Automatic Control

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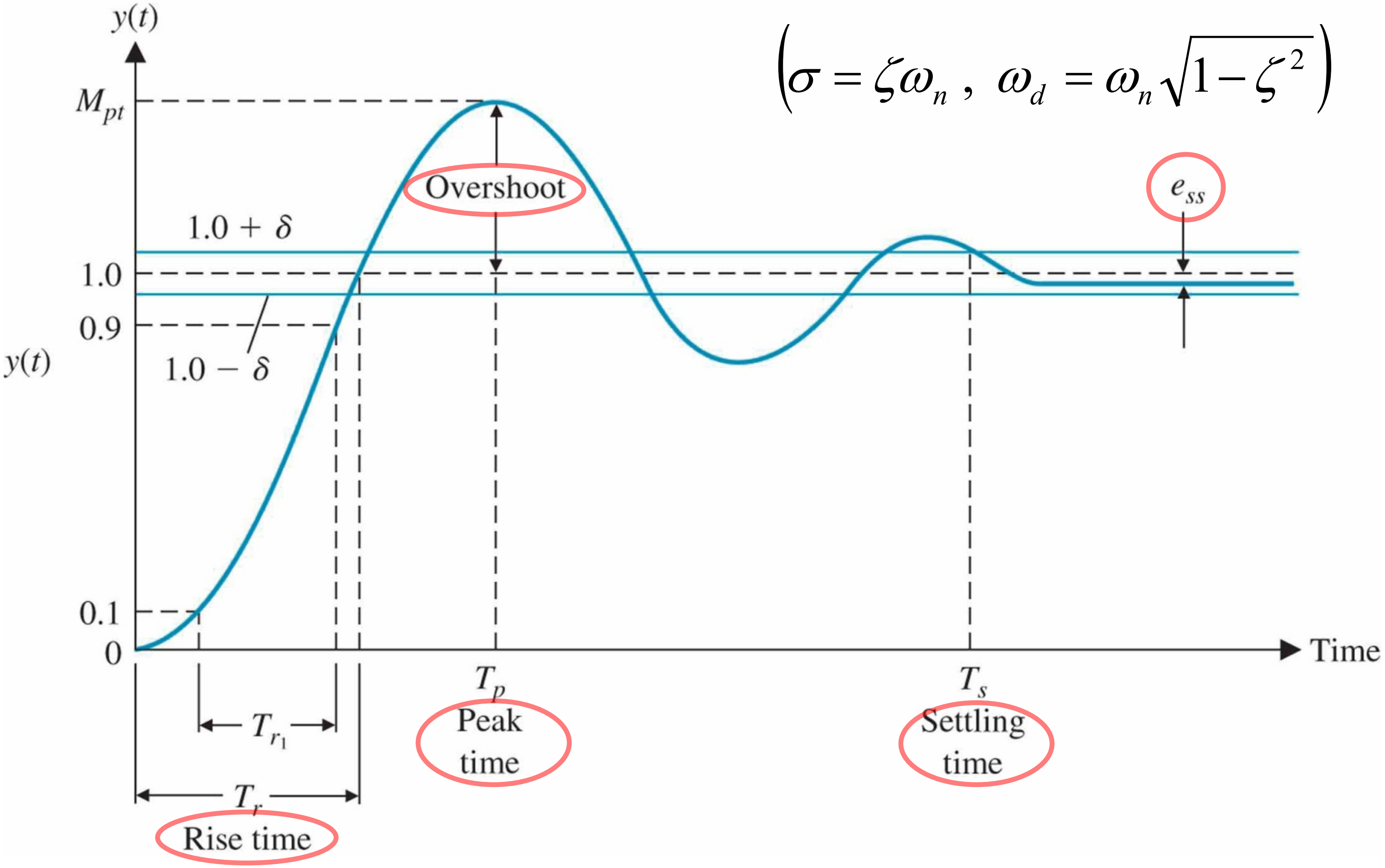
Time Domain Specifications

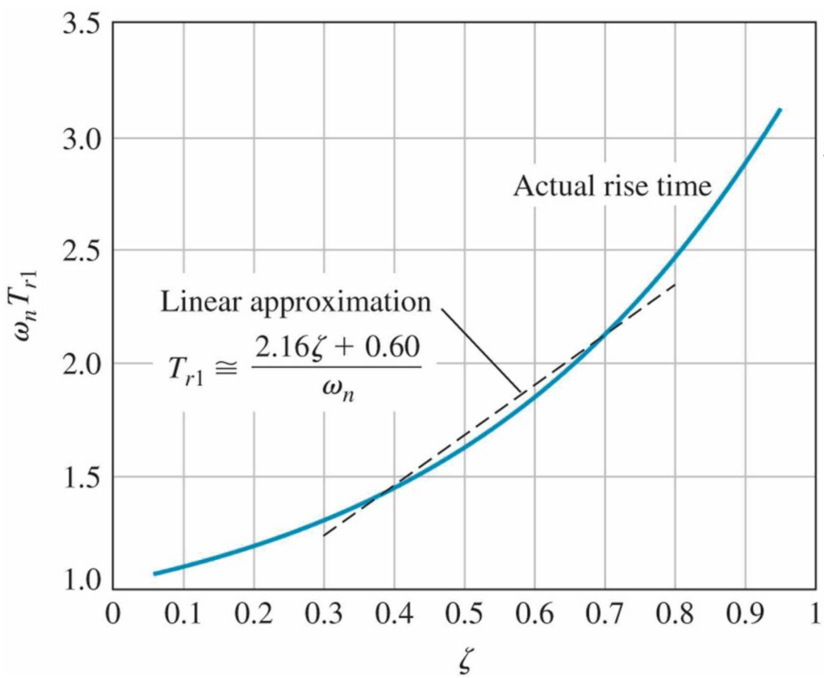
Step Response – Time Domain Specifications

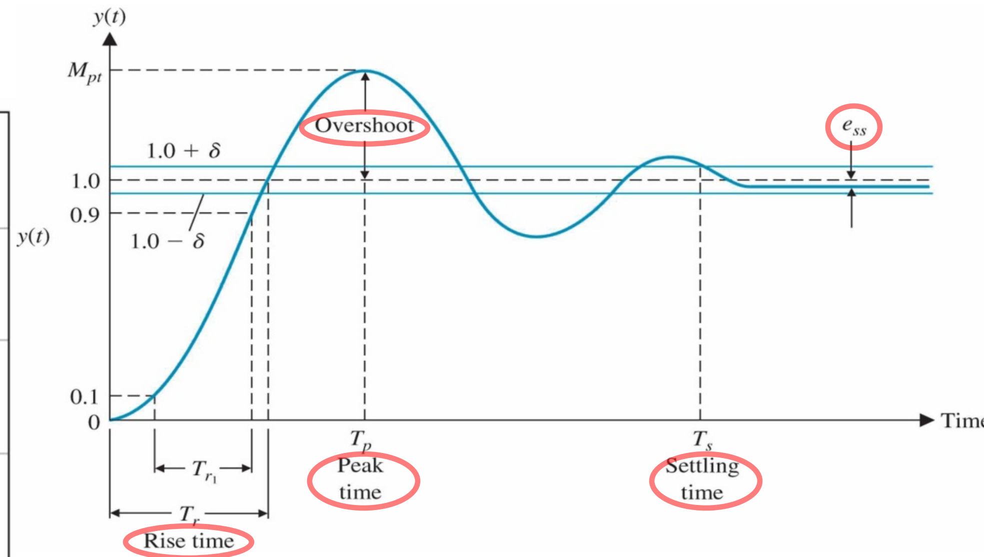
$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$\left(\sigma = \zeta\omega_n\ ,\ \omega_d = \omega_n\sqrt{1-\zeta^2}\right)$$



Rise Time





T_r : 0-100%
T_{r1} : 10-90%

$$t_{r1} \approx \frac{1.8}{\omega_n}$$ when $\zeta = 0.5$ or $$t_{r1} \approx \frac{2.16\zeta + 0.6}{\omega_n}$$

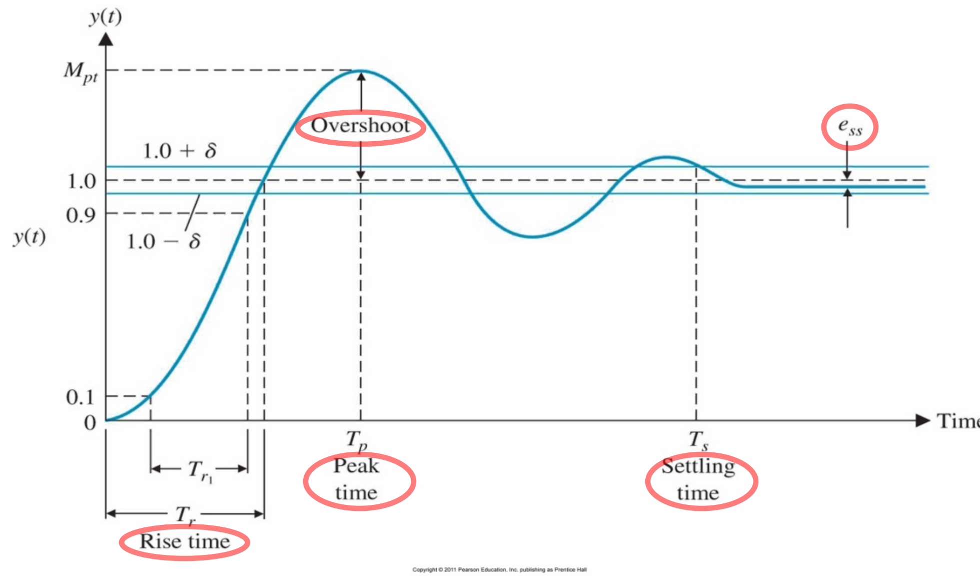
Peak Time

Peak Time (1st time for *dy/dt*=0)

