Automatic Control

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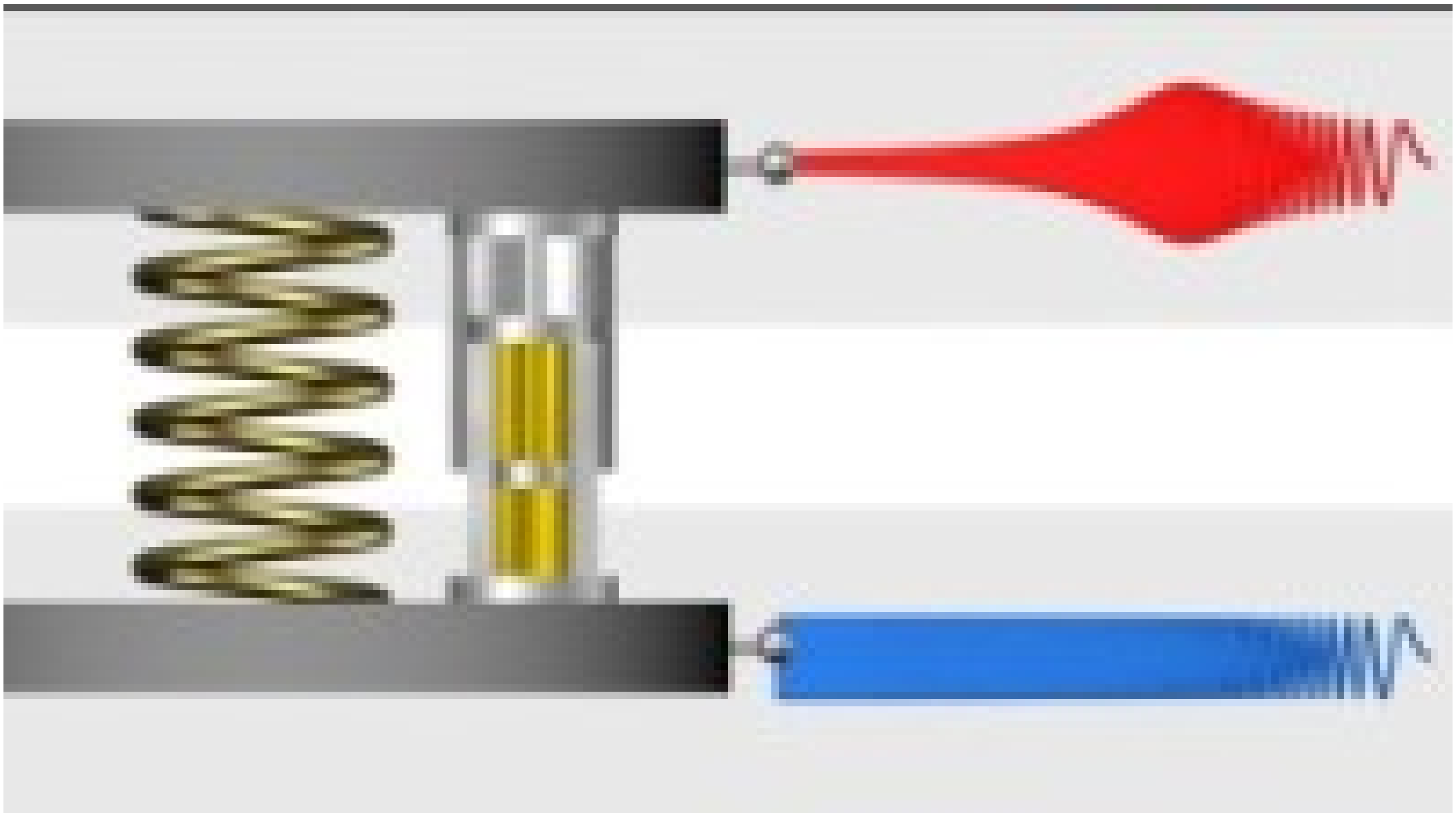
Frequency Response

Bode Plot 1 - Basics

Sinusoidal Input vs Output

https://www.youtube.com/watch?v=cgo_Dh-Bz8c



Observations

- When input is slow
  - Output is almost the same as the input
  - Same magnitude
  - Same phase
- As the input frequency increases
  - Output magnitude increases
  - Phase lags
- At some point
  - Output magnitude is maximum
  - Phase is opposite
- As the input frequency keep increases
  - Output magnitude becomes smaller

