A decision theoretic analysis à la Hartman (1972) and Abel (1983)

1. a Cobb-Douglas production function: $y = \varepsilon k^{\alpha} n^{1-\alpha}$.
2. ε is firm-level productivity, k is capital and n is labor.
3. the capital accumulation equation: $k_{t+1} = (1-\delta)k_t + i_t$.
4. an increasing convex cost of adjustment (with a constant elasticity of β).
5. ε is drawn from $N(\bar{\varepsilon}, \sigma_{\varepsilon}^2)$.
6. The firm's objective at time t is to choose labor $\{n_{t+j}\}_{j=0}^{\infty}$ and investment $\{i_{t+j}\}_{j=0}^{\infty}$ to maximize the following (given k_t):

$$\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} (\varepsilon_{t+j} k_{t+j}^{\alpha} n_{t+j}^{1-\alpha} - \omega n_{t+j} - \gamma i_{t+j}^{\beta})$$

subject to

$$k_{t+j+1} = (1-\delta)k_{t+j} + i_{t+j}$$

Maximising out the optimal labor choice.

1. Taking the first order condition with respect to n yields:

$$\begin{aligned}
(1-\alpha)\varepsilon k^{\alpha} n^{-\alpha} &= \omega \\
\varepsilon k^{\alpha} n^{-\alpha} &= \frac{\omega}{(1-\alpha)} \\
n^{1-\alpha} &= \varepsilon^{\frac{1-\alpha}{\alpha}} k^{1-\alpha} [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} \\
n &= \varepsilon^{\frac{1}{\alpha}} k [\frac{1-\alpha}{\omega}]^{\frac{1}{\alpha}}
\end{aligned}$$

2. Profits for each period as a function of productivity and capital:

$$\begin{aligned}
\max_n [\varepsilon k^{\alpha} n^{1-\alpha} - \omega n] &= \varepsilon k^{\alpha} \varepsilon^{\frac{1-\alpha}{\alpha}} k^{1-\alpha} [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} - \omega \varepsilon^{\frac{1}{\alpha}} k [\frac{1-\alpha}{\omega}]^{\frac{1}{\alpha}} \\
&= \varepsilon^{\frac{1}{\alpha}} k [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} - \varepsilon^{\frac{1}{\alpha}} k [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} (1-\alpha) \\
&= \underbrace{\alpha \varepsilon^{\frac{1}{\alpha}} [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}}}_{\text{the marginal revenue productivity of capital}} \quad k
\end{aligned}$$

3. The objective can be written as:

$$\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} \left[\alpha \varepsilon_{t+j}^{\frac{1}{\alpha}} [\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} k_{t+j} - \gamma i_{t+j}^{\beta} \right]$$

Optimal investment

The first order condition yields:

$$\alpha\,\mathbb{E}[\varepsilon_{t+1}^{\frac{1}{\alpha}}][\frac{1-\alpha}{\omega}]^{\frac{1-\alpha}{\alpha}} + (1-\delta) \quad = \quad (1+r)\beta\gamma I_t^{\beta-1}$$

Focusing on this decision theoretic analysis:

1. A mean-preserving spread in ε increases the left hand side of the equation as $\alpha < 1$. To restore the equality, I_t should increase on the right hand side. This is what Hartman and Abel argue (the so-called OHA effect).
2. Key is that the marginal revenue product of capital is a convex function of future productivity, which is not due to DRS (in fact, the above is CRS), instead it is due to that labor is chosen within period and capital is subject to time-to-build.
3. Due to DRS ($\alpha < 1$), "*firms with good draws expand rapidly and firms with bad draws contract (the classic OHA effect).*" as R1 said. But I think this is not what Hartman and Abel discuss.
4. I demonstrate this in a model with DRS but without labor as below.