$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$

$$\frac{d}{dt}y(t) =$$
$$-\sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$$
$$+e^{-\sigma t}\left(-\omega_d \sin\omega_d t + \sigma\cos\omega_d t\right)$$
$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d} + \omega_d\right)\sin\omega_d t = 0$$

$$\omega_d t_p = k\pi \qquad (k = 0,1,2,\ldots)$$



Maximum overshoot when k = 1

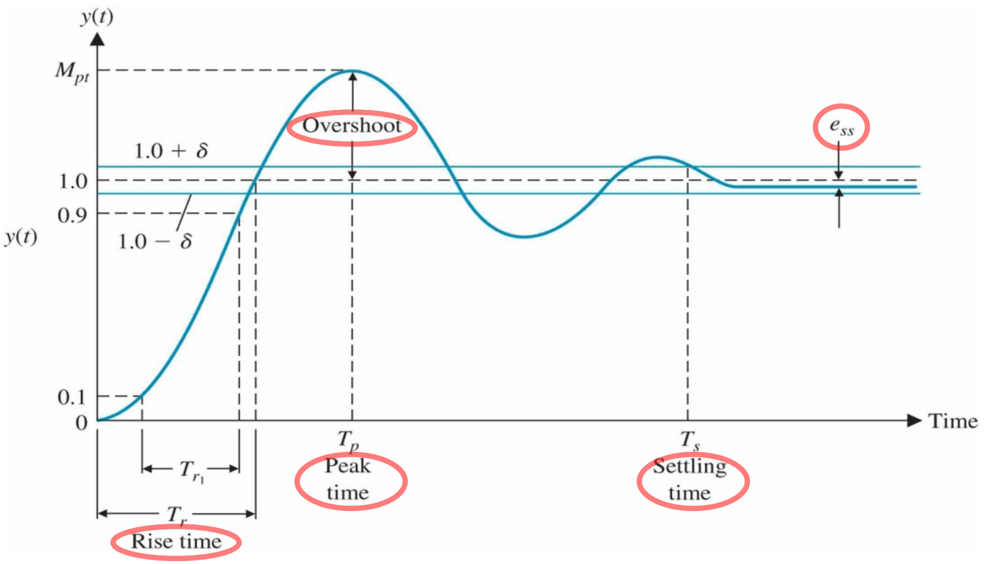
$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Overshoot

At t = t_p

$$y(t_p) = 1 - e^{-\sigma \frac{\pi}{\omega_d}} \left(\cos \omega_d \frac{\pi}{\omega_d} + \frac{\sigma}{\omega_d} \sin \omega_d \frac{\pi}{\omega_d} \right)$$

$$M_{pt} = 1 + e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

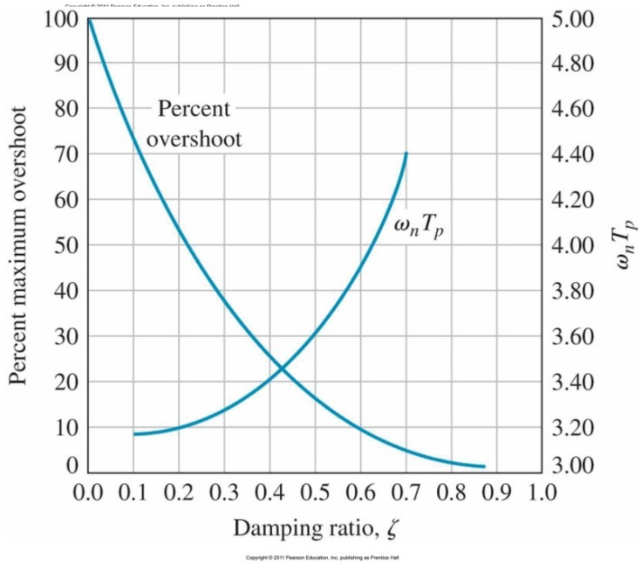


Percent Overshoot

$$P.O. = 100\left(\frac{M_{pt} - F.V.}{F.V.}\right) = 100\left(\frac{1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}} - 1}{1}\right)$$

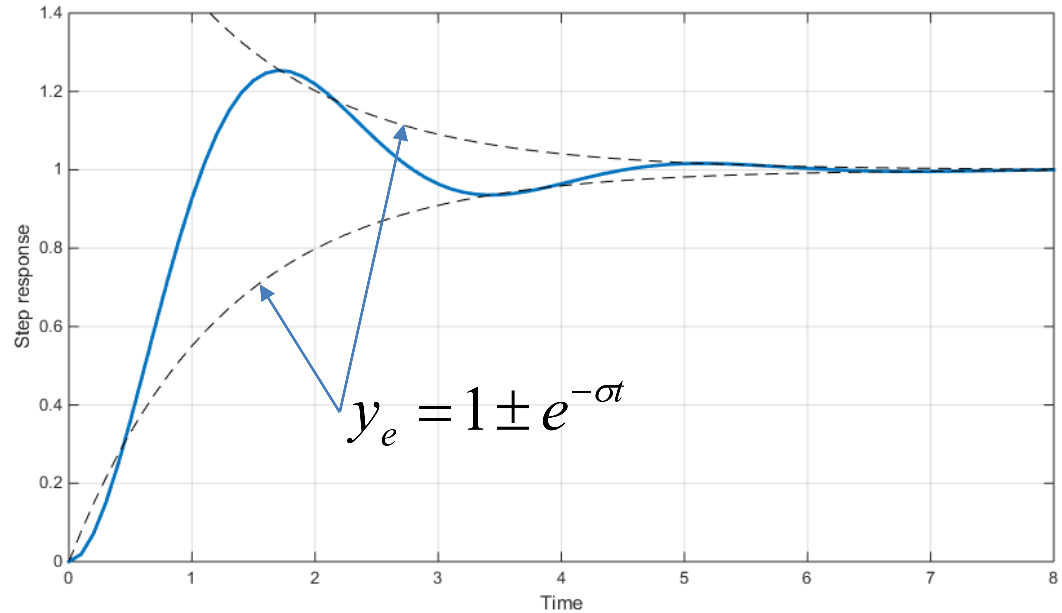
$$= 100 e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$= 100 e^{-\pi \tan\theta}$$



Settling Time

T_s : the time required for the system to settle within a certain percentage of the input amplitude



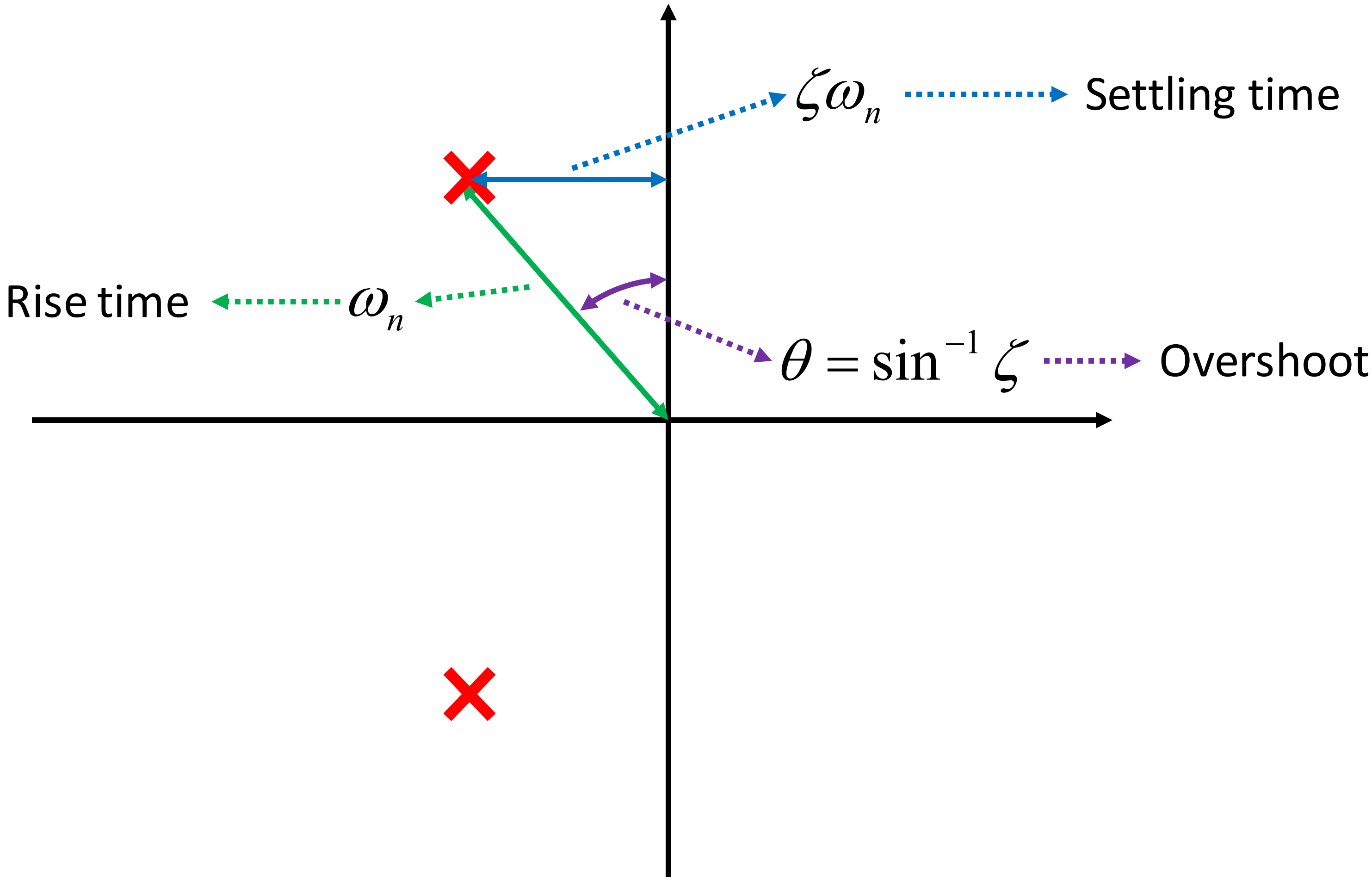
For the 2nd order system, 2% settling time is computed using

