ASE7030 Convex optimization: Homework #1

1) *Intersection of convex sets.* Show that the intersection of convex sets is convex. That is, for convex $\mathcal{C}_1, \dots, \mathcal{C}_n$,
$$\mathcal{C} = \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_n$$
is convex.

2) *Sum of convex functions.* Show that the sum of convex functions is convex. That is, for convex $f_1(x), \dots, f_n(x)$,
$$f(x) = f_1(x) + \cdots + f_n(x)$$
is convex.

3) *Rotated cone.* The following set in $\mathbb{R}^n$,
$$\mathcal{Q}^n_{\text{rot}} = \left\{ x \mid 2x_1x_2 \geq x_3^2 + \cdots + x_n^2,\ x_1, x_2 \geq 0 \right\}$$
is called a *rotated cone* in $\mathbb{R}^n$. Show that the following sets are convex.

   a) $\mathcal{Q}^n_{\text{rot}}$

   b) $\mathcal{C} = \{(t, x) \mid tx \geq 1,\ x \geq 0\}$

   c) $\mathcal{C} = \{(t, x) \mid |t| \leq \sqrt{x},\ x \geq 0\}$

   d) $\mathcal{C} = \left\{(x, y, t) \mid x^T x / y \leq t,\ y > 0\right\}$

4) *Farkas' lemma.* Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Show that exactly one of the following two assertions is true:

   a) There exists an $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$.

   b) There exists a $y \in \mathbb{R}^m$ such that $A^T y \geq 0$ and $b^T y < 0$.

5) *Degenerate quadratic function.* Consider the following quadratic function with $P \geq 0$.
$$f(x) = x^T P x + 2q^T x + r$$

   a) Find the condition on P and q that makes the above function bounded below.

   b) Now, suppose that the above condition holds. Parameterize all solutions that minimize $f(x)$.