First Order System

$$\tau \dot{x} + 1x = u(t) = u_0 \sin \omega t$$

- Assume
  - $x = A \cos \omega t + B \sin \omega t$
  - $\dot{x} = -A\omega \sin \omega t + B\omega \cos \omega t$
- Solve
  - Substitute
    - $-\tau A\omega \sin \omega t + \tau B\omega \cos \omega t + A \cos \omega t + B \sin \omega t = u_0 \sin \omega t$
    - $(-\tau A\omega + B) \sin \omega t + (bA + \tau\omega B) \cos \omega t = u_0 \sin \omega t$
  - Compare
    - $-\tau\omega A + B = u_0$
    - $A + \tau\omega B = 0$
  - Matrix form
    - $\begin{bmatrix} -\tau\omega & 1 \\ 1 & \tau\omega \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$
    - $\begin{bmatrix} A \\ B \end{bmatrix} = -\frac{1}{(\tau\omega)^2+1} \begin{bmatrix} \tau\omega & -1 \\ -1 & -\tau\omega \end{bmatrix} \begin{bmatrix} u_0 \\ 0 \end{bmatrix} = \frac{u_0}{(\tau\omega)^2+1} \begin{bmatrix} -\tau\omega \\ 1 \end{bmatrix}$

First Order System

- Particular solution
  - Rearrange
    - $x = A\cos\omega t + B\sin\omega t = u_0\left(\frac{-\tau\omega}{(\tau\omega)^2+1}\cos\omega t + \frac{1}{(\tau\omega)^2+1}\sin\omega t\right)$
    - $x = u_0\sqrt{\left(\frac{\tau\omega}{(\tau\omega)^2+1}\right)^2 + \left(\frac{1}{(\tau\omega)^2+1}\right)^2}\sin(\omega t + \phi)$
  - Finally
    - $x = \frac{u_0}{\sqrt{(\tau\omega)^2+1}}\sin(\omega t + \phi)$
    - Where, $\tan\phi = -\tau\omega$