$$|y-1| < 0.02 \Rightarrow |y_e - 1| < 0.02 \Rightarrow e^{-\zeta\omega_n t} < 0.02 \Rightarrow t > \frac{-\ln(0.02)}{\zeta\omega_n} = \frac{4}{\zeta\omega_n}$$

$$-\ln(0.01) \approx 4.6$$
$$-\ln(0.02) \approx 4$$

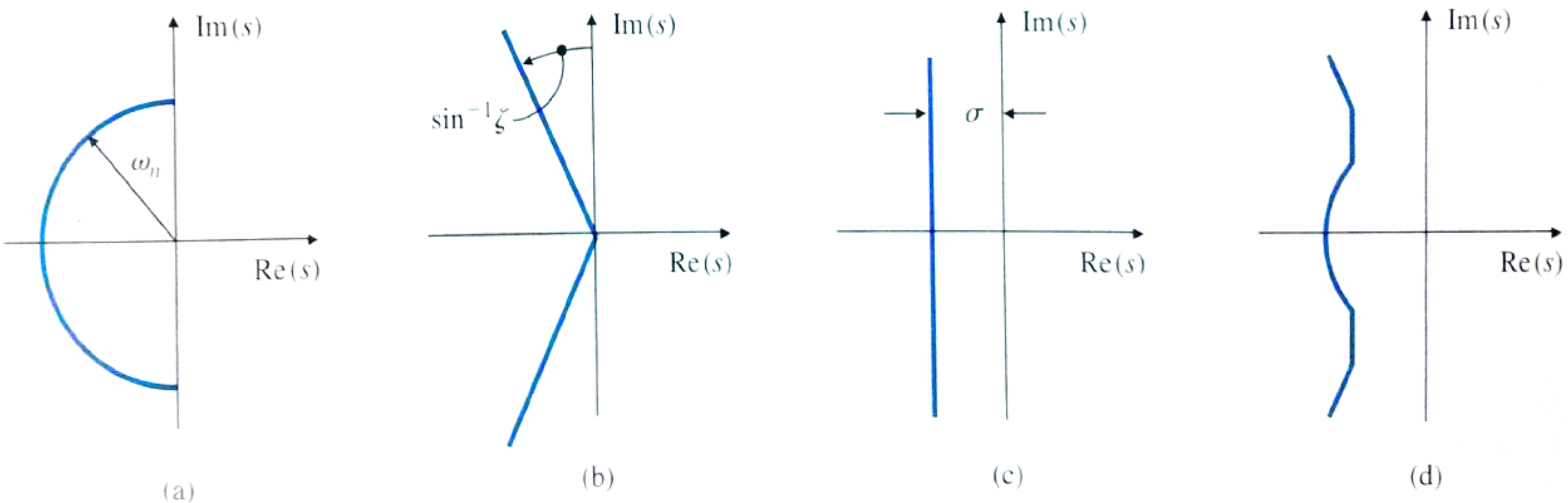
$$t_s = \begin{cases} \dfrac{4}{\zeta\omega_n} & (2\%) \\ \dfrac{4.6}{\zeta\omega_n} & (1\%) \end{cases}$$

Graphical Interpretation

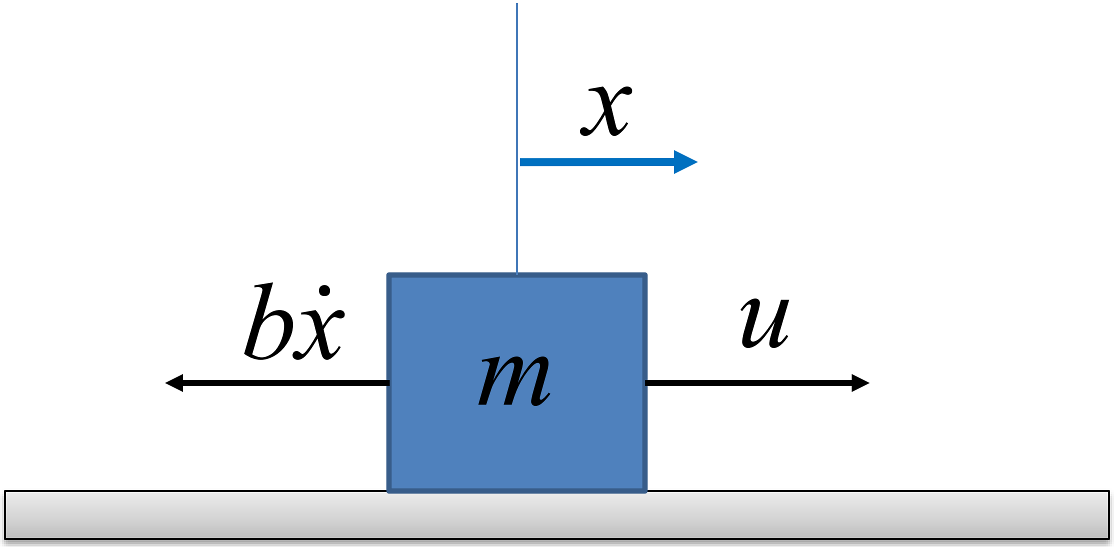


Summary and Graphical Interpretation

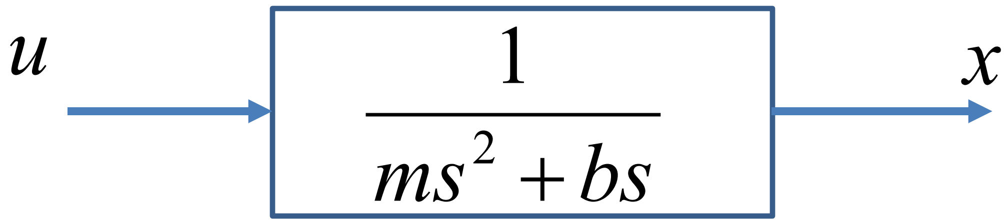
$$t_{r1} \approx \frac{1.8}{\omega_n} \qquad P.O. = 100e^{-\pi \tan\theta} \qquad t_s = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$$



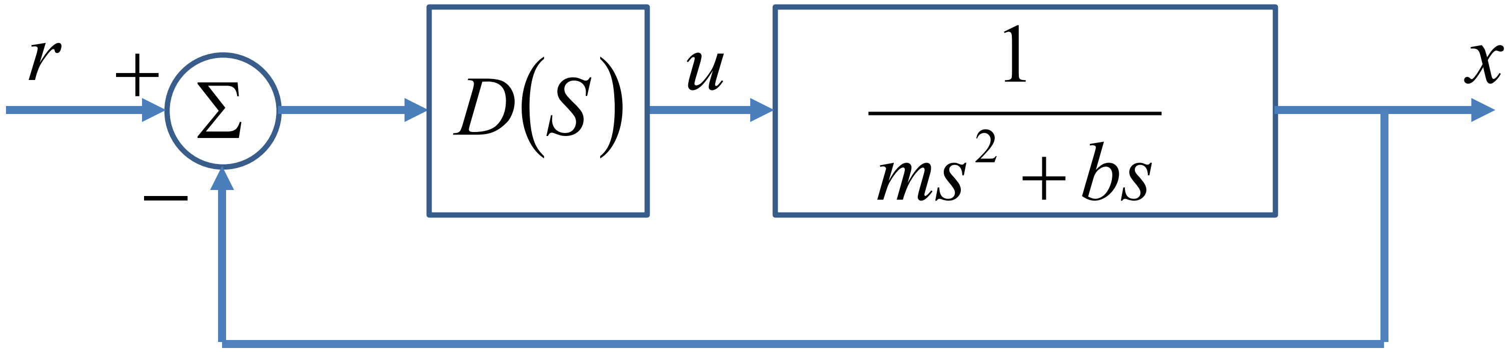
Time Domain Example 1 – Mass with Friction



$$m\ddot{x} + b\dot{x} = u$$



Setup Feedback Control Loop



- Measure the position
- Compare the measurement from what you want
- Generate the force based on the error

