Automatic Control

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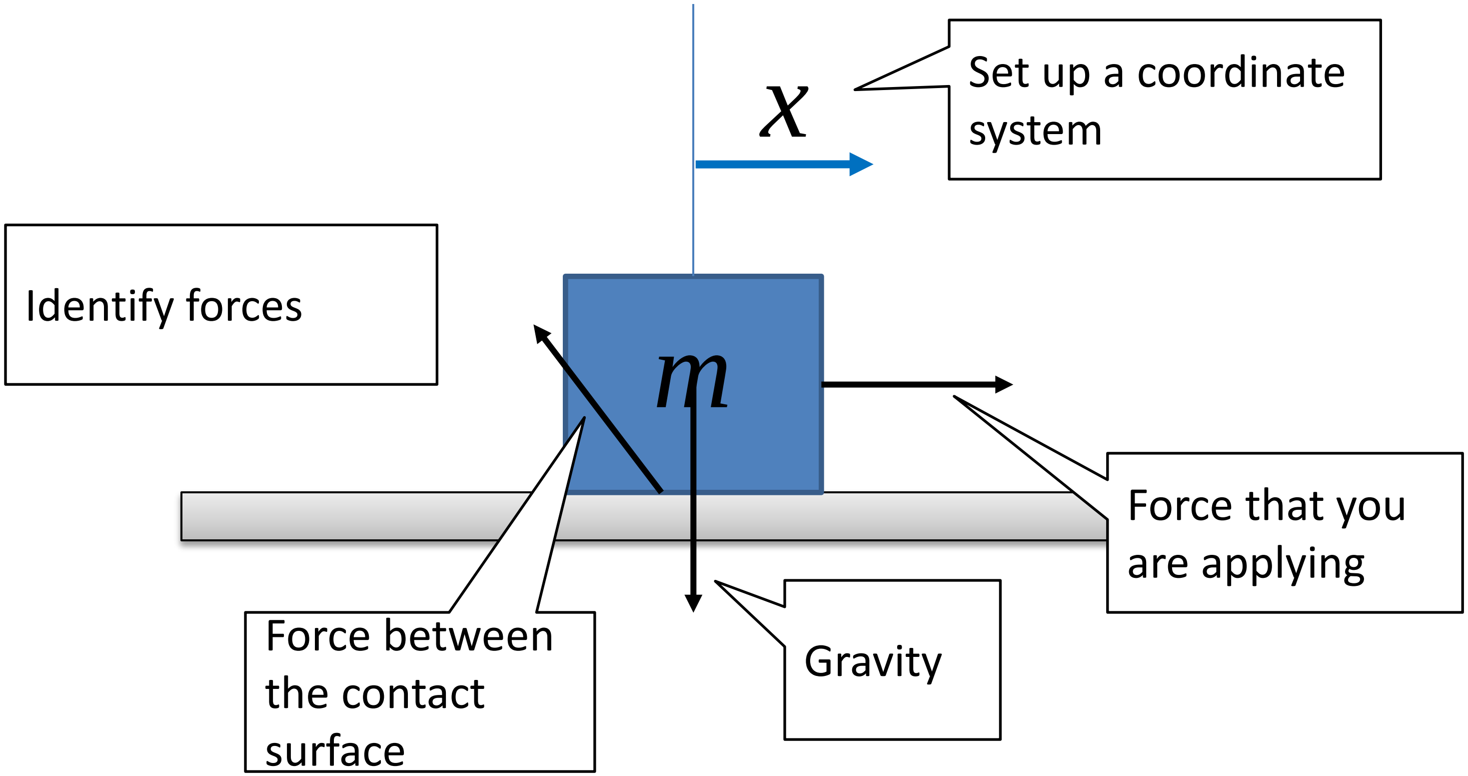
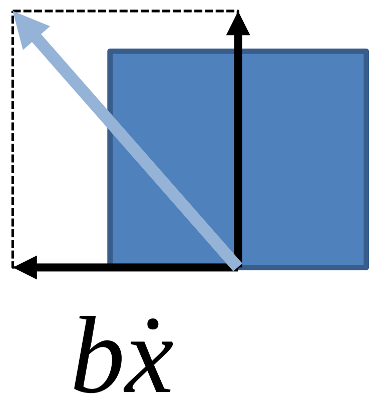
Dynamics

Spring, Mass, and Damper

Cruise-control Model

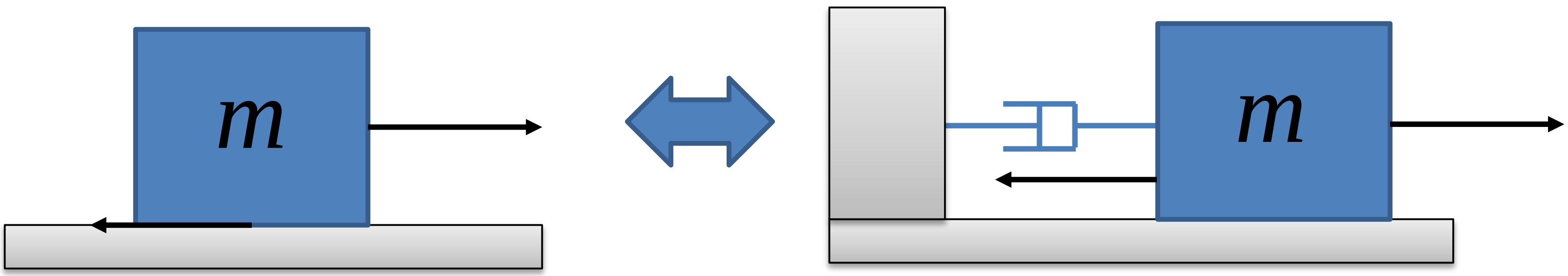


Friction



- Reaction force can be divided into normal and tangential component
- Tangential component
  - If relative motion exist → friction
- Friction
  - Magnitude: proportional to the relative speed
  - Direction: oppose the direction of motion

