Probability and Statistics

Course Description

❑ Probability

- Probability
- Joint probability, Marginal probability, Conditional probability
- Independence
- Bayes theorem

❑ Random Variables

- Random variables
- Distribution functions
- Probability density functions
- Gaussian random variables

❑ Statistics

- Expectation
- Estimation

Probability

Definition of Probability

❑ Probability models

- Systems work in a chaotic environment
- Probability models
  - ✓ Enable the designer to make sense out of the chaos
  - ✓ Build efficient, reliable, and cost-effective systems

❑ Deterministic models

- The **conditions** under which an experiment is carried out determine the exact outcome of the experiment

Definition of Probability

❑ Terms of Probability

- Experiment(실험) 및 Trial(시행)

  동일한 조건하에서 반복할 수 있고, 또 그 결과가 우연에 의해서 지배되지만, 가능한 모든 결과의 집합을 알 수 있는 관찰, 조사, 실험

  Ex) 주사위를 1회 던질 때 나오는 결과를 다음과 같이 정의하자.

  - 1의 눈이 나오는 것을 1
  - 2의 눈이 나오는 것을 2
  - …
  - 6의 눈이 나오는 것을 6

- Sample space(표본공간)

  어떤 시행에서 일어날 수 있는 모든 가능한 결과의 집합

  Ex) $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

- Event(사건)

  표본공간 S의 부분집합

  Ex) $A = \phi,\ \{1\},\ \{2\},\ ...,\ \{6\},\ \{1,2\},\ ...,\ \{1,2,3,4,5,6\}$

Definition of Probability

❑ Terms of Probability

- Discrete sample space(이산표본공간)

Ex) **Two coins tossed:** When two coins are tossed, there are four possible outcomes. Let H_1 and T_1 denote the head and tail on the first coin and H_2 and T_2 denote the head and tail on the second coin respectively. The sample space can be written in the form as

$$S = \{(H_1, H_2),\ (H_1, T_2),\ (T_1, H_2),\ (T_1, T_2)\}$$

Ex) **Two dice tossed:**

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

Definition of Probability

❑ Terms of Probability

- Continuous sample space(연속표본공간)

Ex) **Arrival time**: The experimental setting is a metro (underground) station where trains pass (ideally) with equal intervals. A person enters the station. The experiment is to note the time of arrival past the departure time of the last train. If T is the interval between two consecutive trains, then the sample space for the experiment is the interval [0, T], or

$$S = [0, T] = \{t : \ 0 \le t \le T\}$$

Ex) **Chord length:** Given a circle of radius R, the experiment is to randomly select a chord in that circle. There are many ways to accomplish such a selection. However, the sample space is always the same:

{AB: A and B are points on a given circle}.

One natural random variable defined on this space is the length of the chord. The variable takes on random length on the interval [0, T], where T is the diameter of the circle at hand. The length of a chord AB is zero if the two points happen to coincide.