Time Domain Example 1

Parameter Selection

Select K and p that will result in
- P.O. < 4.3%
- 2% T_s < 4 sec

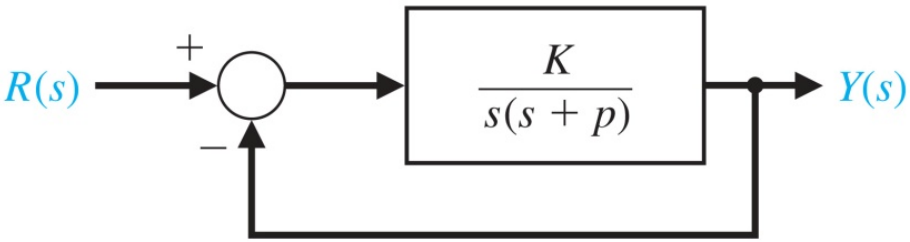


Figure: 05-14

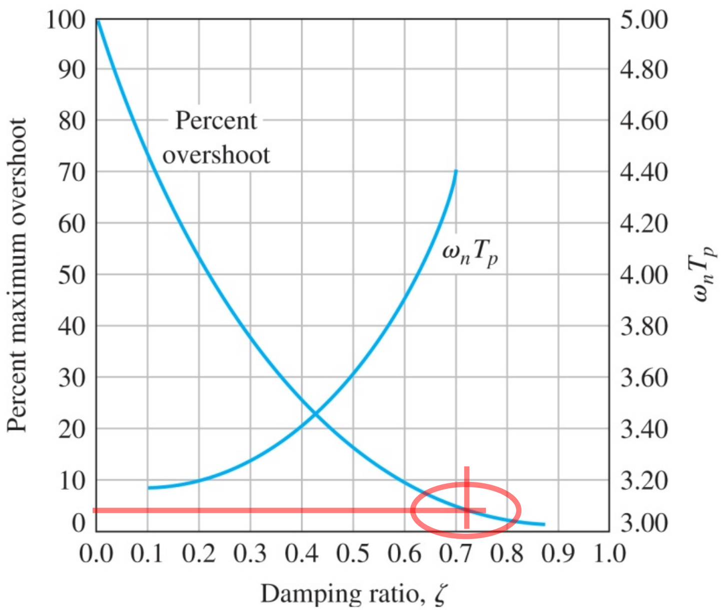


Figure: 05-08

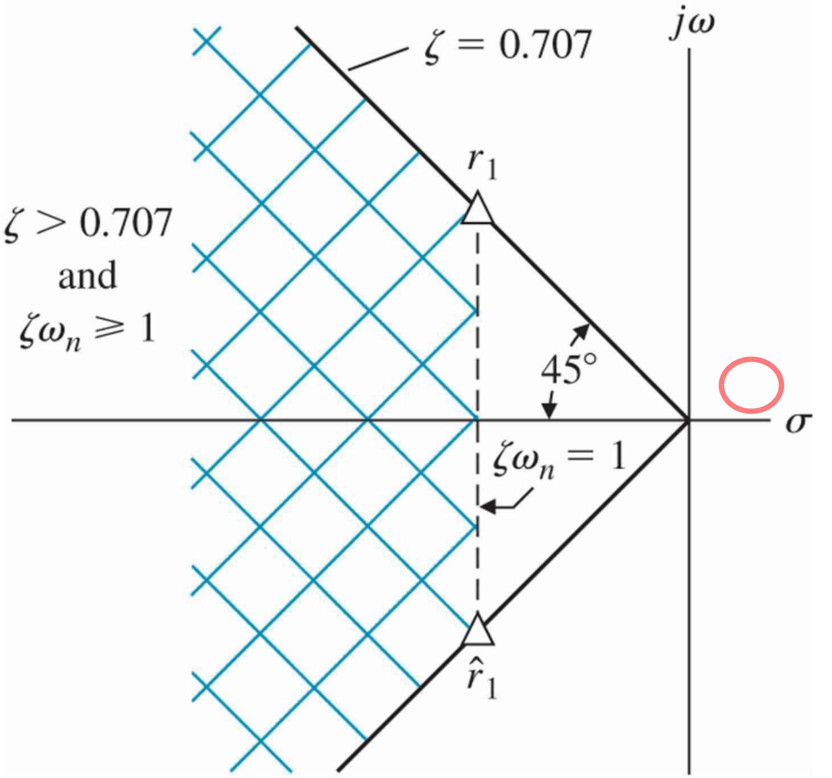
$$T_s = \frac{4}{\zeta\omega_n} \leq 4 \ \text{sec} \Rightarrow \zeta\omega_n \geq 1$$

