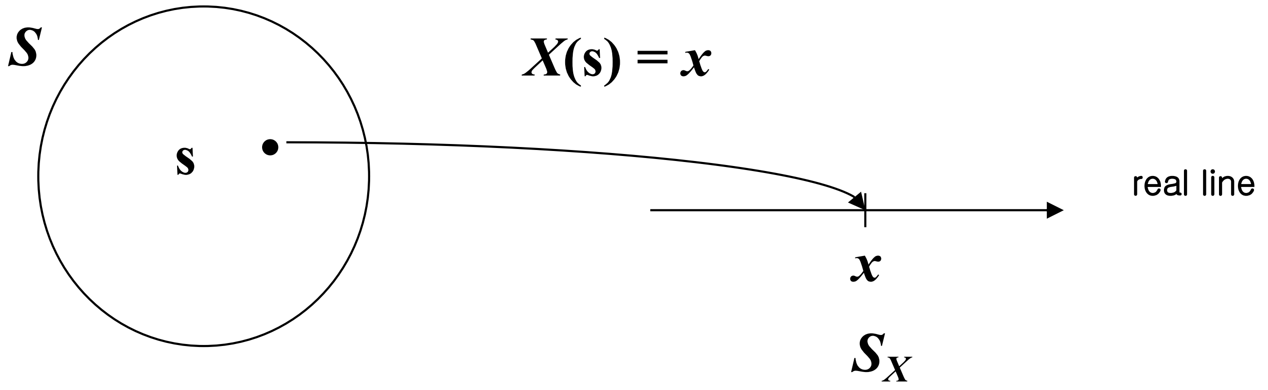
Random Variables

Random Variable

❑ The notion of a Random Variable



– **Random variable *X* : a function that assigns a real number *X(s)* to each outcome *s* in the sample space of a random experiment**

– Specification of a measurement on the outcome of a random experiment

⇨ Define a function on the sample space, i.e. a random variable

– S = the domain of the random variable

S_X = the range of the random variable → a subset of the set of all real numbers

Random Variable

❑ Classification of Random Variables

- Discrete Random Variable
  - R.V. from countable or countably infinite set, $S_X = \{x_1, x_2, \ldots\}$

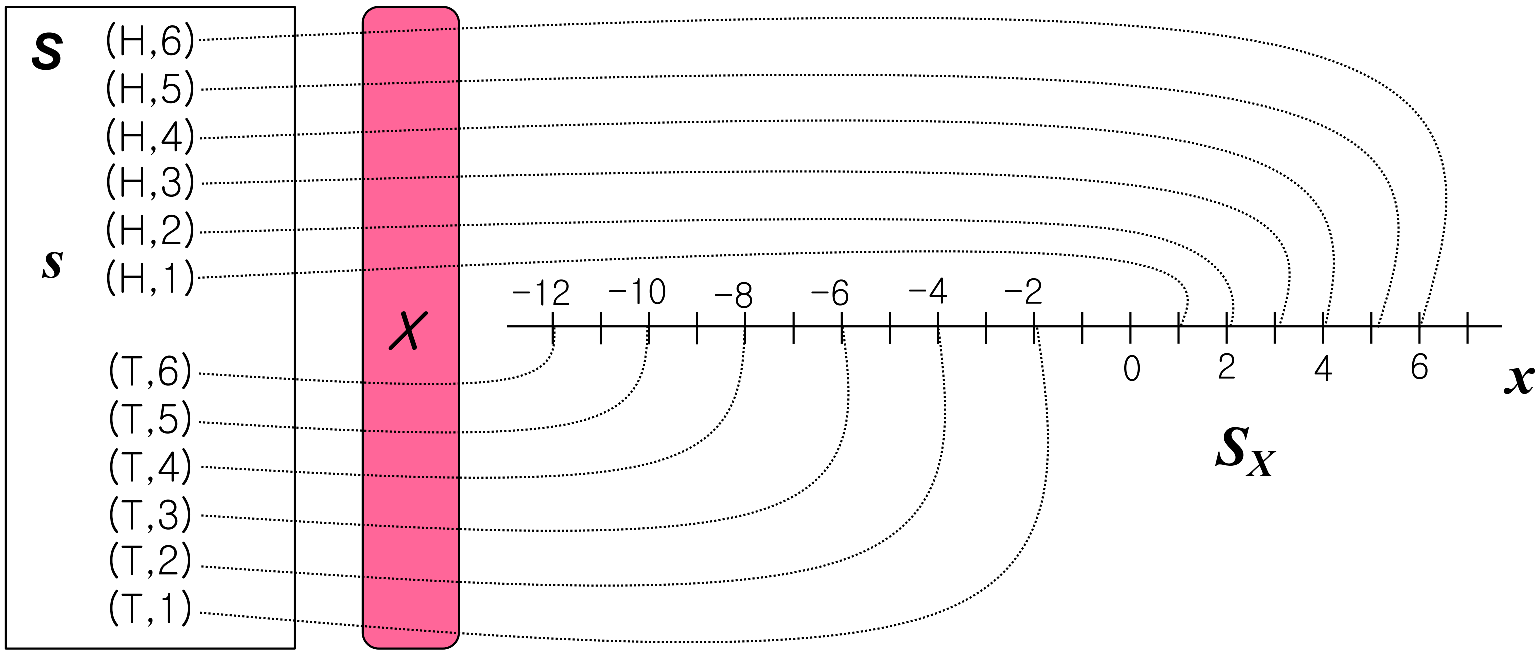
- Continuous Random Variable
  - R.V. whose *cdf* is continuous everywhere and sufficiently smooth.