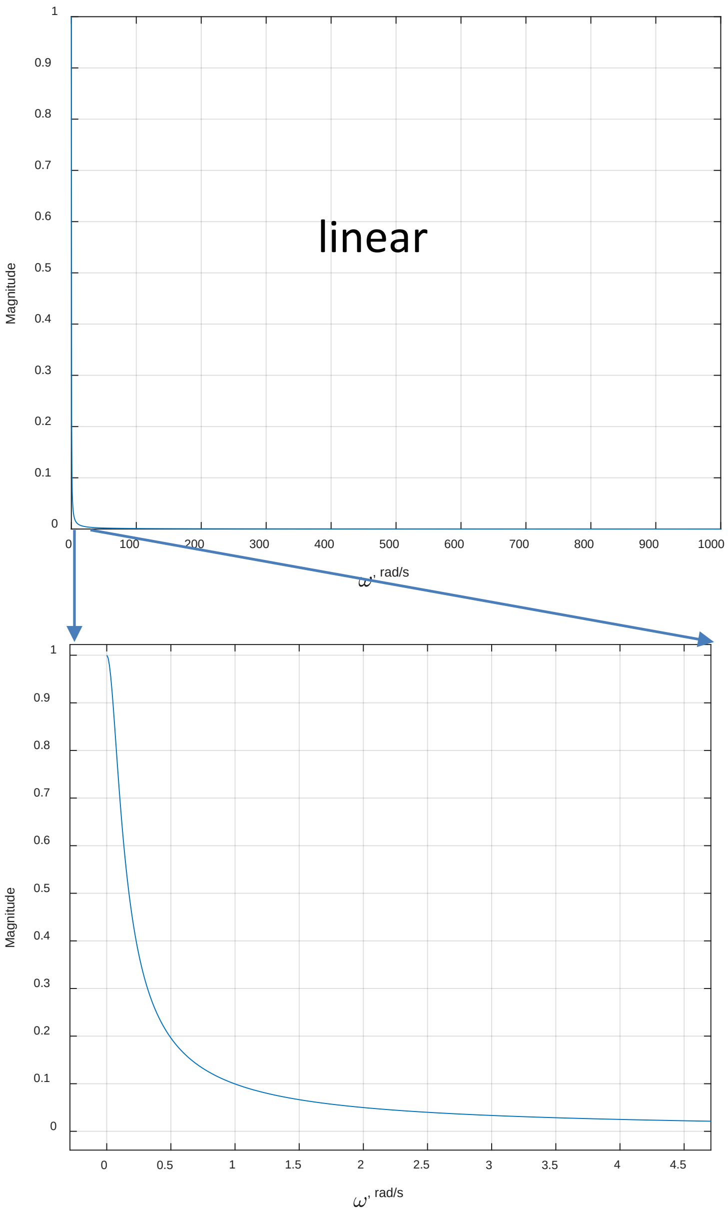
Magnitude and Phase vs Frequency

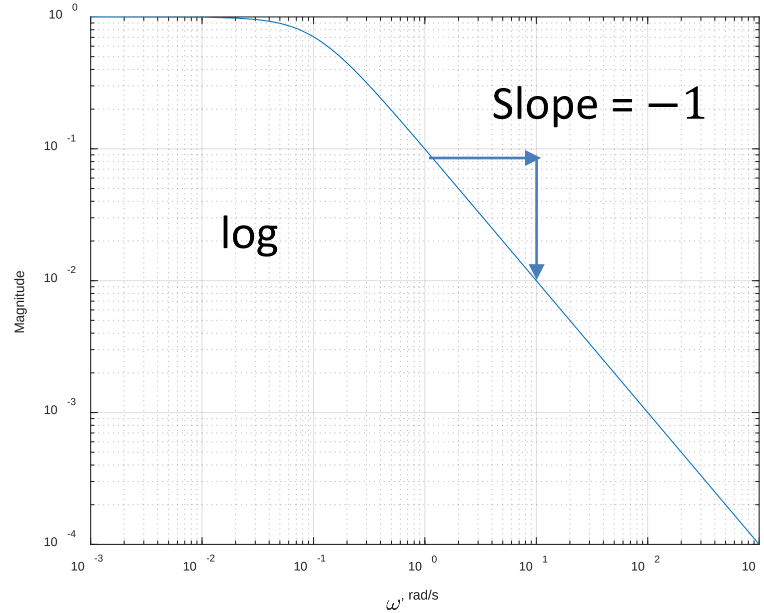
- Magnitude, M
  - Input: u_0
  - Output: $\frac{u_0}{\sqrt{(\tau\omega)^2+1}}$
    - When frequency is very small ($\omega \rightarrow 0$): $M \rightarrow u_0$
    - When frequency is very large ($\omega \rightarrow \infty$): $M \rightarrow \frac{u_0}{\tau\omega}$
- Phase
  - $\tan\phi = -\tau\omega$
    - When frequency is very small ($\omega \rightarrow 0$): $\phi \rightarrow 0$
    - When frequency is very large ($\omega \rightarrow \infty$): $\phi \rightarrow -90°$

Log Scale

- Recall the magnitude relation
  - $M \approx \frac{u_0}{\tau\omega}$ (for large ω)
  - $\log_{10} M = \log_{10} \frac{u_0}{\tau\omega} = \log_{10} \frac{u_0}{\tau} - \log_{10} \omega$
    - $(\log_{10} M) = (-1)(\log_{10} \omega) + const$
    - Straight line with a slope of -1 when $(\log_{10} M)$ is plotted for with respect to $(\log_{10} \omega)$