$$P.O. = 100e^{-\pi \tan\theta} \leq 4.3 \Rightarrow \theta \geq 45^\circ$$

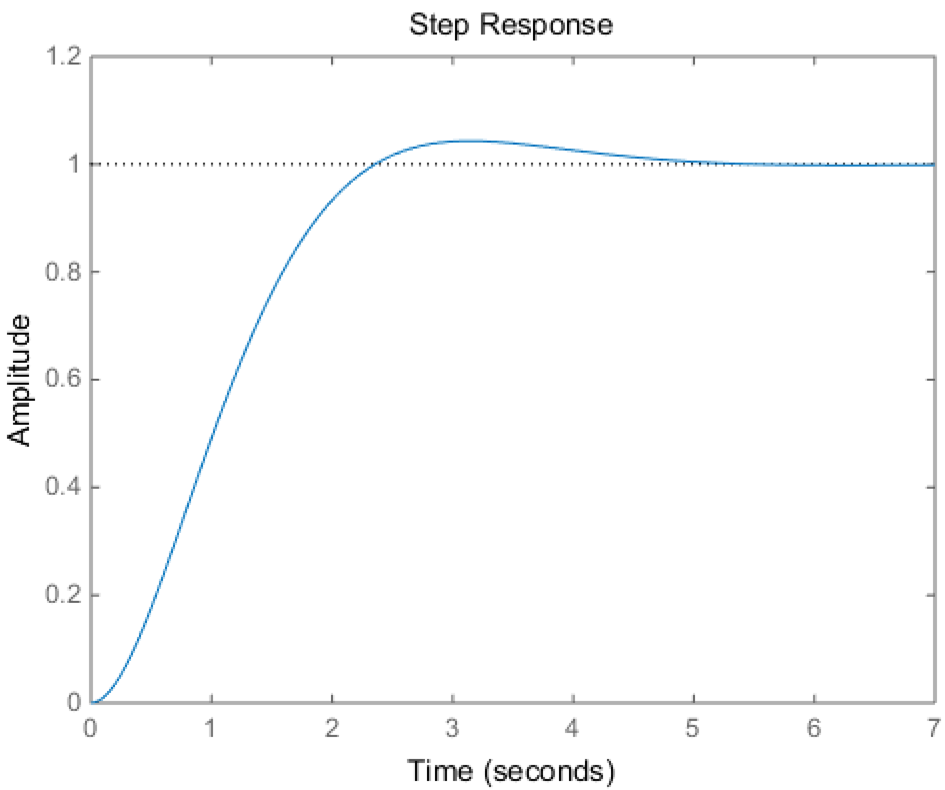
$$\Rightarrow \zeta \geq 0.707$$

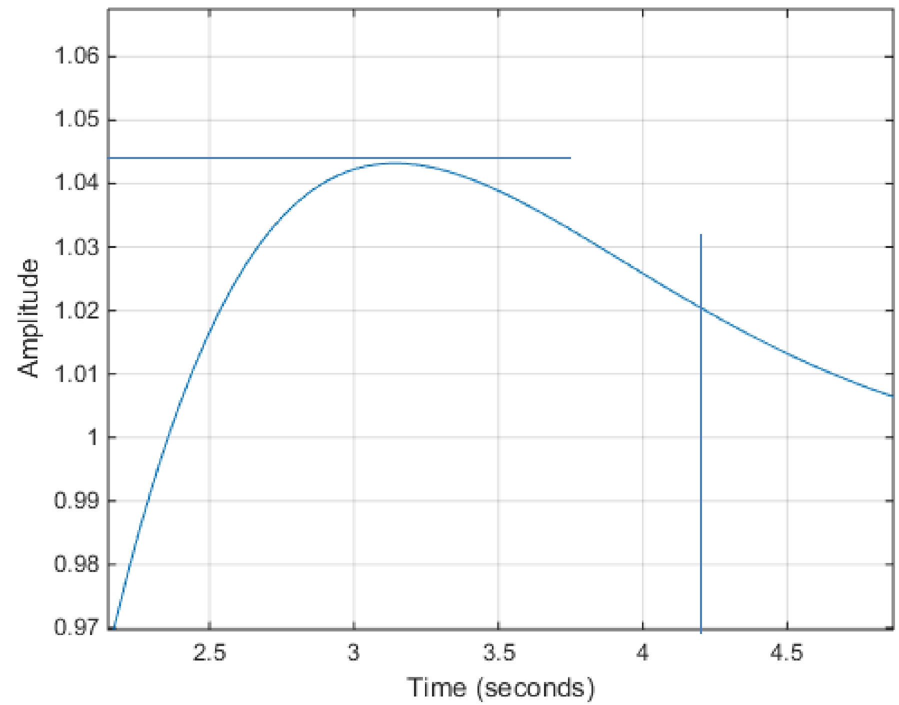
Time Domain Example 1

$$K = \omega_n^2 = 2$$
$$p = 2\zeta\omega_n = 2$$

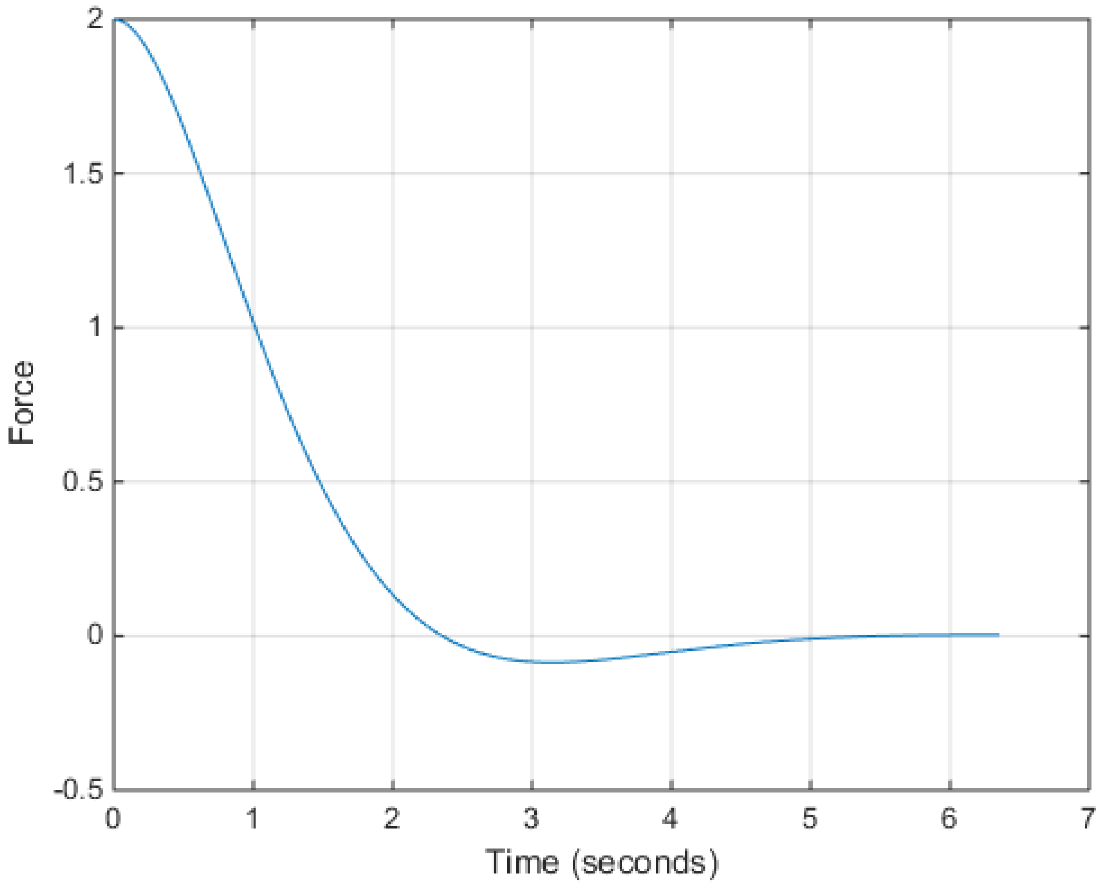


Actual Response





How the Control Force Changes



Effect of Additional Zero

$$H(s) = \frac{\omega_n{}^2\left(\dfrac{s}{\alpha(\zeta\omega_n)} + 1\right)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} = \frac{(s/\alpha\zeta\omega_n + 1)}{(s/\omega_n)^2 + 2\zeta(s/\omega_n) + 1}$$

$$H(s) = \frac{\dfrac{s}{\alpha\zeta} + 1}{s^2 + 2\zeta s + 1} = \frac{1}{s^2 + 2\zeta s + 1} + \left(\frac{1}{\alpha\zeta}\right)\frac{s}{s^2 + 2\zeta s + 1}$$

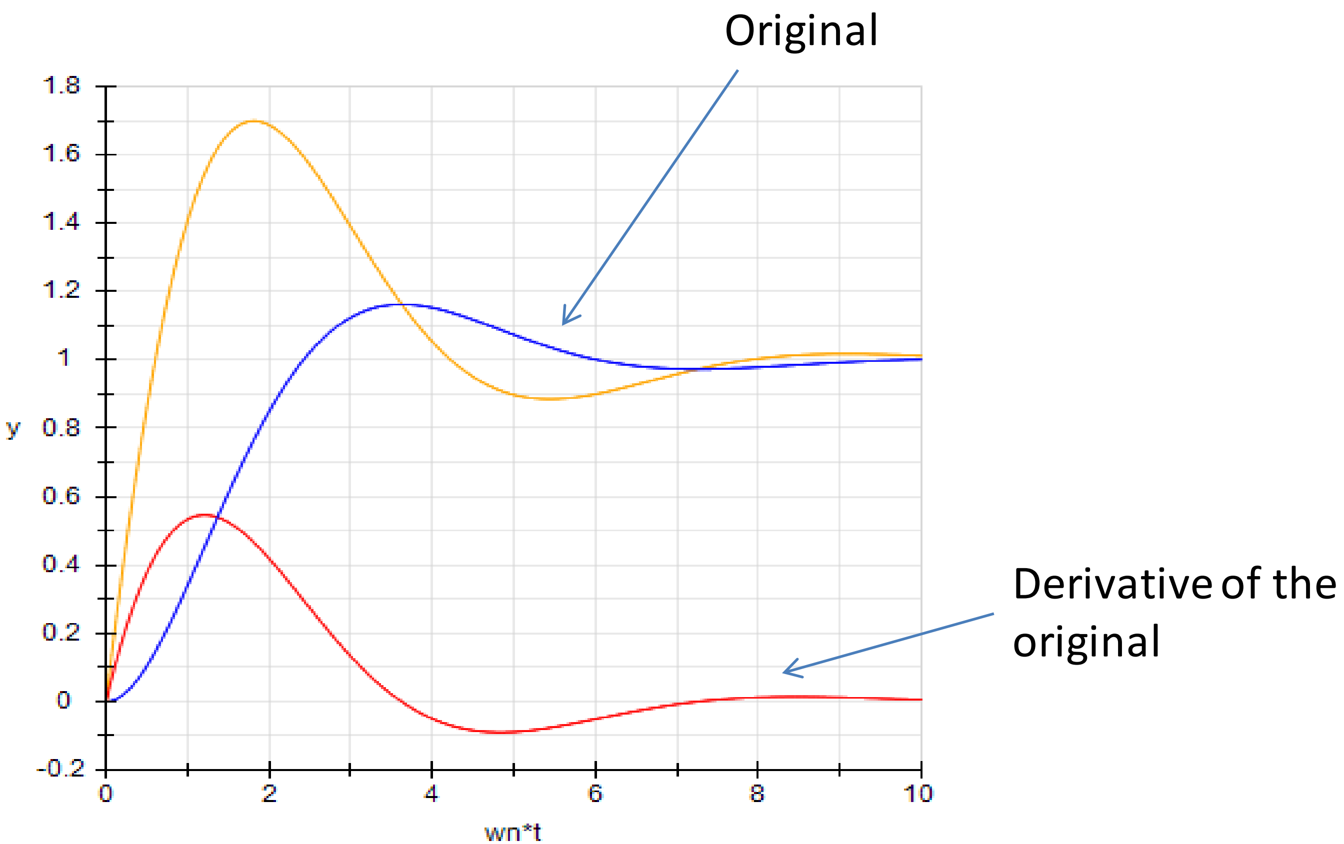
Original

Derivative of the original

Effect of Additional Zero

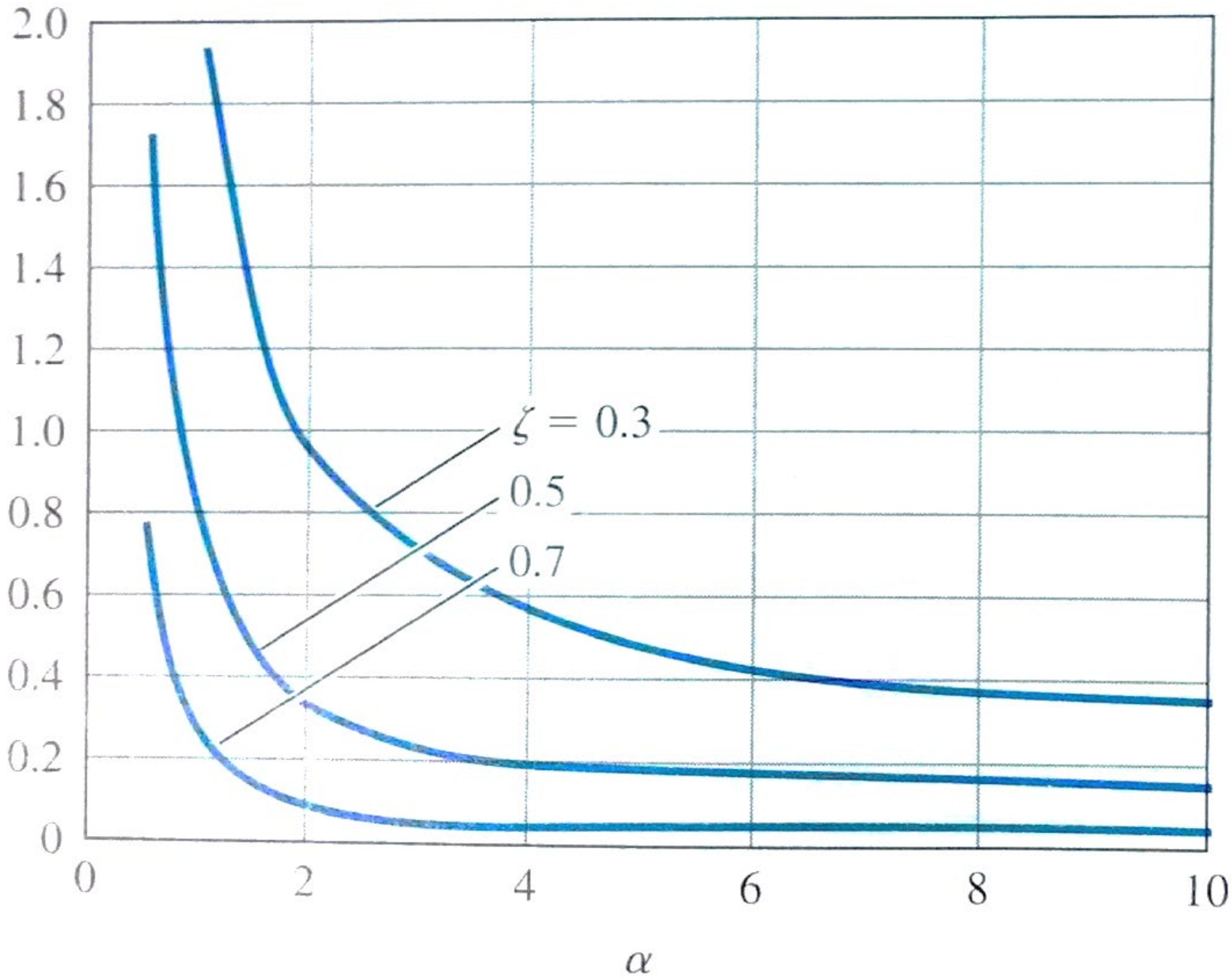
$$H(s) = \frac{1}{s^2 + 2\zeta s + 1} + \left(\frac{1}{\alpha\zeta}\right)\frac{s}{s^2 + 2\zeta s + 1}$$

