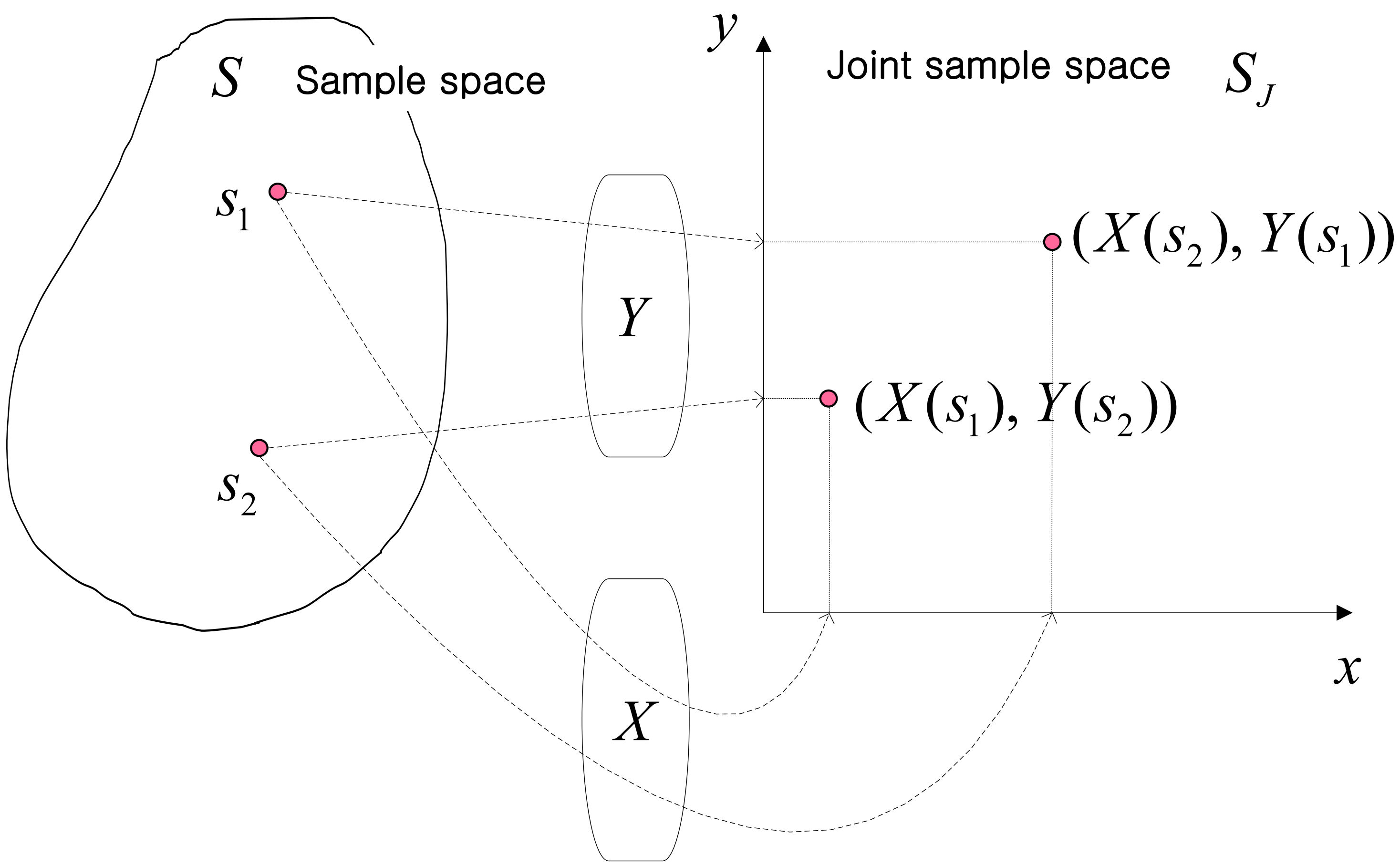
Expectation and Estimation

Multiple Random Variables

❑ **2차원 랜덤변수 사상**



Joint Cumulative Distribution Function

❑ Single Cumulative Distribution Function

$$F_X(x) = P\{X \le x\}$$
$$F_Y(y) = P\{Y \le y\}$$

❑ Joint Cumulative Distribution Function

$$F_{X,Y}(x, y) = P\{X \le x \text{ and } Y \le y\}$$

❑ Ex 4.2-1)