- Mixed Type
  - *Cdf* has jumps on a countable set of points $x_0, x_1, x_2, \ldots$
  - *Cdf* increases continuously over at least one interval of values of x

cdf: cumulative distribution function

Random Variable

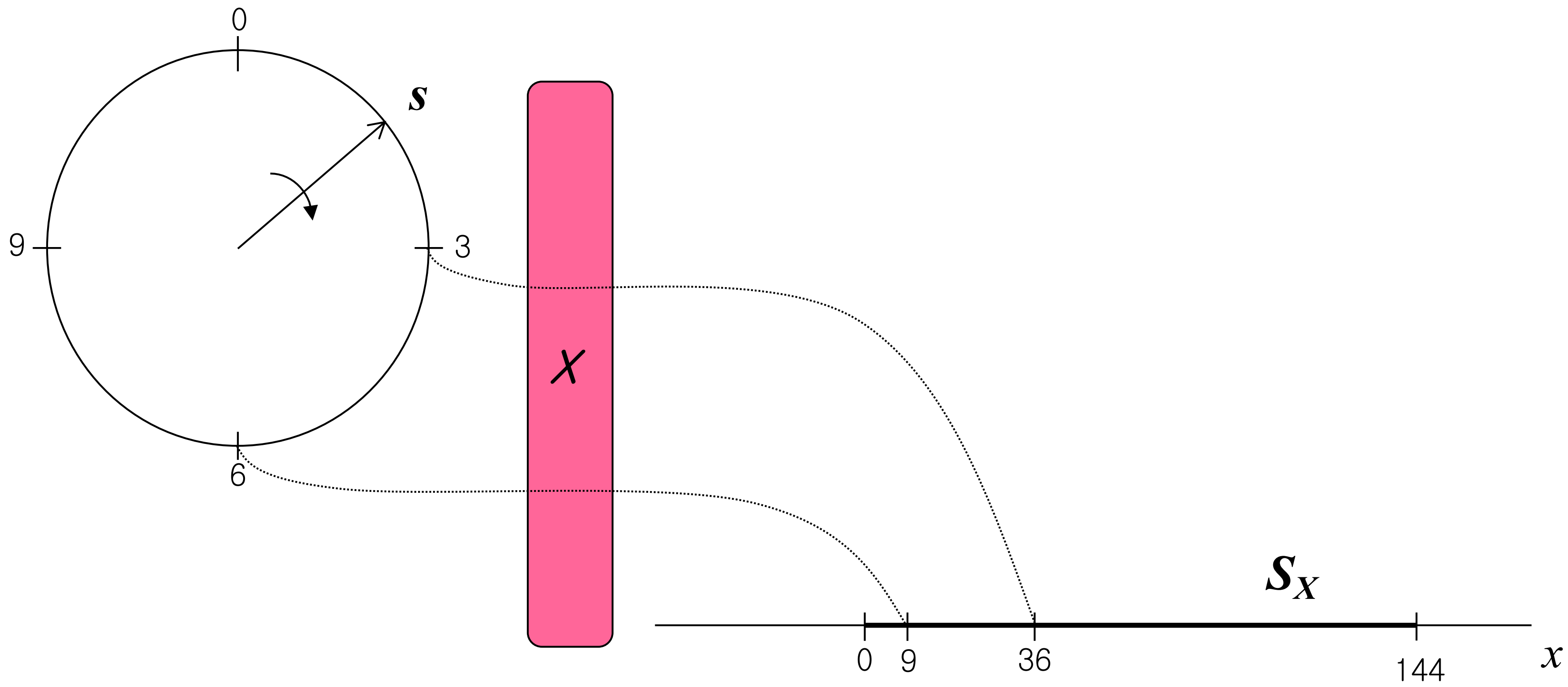
- **Ex) 2.1-1 Discrete experiments**
  - 한 개의 주사위와 한 개의 동전을 던지는 실험
  - 랜덤변수 X 대응(mapping) 규칙
    - 동전 앞면(H) 일 때, 주사위 눈의 수를 양수 값에 대응
    - 동전 뒷면(T) 일 때, 주사위 눈의 2배의 음수 값에 대응



Random Variable

- **Ex) 2.1-2 Continuous experiment**
  - 원판에서 바늘이 회전하는 실험
  - 랜덤변수 X 정의: 대응(mapping) 규칙

$$X(s) = s^2$$



Probability Mass Function (PMF)

❑ Definition

$$p_X(x) = P[X = x] = P[\{x : X(s) = x\}] \text{ for } x \text{ a real number}$$

❑ Properties

- $p_X(x) \geq 0$ for all x
- $\sum_{x \in S_X} p_X(x) = \sum_{\forall k} p_X(x_k) = \sum_{\forall k} P[A_k] = 1$
- $P[X \text{ in } B] = \sum_{x \in B} p_X(x)$ where $B \subset S_X$

Cumulative Distribution Function (CDF)

