

**ASE6029 Linear optimal control: Homework #5**

- 1) *Feedback invariants.* Given a continuous-time linear time invariant (LTI) system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\tag{1}$$

we say that a functional

$$H(x(\cdot), u(\cdot))$$

that involves the system's input and state is a *feedback invariant* for the system if, when computed along a solution to the system (1), its value depends only on the initial condition  $x(0)$  and not on the specific input signal  $u(t)$ .

Show that, for every symmetric matrix  $P$ , the functional

$$H(x(\cdot), u(\cdot)) = - \int_0^\infty (Ax(t) + Bu(t))^T P x(t) + x(t)^T P (Ax(t) + Bu(t)) dt \tag{2}$$

is a feedback invariant for system (1) as long as  $\lim_{t \rightarrow \infty} x(t) = 0$ .

- 2) *Feedback invariants in optimal control.* Suppose that we are able to express a cost function  $J(x(\cdot), u(\cdot))$  to be minimized by an appropriate choice of the input  $u(t)$  in the following form:

$$J(x(\cdot), u(\cdot)) = H(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda(x(t), u(t)) dt. \tag{3}$$

where  $H(x(\cdot), u(\cdot))$  is a feedback invariant and the function  $\Lambda(x(t), u(t))$  has the property that

$$\min_{u(t)} \Lambda(x(t), u(t)) = 0, \quad \text{for all } x(t).$$

In this case, the control

$$u^*(t) = \arg \min_{u(t)} \Lambda(x(t), u(t)),$$

minimizes the functional (3), and the optimal value of (3) is equal to the feedback invariant

$$J(x(\cdot), u(\cdot)) = H(x(\cdot), u(\cdot)).$$

The LQR cost can be expressed as in (3) with the feedback invariant in (2), provided that we choose the matrix  $P$  appropriately. To check that this is so, we add and subtract this feedback invariant to the LQR cost and conclude that

$$\begin{aligned}J_{\text{lqr}}(x(\cdot), u(\cdot)) &= \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt \\ &= H_{\text{lqr}}(x(\cdot), u(\cdot)) + \int_0^\infty \Lambda_{\text{lqr}}(x(t), u(t)) dt.\end{aligned}$$

a) What is  $\Lambda_{\text{lqr}}((x(t), u(t)))$ ? Your answer should be in terms of  $A, B, Q, R$ , and  $P$  (as well as  $x(t)$  and  $u(t)$ ).

b) Separate  $\Lambda_{\text{lqr}}((x(t), u(t)))$  into two parts,

$$\Lambda_{\text{lqr}}((x(t), u(t))) = \Lambda_0((x(t))) + \Lambda_+((x(t), u(t)))$$

where  $\Lambda_0((x(t)))$  depends only on  $x(t)$  and not on  $u(t)$ , and  $\Lambda_+((x(t), u(t)))$  is nonnegative for all  $x(t)$  and  $u(t)$ .

c) Explain how you can choose  $u^*(t)$  and  $P$  such that  $\Lambda_{\text{lqr}}((x(t), u(t))) = 0$  for all  $x(t)$ .

d) What is the optimal LQR cost in this case? What happens if  $P$  is not positive semidefinite?

3) *Controllability, observability, and Hamiltonian.* Consider the following continuous-time LTI system,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}$$

a) Show that when the matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is full column rank, then the matrix

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

is full column rank for every  $\lambda \in \mathbb{C}$ . *Hint: Prove the statement by contradiction.* Your answer should also prove that, when the matrix

$$\mathcal{C} = [B \quad AB \quad \cdots \quad A^{n-1}B]$$

is full row rank, then the matrix

$$[A - \lambda I \quad B]$$

is full row rank for every  $\lambda \in \mathbb{C}$ .

b) Show that if  $v = [v_1^T \quad v_2^T]^T$  with  $v_1, v_2 \in \mathbb{C}^n$  is an eigenvector of a matrix  $H \in \mathbb{R}^{2n \times 2n}$  associated with an eigenvalue  $\lambda = j\omega$  on the imaginary axis, then

$$[v_2^* \quad v_1^*] H v + (H v)^* \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = 0,$$

where  $(\cdot)^*$  denotes the complex conjugate transpose. Notice that the order of the indexes of  $v_1$  and  $v_2$  above is opposite to the order in the definition of  $v$ .

c) Show that if  $v = \begin{bmatrix} v_1^T & v_2^T \end{bmatrix}^T$  with  $v_1, v_2 \in \mathbb{C}^n$  is an eigenvector of

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

associated with an eigenvalue  $\lambda = j\omega$  on the imaginary axis, then

$$B^T v_2 = 0 \quad \text{and} \quad C v_1 = 0.$$

d) Show that, if for every  $\lambda \in \mathbb{C}$

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} \quad \text{is full row rank}$$

and

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} \quad \text{is full column rank}$$

then the Hamiltonian matrix

$$H = \begin{bmatrix} A & -BB^T \\ -C^T C & -A^T \end{bmatrix}$$

cannot have any eigenvalues on the imaginary axis.

The results we've got here imply that, if  $(A, B)$  is controllable and  $(C, A)$  is observable, the Hamiltonian matrix associated with the system has exactly  $n$  stable eigenvalues and  $n$  unstable eigenvalues. Together with what we've discussed in class, this in turn implies that there exists a unique stabilizing  $P$  that makes  $A + BK = A - BR^{-1}B^T P$  a stability matrix (all of whose eigenvalues have strictly negative real parts).