$$S_J = \{(1,1),(2,1),\ (3,3)\} \ \text{ and } \ P(1,1) = 0.2,\ P(2,1) = 0.3,\ P(3,3) = 0.5$$

$$\Rightarrow F_{X,Y}(0,1) = P\{X \le 0,\ Y \le 1\} = 0$$
$$F_{X,Y}(1,1) = P\{X \le 0,\ Y \le 1\} = P(1,1) = 0.2$$
$$F_{X,Y}(2,1) = P\{X \le 2,\ Y \le 1\} = P(1,1) + P(2,1) = 0.5$$
$$F_{X,Y}(2,2) = P\{X \le 2,\ Y \le 2\} = P(1,1) + P(2,1) = 0.5$$
$$F_{X,Y}(3,3) = P\{X \le 3,\ Y \le 3\} = P(1,1) + P(2,1) + P(3,3) = 1.0$$

Joint Cumulative Distribution Function

❑ Properties of Joint Cumulative Distribution Function

- $F_{X,Y}(-\infty,-\infty) = F_{X,Y}(-\infty, y) = F_{X,Y}(x,-\infty) = 0$
- $F_{X,Y}(\infty,\infty) = 1$
- $0 \le F_{X,Y}(x, y) \le 1$
- $F_{X,Y}(x, y)$ is a non-decreasing function.
- $F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$
  $= P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} \ge 0$
- $F_{X,Y}(x,\infty) = F_X(x),\ \ F_{X,Y}(\infty, y) = F_Y(y)$

Let $A = \{X \le x\}$, $B = \{Y \le y\}$, then $F_{X,Y}(x, y) = P\{X \le x,\ Y \le y\} = P[A \cap B]$.

For $\mathrm{y} \to \infty$, $B = \{Y \le \infty\} = S \Rightarrow A \cap B = A$

Therefore, $F_{X,Y}(x,\infty) = P[A \cap S] = P[A] = P\{X \le x\} = F_X(x)$

Joint Probability Density Function

❑ Joint Probability Density Function

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

❑ Properties of Joint pdf

- $f_{X,Y}(x,y) \geq 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
- $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1, \quad F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \qquad (*)$

Joint Probability Density Function

- **Proof of (*)**

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2}_{\triangleq k(\xi_1)} d\xi_1$$

$$= \frac{d}{dx}\left[K(x) - K(-\infty)\right], \quad K = \int k(\xi_1)d\xi_1$$

$$= k(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,\xi_2)d\xi_2 = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

In similar way, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

Conditional Probability

❑ Conditional Probability of X on B

$$F_X(x \mid B) = P\{X \leq x \mid B\} = \frac{P\{X \leq x \bigcap B\}}{P[B]} \text{ for } P[B] \neq 0$$

$$\Rightarrow f_X(x \mid B) = \frac{dF_X(x \mid B)}{dx}$$

❑ Conditional Probability of X

Let $B = \{y - \Delta y < Y \leq y + \Delta y\}$

then $F_X(x \mid B) = F_X(x \mid y - \Delta y < Y \leq y + \Delta y)$

$$= \frac{P[X \leq x \bigcap \{y - \Delta y < Y \leq y + \Delta y\}]}{P\{y - \Delta y < Y \leq y + \Delta y\}}.$$

Conditional Probability Density Function

❑ Conditional Probability of X

Recall that $P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2.$

$$F_X(x\,|\,B) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi)\,d\xi}$$

$$\approx \frac{\int_{-\infty}^{x} f_{X,Y}(\xi_1,y)\,d\xi_1 \cdot 2\Delta y}{f_Y(y)\cdot 2\Delta y}$$

If $\Delta y \to 0,\ F_X(x\,|\,B) \to F_X(x\,|\,Y=y) = \dfrac{\int_{-\infty}^{x} f_{X,Y}(\xi_1,y)\,d\xi_1}{f_Y(y)}.$

$$\Rightarrow f_X(x\,|\,Y=y) = \frac{dF_X(x\,|\,Y=y)}{dx} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Simply, $f_X(x\,|\,y) = f_X(x\,|\,Y=y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)},\ f_Y(y\,|\,x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

Conditional Probability Density Function

❑ Statistical Independence

It is said that A is statistically independent on B,
if $P[A \cap B] = P[A]P[B]$.