Example - Magnitude



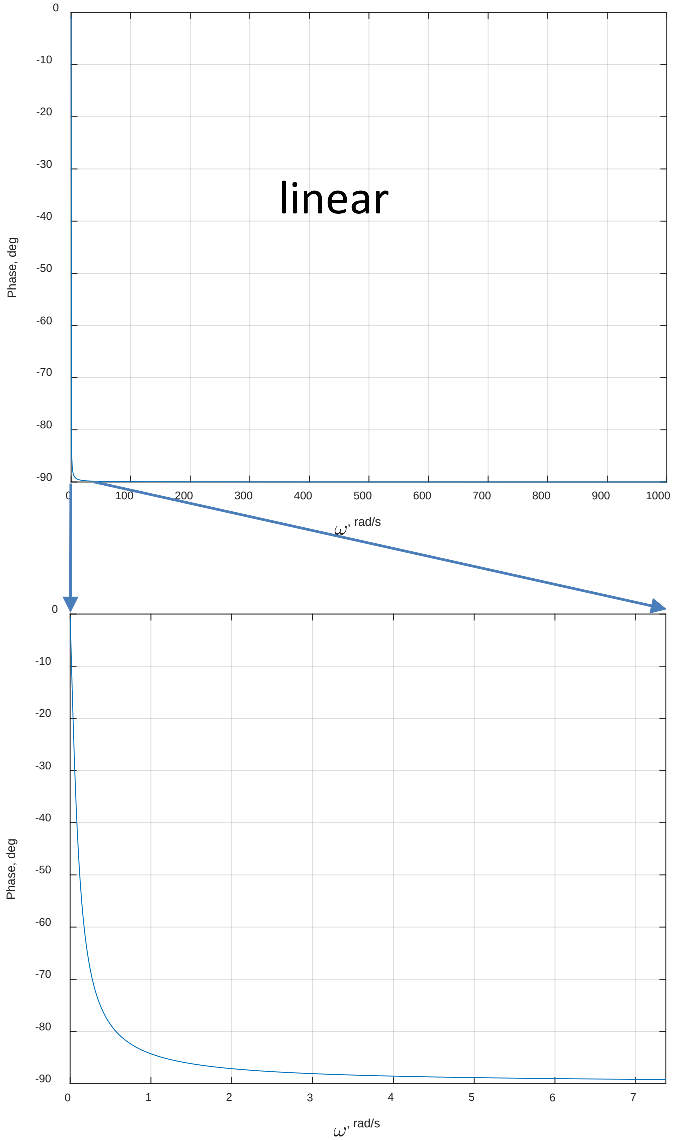


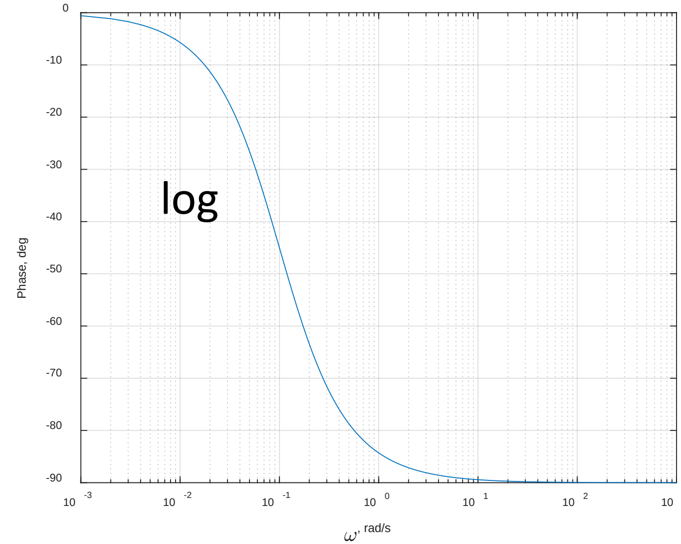
$$\tau = 10$$
$$u_0 = 1$$

$$10\dot{x} + x = \sin \omega t$$

$$\Rightarrow \quad x = \frac{1}{\sqrt{(\tau\omega)^2 + 1}} \sin(\omega t + \phi)$$

Example – Phase





$$\tan\phi = -\tau\omega$$

2nd Order System

$$\frac{\ddot{x}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{x} + 1x = u(t) = u_0 \sin \omega t$$

- Assume
  - $x = A\cos\omega t + B\sin\omega t$
  - $\dot{x} = -A\omega\sin\omega t + B\omega\cos\omega t$
  - $\ddot{x} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

- Solve
  - Substitute
    - $\left(-A\frac{\omega^2}{\omega_n^2}\cos\omega t - B\frac{\omega^2}{\omega_n^2}\sin\omega t\right) + 2\zeta\left(-A\frac{\omega}{\omega_n}\sin\omega t + B\frac{\omega}{\omega_n}\cos\omega t\right) + (A\cos\omega t + B\sin\omega t) = u_0\sin\omega t$
    - $\left(-B\frac{\omega^2}{\omega_n^2} - A2\zeta\frac{\omega}{\omega_n} + B\right)\sin\omega t + \left(-A\frac{\omega^2}{\omega_n^2} + B2\zeta\frac{\omega}{\omega_n} + A\right)\cos\omega t = u_0\sin\omega t$
  - Compare
    - $\left(-B\frac{\omega^2}{\omega_n^2} - A2\zeta\frac{\omega}{\omega_n} + B\right) = \left(B\left(1 - \frac{\omega^2}{\omega_n^2}\right) - A2\zeta\frac{\omega}{\omega_n}\right) = u_0$
    - $\left(-A\frac{\omega^2}{\omega_n^2} + B2\zeta\frac{\omega}{\omega_n} + A\right) = \left(A\left(1 - \frac{\omega^2}{\omega_n^2}\right) + B2\zeta\frac{\omega}{\omega_n}\right) = 0$

2nd Order System

$$\frac{\ddot{x}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{x} + x = u(t) = u_0 \sin \omega t$$

- Solve
  - Matrix form (let $\lambda = \frac{\omega}{\omega_n}$)
    - $\begin{bmatrix} -2\zeta\lambda & 1-\lambda^2 \\ 1-\lambda^2 & 2\zeta\lambda \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \end{bmatrix}$
    - $\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{-(2\zeta\lambda)^2-(1-\lambda^2)^2}\begin{bmatrix} 2\zeta\lambda & -(1-\lambda^2) \\ -(1-\lambda^2) & -2\zeta\lambda \end{bmatrix}\begin{bmatrix} u_0 \\ 0 \end{bmatrix} = \frac{u_0}{(1-\lambda^2)^2+(2\zeta\lambda)^2}\begin{bmatrix} -2\zeta\lambda \\ 1-\lambda^2 \end{bmatrix}$
- Particular solution
  - Rearrange
    - $x = A\cos\omega t + B\sin\omega t = u_0\left(\frac{-2\zeta\lambda}{(1-\lambda^2)^2+(2\zeta\lambda)^2}\cos\omega t + \frac{1-\lambda^2}{(1-\lambda^2)^2+(2\zeta\lambda)^2}\sin\omega t\right)$
    - $x = u_0\sqrt{\left(\frac{-2\zeta\lambda}{(1-\lambda^2)^2+(2\zeta\lambda)^2}\right)^2 + \left(\frac{1-\lambda^2}{(1-\lambda^2)^2+(2\zeta\lambda)^2}\right)^2}\sin(\omega t+\phi) = \frac{u_0}{\sqrt{(1-\lambda^2)^2+(2\zeta\lambda)^2}}\sin(\omega t+\phi)$
    - $\tan\phi = \frac{-2\zeta\lambda}{1-\lambda^2}$