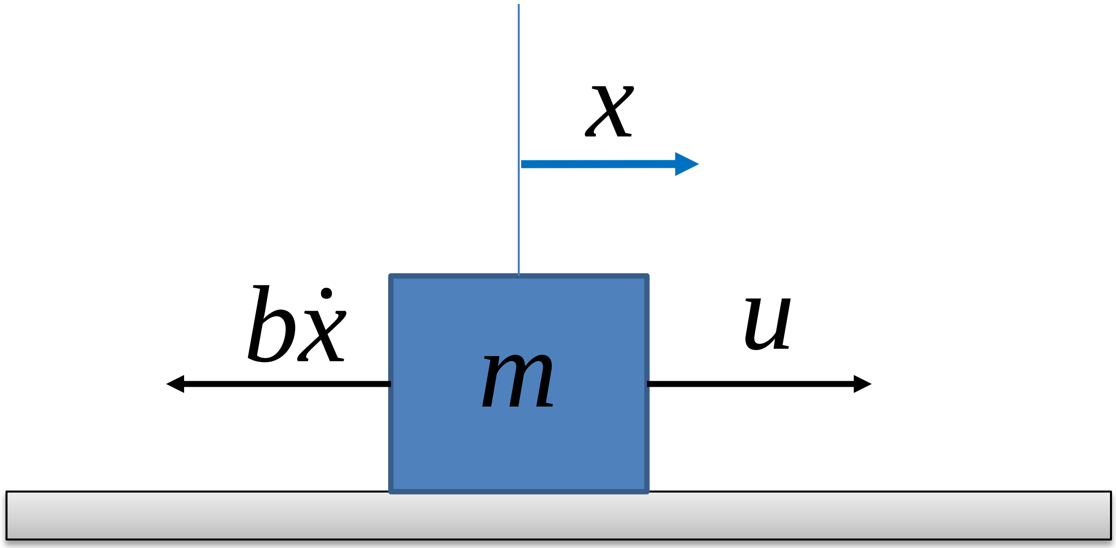
Damper



- Provide damping force
  - Proportional to the relative speed

- In terms of modeling, it is equivalent to friction

Equation of Motion



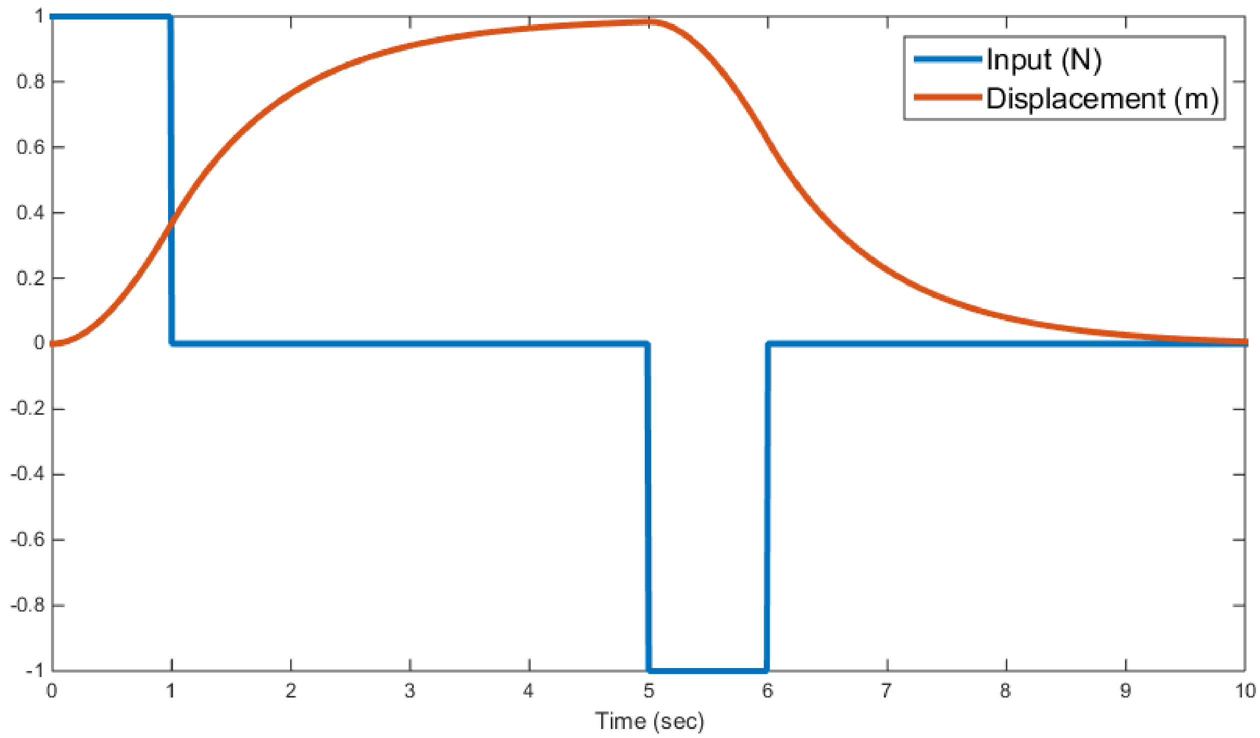
$$\sum \mathbf{F} = u - b\dot{x} = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} = u$$