Consider the economy under DRS without labor

1. A decision theoretic one firm model: $y = \varepsilon k^{\alpha}$
2. ε is firm-level productivity and k is capital.
3. the capital accumulation equation: $k_{t+1} = (1-\delta)k_t + i_t$.
4. ε is drawn from $N(\bar{\varepsilon}, \sigma_{\varepsilon}^2)$
5. The firm's objective at time t is to choose $\{i_{t+j}\}_{j=0}^{\infty}$ to maxmize the following (given k_t):
$$\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} (\varepsilon_{t+j} k_{t+j}^{\alpha} - i_{t+j})$$
subject to
$$k_{t+j+1} = (1-\delta)k_{t+j} + i_{t+j}$$

Optimal investment

The first order condition yields:

$$\alpha E[\varepsilon_{t+1}]k_{t+1}^{\alpha-1} \quad = \quad r+\delta$$

Focusing on this decision theoretic analysis:

1. A mean-preserving spread in ε has no effect on the left hand side. So, there is no OHA effect.
2. $k_{t+1} = \left(E[\varepsilon_{t+1}]\frac{\alpha}{(r+\delta)}\right)^{\frac{1}{1-\alpha}}$. Due to DRS ($\alpha < 1$), "*firms with good draws expand rapidly and firms with bad draws contract.*" as R1 said. This is true but this is not about the OHA effect. Hartman and Abel's argument is about $\frac{\partial k_{t+1}}{\partial \sigma_\varepsilon^2} > 0$. Here, $\frac{\partial k_{t+1}}{\partial \sigma_\varepsilon^2} = 0$
3. Arellano, Bai, and Kehoe (2019) also say like "an increase in the volatility of persistent productivity shocks would actually lead to an increase in aggregate employment . . . , and resources would be disproportionately reallocated to the most productivity firms, causing boom." This argument is about $\frac{\partial^2 k_{t+1}}{\partial E[\varepsilon_{t+1}]^2} > 0$.
4. Gourio (2008) calls this the standard reallocation effect whereby higher variance in productivity leads to higher TFP to the extent determined by the amount of decreasing returns to scale, distinguishing from a Jensen effect, which is about $\frac{\partial E[\varepsilon_{t+1}]}{\partial \sigma_\varepsilon^2} > 0$.
5. In general, an increase in σ_ε^2 may not be a mean-preserving-spread in ε. In particular, when ε follows log-normal as in the literature, an increase in σ_ε^2 involve with a Jensen's effect ($\leftarrow$ need to check the original like Nick or Abel). We will see this below. As log-normality renders aggregation with analytical expressions, we also aggregate this.

Consider the economy under DRS and persistent log-normal productivity (without labor), like in Gourio (2008)

1. unit measure firms: $y = f(z, k) = zk^{\alpha}$, and we aggregate so that we can study aggregate output and TFP (so, no longer decision theoretic).
2. $\log z_{t+1} = \rho \log z_t + (1-\rho)\mu + \sigma_{\varepsilon}\varepsilon_{t+1}$ with ε_t from iid normal distribution with zero mean and unit variance.
3. the capital accumulation equation (time-to-build): $k_{t+1} = (1-\delta)k_t + i_t$.
4. The firm's objective at time t is to choose $\{i_{t+j}\}_{j=0}^{\infty}$ to maxmize the following (given k_t):

$$\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} (z_{t+j} k_{t+j}^{\alpha} - i_{t+j})$$

subject to

$$k_{t+j+1} = (1-\delta)k_{t+j} + i_{t+j}$$

Optimal investment

1. The first order condition yields:

$$\underbrace{\alpha E_t[z_{t+1}]k_{t+1}^{\alpha-1}}_{\text{the expected marginal product of capital}} = \underbrace{r+\delta}_{\text{the user cost of capital}}$$

2. Rearranging this and taking log:

$$k_{t+1}^{1-\alpha} = E_t[z_{t+1}]\frac{\alpha}{(r+\delta)}$$

$$\log k_{t+1} = \frac{1}{1-\alpha}\log E_t[z_{t+1}] - \frac{1}{1-\alpha}\log\frac{(r+\delta)}{\alpha}$$

3. Given $\log E_t[z_{t+1}] = \rho \log z_t + (1-\rho)\mu + \frac{\sigma_\varepsilon^2}{2}$, we can see k_{t+1} increases with σ_ε^2 but this is just because of log-normal productivity distribution, not because of decreasing returns to scale.