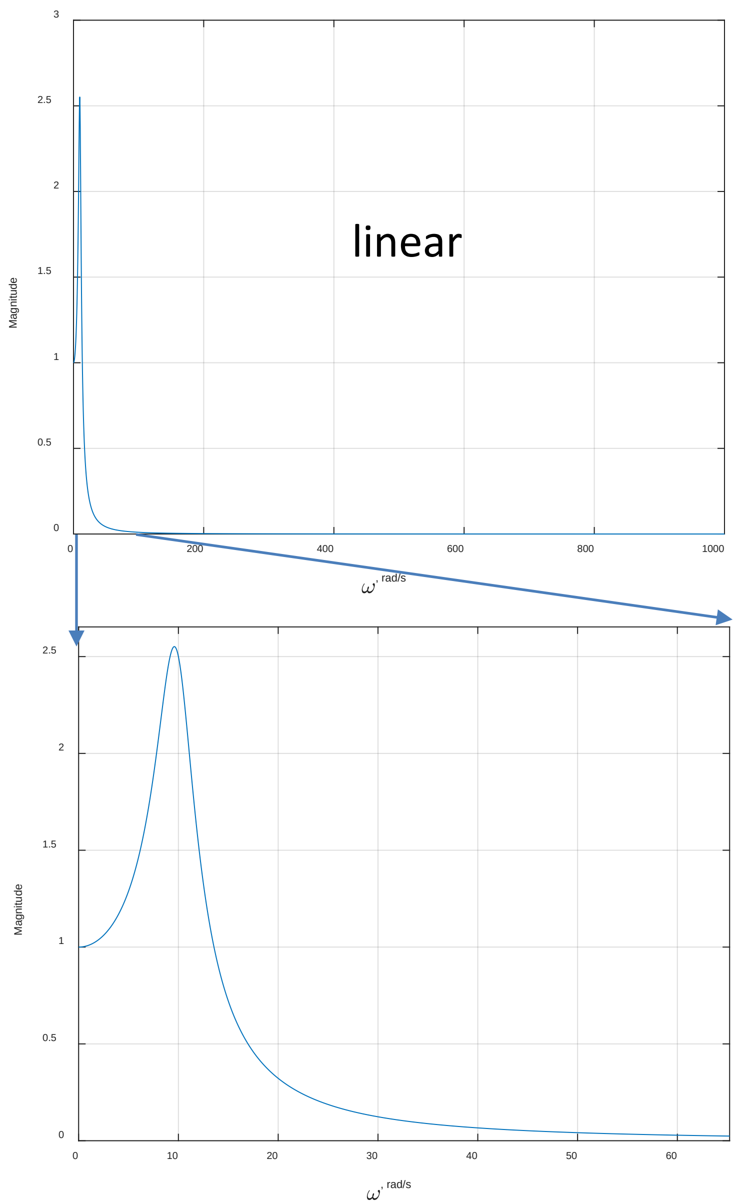
Magnitude and Phase vs Frequency

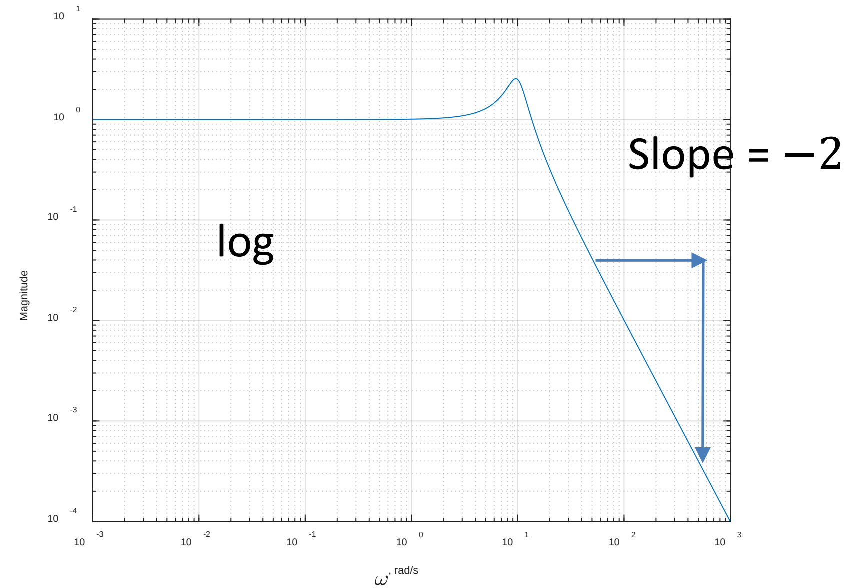
- Magnitude, M
  - Input: u_0
  - Output: $\frac{u_0}{\sqrt{(1-\lambda^2)^2+(2\zeta\lambda)^2}}$
    - When frequency is very small ($\omega \to 0$): $M \to u_0$
    - When frequency is very large ($\omega \to \infty$): $M \to \frac{u_0}{\lambda^2} = \frac{u_0}{(\omega/\omega_n)^2}$
- Phase
  - $\tan\phi = \frac{-2\zeta\lambda}{1-\lambda^2}$ $(\phi = \mathrm{atan2}(-2\zeta\lambda, 1-\lambda^2))$
    - When frequency is very small ($\omega \to 0$): $\phi \to 0$
    - When frequency is very large ($\omega \to \infty$): $\phi \to -180°$