Example

$$m = 1\,\text{kg}$$

$$b = 1\,\text{kg/s}$$

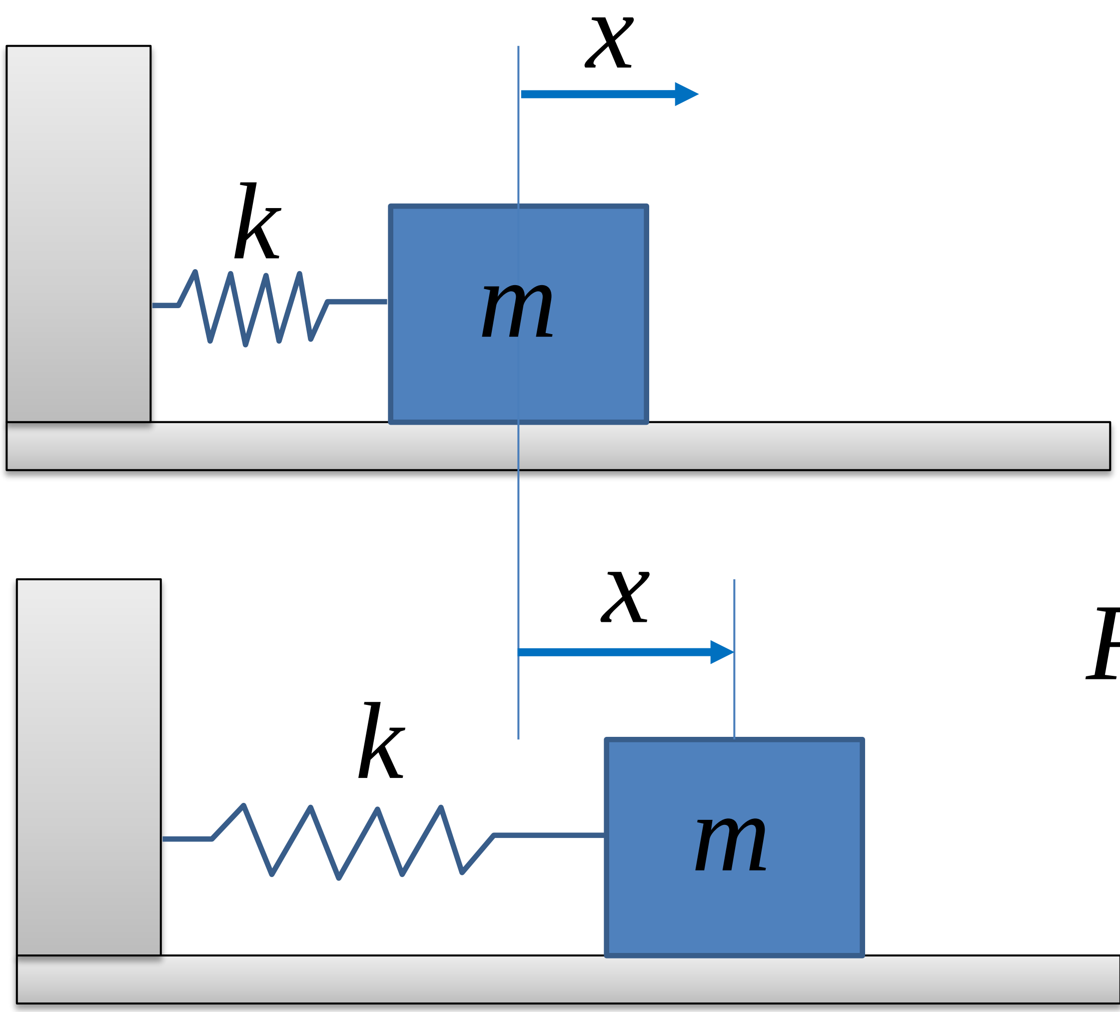


Numerical Simulation Tool

- Matlab
  - Industry standard
  - Too expensive?
  - License for Inha students

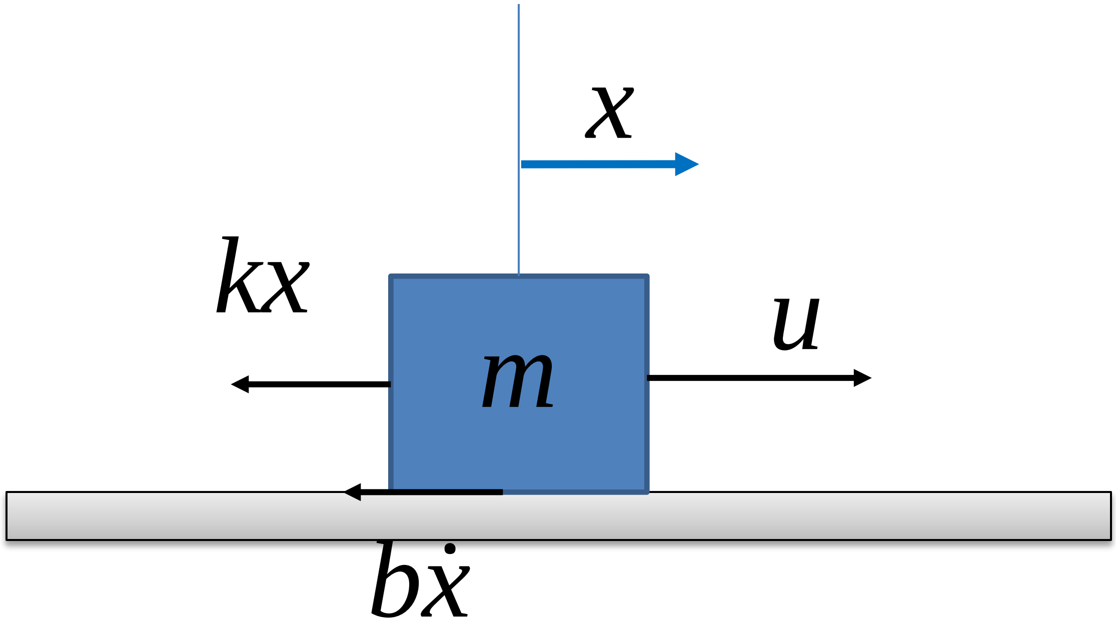
- Octave
  - Open source, free

Mass – Spring System



$$F_s = -kx$$

Mass – Spring System with Friction



$$\sum \mathbf{F} = u - b\dot{x} - kx = m\ddot{x}$$

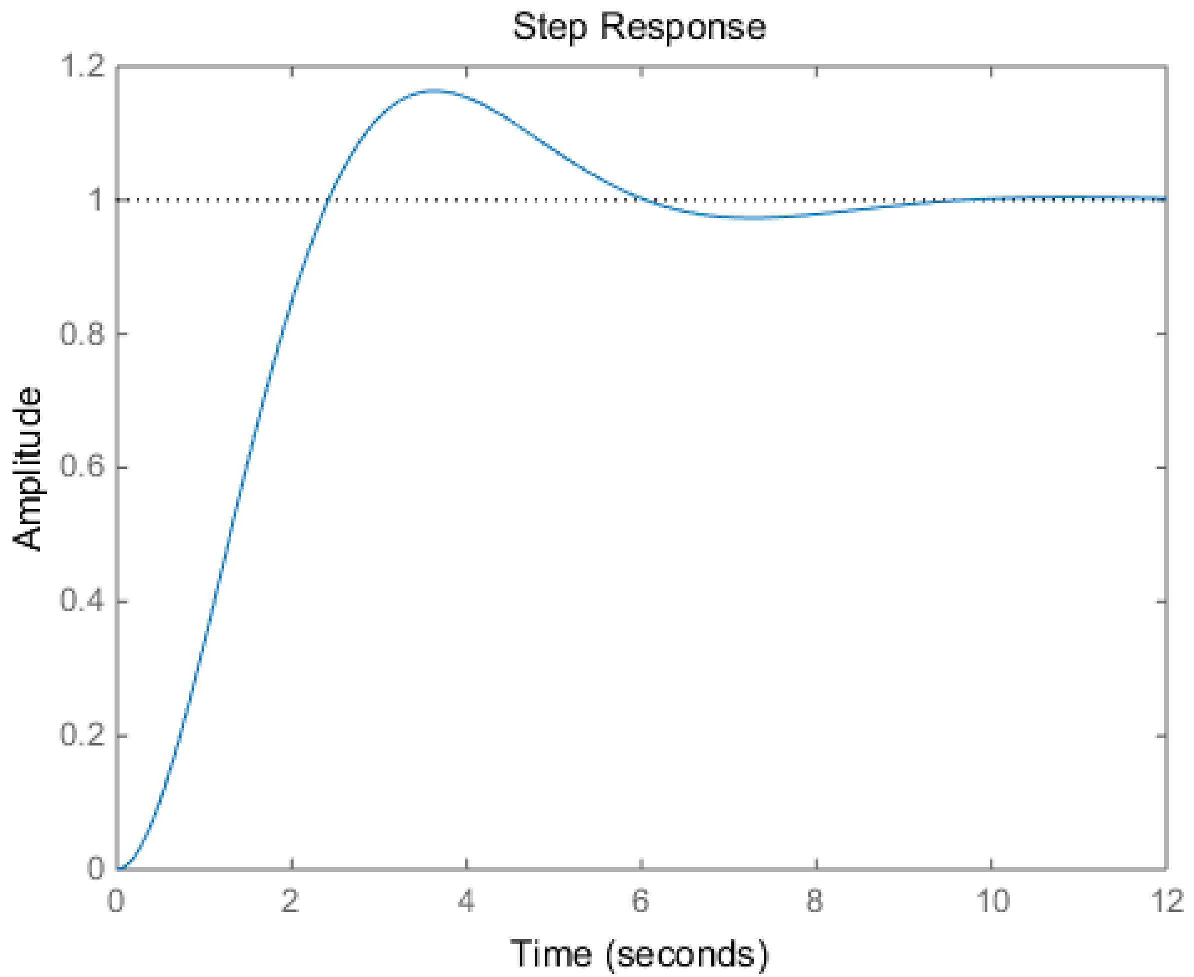
$$m\ddot{x} + b\dot{x} + kx = u$$

