1. Vectors

Outline

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Addition and scalar multiplication

Inner product

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Vectors

- a *vector* is an ordered list of numbers
- written as

$$\begin{bmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{bmatrix} \quad \text{or} \quad \begin{pmatrix} -1.1 \\ 0.0 \\ 3.6 \\ -7.2 \end{pmatrix}$$

  or $(-1.1, 0, 3.6, -7.2)$
- numbers in the list are the *elements* (*entries*, *coefficients*, *components*)
- number of elements is the *size* (*dimension*, *length*) of the vector
- vector above has dimension 4; its third entry is 3.6
- vector of size n is called an n-*vector*
- numbers are called *scalars*

Vectors via symbols

- we'll use symbols to denote vectors, *e.g.*, a, X, p, β, E^{aut}
- other conventions: $\mathbf{g}$, $\vec{a}$
- ith element of n-vector a is denoted a_i
- if a is vector above, $a_3 = 3.6$
- in a_i, i is the *index*
- for an n-vector, indexes run from $i = 1$ to $i = n$
- *warning:* sometimes a_i refers to the ith vector in a list of vectors
- two vectors a and b *of the same size* are *equal* if $a_i = b_i$ for all i
- we overload $=$ and write this as $a = b$

Block vectors

- suppose b, c, and d are vectors with sizes m, n, p
- the *stacked vector* or *concatenation* (of b, c, and d) is

$$a = \begin{bmatrix} b \\ c \\ d \end{bmatrix}$$

- also called a *block vector*, with (block) entries b, c, d
- a has size $m + n + p$

$$a = (b_1, b_2, \ldots, b_m, c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_p)$$

Zero, ones, and unit vectors

- n-vector with all entries 0 is denoted 0_n or just 0
- n-vector with all entries 1 is denoted $\mathbf{1}_n$ or just $\mathbf{1}$
- a *unit vector* has one entry 1 and all others 0
- denoted e_i where i is entry that is 1
- unit vectors of length 3:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Sparsity

- a vector is *sparse* if many of its entries are 0
- can be stored and manipulated efficiently on a computer
- $\mathbf{nnz}(x)$ is number of entries that are nonzero
- examples: zero vectors, unit vectors

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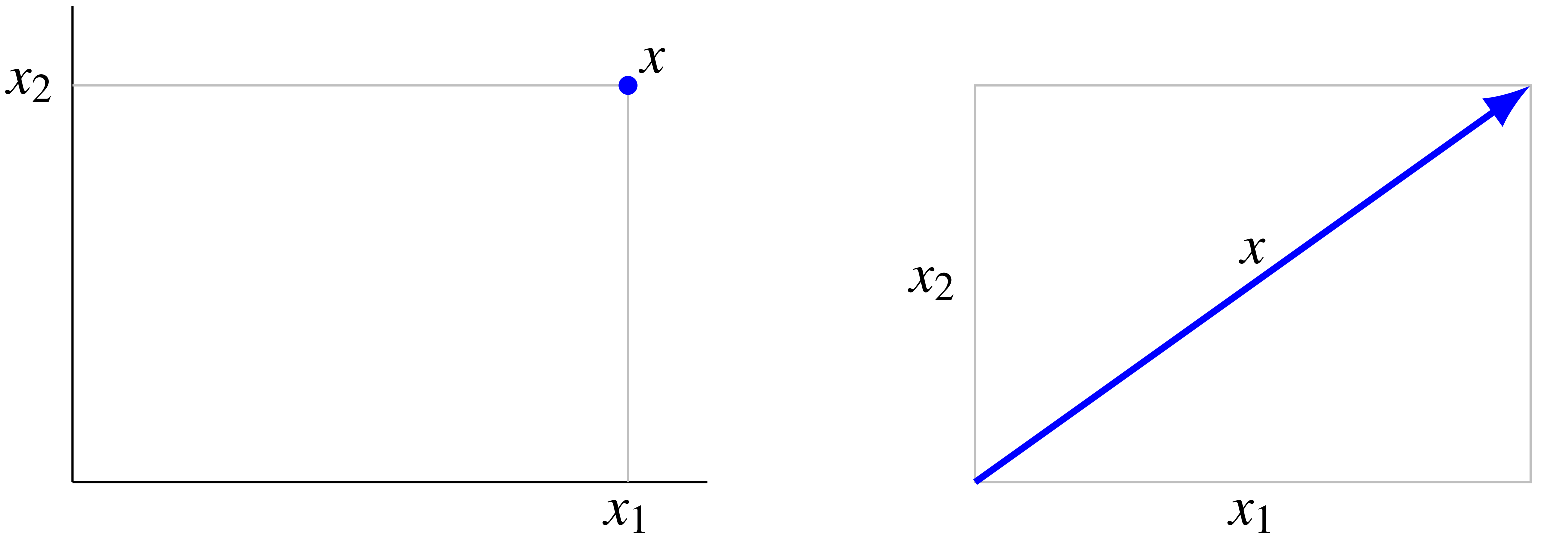
Location or displacement in 2-D or 3-D