Log Scale

- Recall the magnitude relation
  - $M \approx \frac{u_0}{(\omega/\omega_n)^2}$ (for large ω)
  - $\log_{10} M = \log_{10} \frac{u_0}{(\omega/\omega_n)^2} = \log_{10} \omega_n^2 u_0 - 2\log_{10} \omega$
    - $(\log_{10} M) = (-2)(\log_{10} \omega) + const$
    - Straight line with a slope of -2 when $(\log_{10} M)$ is plotted for with respect to $(\log_{10} \omega)$

Example - Magnitude



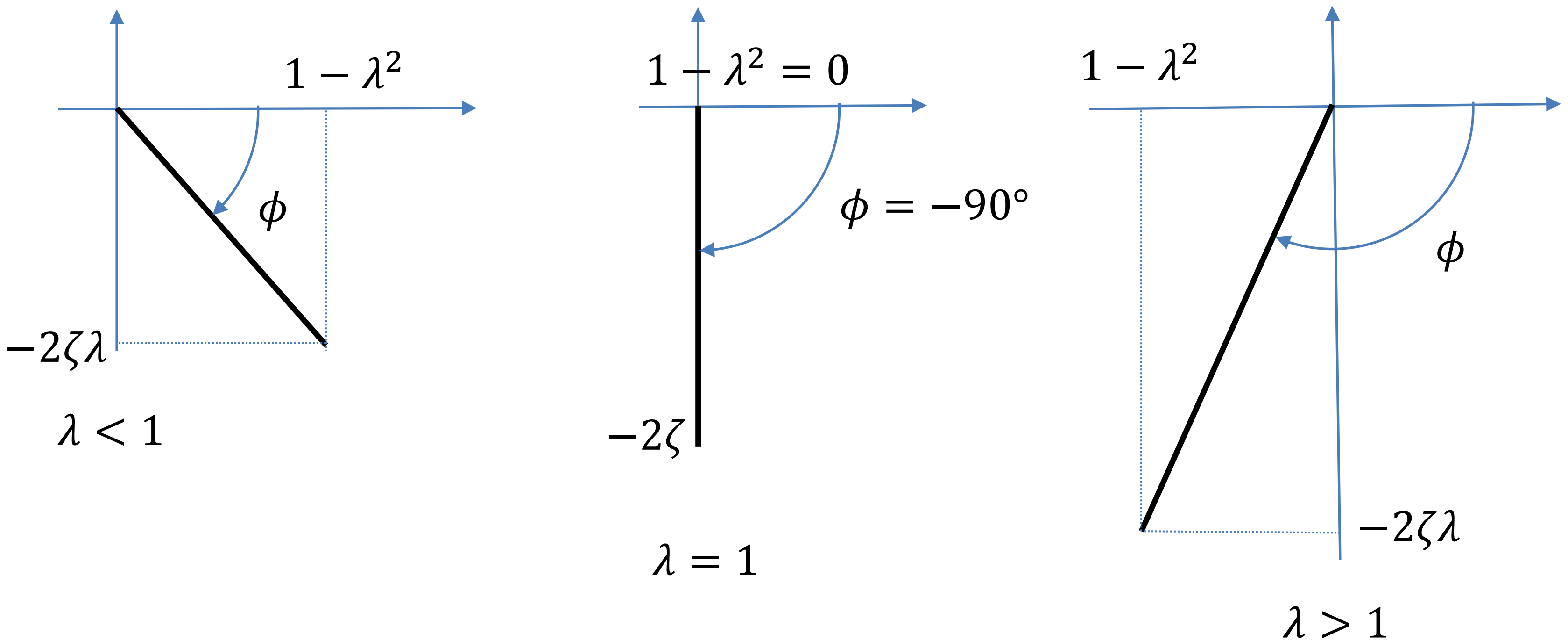


$$\omega_n = 10$$
$$\zeta = 0.2$$

$$\frac{\ddot{x}}{100} + 0.4\,\frac{\dot{x}}{10} + \; x = \sin \omega t$$

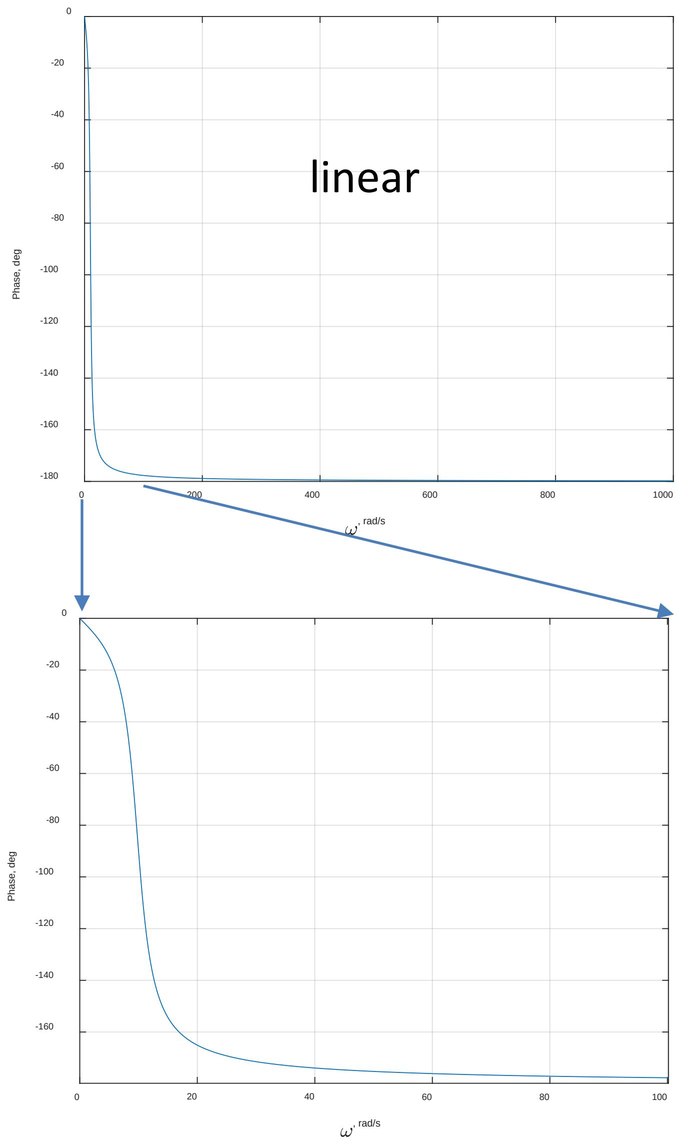
$$x = \frac{1}{\sqrt{(1-(\omega/10)^2)^2 + (0.4 \cdot \omega/10)^2}} \sin(\omega t + \phi)$$

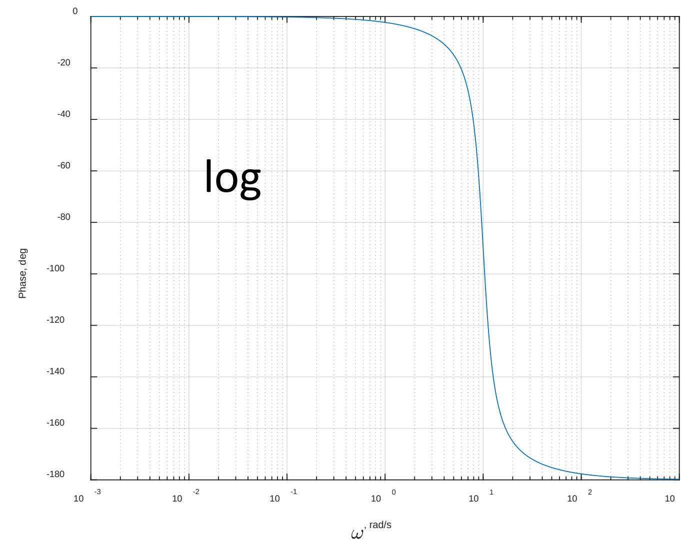
Phase



$1 - \lambda^2$