Example

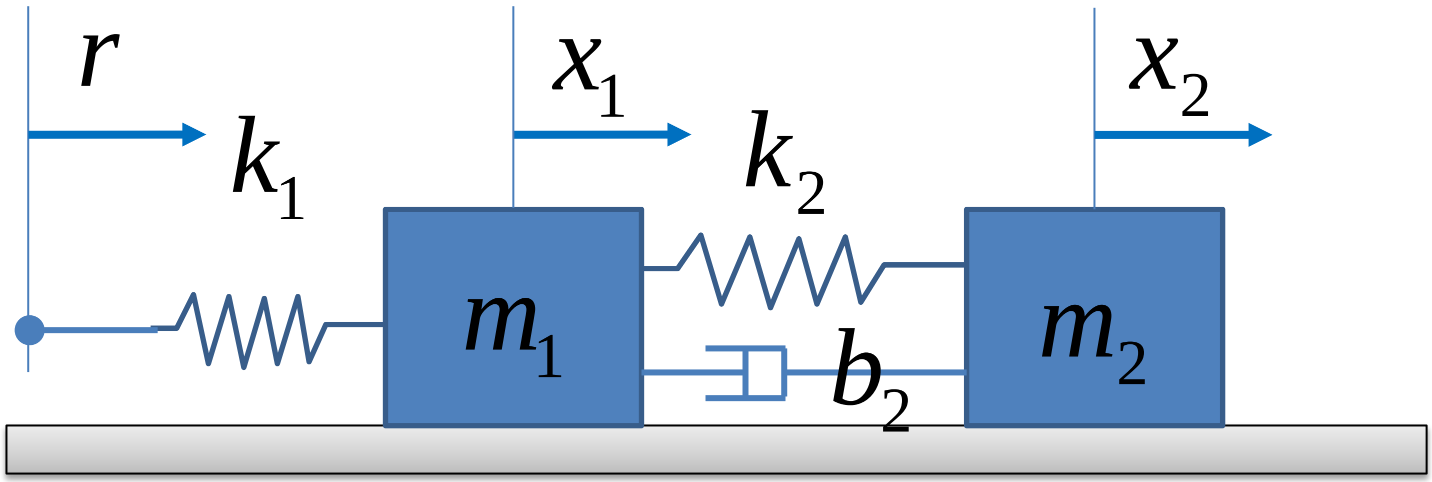
$$m = 1\,\mathrm{kg}$$

$$b = 1\,\mathrm{kg/s}$$

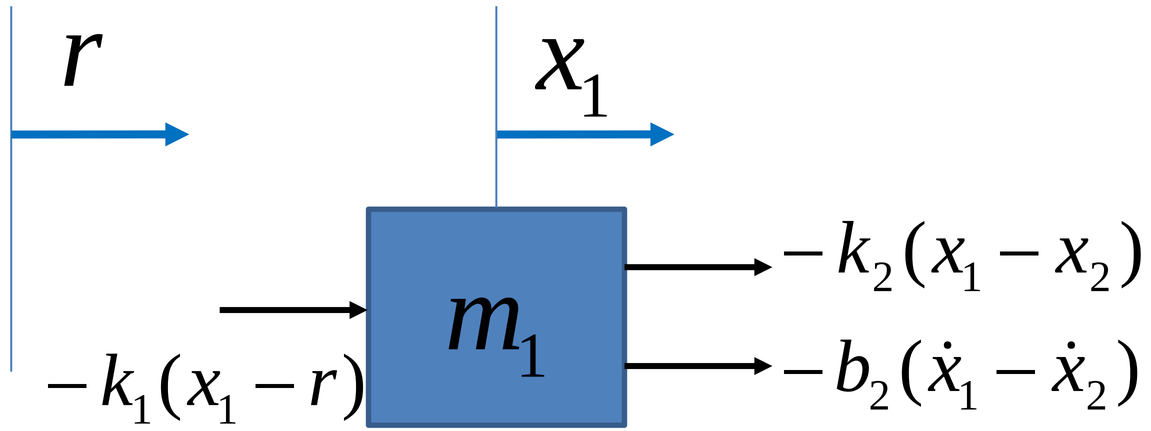
$$k = 1\,\mathrm{N/m}$$



A Two-Mass System



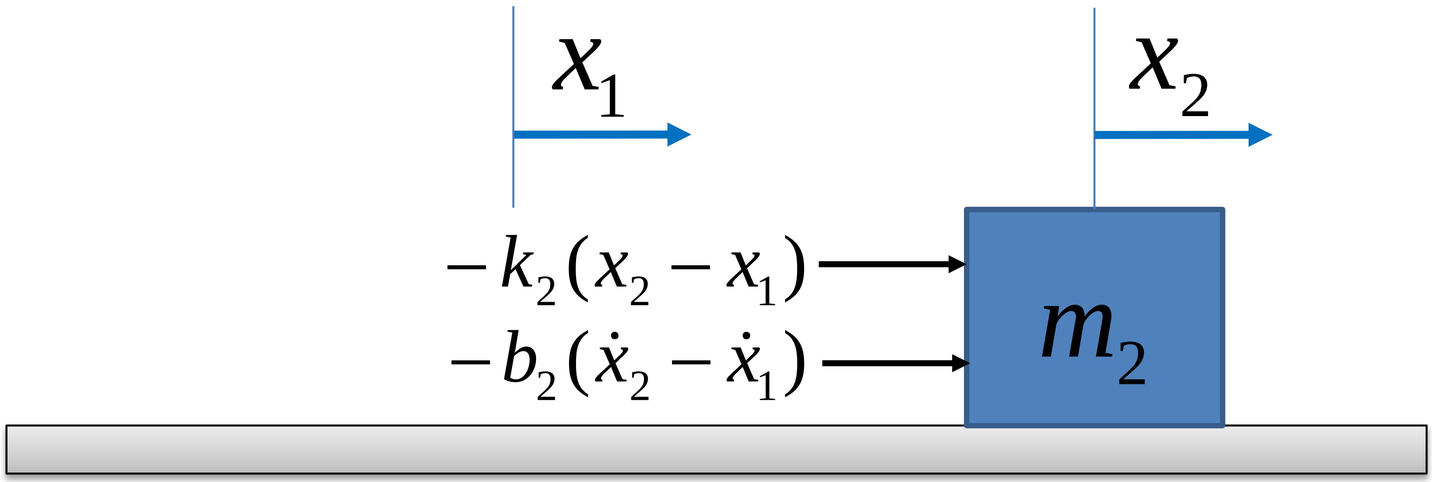
A Two-Mass System



$$-k_1\left(x_1 - r\right) - k_2\left(x_1 - x_2\right) - b_2\left(\dot{x}_1 - \dot{x}_2\right) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + b_2\left(\dot{x}_1 - \dot{x}_2\right) + k_2\left(x_1 - x_2\right) + k_1x_1 = k_1r$$