Increase overshoot



Effect of Additional Zero

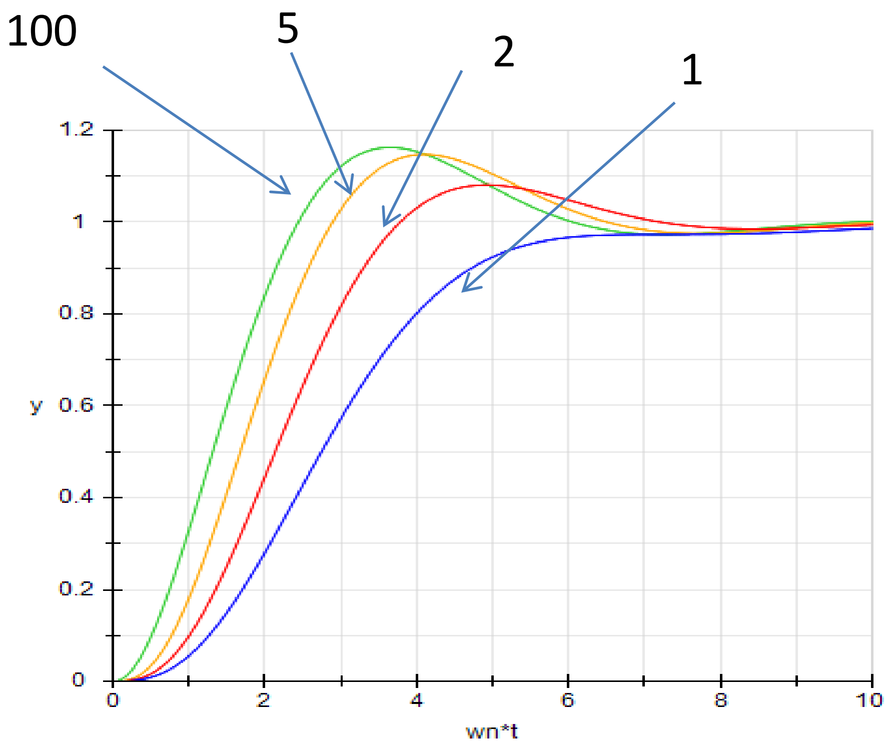
Overshoot



Effect of Additional Pole

$$H(s)=\frac{1}{\left(\frac{s}{\alpha(\zeta\omega_n)}+1\right)\left((s/\omega_n)^2+2\zeta(s/\omega_n)+1\right)}$$

Increase rise time
Decrease overshoot



Effect of Additional Pole

$$T(s) = \frac{1}{(s^2 + 2\zeta s + 1)(\gamma s + 1)}$$

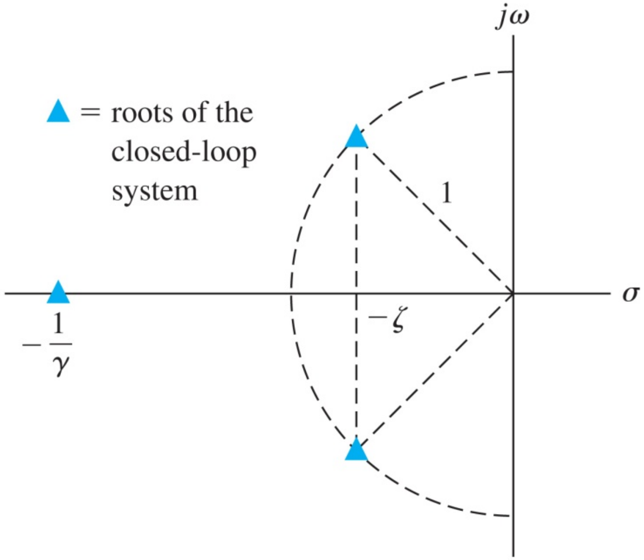
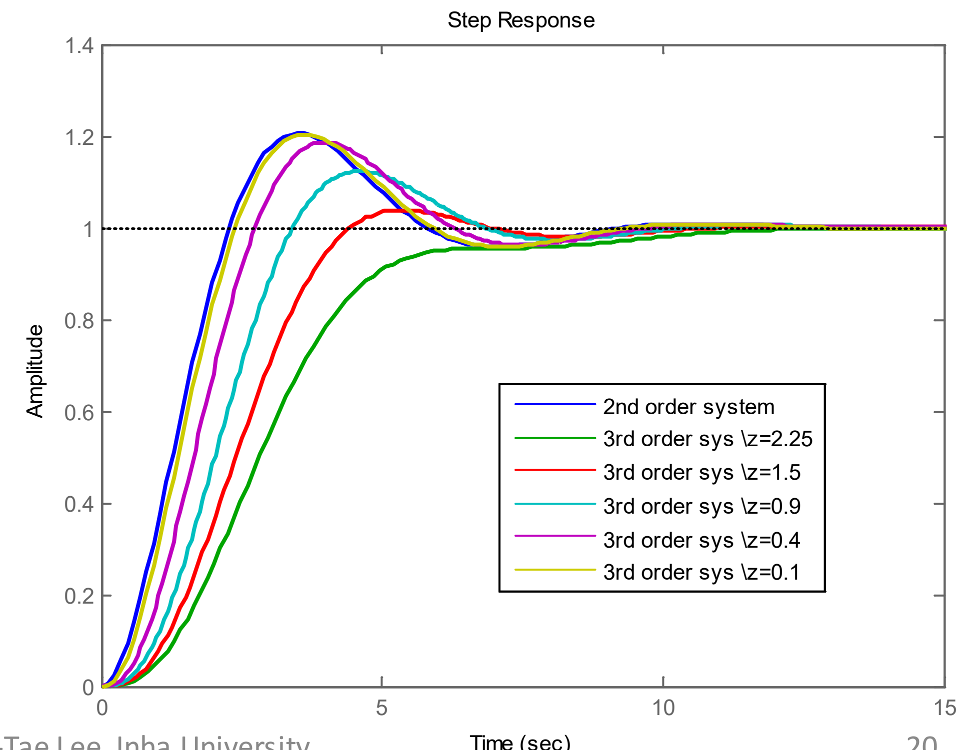


Figure: 05-12

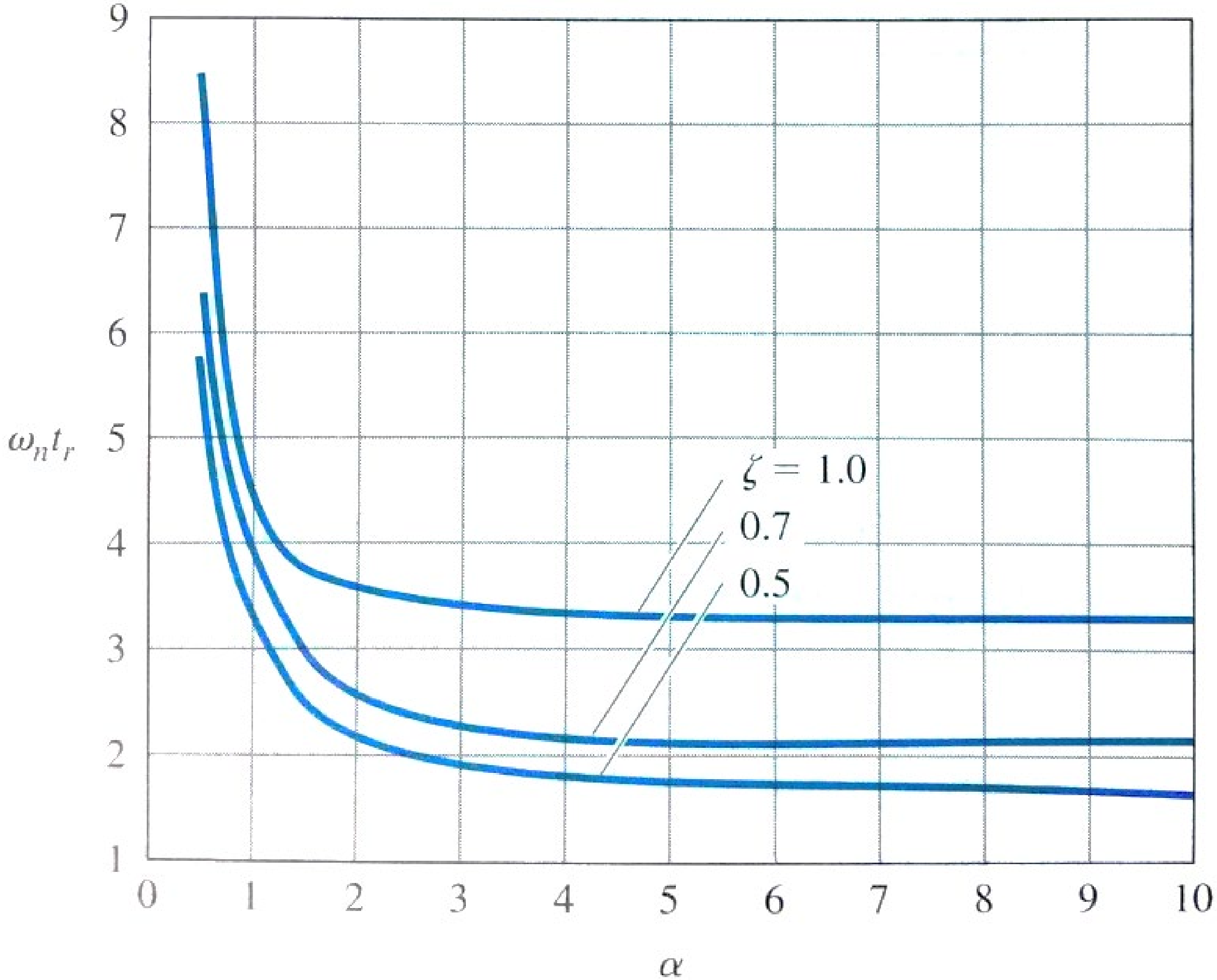
The performance (P.O, Ts,…) are adequately represented by the 2nd order system curve when $|1/\gamma| \geq 10|\zeta\omega_n|$