Let $A = \{X \le x\}$, $B = \{Y \le y\}$, then $P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$.

$\Rightarrow \; F_{X,Y}(x,y) = F_X(x)F_Y(y)$

$$F_X(x \mid Y \le y) = \frac{P\{X \le x \cap Y \le y\}}{P\{Y \le y\}} = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x)F_Y(y)}{F_Y(y)}$$

$$= F_X(x)$$

In similar way, $F_Y(y \mid X \le x) = F_Y(y)$

Also, $f_X(x \mid Y \le y) = f_X(x), \quad f_Y(y \mid X \le x) = f_Y(y)$

Conditional Probability Density Function

❑ Example

Let the joint density of two random variables X and Y be given by

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{6}(x+4y), & 0<x<2,\ 0<y<1\\ 0, & \text{otherwise}\end{cases}$$

1) $f_{X,Y}(x,y)\geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy=\int_0^1\int_0^2 \frac{1}{6}(x+4y)dxdy$$

$$=\int_0^1\left[\frac{1}{6}(\frac{1}{2}x^2+4xy)\right]_{x=0}^{x=2}dy=\int_0^1\frac{1}{6}(2+8y)dy$$

$$=\left[\frac{1}{6}(2y+4y^2)\right]_{y=0}^{y=1}=1$$

$\Rightarrow$ Probability Density function

Conditional Probability Density Function

- Example

$$2)\ f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy=\int_0^1 \frac{1}{6}(x+4y)dy=\frac{1}{6}(xy+2y^2)\Big]_0^1=\frac{1}{6}(x+2)$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx=\int_0^2 \frac{1}{6}(x+4y)dx=\frac{1}{6}(\frac{1}{2}x^2+4xy)\Big]_0^2=\frac{1}{6}(2+8y)=\frac{1}{3}(1+4y)$$

$$f_X(x\,|\,y)=f_X(x\,|\,Y=y)=\begin{cases}\dfrac{f_{X,Y}(x,y)}{f_Y(y)}=\dfrac{\frac{1}{6}(x+4y)}{\frac{1}{3}(1+4y)}=\dfrac{x+4y}{2(1+4y)}, & 0<x<2\\ 0, & \text{otherwise}\end{cases}$$

$$3)\ F_X(1\,|\,0.5)=F_X(x\,|\,Y=0.5)=\frac{\int_{-\infty}^{x} f_{X,Y}(\xi_1,y)d\xi_1}{f_Y(y)}=\frac{\int_{-\infty}^{1} f_{X,Y}(\xi_1,0.5)d\xi_1}{f_Y(0.5)}$$

$$=\frac{\int_0^1 \frac{1}{6}(x+2)dx}{1}=\frac{1}{6}(\frac{1}{2}x^2+2x)\Big]_0^2=\frac{5}{12}$$

Expected Value

❑ Expected value or Mean of a discrete random variable X

$$m_X = E[X] = \sum_{x \in S_X} x p_X(x) = \sum_k x_k p_X(x_k)$$

- The expected value (or expectation) refers, intuitively, to the value of a random variable one would "expect" to find if one could repeat the random variable process an infinite number of times and take the average of the values obtained.

- The expected value is a weighted average of all possible values.