A Two-Mass System



$$-k_2\left(x_x - x_1\right) - b_2\left(\dot{x}_2 - \dot{x}_1\right) = m_2\ddot{x}_2$$

$$m_2\ddot{x}_2 + b_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) = 0$$

Observations

- Forces acting on me are functions of
  - My position and velocity
  - Positions and velocities of objects connected to me

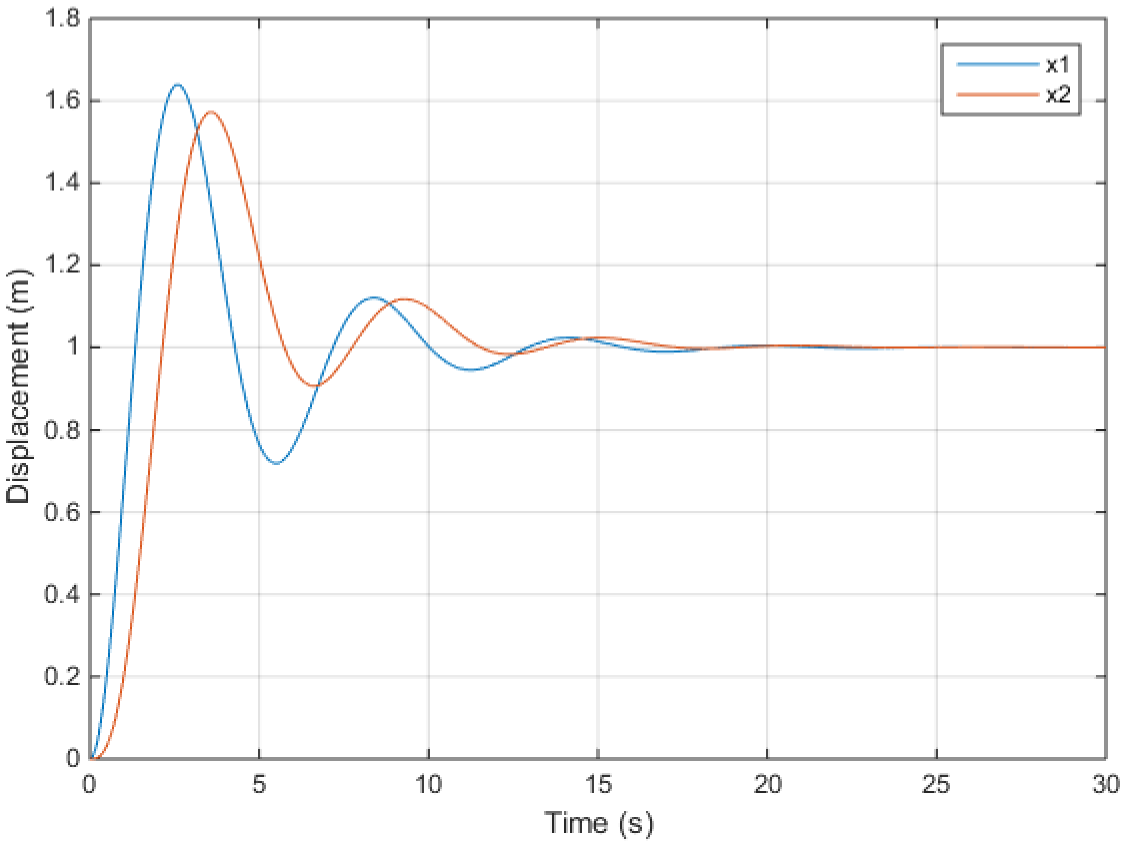
- Spring forces and damping forces act in directions
  - **Opposite: my** movement
  - **Same:** movements of **others**