❑ Definition

$$F_X(x) = P[X \leq x], \qquad x \in \mathbb{R}$$

❑ Properties

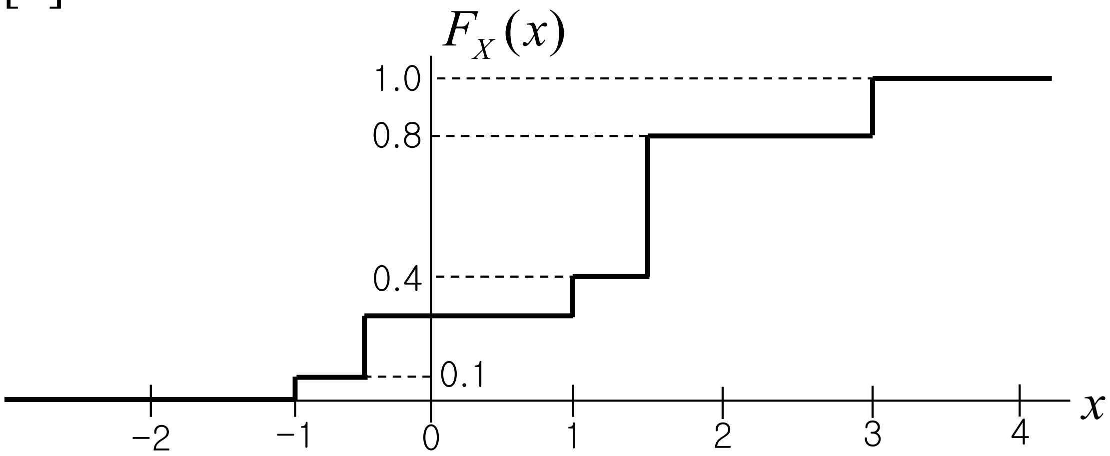
- $F_X(-\infty) = 0$
- $F_X(\infty) = 1$
- $0 \leq F_X(x) \leq 1$
- $F_X(x_1) \leq F_X(x_2) \quad \text{if } x_1 < x_2$: nondecreasing function
- $P[\{x_1 < X \leq x_2\}] = F_X(x_2) - F_X(x_1)$
- $F_X(x^+) = F_X(x)$

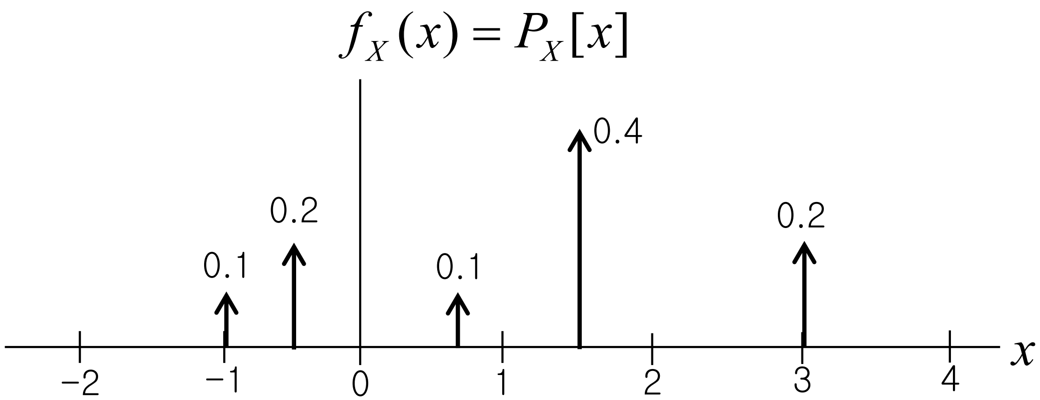
Cumulative Distribution Function (CDF)

❑ Ex) 2.2-1 CDF of a discrete event

- $X = \{-1, -0.5, 0.7, 1.5, 3\}$
- $P[x=-1]=0.1,\ P[x=-0.5]=0.2, P[x=0.7]=0.1, P[x=1.5]=0.4, P[x=3]=0.2$

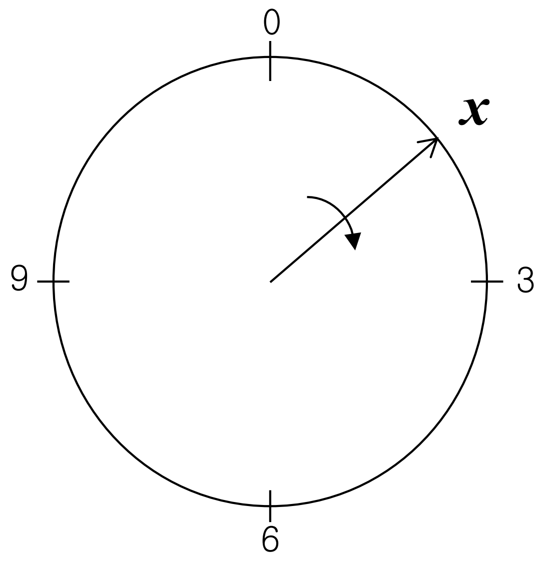
$\Rightarrow \sum_{\forall x \in X} P[x] = 1$