dominant roots

When z = 0.45, $s = -0.45 \pm j0.893$



Effect of Additional Pole

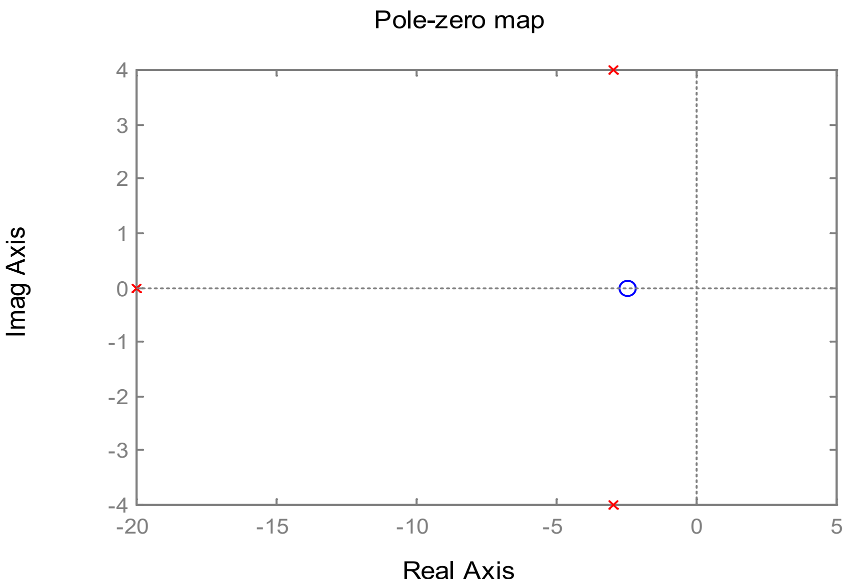


Example

$$T(s) = \frac{200(s+2.5)}{(s^2+6s+25)(s+20)}$$

Poles: $s = -3 \pm 4j, -20$ $\quad \frac{20}{3} \approx 6.6 > 5$ Can be ignored

Zeroes: $s = -2.5$ $\quad \frac{2.5}{3} \approx 0.83 << 5$ Cannot be ignored



$$T(s) \approx \frac{200(s+2.5)}{(s^2+6s+25)(20)} = \frac{10(s+2.5)}{(s^2+6s+25)}$$

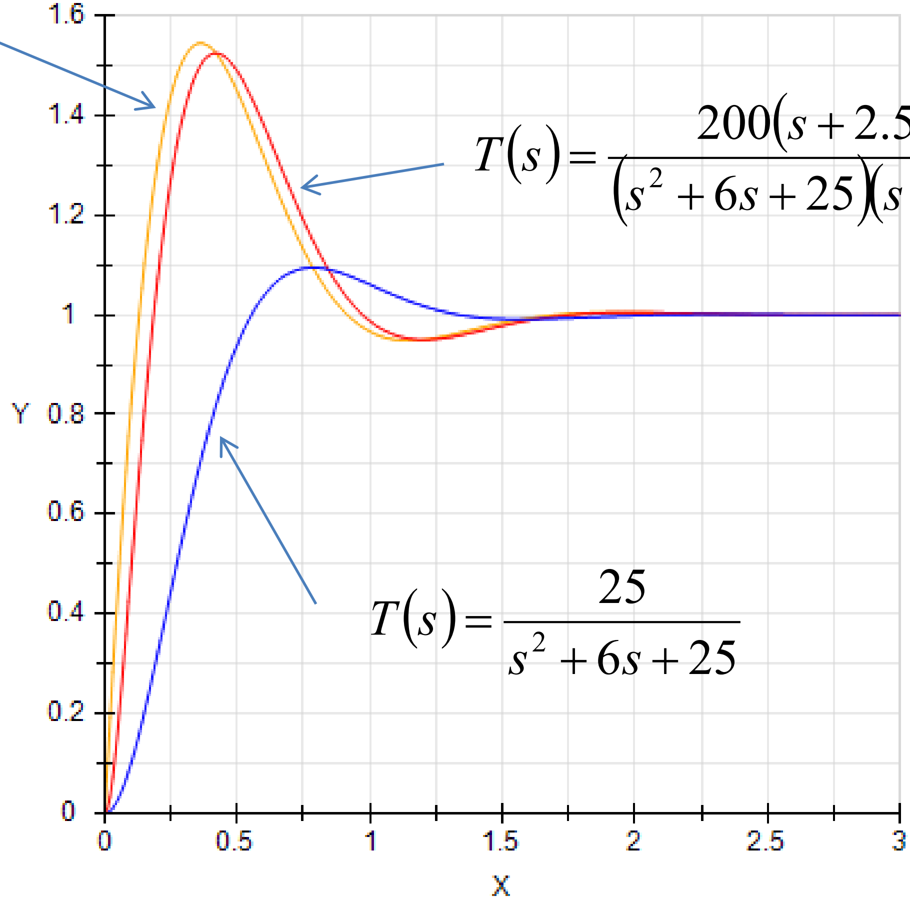
Time Response

$$T(s) = \frac{10(s+2.5)}{(s^2+6s+25)}$$

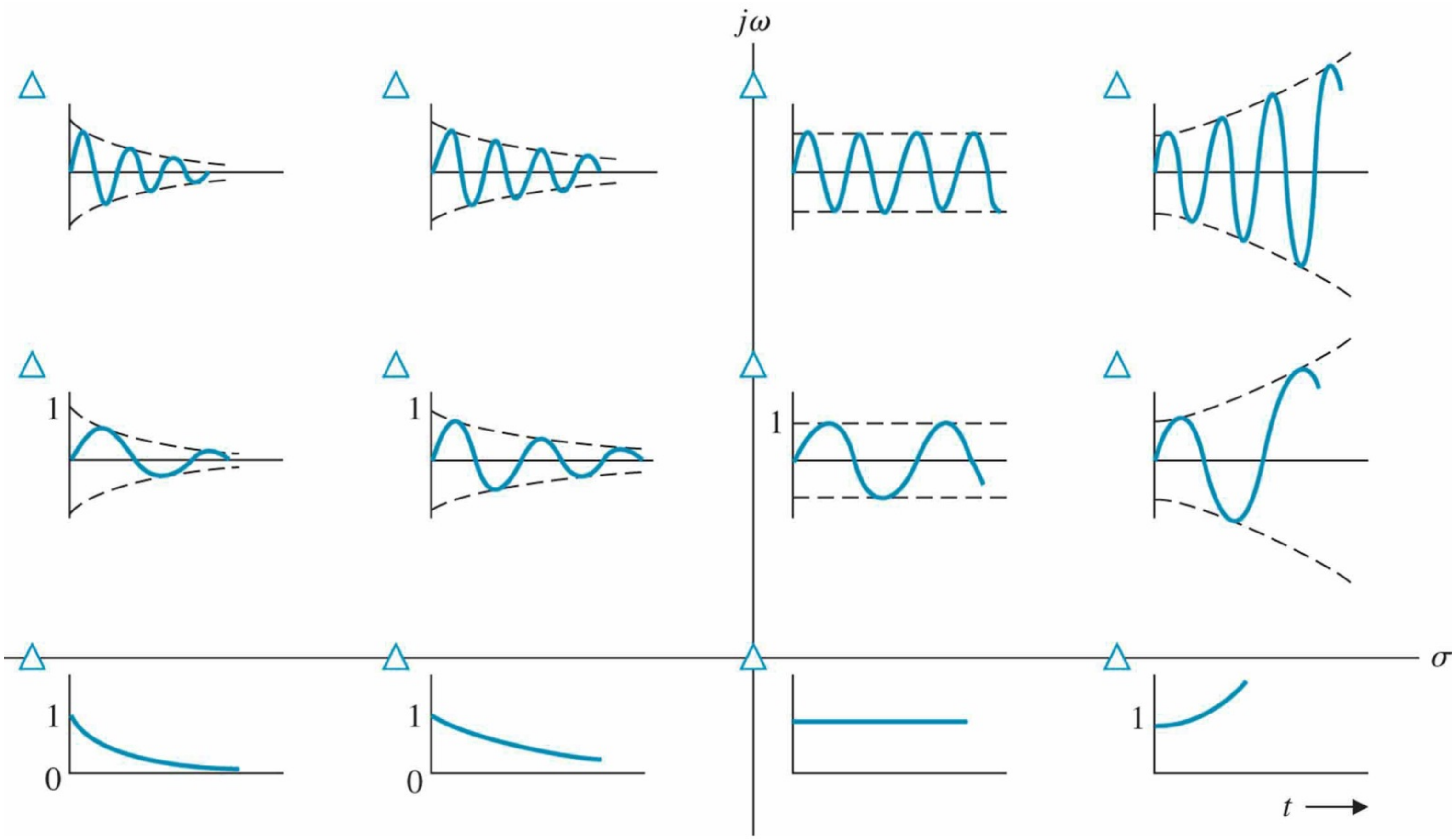
$$T(s) = \frac{200(s+2.5)}{(s^2+6s+25)(s+20)}$$