$$m_1\ddot{x}_1 + b_2\left(\dot{x}_1 - \dot{x}_2\right) + k_2\left(x_1 - x_2\right) + k_1 x_1 = k_1 r$$

Same sign!

Step Response 1

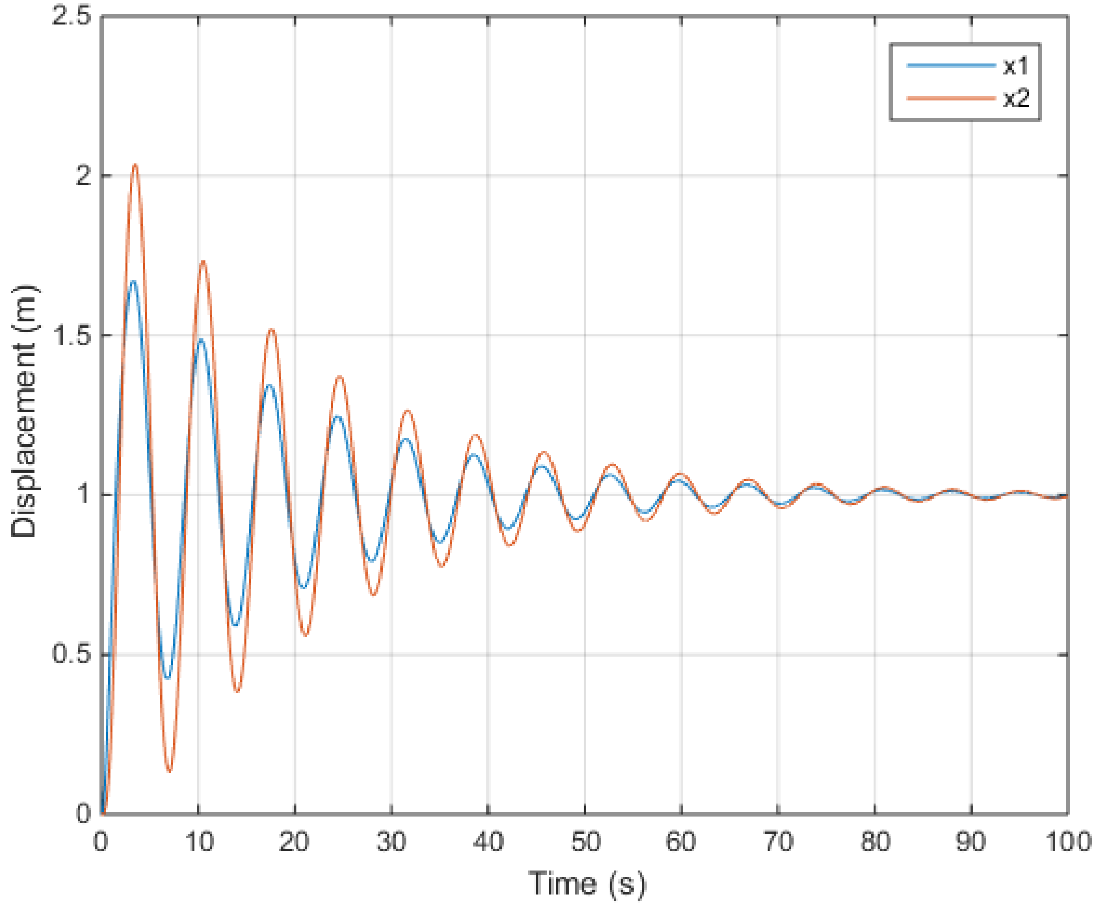
$m_1 = 1$
$k_1 = 2$
$m_2 = 1$
$b_2 = 1$
$k_2 = 0.2$