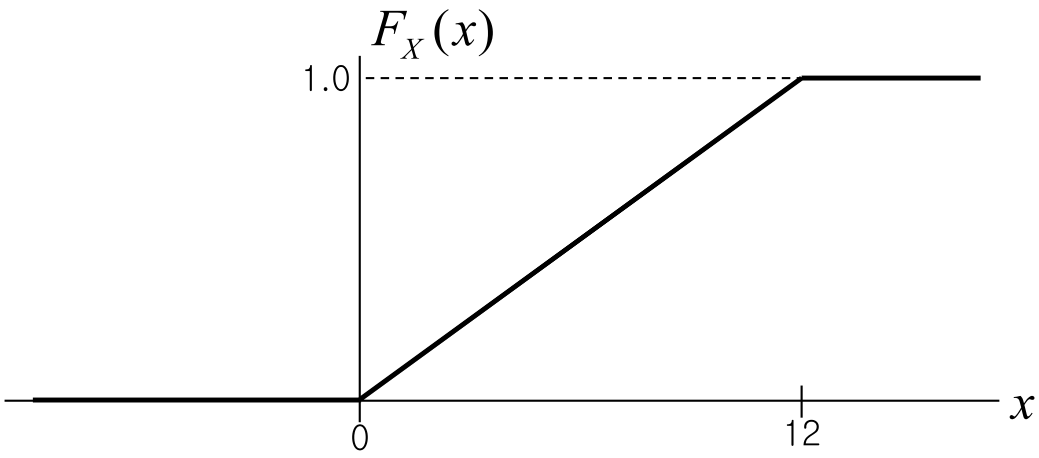


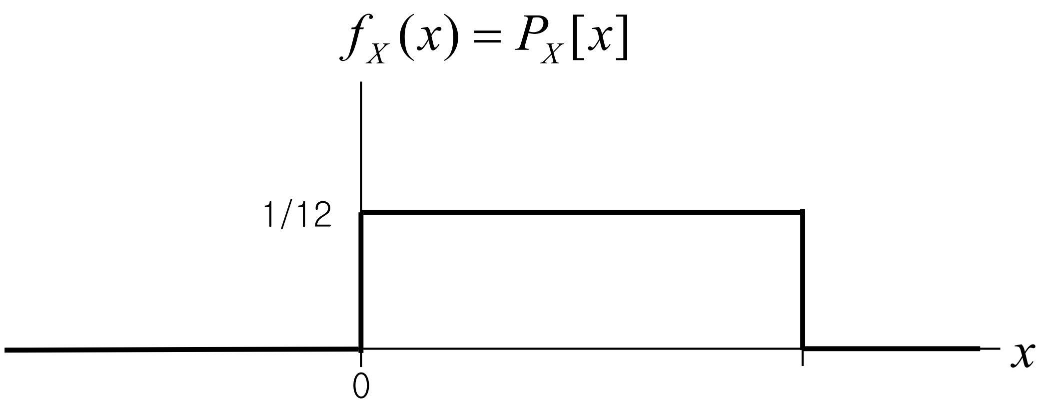


❑ Ex) 2.2-1 CDF of a continuous event

- 0-12가 적힌 원판에서 바늘이 회전하는 실험







Probability Density Function (PDF)

❑ Definition

$$f_X(x) = \frac{dF_X(x)}{dx}$$

❑ Properties

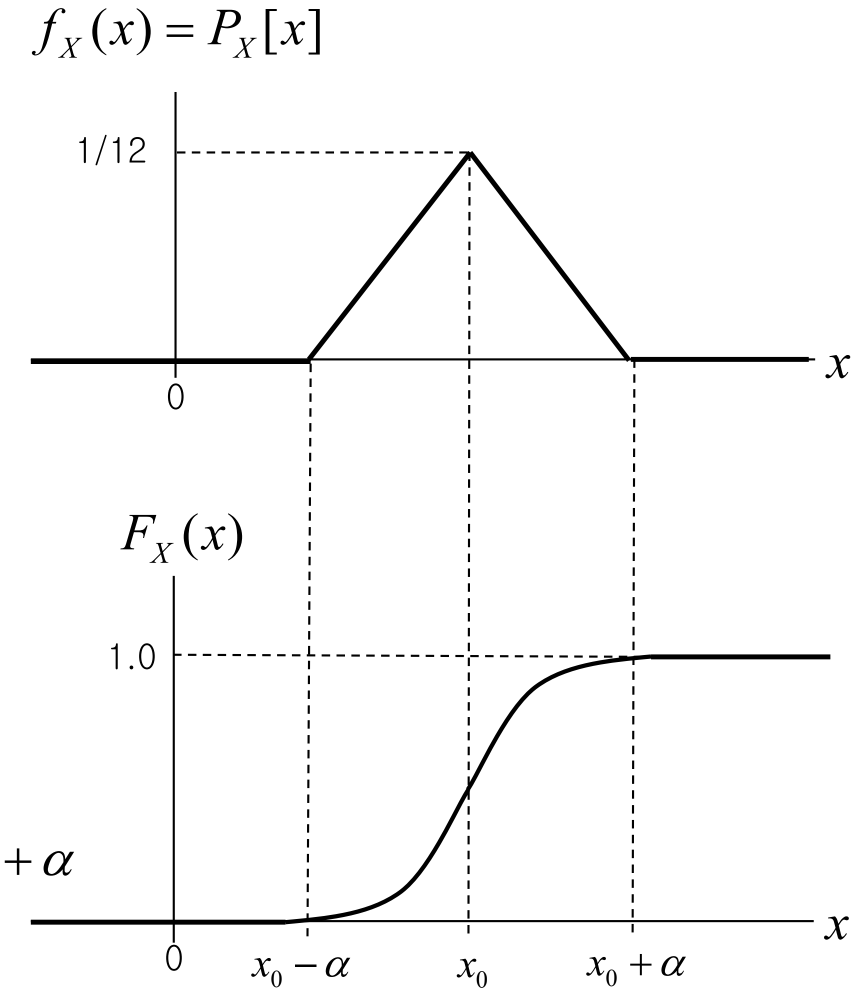
- $0 \le f_X(x)$
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $F_X(x) = \int_{-\infty}^{x} f_X(\xi)d\xi$
- $P[\{x_1 < X \le x_2\}] = \int_{x_1}^{x_2} f_X(\xi)d\xi$

Probability Density Function (PDF)

- **Ex) 2.3-1**

$$f_X(x) = \begin{cases} 0, \; x < x_0 - \alpha \quad \text{or} \quad x_0 + \alpha \le x \\ \dfrac{1}{\alpha^2}(x - x_0 + \alpha), \; x_0 - \alpha \le x < x_0 \\ \dfrac{1}{\alpha} - \dfrac{1}{\alpha^2}(x - x_0), \; x_0 \le x < x_0 + \alpha \end{cases}$$

$$F_X(x) = \begin{cases} 0, \; x < x_0 - \alpha \\ \dfrac{1}{2\alpha^2}(x - x_0 + \alpha)^2, \; x_0 - \alpha \le x < x_0 \\ \dfrac{1}{2} + \dfrac{1}{\alpha}(x - x_0) - \dfrac{1}{2\alpha^2}(x - x_0)^2, \; x_0 \le x < x_0 + \alpha \\ 1, \; x_0 + \alpha \le x \end{cases}$$