Moments used below

$$\begin{aligned}
\mathbb{E}[\log z] &= \mu \\
Var[\log z] &= \frac{\sigma_\varepsilon^2}{1-\rho^2} \\
\mathbb{E}[\log E_t[z_{t+1}]] &= \mu + \frac{\sigma_\varepsilon^2}{2} \\
Var[\log E_t[z_{t+1}]] &= \rho^2 Var[\log z] = \frac{\rho^2\sigma_\varepsilon^2}{1-\rho^2} \\
\mathbb{E}[\log k] &= \frac{1}{1-\alpha}\mathbb{E}[\log E_t[z_{t+1}]] - \frac{1}{1-\alpha}\log\frac{(r+\delta)}{\alpha} \\
&= \frac{1}{1-\alpha}(\mu + \frac{\sigma_\varepsilon^2}{2}) - \frac{1}{1-\alpha}\log\frac{(r+\delta)}{\alpha} \\
Var[\log k] &= [\frac{1}{1-\alpha}]^2 Var[\log E_t[z_{t+1}]] \\
&= [\frac{\rho}{1-\alpha}]^2 \frac{\sigma_\varepsilon^2}{1-\rho^2} \\
Cov(\log z, \log k) &= \frac{\rho^2}{1-\alpha}\frac{\sigma_\varepsilon^2}{1-\rho^2}
\end{aligned}$$

Aggregation

$$
\begin{aligned}
\log Y &= \log \int \int z k^{\alpha} d\mu(z,k) \\
&= \log \mathbb{E}[exp(\log z + \alpha \log k)] \\
&= \mathbb{E}[\log z] + \frac{1}{2} Var[\log z] + \alpha\, \mathbb{E}[\log k] + \frac{\alpha^2}{2} Var[\log k] + \alpha\, Cov(\log z, \log k) \\
&= \mu + \underbrace{\frac{1}{2}\frac{\sigma_{\varepsilon}^2}{(1-\rho)}}_{\text{direct Jensen effect}} + \underbrace{\frac{\alpha}{1-\alpha}(\mu + \frac{\sigma_{\varepsilon}^2}{2})}_{\text{indirect Jensen effect through capital choice under DRS}} - \frac{\alpha}{1-\alpha} \log \frac{(r+\delta}{\alpha} \\
&+ \underbrace{\frac{\alpha^2}{2}\left[\frac{\rho}{1-\alpha}\right]^2 \frac{\sigma_{\varepsilon}^2}{1-\rho^2}}_{\text{reallocation effect}} + \underbrace{\frac{\rho^2}{1-\alpha}\frac{\sigma_{\varepsilon}^2}{1-\rho^2}}_{\text{Olley-Pakes covariance term}}
\end{aligned}
$$

The sign of $\frac{\partial \log Y}{\partial \sigma_{\varepsilon}^2}$ is positive.

Detailed derivation (Gourio, 2008)