2-vector (x_1, x_2) can represent a location or a displacement in 2-D



More examples

- color: $(\mathrm{R}, \mathrm{G}, \mathrm{B})$
- quantities of n different commodities (or resources), *e.g.*, bill of materials
- portfolio: entries give shares (or $ value or fraction) held in each of n assets, with negative meaning short positions
- cash flow: x_i is payment in period i to us
- audio: x_i is the acoustic pressure at sample time i (sample times are spaced 1/44100 seconds apart)
- features: x_i is the value of ith *feature* or *attribute* of an entity
- customer purchase: x_i is the total $ purchase of product i by a customer over some period
- word count: x_i is the number of times word i appears in a document

Word count vectors

- a short document:

  > **Word** count vectors are used **in** computer based **document** analysis. Each entry of the **word** count vector is the **number** of times the associated dictionary **word** appears **in** the **document**.

- a small dictionary (left) and word count vector (right)

$$\begin{array}{l} \text{word} \\ \text{in} \\ \text{number} \\ \text{horse} \\ \text{the} \\ \text{document} \end{array} \qquad \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}$$

- dictionaries used in practice are much larger

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Vector addition

- n-vectors a and b can be added, with sum denoted $a + b$
- to get sum, add corresponding entries:

$$\begin{bmatrix} 0 \\ 7 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix}$$

- subtraction is similar

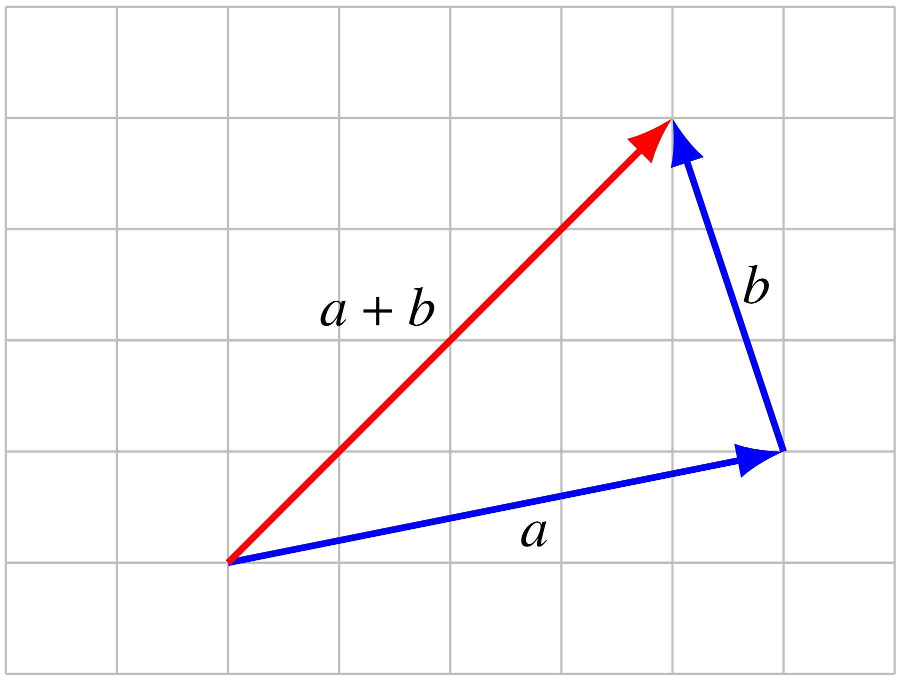
Properties of vector addition

- *commutative*: $a + b = b + a$
- *associative*: $(a + b) + c = a + (b + c)$
  (so we can write both as $a + b + c$)
- $a + 0 = 0 + a = a$
- $a - a = 0$