Gaussian Random Variable

❑ Definition

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

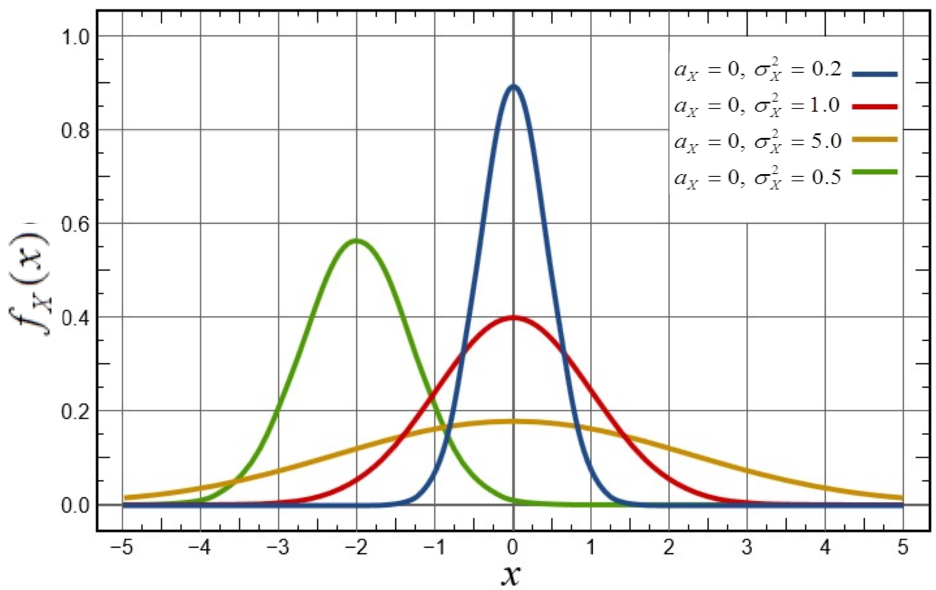
❑ Properties

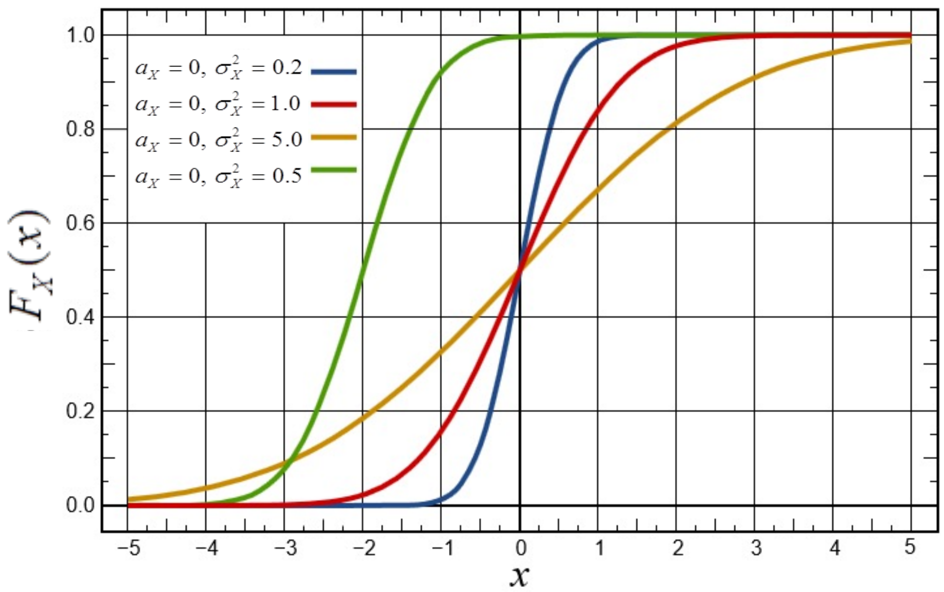
- Most widely discovered nature
- Normal density(distribution), denoted by $X \sim N(a_X, \sigma_X^2)$
- $F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-\frac{(\xi-a_X)^2}{2\sigma_X^2}} d\xi$: No closed-form solution
- Normalized gaussian density function: $a_X = 0, \sigma_X = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-0.5x^2}$$

$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-0.5\xi^2} d\xi$: Error function, usually given by table