ϕ
$-2\zeta\lambda$
$\lambda < 1$
$1 - \lambda^2 = 0$
$\phi = -90°$
-2ζ
$\lambda = 1$
$1 - \lambda^2$
ϕ
$-2\zeta\lambda$
$\lambda > 1$

Example – Phase





$$\tan\phi = \frac{-2\zeta\lambda}{1-\lambda^2}$$

Recall Convolution

- Convolution
  - Definition
    - $f * g = \int_0^t f(t-\tau)g(\tau)d\tau$
  - Characteristics
    - $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$

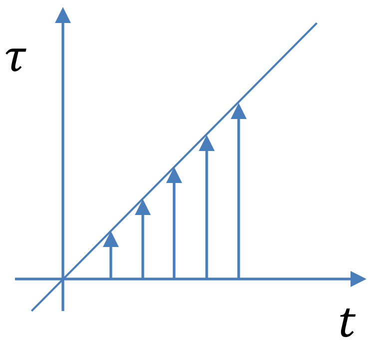
- How?
  - $\mathcal{L}(f * g) = \mathcal{L}\left(\int_0^t f(t-\tau)g(\tau)d\tau\right)$

$$= \int_{t=0}^{t=\infty} \left(\int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)d\tau\right) e^{-st} dt$$

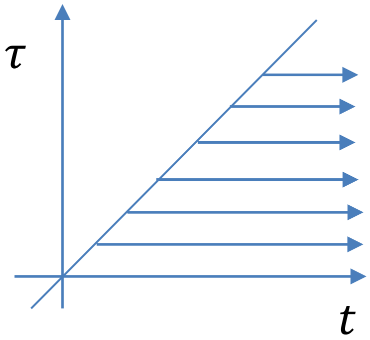
$$= \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)e^{-st} d\tau\, dt$$