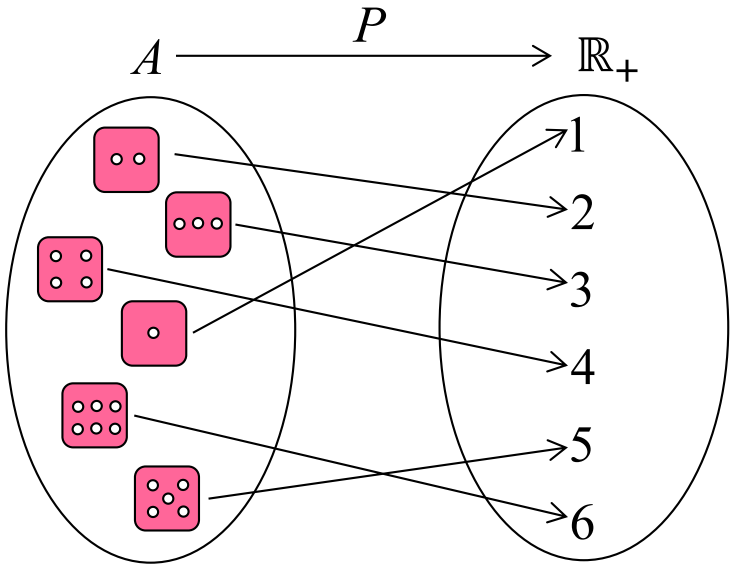
Definition of Probability

❑ Definition of Probability

- 표본공간 S에서 정의된 각 사건에 확률(Probability)라고 불리는 음이 아닌 수를 할당

- 확률은 하나의 함수 $P : A \longrightarrow \mathbb{R}_+ \ (\geq 0)$

- 사건 A의 확률 $P[A]$



- 확률의 공리

$$Axiom\ 1)\quad P[A] \geq 0$$

$$Axiom\ 2)\quad P[S] = 1$$

$$Axiom\ 3)\quad P\left[\bigcup_{n=1}^{N} A_n\right] = \sum_{n=1}^{N} P[A_n] \ \text{ if } \ A_m \cap A_n = \phi \ \text{ for } \ m \neq n \ \text{ and } \ A_n, A_m \subset S$$

Note) **공리**(axiom): 증명을 필요로 하지 않거나 증명할 수 없지만 직관적으로 자명한 진리의 명제인 동시에 다른 명제들의 전제가 되는 명제

Definition of Probability

❑ Ex) 1.3-1

- 0부터 100까지의 점들이 표시된 Fair한 회전판을 돌려서 어떤 수 x를 구하는 실험
- 표본공간 $S = \{0 < x \le 100\}$
- $A = \{x_1 < x \le x_2\} \rightarrow P[A] = \dfrac{x_2 - x_1}{100} \ge 0$: Axiom 1 is satisfied.
- $If\ \ x_2 = 100\ and\ \ x_1 = 0,\ then\ P[A] = 1$: Axiom 2 is satisfied.
- $A_n = \{x_{n-1} < x \le x_n\},\ x_n = \dfrac{100n}{N},\ n = 1, 2, \ldots, N,\ x_0 = 0 \ \Rightarrow P(A_n) = \dfrac{1}{N}$

  $\Rightarrow P\left[\bigcup_{n=1}^{N} A_n\right] = \sum_{n=1}^{N} P[A_n] = \sum_{n=1}^{N} \dfrac{1}{N} = 1$: Axiom 3 is satisfied.

Definition of Probability

❑ Mathematical probability

- $P[A] = \dfrac{n(A)}{n(S)} = \dfrac{\text{Number of elementary events belong to } A}{\text{Number of entire elementary events}}$

- Ex) 1.3-2, 두 주사위를 던져서 나오는 수의 합을 관찰하는 시험

$$A = \{x_1 + x_2 = 7\}$$
$$B = \{8 < x_1 + x_2 \le 11\}$$
$$C = \{10 < x_1 + x_2\}$$

$$S = \begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

$$\Rightarrow P[A] = \frac{6}{36} = \frac{1}{6},\ P[B] = \frac{9}{36} = \frac{1}{4},\ P[C] = \frac{3}{36} = \frac{1}{12}$$

Definition of Probability

❑ Probability in a relative frequency

- 동일한 시행을 n번 반복하여, 사건 A가 일어난 횟수를 r이라 할 때, n을 충분히 크게 하면 상대도수 r/n은 일정한 값 p에 가까워진다. 이 p를 사건 A의 통계적 확률 또는 경험적 확률이라 한다.

$$P[A] = \lim_{n \to \infty} \frac{r}{n}$$

- Ex) 어느 고등학교 학생 2,000명의 혈액형을 조사한 결과 A형인 학생이 820명이었다고 한다. 이학교의 학생 1명을 임의로 뽑을 때, 그 학생의 혈액형이 A형일 확률은?

$$P[A] = \frac{820}{2,000} = 0.41$$

Joint Probability

❑ Definition of joint probability

- Joint probability is the probability that two events will occur simultaneously.

$$P[A \cap B] = P[A] + P[B] - P[A \cup B]$$

$$\text{or} \quad P[A \cup B] = P[A] + P[B] - P[A \cap B] \leq P[A] + P[B]$$

$$\text{If } A \text{ and } B \text{ are disjoint, } P[A \cap B] = 0. \Rightarrow P[A \cup B] = P[A] + P[B]$$