these are easy and boring to verify

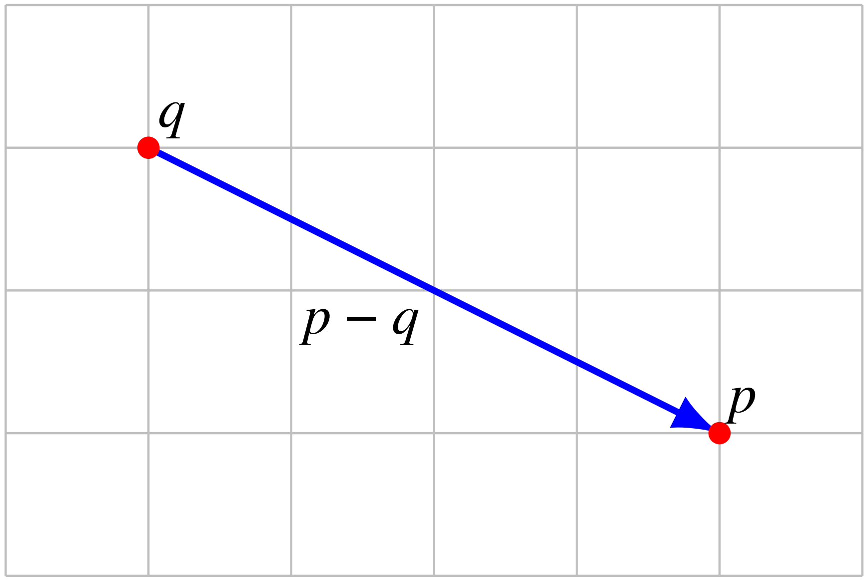
Adding displacements

if 3-vectors a and b are displacements, $a + b$ is the sum displacement



Displacement from one point to another

displacement from point q to point p is $p - q$



Scalar-vector multiplication

- scalar β and n-vector a can be multiplied

$$\beta a = (\beta a_1, \ldots, \beta a_n)$$

- also denoted $a\beta$
- example:

$$(-2)\begin{bmatrix} 1 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -18 \\ -12 \end{bmatrix}$$

Properties of scalar-vector multiplication

- associative: $(\beta\gamma)a = \beta(\gamma a)$
- left distributive: $(\beta + \gamma)a = \beta a + \gamma a$
- right distributive: $\beta(a + b) = \beta a + \beta b$