Gaussian Random Variable

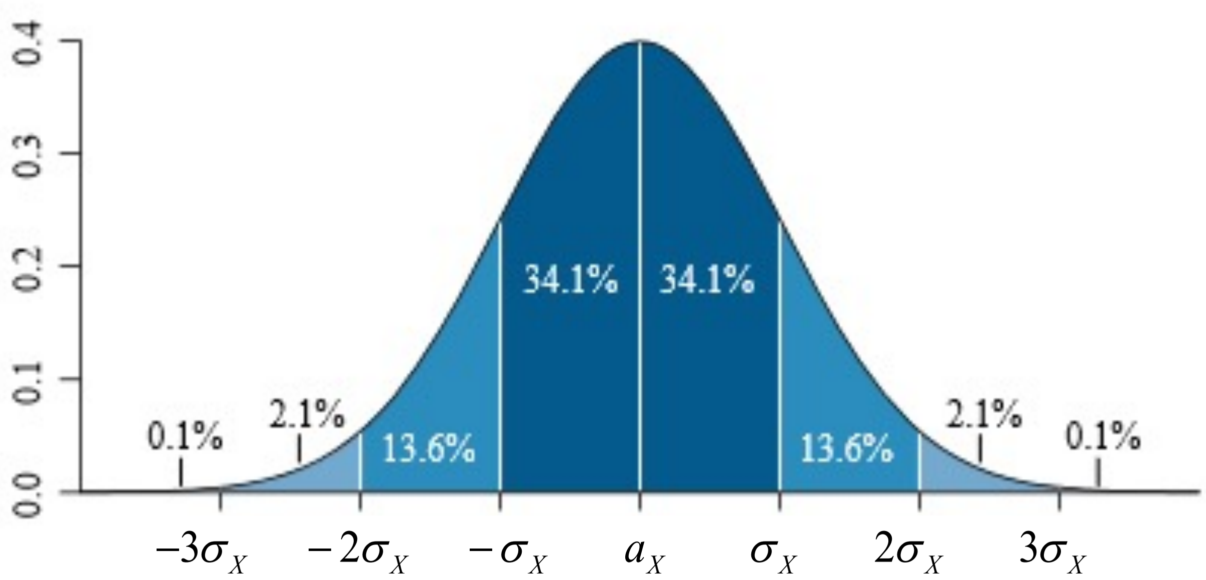




Gaussian Random Variable

❑ Percentage of area

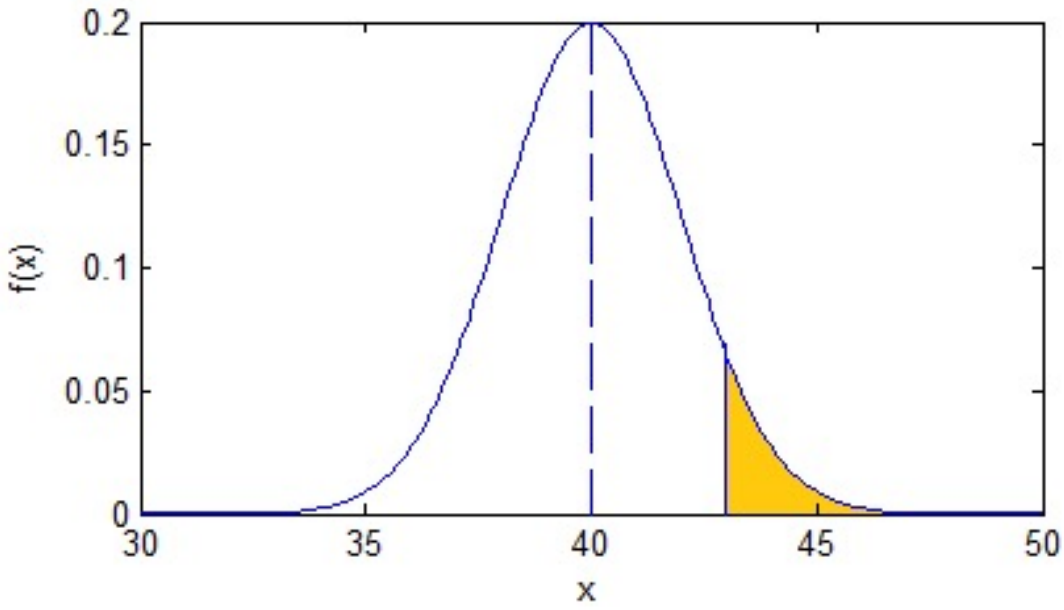
- Percent: 99.73% 99% 95.45% 95% 90% 80% 68.37%
- 표준편차: 3.00 2.58 2.00 1.96 1.645 1.28 1.00



Gaussian Random Variable

❑ Ex) 2.4-2

- 평균저항이 40Ω이고, 표준편차가 2Ω인 저항기를 만드는 기계가 있다. 저항이 정규분포를 따른다고 가정할 때, 43Ω이 넘는 저항을 가지게 되는 저항기는 몇 퍼센트인가?



$$z = \frac{43-40}{2} = 1.5$$

$$\begin{aligned} P(X > 43) &= P(Z > 1.5) \\ &= 1 - P(Z < 1.5) = 1 - 0.9332 \\ &= 0.0668 = 6.68\% \end{aligned}$$

Central limit theorem (CLT)

❑ Central limit theorem

- Given <u>a distribution</u> with a mean μ and variance σ^2, the sampling distribution of the mean approaches a Gaussian distribution with a mean(μ) and a variance σ^2/n as the sample size n increases.
- In other words, (Lindeberg–Lévy CLT) Suppose $\{x_1, x_2, \cdots\}$ is a sequence of independently and identically distributed random variables with $E[x_i] = \mu$ and $Var[x_i] = \sigma^2 < \infty$. Then,

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i \sim N(\mu, \frac{\sigma^2}{n}) \text{ as } n \to \infty$$

or

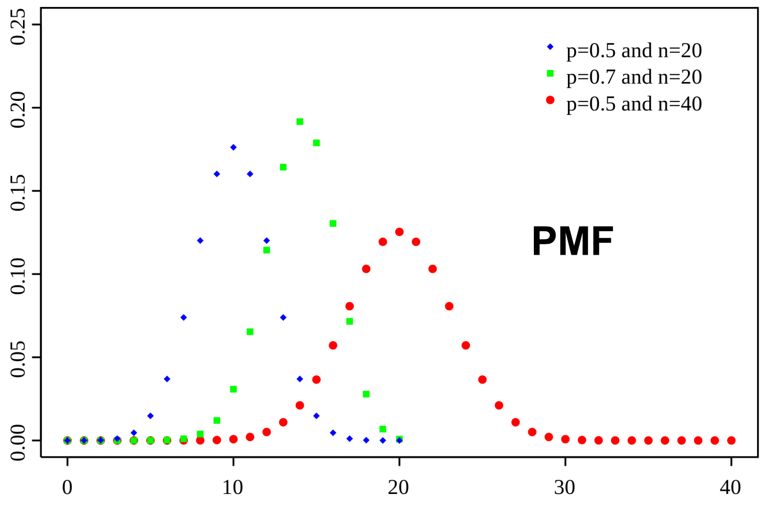
$$\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) - \mu\right] \sim N(0, \sigma^2) \text{ as } n \to \infty$$