$$T(s) = \frac{25}{s^2+6s+25}$$

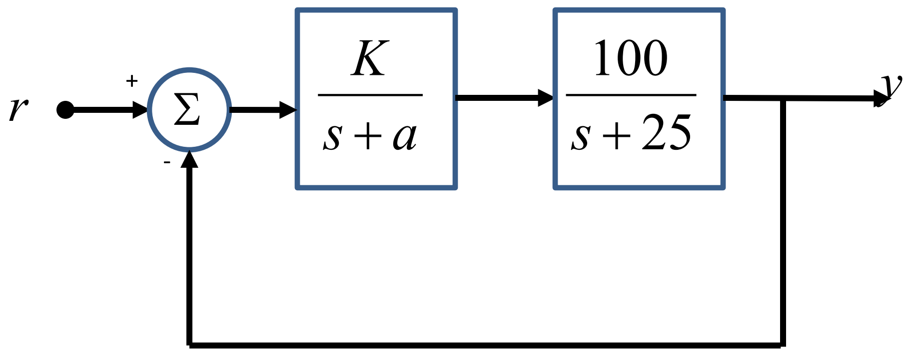
$$\omega_n = 5, \zeta = 0.6$$



The s-Plane Root Location and the Transient Response



Example



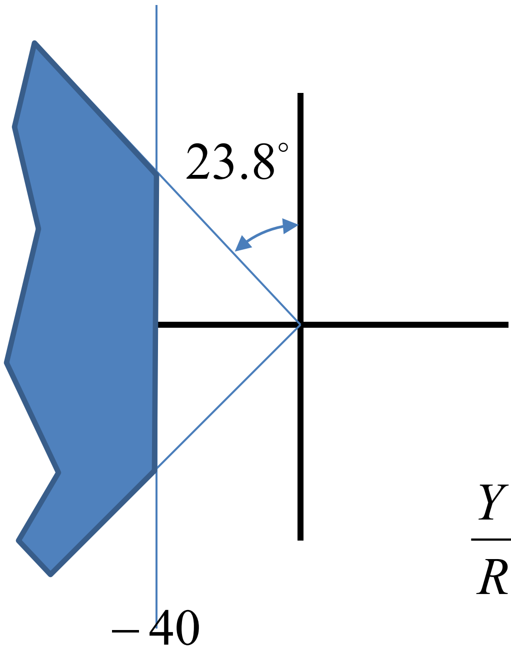
Overshoot < 25%

2% settling time < 0.1

Overshoot: $e^{-\pi \tan\theta} < 0.25$ ⇨ $\theta > 23.8^\circ$ ⇨ $\zeta > 0.4$

Settling time: $\frac{4}{\zeta\omega_n} < 0.1$ ⇨ $\zeta\omega_n > 40$

Example



Pick $\zeta = 0.4$ $\omega_n = 100$

$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s+a}\frac{100}{s+25}}{1+\frac{K}{s+a}\frac{100}{s+25}} = \frac{100K}{s^2+(25+a)s+25a+100K}$$

$$2\zeta\omega_n = 25 + a$$

$$a = 2\zeta\omega_n - 25 = 80 - 25 = 55$$

$$\omega_n^{\ 2} = 25a + 100K$$

$$K = \frac{\omega_n^{\ 2} - 25a}{100} = \frac{100^2 - 25\times 55}{100} = 86$$

Exercise #1

- Sketch the area on the complex plane to satisfy the given specifications.
  - Rise time > 0.9 seconds
  - Percentage overshoot < 16.3%
  - Settling time < 2.67

Exercise #2

- Assume your poles are s=-1.2+1.8j and s=-1.2-1.8j
- Among the conditions in Exercise #1, what are the ones that are satisfied?
  - Rise time > 0.9 seconds
  - Percentage overshoot < 16.3%
  - Settling time < 2.67

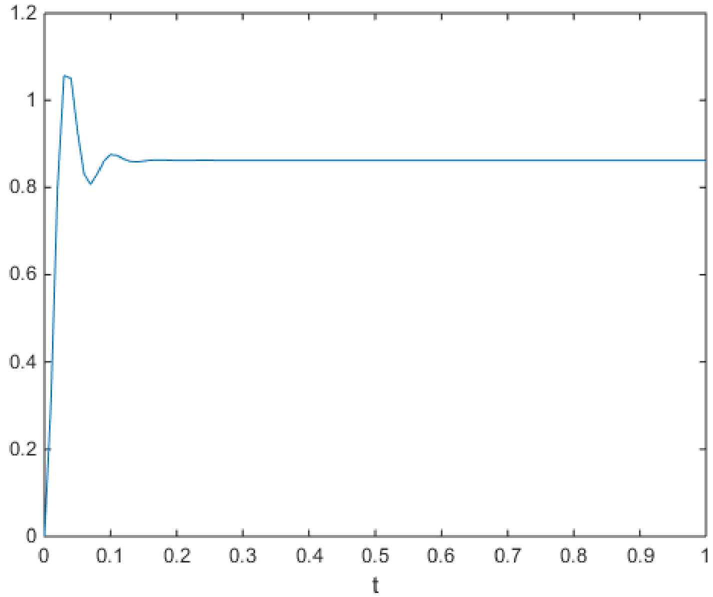
Matlab Example

Check!

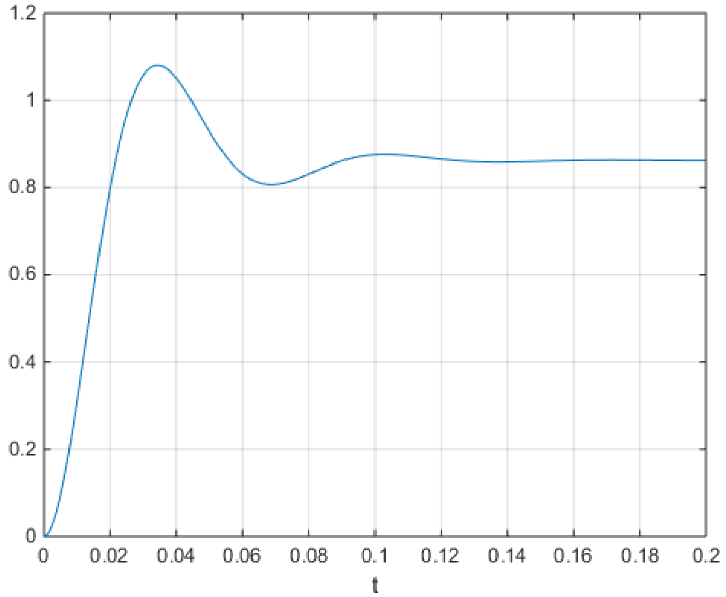
```
a=55;
K=86;

num = 100*K;
den = [1 (25+a) (25*a+100*K)];

t = 0:0.01:1;

y = step(num, den, t);

figure(1)
plot(t, y)
xlabel('t');

t = 0:0.001:0.2;
y = step(num, den, t);

figure(2)
plot(t, y); xlabel('t'); grid on;
```

```
ys = 100*K/(25*a+100*K);
ym = max(y);
Mp = (ym-ys)/ys*100;

ysu = ones(size(t))*ys*1.02;
ysl = ones(size(t))*ys*0.98;

figure(3)
plot(t, y, t, ysu, t, ysl)
axis([0.075 0.125 0.8 0.9])
xlabel('t'); grid on;

Mp
stepinfo(tf(num, den))
```

Plots

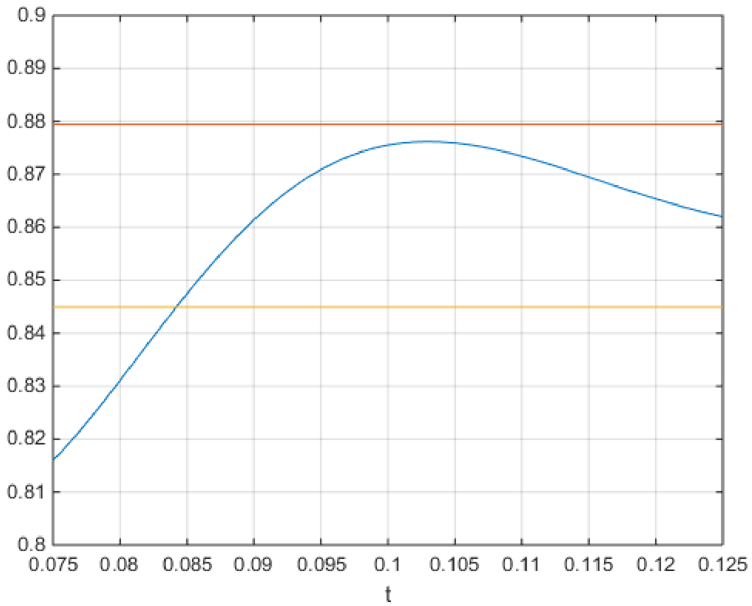


figure(1)



figure(2)

Plots



figure(3)

```
>> Mp

Mp =

   25.3171

>> stepinfo(tf(num, den))

ans =

        RiseTime: 0.0147
    SettlingTime: 0.0842
     SettlingMin: 0.7802
     SettlingMax: 1.0805
       Overshoot: 25.3253
      Undershoot: 0
            Peak: 1.0805
        PeakTime: 0.0345
```

output