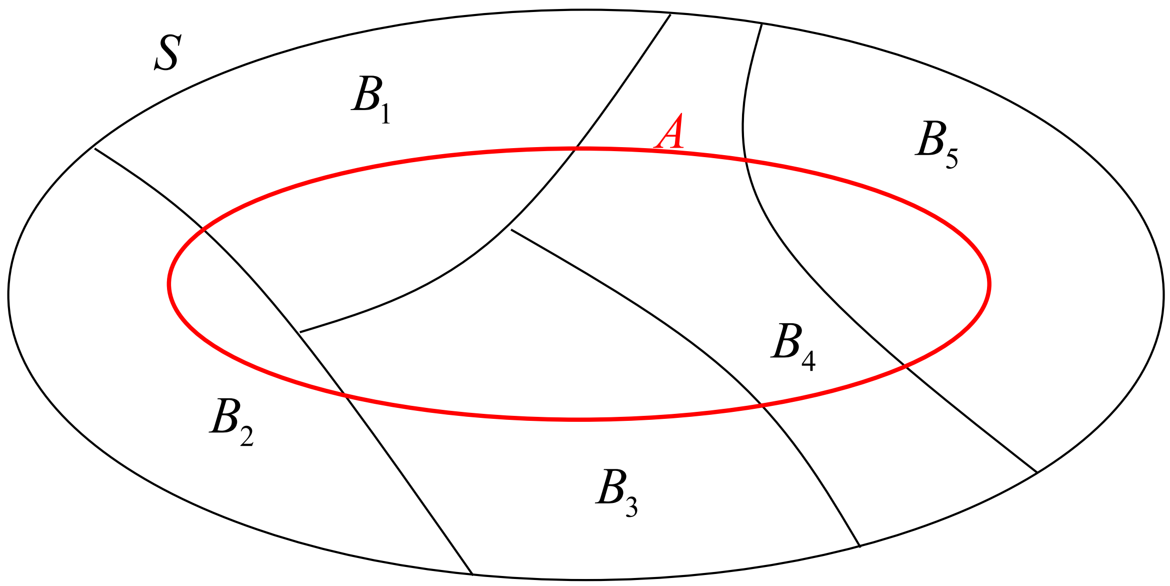
Joint Probability

❑ Definition of marginal probability

Suppose that a sample space S is partitioned by the events

$$\{B_1, B_2, ..., B_n\}$$

where $S = \bigcup_{i=1}^{n} B_i$ *and* $B_i \cap B_j = \phi, \ i \neq j.$



Joint Probability

- **Definition of marginal probability**

Given $P[A \cap B_i]$, $i = 1, \ldots, m$, the event A can be written as

$$A = \bigcup_{i=1}^{m} (A \cap B_i)$$

where $(A \cap B_i)$, $i = 1, \ldots, m$ are disjoint each other, i.e.,

$$(A \cap B_i) \cap (A \cap B_j) = \phi, \ i \neq j.$$

Then,

(1) $$P[A] = P\left[\bigcup_{i=1}^{m} (A \cap B_i)\right] = \sum_{i=1}^{m} P[A \cap B_i].$$

Here, $P[A \cap B_i]$ is called the joint probability of the events A and B_i.

$P[A]$, the sum of joint probabilities, is called the marginal probability.

Conditional Probability

❑ Definition of conditional probability

The conditional probability is the probability if an occurrence of one event is subject to the hypothesis of the occurrence of another event. For $P[A \mid B]$, we consider the event $A \cap B$ in the sample space S. Construct the conditional probability by the frequency method

$$P[B] = \frac{n(B)}{n(S)}$$

Then, $P[A \mid B] = \dfrac{n(A \cap B)}{n(B)} = \dfrac{n(A \cap B)/n(S)}{n(B)/n(S)}$

