_\ell}{h_2}\int_{B_{2i}^l}^{\bar{b}}\frac{K\left(\frac{b-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\tilde{\ell}\in\mathcal{L}_I}\lambda_{\tilde{\ell}}\overline{K}_{2\tilde{\ell}}(b)}db\right\}}_{(\hat{G}_2^l)^{I-1}G_{2|1}^w}\frac{1}{I-1},\end{aligned}\tag{50}$$

where Eq(50) uses Eq(20) and Eq(29), and also, $f(x),g(x)$ inside Eq(48) is $f(x)=\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\lambda_\ell K\left(\frac{x-B_{2\ell}^{max}}{h_2}\right)\left(\sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell}{\partial b_{1i}}\overline{K}_{2\ell}(x)\right)}{(\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(x))^2}$ and $g(x)=\frac{\sum_{\{\ell\in\mathcal{L}_I:W_{1\ell}\neq W_{2\ell}\}}\frac{\partial\lambda_\ell}{\partial b_{1i}}K\left(\frac{x-B_{2\ell}^{max}}{h_2}\right)}{\sum_{\ell\in\mathcal{L}_I}\lambda_\ell\overline{K}_{2\ell}(x)}$.
I am not so sure whether Eq(49), which is ② itself, can be computed.

Assume that Eq(43) and Eq(49), which is ① and ② respectively, can be computed numerically. Then, we can compute $\xi_1(b_{1i})$, and the rest steps follow exactly the same as in (v) from OnIdentification 1.4.