❑ Ex) 주사위를 1회 던졌을 때 나타나는 눈의 기대값

$$X = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\},\ p_X(x_i) = 1/6,\ i = 1, 2, \ldots, 6$$

$$m_X = E[X] = \sum_k x_k p_X(x_k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

Expected Value

- **Expected value a random variable X(General meaning)**

$$m_X = E\left[X\right] = \int_{-\infty}^{\infty} x f_X(x)dx$$

- **Ex 3.1-2)**

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b}, & x > a \\ 0, \ x < a \end{cases}$$

$$E[X] = \frac{1}{b}\int_a^{\infty} x e^{-\frac{(x-a)}{b}} dx = \frac{1}{b} e^{\frac{a}{b}} \int_a^{\infty} x e^{-\frac{x}{b}} dx = \left(\frac{1}{b} e^{\frac{a}{b}}\right) \underbrace{\left[e^{-\frac{x}{b}} \left(\frac{x}{-\frac{1}{b}} - \frac{1}{\frac{1}{b^2}} \right) \right]_a^{\infty}}_{\text{from C-46}}$$

$$= \left(\frac{1}{b} e^{\frac{a}{b}}\right) e^{-\frac{a}{b}} \left[ab + b^2\right] = a + b$$

Expected Value

- Ex) Mean of normal pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

$$E[X] = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} [x e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} - a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} + a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}] dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} [\underbrace{\int_{-\infty}^{\infty} (x-a_X) e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx}_{=0 \because odd\ \ function} + \int_{-\infty}^{\infty} a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx]$$

$$= a_X \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx = a_X \underbrace{\int_{-\infty}^{\infty} f_X(x) dx}_{=1} = a_X$$

Moments

❑ Expected value functions of a random variable X

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

❑ Moments about the origin

Let $g(X) = X^n, \quad n = 0, 1, 2, \ldots$

then the n-th moment of X is given by $m_n = E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)dx$

❑ Central Moments

Let $g(X) = (X - \bar{X})^n, \quad n = 0, 1, 2, \ldots$

where $\bar{X}$ is the mean of X, then the n-th central moment of X

is given by $\mu_n = E[(X - \bar{X})^n] = \int_{-\infty}^{\infty} (x - \bar{X})^n f_X(x)dx.$

$cf)\ \mu_0 = E[(X - \bar{X})^0] = \int_{-\infty}^{\infty} f_X(x)dx = 1$

$\mu_1 = E[(X - \bar{X})] = \int_{-\infty}^{\infty} (x - \bar{X}) f_X(x)dx = 0$

Moments

❑ Properties of Mean

- $E\left[g(X)+h(X)\right]=E\left[g(X)\right]+E\left[h(X)\right]$
- $E\left[ag(X)\right]=aE\left[g(X)\right]$
- $E\left[g(X)+c\right]=E\left[g(X)\right]+c$
- $E\left[c\right]=c$
- $E\left[\sum g_k(X)\right]=\sum E\left[g_k(X)\right]$

Moments

- Ex) 주사위를 1회 던졌을 때 나타나는 눈의 모멘트

$$X=\{1,\ 2,\ 3,\ 4,\ 5,\ 6\},\ p_X(x_i)=1/6,\ i=1,2,\ldots,6$$

$$\text{Mean (the 1}^{st}\text{ moment)}:\ \bar{X}=E[X]=\sum_k x_k p_X(x_k)=1\cdot\frac{1}{6}+\ldots+6\cdot\frac{1}{6}=\frac{21}{6}=\frac{7}{2}$$

$$\text{The 2}^{nd}\text{ moment}:\ E[X^2]=\sum_k k^2 p_X(x_k)=1^2\cdot\frac{1}{6}+2^2\cdot\frac{1}{6}+\ldots+6^2\cdot\frac{1}{6}=\frac{91}{6}$$

Variance(the 2^{nd} central moment) :

$$E[(X-\bar{X})^2]=\sum_k\left(k-\frac{7}{2}\right)^2 p_X(x_k)\Leftarrow\text{Too tedious}$$

$$=E[X^2-2X\bar{X}+\bar{X}^2]$$

$$=E[X^2]-2\bar{X}\,E[X]+\bar{X}^2=E[X^2]-\bar{X}^2$$

$$=\frac{91}{6}-\left(\frac{7}{2}\right)^2$$

Moments

- **Ex) Gaussian Random Variable**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

$$\text{Mean (the } 1^{st} \text{ moment)}: \ \bar{X} = E[X] = a_X$$

$$\text{Variance(the } 2^{nd} \text{ central moment)}: \ E[(X-\bar{X})^2] = \sigma_X^2$$

$\Rightarrow$ The pdf of the gaussian RV is represented by its mean and variance.