$$= \frac{P[A \cap B]}{P[B]}$$

Therefore,

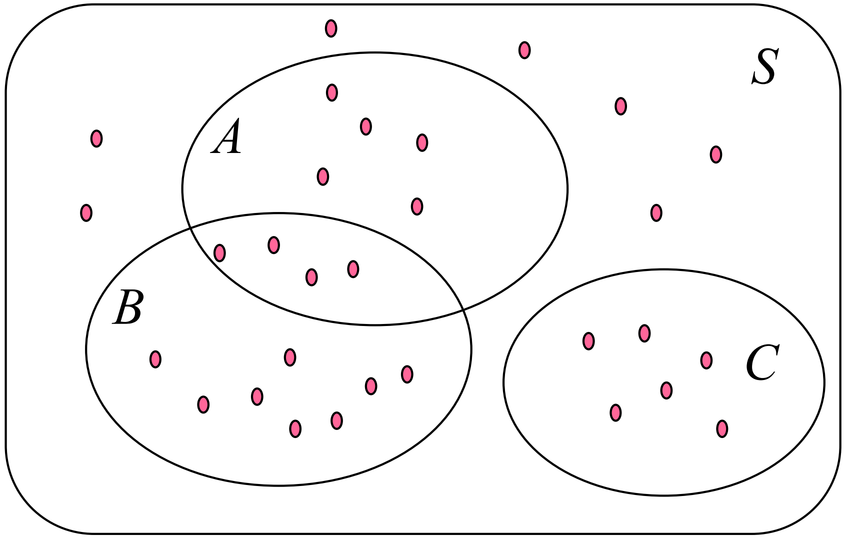
(2) $$P[A \cap B] = P[A \mid B]P[B].$$

Also, from (1) and (2), we see that

$$P[A] = \sum_{i=1}^{m} P[A \cap B_i] = \sum_{i=1}^{m} P[A \mid B_i]P[B_i] : \text{ Theorem on tatal probability}$$

Conditional Probability

- Ex)



$$n(S) = 30$$
$$n(A) = 9$$
$$n(B) = 12$$
$$n(A \cap B) = 4$$
$$n(C) = 6$$

$$P[A] = \frac{n(A)}{n(S)} = \frac{3}{10},\ P[B] = \frac{n(B)}{n(S)} = \frac{2}{5},\ P[C] = \frac{n(C)}{n(S)} = \frac{1}{5},\ P[A \cap B] = \frac{n(A \cap B)}{n(S)} = \frac{2}{15}$$

$$\Rightarrow P[B \mid A] = \frac{P[A \cap B]}{P[A]} = \frac{2/15}{3/10} = \frac{4}{9} \quad \left(\Rightarrow P[A \cap B] = P[B \mid A]P[A] = \frac{4}{9} \times \frac{3}{10} = \frac{2}{15} \right)$$

$P[A \cap C] = 0$: A and C are disjoint.

$P[A \cap C] \neq P[A]P[C]$: A and C are not independent.

Conditional Probability

❑ Ex)

Select a ball from an urn containing balls numbered 1 to 4
(1,2: Black, 3,4: White)

Sample space $S = \{(1,b),\ (2,b),\ (3,w),\ (4,w)\}$: Equally likely outcomes

- $A = \{(1,b),\ (2,b)\}$: Black ball selected
- $B = \{(2,b),\ (4,w)\}$: Even-numbered ball selected
- $C = \{(3,w),\ (4,w)\}$: Number of ball is greater than 2

$$P[A \mid B] = P[A \cap B] / P[B] = 0.25 / 0.5 = 0.5 = P[A]$$
$$P[A \mid C] = P[A \cap C] / P[C] = 0.0 / 0.5 = 0$$

Conditional Probability

❑ Ex) Theorem on Total Probability

Two black balls and three white balls are in an urn. Two balls are sequentially selected at random without replacement. Find the probability of the event W_2 that the second ball is white. $W_2 = \{(b,w),\ (w,w)\}$

- $B_1 = \{(b,b),\ (b,w)\}$: The first ball is black.
- $W_1 = \{(w,b),\ (w,w)\}$: The first ball is white.

$\Rightarrow B_1, W_1$ form a partition of S.