Order of Integration

$$\int_{t=0}^{t=\infty}\int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)e^{-st}\,d\tau\,dt$$



$$\int_{\tau=0}^{\tau=\infty}\int_{t=\tau}^{t=\infty} f(t-\tau)g(\tau)e^{-st}\,dt\,d\tau$$



Change Order and Variable

$$\mathcal{L}(f * g) = \int_{t=0}^{t=\infty} \int_{\tau=0}^{\tau=t} f(t-\tau)g(\tau)e^{-st} d\tau\, dt$$

$$= \int_{\tau=0}^{\tau=\infty} \int_{t=\tau}^{t=\infty} f(t-\tau)g(\tau)e^{-st} dt\, d\tau$$

Substitute $\hat{t} = t - \tau$ $\Rightarrow$

$$dt = d\hat{t}$$
$$t = \tau \;\rightarrow \hat{t} = 0$$
$$t = \infty \;\rightarrow \hat{t} = \infty$$

$$\int_{\tau=0}^{\tau=\infty} \int_{t=\tau}^{t=\infty} f(t-\tau)g(\tau)e^{-st} dt\, d\tau = \int_{\tau=0}^{\tau=\infty} \int_{\hat{t}=0}^{\hat{t}=\infty} f(\hat{t})g(\tau)e^{-s(\hat{t}+\tau)} d\hat{t}\, d\tau$$

$$= \int_{\tau=0}^{\tau=\infty} \int_{\hat{t}=0}^{\hat{t}=\infty} \left(f(\hat{t})e^{-s\hat{t}}\right)\left(g(\tau)e^{-s\tau}\right) d\hat{t}\, d\tau$$

$$= \left(\int_{\hat{t}=0}^{\hat{t}=\infty} \left(f(\hat{t})e^{-s\hat{t}}\right) d\hat{t}\right)\left(\int_{\tau=0}^{\tau=\infty} \left(g(\tau)e^{-s\tau}\right) d\tau\right) = \mathcal{L}(f)\mathcal{L}(g)$$

Output for a Sinusoidal Input

- Transfer function expression
  - $Y(s) = H(s)U(s)$
- Sinusoidal input in time domain
  - $u(t) = e^{j\omega t}$
- Output in time domain using convolution
  - $y(t) = h * u = \int_0^t h(\tau)u(t-\tau)d\tau$

Output for a Sinusoidal Input

$$y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{j\omega(t-\tau)}d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$= e^{j\omega t}\int_{-\infty}^{\infty} h(\tau)e^{-(j\omega)\tau}d\tau = H(j\omega)e^{j\omega t}$$

$$y(t) = H(j\omega)e^{j\omega t}$$

Magnitude and Phase

- Magnitude
  - $|y(t)| = \left|H(j\omega)e^{j\omega t}\right| = |H(j\omega)|\left|e^{j\omega t}\right| = |H(j\omega)|$
  - Magnitude is scaled by the magnitude of the transfer function evaluated at $j\omega$

- Phase
  - $\angle y(t) = \angle\left(H(j\omega)e^{j\omega t}\right) = \angle H(j\omega) + \omega t$
  - Phase of the transfer function evaluated at $j\omega$ is added to the phase of the input

Verification with a 1st Order System

- ODE expression
  - $\tau\dot{x} + bx = u(t) = u_0 \sin \omega t$
- Transfer function
  - $(\tau s + 1)X(s) = U(s)$
  - $G(s) = \frac{1}{\tau s+1}$
  - $G(j\omega) = \frac{1}{j\tau\omega+1}$
    - $|G(j\omega)| = \left|\frac{1}{j\tau\omega+1}\right| = \frac{1}{|j\tau\omega+1|} = \frac{1}{\sqrt{(\tau\omega)^2+1}}$
    - $\angle G(j\omega) = \angle\frac{1}{j\tau\omega+1} = \angle 1 - \angle(j\tau\omega + 1) = -\tan^{-1}\tau\omega$

Verification with a 2nd Order System

- ODE expression
  - $\frac{\ddot{x}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{x} + 1x = u(t) = u_0 \sin \omega t$
- Transfer function
  - $(\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1)X(s) = U(s)$
  - $G(s) = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$ (this is the same as $\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$)
  - $G(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right) + 1} = \frac{1}{(1-\lambda^2) + j(2\zeta\lambda)}$
    - $|G(j\omega)| = \left|\frac{1}{(1-\lambda^2) + j(2\zeta\lambda)}\right| = \frac{1}{|(1-\lambda^2) + j(2\zeta\lambda)|} = \frac{1}{\sqrt{(1-\lambda^2)^2 + (2\zeta\lambda)^2}}$
    - $\angle G(j\omega) = \angle \frac{1}{(1-\lambda^2) + j(2\zeta\lambda)} = \angle 1 - \angle\left((1-\lambda^2) + j(2\zeta\lambda)\right) = -\tan^{-1}\frac{2\zeta\lambda}{1-\lambda^2}$