Moments

❑ Joint moments about the origin

For two random variables X and Y,

the joint moment is given by $m_{nk} = E\left[X^n Y^k\right] = \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)dxdy$,

where $n+k$ is the order of the joint moment.

❑ Correlation

$$R_{XY} = m_{11} = E\left[XY\right] = \int_{-\infty}^{\infty} x\, y\, f_{X,Y}(x,y)dxdy$$

If $R_{XY} = E[X]E[Y]$ is satisfied, we say that there is no correlation between X and Y. Or, it is said that X is statistically indpendent on Y.

If $R_{XY} = 0$, two random variables X and Y are orthogonal each other.

Moments

- **Joint central moments**

For two random variables X and Y,

$$\mu_{nk} = E\left[(X-\bar{X})^n (Y-\bar{Y})^k\right] = \int_{-\infty}^{\infty} (x-\bar{X})^n (y-\bar{Y})^k f_{X,Y}(x,y)dxdy$$

is called the joint central moment.

Moments

- **Covariance**

$$C_{XY} = \mu_{11} = E\left[(X-\bar{X})(Y-\bar{Y})\right] = \int_{-\infty}^{\infty}(x-\bar{X})(y-\bar{Y})\, f_{X,Y}(x,y)dxdy$$

$$= \underbrace{\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy}_{=E[XY]} - \underbrace{\bar{X}\int_{-\infty}^{\infty} y\, f_{X,Y}(x,y)dxdy}_{=E[\bar{X}Y]=\bar{X}E[Y]=\bar{X}\bar{Y}} - \underbrace{\bar{Y}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)dxdy}_{=E[X\bar{Y}]=\bar{Y}E[X]=\bar{X}\bar{Y}}$$

$$+ \bar{X}\bar{Y}\underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy}_{=1} = R_{XY} - \bar{X}\bar{Y}$$

If $C_{XY} = 0(\Rightarrow R_{XY} = E[X]E[Y])$, there is no correlation or independent each other between X and Y.

If X and Y are orthogonal($R_{XY} = 0$), $C_{XY} = -E[X]E[Y]$.

Moments

- **Correlation coefficient**

$$\rho = E\left[\frac{(X-\bar{X})}{\sigma_X} \frac{(Y-\bar{Y})}{\sigma_Y} \right]$$

where $\sigma_X^2 = E[(X-\bar{X})^2], \sigma_Y^2 = E[(Y-\bar{Y})^2].$

HW) Prove $-1 \le \rho \le 1.$

Multivariate random variables

❑ Multivariate random variables (Random vectors)

- Vectors with random variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Mean

$$\mu = E[x] = \begin{bmatrix} E[x_1] \\ E[x_2] \\ \vdots \\ E[x_n] \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}$$

- Covariance

$$\Sigma = E[(x-\mu)(x-\mu)^T] =$$

$$\begin{bmatrix} E[(x_1-\mu_1)(x_1-\mu_1)] & E[(x_1-\mu_1)(x_2-\mu_2)] & \cdots & E[(x_1-\mu_1)(x_n-\mu_n)] \\ E[(x_2-\mu_2)(x_1-\mu_1)] & E[(x_2-\mu_2)(x_2-\mu_2)] & \cdots & E[(x_2-\mu_2)(x_n-\mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ E[(x_n-\mu_n)(x_1-\mu_1)] & E[(x_n-\mu_n)(x_2-\mu_2)] & \cdots & E[(x_n-\mu_n)(x_n-\mu_n)] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

Multivariate random variables

❑ Mean and covariance under linear transformation

- Suppose the mean and the covariance of x is given by

Mean: $E[x] = \bar{x}$

Covariance: $E[(x-\bar{x})(x-\bar{x})^T] = \Sigma$

- A new variable via linear transformation $y = Ax$. Then,

Mean:

$$\begin{aligned} E[y] &= E[Ax] \\ &= AE[x] \\ &= A\bar{x} \end{aligned}$$

Covariance:

$$\begin{aligned} E[(y-\bar{y})(y-\bar{y})^T] &= E[(Ax-A\bar{x})(Ax-A\bar{x})^T] \\ &= E[A(x-\bar{x})(x-\bar{x})^T A^T] \\ &= AE[(x-\bar{x})(x-\bar{x})^T]A^T \\ &= A\Sigma A^T \end{aligned}$$