- $n(b,b) = 2,\ n(b,w) = 6 \quad \Rightarrow \quad n(B_1) = 8,\ \ n(W_2 \mid B_1) = 6/8 = 3/4$
- $n(w,b) = 6,\ n(w,w) = 6 \quad \Rightarrow \quad n(W_1) = 12,\ \ n(W_2 \mid W_1) = 6/12 = 1/2$
- $n(S) = 20$

$$\therefore\ P[W_2] = P[W_2 \mid B_1]P[B_1] + P[W_2 \mid W_1]P[W_1]$$

$$= \frac{3}{4}\cdot\frac{8}{20} + \frac{1}{2}\cdot\frac{12}{20} = \frac{3}{4}\cdot\frac{2}{5} + \frac{1}{2}\cdot\frac{3}{5} = \frac{3}{5}$$

Independence of Events

❑ Definition

If $P[A \cap B] = P[A]P[B]$, *then* A and B are independent.

$$\Leftarrow \; P[A] = P[A \mid B] = \frac{P[A \cap B]}{P[B]}$$

- implies that $P[A] = P[A \mid B]$, $P[B] = P[B \mid A]$

- implies also that $P[A] \neq 0$ and $P[B] \neq 0$

Two events A and B are mutually exclusive or disjoint

if $P[A] \neq 0$. $P[B] \neq 0$, $P[A \cap B] = 0$.

$\Rightarrow$ A and B cannot be independent.

$\because P[A \cap B] = P[A]P[B] = 0 \;\Leftarrow$ contradiction

Independence of Events

❑ Definition

$$P[A] = P[A \mid B]$$

- When the proportion of outcomes in S that lead to the occurence of A is equal to the proportion of outcomes in B that lead to A.
- Knowledge of the occurence of B does not alter the probability of occurrence of A.

Bayes' Rule

❑ Definition

Note that $P[A \cap B] = P[A \mid B]P[B] = P[B \mid A]P[A]$.

Hence,

(3) $$P[B \mid A] = \frac{P[A \mid B]P[B]}{P[A]}$$

Also, we see that

$$P[B_i \mid A] = \frac{P[A \mid B_i]P[B_i]}{P[A]}.$$

If $B_j,\ j = 1,\ldots,m$ are disjoint each other and form $S = \sum_{j=1}^{m} B_j$, then we have the following from the theorem on total probability

(4) $$P[B_i \mid A] = \frac{P[A \mid B_i]P[B_i]}{\sum_{j=1}^{m} P[A \mid B_j]P[B_j]}$$: Bayes' Rule

Bayes’ Rule

❑ Meanings of Bayes Rule

$$P[B_i \mid A] = \frac{P[A \mid B_i]P[B_i]}{\sum_{j=1}^{m} P[A \mid B_j]P[B_j]}$$

$P[B_i]$: *a priori* knowledge(probabilities) (unconditional probabilities),
the probabilies of the events before the experiment is performed

$P[B_i \mid A]$: *a posteriori* knowledge(probabilities) (conditional probabilities),
the probabilies of the events in the partition $P[B_j \mid A]$ given this additional information

Bayes' Rule provides a basis of the filtering theory to estimate most probable states under noisy circumstances.

Bayes' Rule

❑ Ex) Communication System Error

Communication from a transmitter to a receiver is done by the binary digit, 0 or 1. There are communication errors in change of 0 and 1. For example, 10010110 $\rightarrow$ 10110010.

Suppose B_1 is the event with the symbol 1 and B_2 is the event with the symbol 0 before communication. And, A_1 and A_2 are the events with the symbol 1 and 0 after communication, respectively.

Assume that $P[B_1] = 0.6$ and $P[B_2] = 0.4$.

And, we already know the following information from lots of experiments;

$P[A_1 \mid B_1] = 0.9$ and $P[A_2 \mid B_1] = 0.1$

$P[A_1 \mid B_2] = 0.1$ and $P[A_2 \mid B_2] = 0.9$.