- Noticeable relative movement between two bodies
- Higher damping effect

Step Response 2

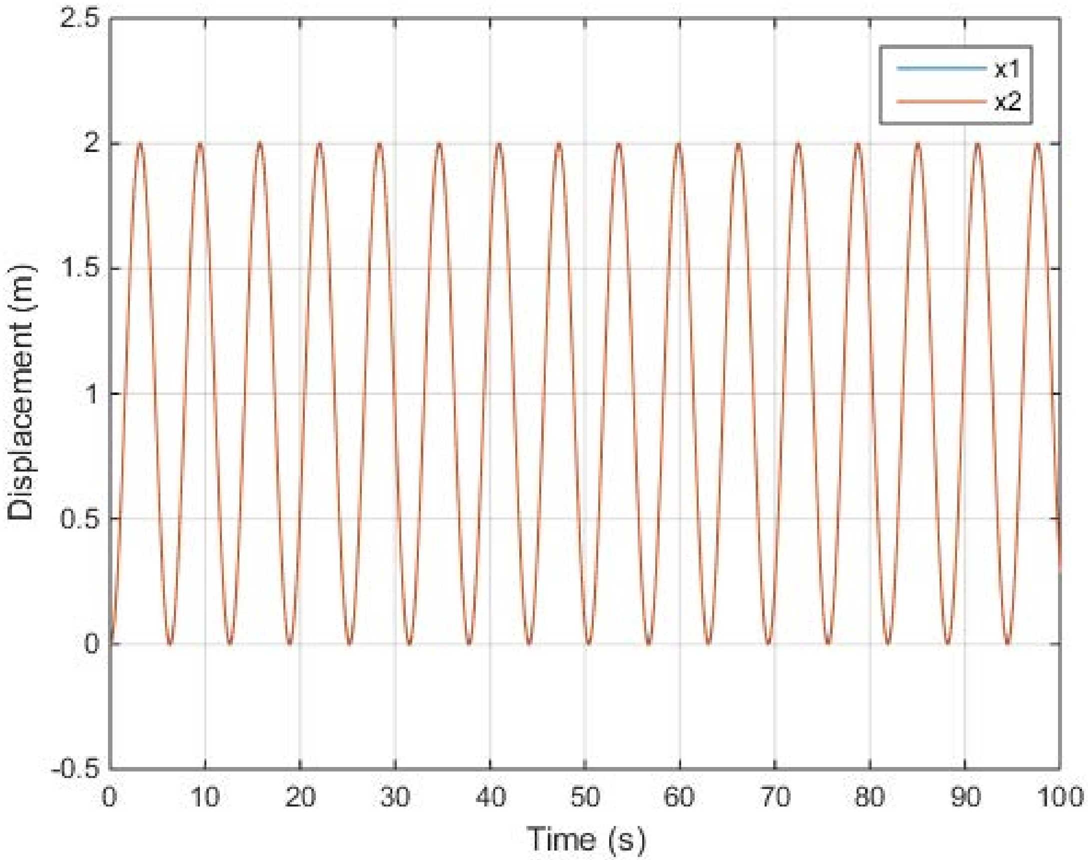
$m_1 = 1$
$k_1 = 2$
$m_2 = 1$
$b_2 = 1$
$k_2 = 2$



- Movement more in phase
- Lower damping

Step Response 3

$m_1 = 1$
$k_1 = 2$
$m_2 = 1$
$b_2 = 1$
$k_2 = 100$



- Moves like one body
- No damping

State-Space Form

$$m_1\ddot{x}_1 + b_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) + k_1 x_1 = k_1 r$$

$$m_2\ddot{x}_2 + b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0$$

Equation of motion gets complicated.

$$\Rightarrow \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

State-space form

State-Space Form – Single Mass

$$m\ddot{x} + b\dot{x} + kx = u \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{x} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix}, \dot{\mathbf{x}} = \begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix}$$

$$\ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x + \frac{1}{m}u$$

$$\dot{x} = 1\dot{x} + 0x + 0u$$

$$\begin{bmatrix} \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{b}{m} & -\frac{k}{m} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u$$

State-Space Form

- 1. Set up the state vector
  - State vector consists of
    - All the variables
    - First derivatives of the variables
  - Fix the order
- 2. Express the derivative of the state vector
  - With a linear combination of the state vector

State-Space Form – 2 mass example

- Set up state vector
  - Variables x_1, x_2
  - Derivatives of the variables $\dot{x}_1, \dot{x}_2$
  - Fix order

$$\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} \begin{matrix} \leftarrow 1 \\ \leftarrow 2 \\ \leftarrow 3 \\ \leftarrow 4 \end{matrix} \qquad \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

Ordering is arbitrary

State-Space Form – 2 mass example

- Express the derivative of each element of the state vector using the state vector

1st element: $\dot{x}_1$

Derivative of the 1st element: $\ddot{x}_1$

Express $\ddot{x}_1$ using $\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix}$

How??

1. *See if it is already in the state vector*
2. *If you can't find it, you have to use the equation of motion*

State-Space Form – 2 mass example

Applicable equation of motion

$$m_1\ddot{x}_1 + b_2\left(\dot{x}_1 - \dot{x}_2\right) + k_2\left(x_1 - x_2\right) + k_1 x_1 = k_1 r$$

$$\ddot{x}_1 = -\frac{b_2}{m_1}\dot{x}_1 - \left(\frac{k_2}{m_1} + \frac{k_1}{m_1}\right)x_1 + \frac{b_2}{m_1}\dot{x}_2 + \frac{k_2}{m_1}x_2 + \frac{k_1}{m_1}r$$

1 2 3 4

State-Space Form – 2 mass example

- Express the derivative of each element of the state vector using the state vector

2nd element: x_1

Derivative of the 1st element: $\dot{x}_1$

Express $\dot{x}_1$ using $\begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix}$

How??

1. *See if it is already in the state vector*
2. *If you can't find it, you have to use the equation of motion*

State-Space Form – Two Mass System

$$m_1\ddot{x}_1 + b_2\left(\dot{x}_1 - \dot{x}_2\right) + k_2\left(x_1 - x_2\right) + k_1 x_1 = k_1 r$$

$$m_2\ddot{x}_2 + b_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) = 0$$

$$\ddot{x}_1 = -\frac{b_2}{m_1}\dot{x}_1 - \left(\frac{k_2}{m_1} + \frac{k_1}{m_1}\right)x_1 + \frac{b_2}{m_1}\dot{x}_2 + \frac{k_2}{m_1}x_2 + \frac{k_1}{m_1}r$$

$$\dot{x}_1 = 1\dot{x}_1 + 0x_1 + 0\dot{x}_2 + 0x_2 + 0r$$

$$\ddot{x}_2 = \frac{b_2}{m_2}\dot{x}_1 + \frac{k_2}{m_2}x_1 - \frac{b_2}{m_2}\dot{x}_2 - \frac{k_2}{m_2}x_2$$