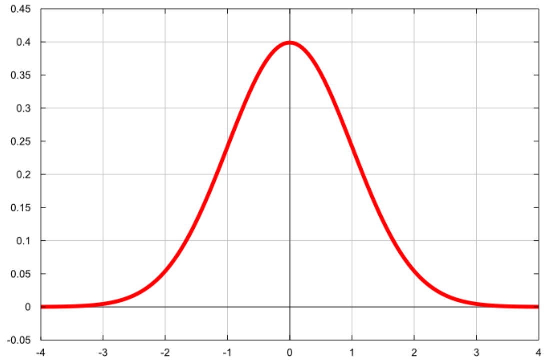
Multivariate Gaussian Random Variables

❑ Multivariate Gaussian Probability Density Function

- Univariate Gaussian pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

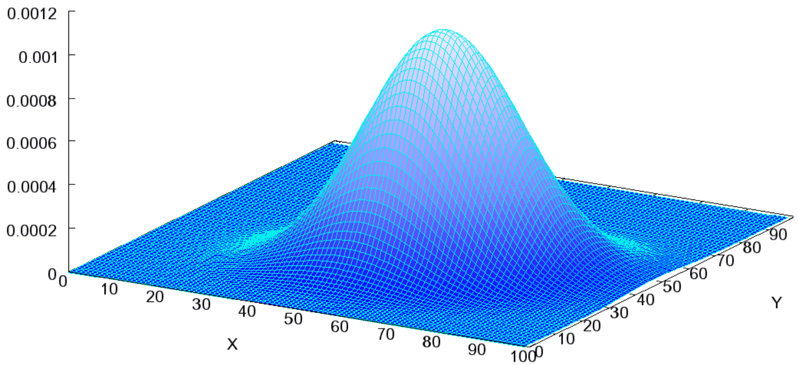
$$x \sim N(\mu, \sigma^2)$$

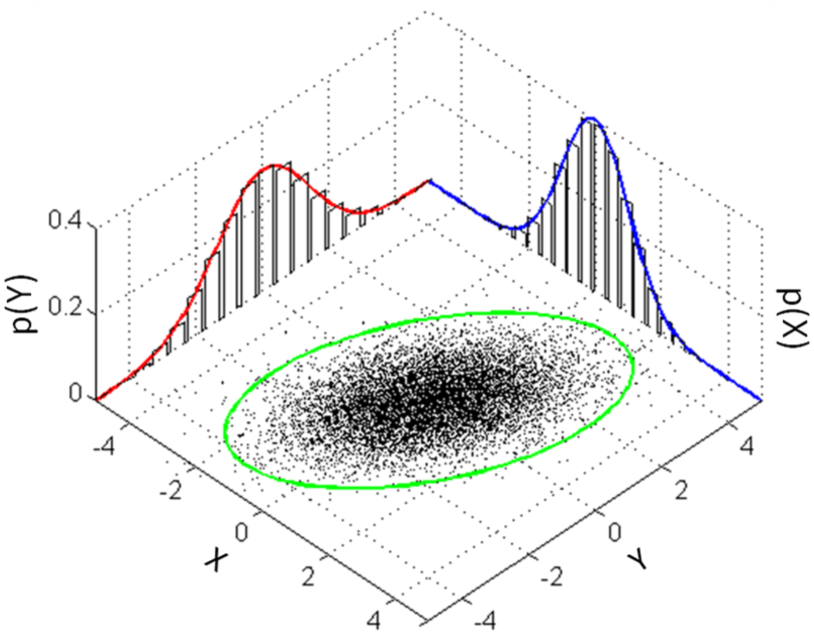


- Multivariate Gaussian pdf

$$f_X(x) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} e^{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)}$$