Binomial distribution

❑ PMF

$$p(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

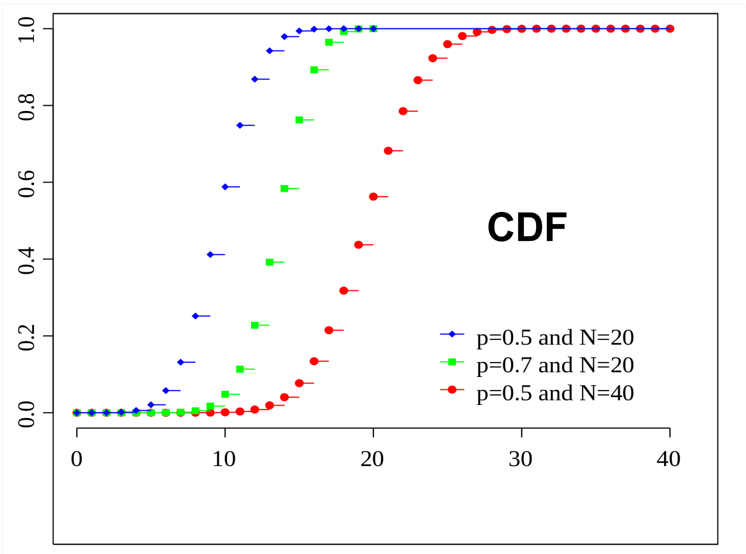
mean : np
variance : $np(1-p)$



❑ Example

- Suppose a biased comes up heads with probability 0.3 when tossed. The probability of seeing exactly 4 heads in 6 tosses is:

$$p(4; 6, 0.3) = \binom{6}{4} 0.3^4 (0.7)^2$$
$$= 0.059535$$

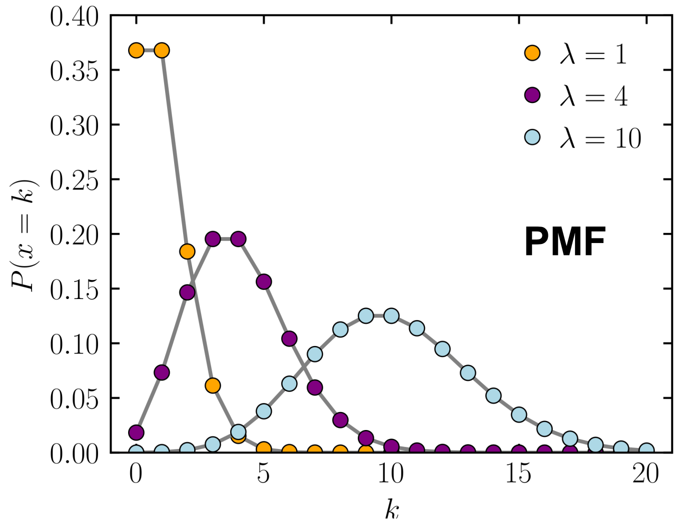


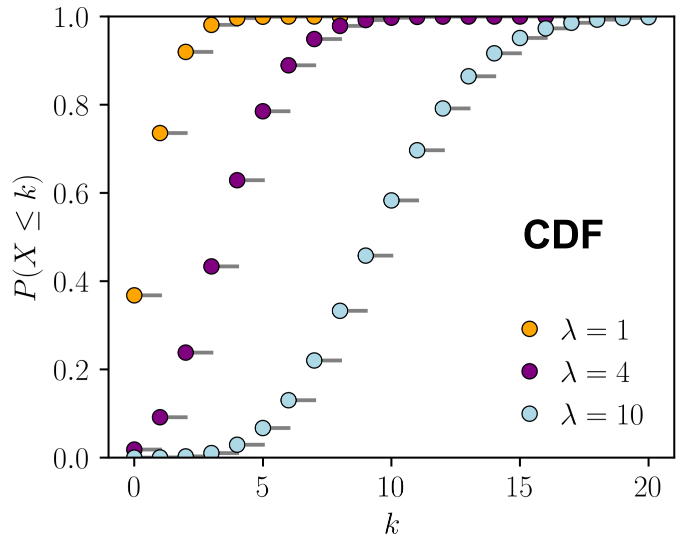
Poisson distribution

❑ PMF

$$p(k;\lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

mean : λ
variance : λ





❑ Examples

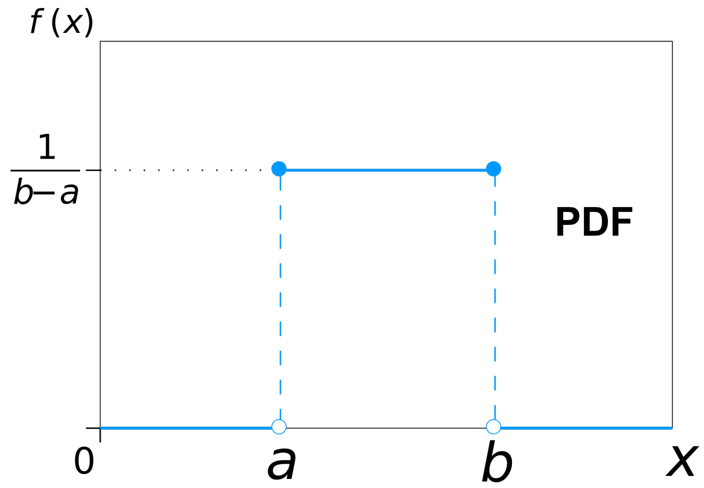
- The number of meteorites greater than 1 meter diameter that strike Earth in a year
- The number of patients arriving in an emergency room between 10 and 11pm
- The number of laser photons hitting a detector in a particular time interval