$$\dot{x}_2 = 0\dot{x}_1 + 0x_1 + 1\dot{x}_2 + 0x_2 + 0r$$

State-Space Form – 2 mass example

$$
\begin{array}{cccccccccccc}
\ddot{x}_1 & = & -\frac{b_2}{m_1} & \dot{x}_1 & -\left(\frac{k_2}{m_1}+\frac{k_1}{m_1}\right) & x_1 & +\frac{b_2}{m_1} & \dot{x}_2 & +\frac{k_2}{m_1} & x_2 & +\frac{k_1}{m_1} & r \\
\dot{x}_1 & = & 1 & \dot{x}_1 & +0 & x_1 & +0 & \dot{x}_2 & +0 & x_2 & +0 & r \\
\ddot{x}_2 & = & \frac{b_2}{m_2} & \dot{x}_1 & +\frac{k_2}{m_2} & x_1 & -\frac{b_2}{m_2} & \dot{x}_2 & -\frac{k_2}{m_2} & x_2 & +0 & r \\
\dot{x}_2 & = & 0 & \dot{x}_1 & +0 & x_1 & +1 & \dot{x}_2 & +0 & x_2 & +0 & r \\
 & & & 1 & & 2 & & 3 & & 4 & & \mathrm{u}
\end{array}
$$

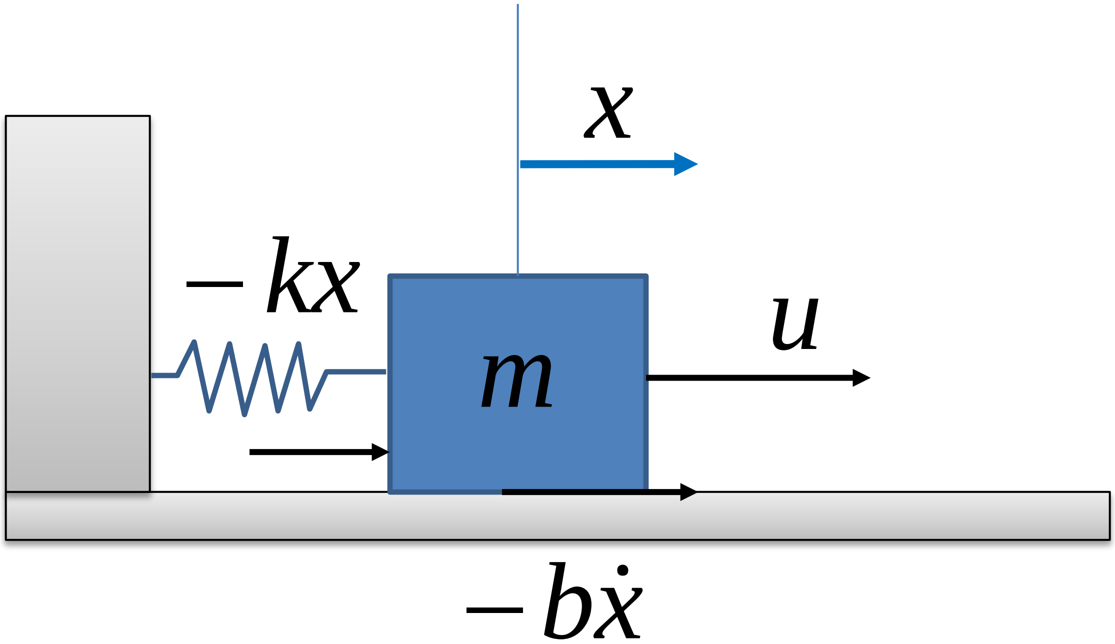
In many cases, you will have lots of 0s.

State-Space Form

$$\begin{bmatrix} \ddot{x}_1 \\ \dot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{b_2}{m_1} & -\left(\frac{k_2}{m_1}+\frac{k_1}{m_1}\right) & \frac{b_2}{m_1} & \frac{k_2}{m_1} \\ 1 & 0 & 0 & 0 \\ \frac{b_2}{m_2} & \frac{k_2}{m_2} & -\frac{b_2}{m_2} & -\frac{k_2}{m_2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{k_1}{m_1} \\ 0 \\ 0 \\ 0 \end{bmatrix} r$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

Control Example



$$\sum \mathbf{F} = u - b\dot{x} - kx = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = u$$

What I want?

- *The mass to be at 1*

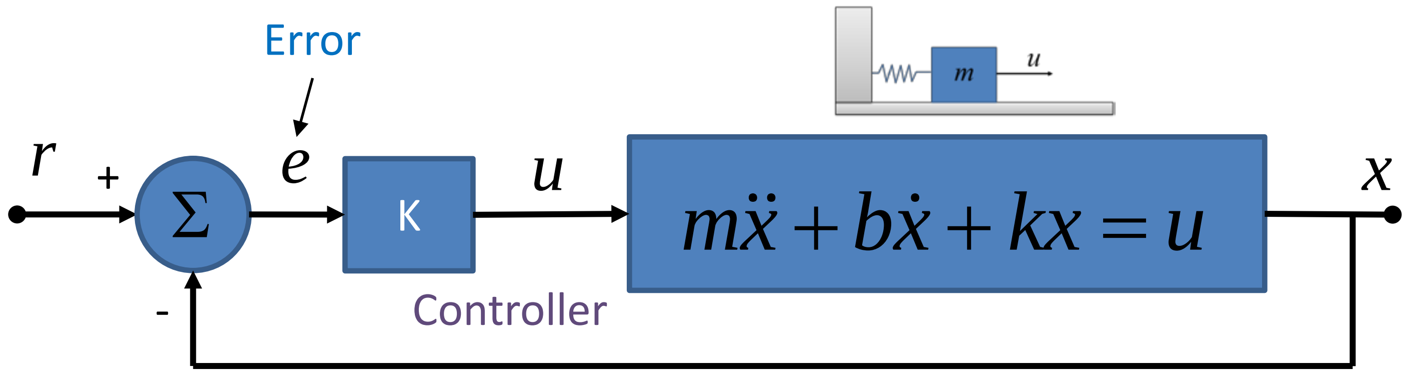
What I can do?

- *Adjust the force,* u

What am I going to do?

- Apply force, u*, proportional to the position error*

Set Up a Feedback Loop



$$m = 1\,\mathrm{kg}$$

$$b = 1\,\mathrm{kg/s}$$

$$k = 0.2\ \mathrm{N/m}$$

Effect of Feedback Control