$$x \sim N(\mu, \Sigma)$$





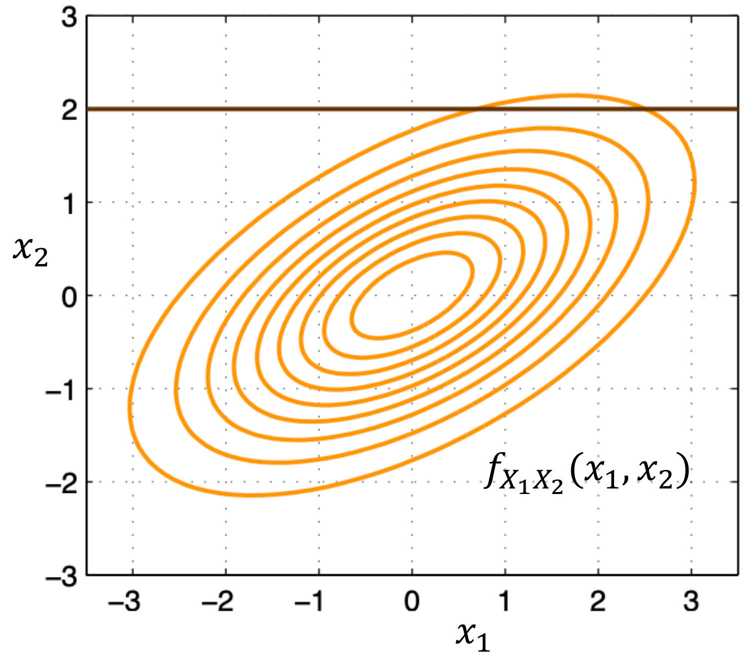
Multivariate Gaussian Random Variables

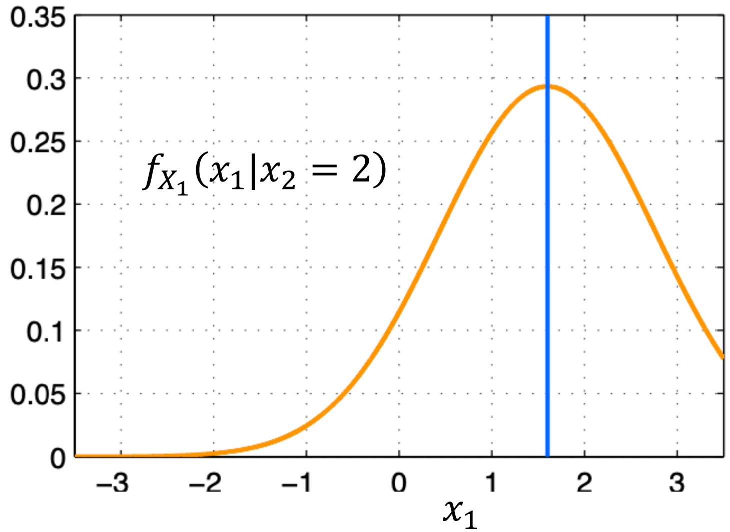
❑ Conditioning

- Given a joint pdf $f_{X_1X_2}(x_1, x_2)$ on R^2

- Measure x_2;
  we would like to find
  the conditional pdf of x_1,

- For example,
  when we know $x_2 = 2$,
  what is $f_{X_1}(x_1|x_2 = 2)$?





Multivariate Gaussian Random Variables

❑ Conditioning

- Suppose $x \sim N(0, \Sigma)$ with

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$$

- Suppose we measure $x_2 = y$.

- Conditional pdf x_1 given $x_2 = y$:

$$f_{X_1}(x_1 | x_2 = y)$$
$$= \frac{1}{\sqrt{(2\pi)^n \left|\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right|}} e^{-\frac{1}{2}\left(x - \Sigma_{12}\Sigma_{22}^{-1}y\right)^T \left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T\right)^{-1}\left(x - \Sigma_{12}\Sigma_{22}^{-1}y\right)}$$

$$\Leftrightarrow \; f_{X_1}(x_1 | x_2 = y) \sim N(\Sigma_{12}\Sigma_{22}^{-1}y,\ \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)$$

$$\Longrightarrow E[x_1 | x_2 = y] = \Sigma_{12}\Sigma_{22}^{-1}y \qquad \mathrm{cov}[x_1 | x_2 = y] = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T$$