moments

$$
\begin{aligned}
\log k_{t+1} &= \frac{1}{1-\alpha} \log \mathbb{E}_t\, z_{t+1} - \frac{1}{1-\alpha} \log \frac{(r+\delta)}{\alpha} \\
\log \mathbb{E}_t\, z_{t+1} &= \rho \log z_t + (1-\rho)\mu + \frac{\sigma^2}{2} \\
\log k'(k,z) &= \frac{1}{1-\alpha}(\rho \log z + (1-\rho)\mu + \frac{\sigma^2}{2}) - \frac{1}{1-\alpha} \log \frac{(r+\delta)}{\alpha} \\
\mathbb{E}(\log z) &= \mu \\
Var(\log z) &= \frac{\sigma^2}{1-\rho^2} \\
\mathbb{E}(\log k) &= \frac{1}{1-\alpha}(\mu + \frac{\sigma^2}{2}) - \frac{1}{1-\alpha} \log(\frac{r+\delta}{\alpha}) \\
Var(\log k) &= \frac{\rho^2\sigma^2}{(1-\alpha)^2(1-\rho^2)} = \frac{\rho^2}{(1-\alpha)^2} Var(\log z) \\
Cov(\log z, \log k) &= \frac{1}{1-\alpha} \frac{\rho^2\sigma^2}{(1-\rho^2)} = \frac{\rho^2}{1-\alpha} Var(\log z)
\end{aligned}
$$

Detailed derivation - Aggregation (Gourio, 2008)

$$
\begin{aligned}
\log Y &= \log \int\int z k^{\alpha} d\mu(z,k) = \log \mathbb{E}[exp(\log z + \alpha \log k)] \\
&= \mathbb{E}[\log z] + \frac{1}{2} Var[\log z] + \alpha\, \mathbb{E}[\log k] + \frac{\alpha^2}{2} Var[\log k] + \alpha\, Cov(\log z, \log k) \\
\log K &= \log \int\int k d\mu(z,k) = \log \mathbb{E}[exp(\log k)] = \mathbb{E}[\log k] + \frac{1}{2} Var[\log k] \\
\log TFP &= \log Y - \alpha \log K = \mathbb{E}[\log z] + \frac{1}{2} Var[\log z] + \frac{\alpha^2 - \alpha}{2} Var[\log k] + \alpha\, Cov(\log z, \log k \\
&= \underbrace{\log \mathbb{E}[z]}_{\text{``Jensen effect''}} + \frac{\alpha^2 - \alpha}{2} \frac{\rho^2}{(1-\alpha)^2} Var[\log z] + \alpha \frac{\rho^2}{(1-\alpha)} Var[\log z] \\
&= \log \mathbb{E}[z] + \underbrace{\frac{\rho^2 \alpha}{(1-\alpha)} \frac{var(\log z)}{2}}_{\text{``Reallocation effect'' (Houthakker, 1955-56; Gilchrist and Williams, 2000)}}
\end{aligned}
$$

Numerical derivation

The firm's objective at time t is to choose labor $\{n_{t+j}\}_{j=0}^{\infty}$ and investment $\{i_{t+j}\}_{j=0}^{\infty}$ to maximize the following (given k_t):

$$\mathbb{E}_t \sum_{j=0}^{\infty} \frac{1}{(1+r)^j} (\varepsilon_{t+j} k_{t+j}^{\alpha} - i_{t+j})$$

subject to

$$k_{t+j+1} = (1-\delta) k_{t+j} + i_{t+j}$$

This leads to the following optimality condition.

$$k_{t+1}^{1-\alpha} = E_t[z_{t+1}] \frac{\alpha}{(r+\delta)}$$

$$k_{t+1}^{1-\alpha} = \rho \log z_t + (1-\rho)\mu + \frac{\sigma_{\varepsilon}^2}{2} \frac{\alpha}{(r+\delta)}$$

Instead, we will obtain the solution numerically by solving the below Bellman equation.

$$\begin{aligned} v(z,k) &= \max_{i,k'} \left[zk^{\alpha} - i + \frac{1}{(1+r)} v(z',k') \right] \\ \text{subject to} \quad k' &= (1-\delta)k + i \end{aligned}$$

Numerical derivation - continued

1. Decisions: solve the Bellman equation to get the policy function: $k' = k'(z,k)$.

2. Distribution: find a stationary distribution of firms: $\mu(z,k)$.

3. Aggregation: aggregate the policy function over the distribution to obtain aggregate variables.
   - $K = \int\int k * d\mu(z,k)$
   - $Y = \int\int y * d\mu(z,k)$
   - $TFP = Y/K^{\alpha}$