these equations look innocent, but be sure you understand them perfectly

Linear combinations

- for vectors $a_1, \ldots, a_m$ and scalars $\beta_1, \ldots, \beta_m$,

$$\beta_1 a_1 + \cdots + \beta_m a_m$$

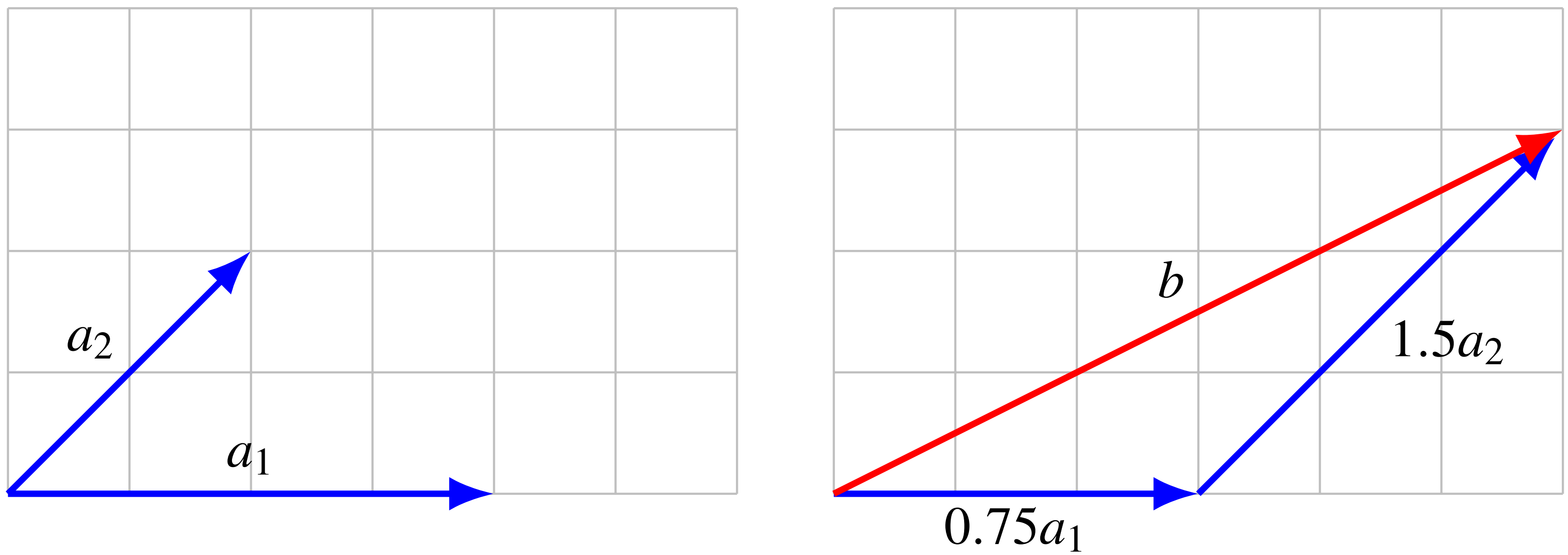
  is a *linear combination* of the vectors

- $\beta_1, \ldots, \beta_m$ are the *coefficients*
- a *very* important concept
- a simple identity: for any n-vector b,

$$b = b_1 e_1 + \cdots + b_n e_n$$

Example

two vectors a_1 and a_2, and linear combination $b = 0.75a_1 + 1.5a_2$



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Inner product

- *inner product* (or *dot product*) of n-vectors a and b is

$$a^T b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

- other notation used: $\langle a, b\rangle$, $\langle a|b\rangle$, (a, b), $a \cdot b$

- example:

$$\left[\begin{array}{r} -1 \\ 2 \\ 2 \end{array}\right]^T \left[\begin{array}{r} 1 \\ 0 \\ -3 \end{array}\right] = (-1)(1) + (2)(0) + (2)(-3) = -7$$

Properties of inner product

- $a^T b = b^T a$
- $(\gamma a)^T b = \gamma(a^T b)$
- $(a+b)^T c = a^T c + b^T c$

can combine these to get, for example,

$$(a+b)^T(c+d) = a^T c + a^T d + b^T c + b^T d$$

General examples

- $e_i^T a = a_i$ (picks out ith entry)
- $\mathbf{1}^T a = a_1 + \cdots + a_n$ (sum of entries)
- $a^T a = a_1^2 + \cdots + a_n^2$ (sum of squares of entries)

Examples

- w is weight vector, f is feature vector; $w^T f$ is score
- p is vector of prices, q is vector of quantities; $p^T q$ is total cost
- c is cash flow, d is discount vector (with interest rate r):

  $$d = (1, 1/(1+r), \ldots, 1/(1+r)^{n-1})$$

  $d^T c$ is net present value (NPV) of cash flow
- s gives portfolio holdings (in shares), p gives asset prices; $p^T s$ is total portfolio value

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Flop counts

- computers store (real) numbers in *floating-point format*
- basic arithmetic operations (addition, multiplication, ...) are called *floating point operations* or flops
- complexity of an algorithm or operation: total number of flops needed, as function of the input dimension(s)
- this can be *very grossly approximated*
- crude approximation of time to execute: (flops needed)/(computer speed)
- current computers are around 1Gflop/sec (10^9 flops/sec)
- but this can vary by factor of 100

Complexity of vector addition, inner product

- $x + y$ needs n additions, so: n flops
- $x^T y$ needs n multiplications, $n - 1$ additions so: $2n - 1$ flops
- we simplify this to $2n$ (or even n) flops for $x^T y$
- and much less when x or y is sparse