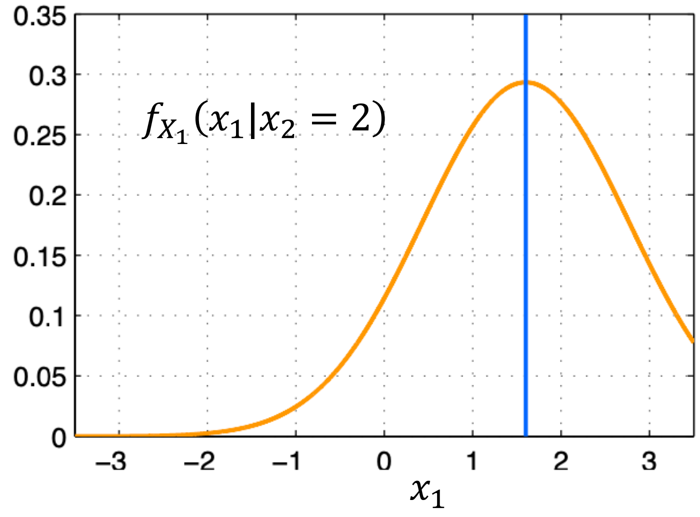
Multivariate Gaussian Random Variables

❑ Conditioning

- Example

  $x \sim N(0, \Sigma)$ with

  $$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

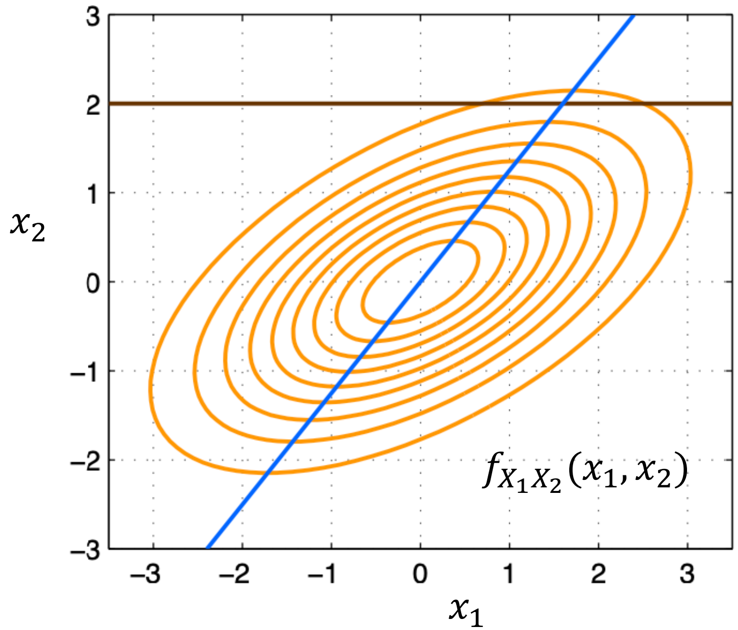


$$\Sigma_{12}\Sigma_{22}^{-1}y = 0.8y$$

$$\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{T} = 1.36$$

$$f_{X_1}(x_1|y) \sim N(0.8y,\ 1.36)$$

$$f_{X_1}(x_1|y = 2) \sim N(1.6,\ 1.36)$$



Multivariate Gaussian Random Variables

❑ Estimation

- Suppose $x \sim N(0, \Sigma_x)$, $w \sim N(0, \Sigma_w)$ uncorrelated, and a measurement $y = Ax + w$ is given:
- We want the best estimate on x given y, i.e., $E[x|y]$
- Let $z = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$
- Then

$$\Sigma_z = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_w \end{bmatrix} \begin{bmatrix} I & A^T \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} \Sigma_x & \Sigma_x A^T \\ A\Sigma_x & A\Sigma_x A^T + \Sigma_w \end{bmatrix}$$

$$\Longrightarrow E[x|y] = \Sigma_x A^T (A\Sigma_x A^T + \Sigma_w)^{-1} y$$

- If $\Sigma_w = I$ and $\Sigma_x \longrightarrow \infty$, this approaches to the least squares approximate solution

$$E[x|y] \longrightarrow (A^T A)^{-1} A^T y$$