

**ASE7030 Convex optimization: Homework #2**

- 1) *Least squares solution using SVD.* Consider a tall rank deficient matrix  $A \in \mathbb{R}^{m \times n}$  with  $m > n$  is given and its singular value decomposition is given by:

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T$$

where  $\text{rank } A = r < n$ ,  $U_1 \in \mathbb{R}^{m \times r}$ ,  $\Sigma_1 = R^{r \times r}$ , and  $V_1 \in \mathbb{R}^{n \times r}$ .

Show that all solutions for:

$$\text{minimize} \quad \|Ax - b\|^2$$

is parameterized as follows with some  $z \in \mathbb{R}^{n-r}$ :

$$x^{\text{lsqr}} = V_1 \Sigma_1^{-1} U_1^T b + V_2 z$$

- 2) *Least norm solution using SVD.* Consider a wide rank deficient matrix  $A \in \mathbb{R}^{m \times n}$  with  $m < n$  is given and its singular value decomposition is given by:

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} = U_1 \Sigma_1 V_1^T$$

where  $\text{rank } A = r < m$ ,  $U_1 \in \mathbb{R}^{m \times r}$ ,  $\Sigma_1 = R^{r \times r}$ , and  $V_1 \in \mathbb{R}^{n \times r}$ . Suppose that  $b \in \text{range } A$  so there exists at least one  $x$  that satisfy  $Ax = b$ ; in fact it has infinitely many solutions.

- a) Show that all solutions for  $Ax = b$  is parameterized by:

$$x^* = V_1 \Sigma_1^{-1} U_1^T b + V_2 z$$

- b) Show that, among all solutions that satisfy  $Ax = b$ , the one with smallest  $\|x\|^2$  is uniquely given by:

$$x^{\text{ln}} = V_1 \Sigma_1^{-1} U_1^T b$$

In other words, the optimal solution for the following:

$$\begin{aligned} &\text{minimize} \quad \|x\|^2 \\ &\text{subject to} \quad Ax = b \end{aligned}$$

is unique, and is equal to  $x^{\text{ln}}$ .