Uniform distribution

❑ PDF

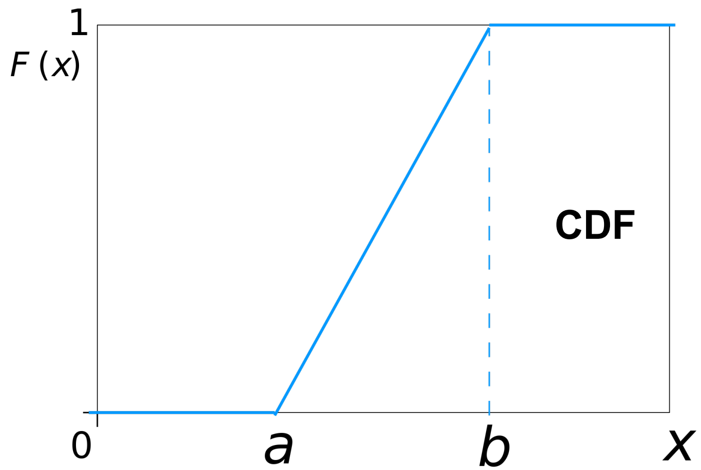
$$f(x;a,b) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

mean : $(a+b)/2$
variance : $(b-a)^2/12$



❑ Examples

- A deck of cards
- Sampling from an arbitrary distribution
- Signal quantization error
- Perfect random number generator
- Probability of guessing an exact time at any moment



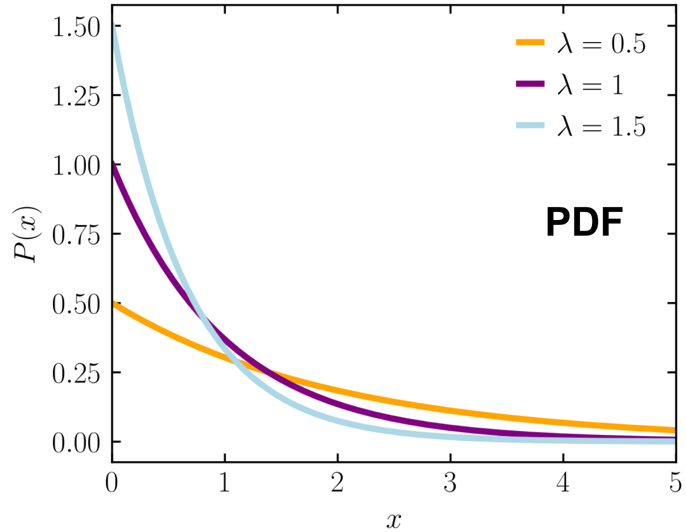
Exponential distribution

❑ Definition

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

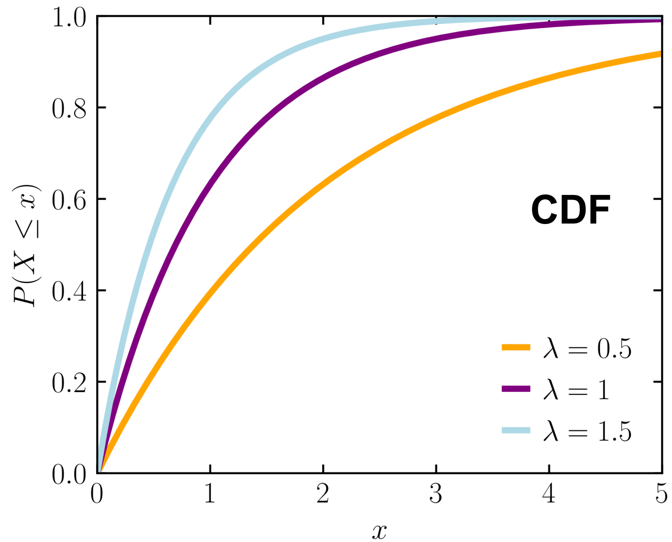
mean : $1/\lambda$

variance : $1/\lambda^2$



❑ Examples

- The amount of time until some specific event occurs
- How long it takes for a bank teller to serve a customer / the amount of time, in months, a car battery lasts / the amount of money customers spend in one trip to the supermarket
- "There are fewer large values and more small values."



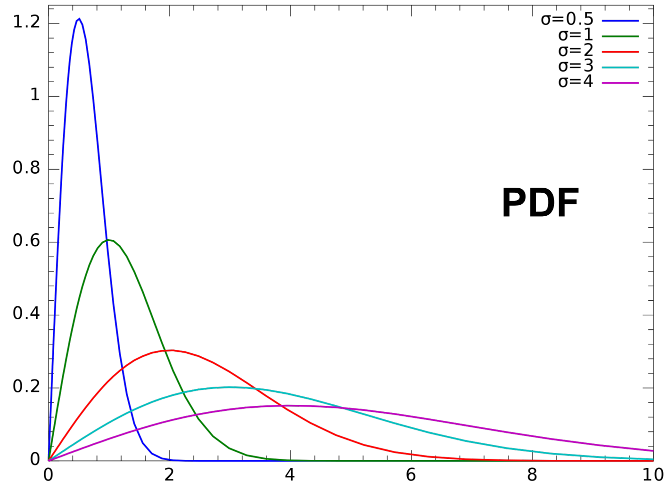
Rayleigh distribution

❑ Definition

$$f(x;\sigma) = \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} \qquad \text{for } x \geq 0$$

mean : $\sigma\sqrt{\pi/2}$

variance : $\frac{4-\pi}{2}\sigma^2$



❑ Examples

- The overall magnitude of a 2D vector, when each component follows normal distribution
- The Rayleigh distribution is used for c alculating the circular error probable - a measure of a weapon's precision
- The significant wave height approxima tely follows a Rayleigh distribution