Bayes' Rule

❑ Ex) Communication System Error

Then, from the theorm on total probability

$$P[A_1] = P[A_1 \mid B_1]P[B_1] + P[A_1 \mid B_2]P[B_2] = 0.9 \cdot 0.6 + 0.1 \cdot 0.4 = 0.58$$

$$P[A_2] = P[A_2 \mid B_1]P[B_1] + P[A_2 \mid B_2]P[B_2] = 0.1 \cdot 0.6 + 0.9 \cdot 0.4 = 0.42$$

The a posteriori probability for correct communication becomes

$$P[B_1 \mid A_1] = \frac{P[A_1 \mid B_1]P[B_1]}{P[A_1]} = \frac{0.9 \cdot 0.6}{0.58} \approx 0.931$$

$$P[B_2 \mid A_2] = \frac{P[A_2 \mid B_2]P[B_2]}{P[A_2]} = \frac{0.9 \cdot 0.4}{0.42} \approx 0.857$$

and the probabilities of false communication are

$$P[B_1 \mid A_2] = \frac{P[A_2 \mid B_1]P[B_1]}{P[A_2]} = \frac{0.1 \cdot 0.6}{0.42} \approx 0.143$$

$$P[B_2 \mid A_1] = \frac{P[A_1 \mid B_2]P[B_2]}{P[A_1]} = \frac{0.1 \cdot 0.4}{0.58} \approx 0.069$$

Bayes' Rule

❑ Ex) Estimation of an integer variable

(Case 1)

Suppose that x is an unknown integer, which is to be estimated by using some measurements.

Define event B_n as follows;

$$\text{Event } B_n : \;\; x = n$$

In the beginning, the only information we have is that x is bounded in [-2, 2]. (Of course this information may not be true)

So we assign the *a priori* probabilities of each event as

$P[B_{-2}] = 0.2,\;\; P[B_{-1}] = 0.2,\;\; P[B_0] = 0.2,\; P[B_1] = 0.2,\; P[B_2] = 0.2$

and $P[B_n] = 0$ if $x \geq 3$ or $x \leq -3$.

Bayes' Rule

❑ Ex) Estimation of an integer variable

The unknown x is measured by using a sensor which has a model given by

$$\text{Measurement:} \quad z = x + v$$

where v is the sensor noise of the following statistical properties.

$P[v=-1]=0.2,\ P[v=0]=0.6,\ P[v=1]=0.2,$ and $P[|v|>1]=0.$

We also define event A_m as

$$\text{Event } A_m: \quad z = m$$

Conditional probability $P[A_m \mid B_n]$ can be calculated from the statistical properties of v; for example

$P[A_0 \mid B_{-1}] = P[z=0 \mid x=-1] = P[v=1] = 0.2$

$P[A_2 \mid B_0] = P[z=2 \mid x=0] = P[v=2] = 0$

$P[A_1 \mid B_1] = P[z=1 \mid x=1] = P[v=0] = 0.6$

Bayes' Rule

❑ Ex) Estimation of an integer variable

Suppose that the first measurement is A_1. Then, the conditional probability $P[B_n \mid A_1]$ is calculated by applying the Bayes' Rule of Eq.(4):

$$P[B_n \mid A_1] = \frac{P[A_1 \mid B_n]P[B_n]}{\sum_{j=1}^{m} P[A_1 \mid B_j]P[B_j]}$$

$$\begin{aligned}
\text{Denominator} = P[A_1] &= \sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j] \\
&= P[A_1 \mid B_{-2}]P[B_{-2}] + P[A_1 \mid B_{-1}]P[B_{-1}] + \ldots + P[A_1 \mid B_2]P[B_2] \\
&= P[v=3]P[B_{-2}] + P[v=2]P[B_{-1}] + P[v=1]P[B_0] \\
&\quad + P[v=0]P[B_1] + P[v=-1]P[B_2] \\
&= 0 \cdot 0.2 + 0 \cdot 0.2 + 0.2 \cdot 0.2 + 0.6 \cdot 0.2 + 0.2 \cdot 0.2 \\
&= 0.2
\end{aligned}$$

Bayes' Rule

❑ Ex) Estimation of an integer variable

Hence, $$P[B_{-2} \mid A_1] = \frac{P[A_1 \mid B_{-2}]P[B_{-2}]}{\sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j]} = \frac{0 \cdot 0.2}{0.2} = 0$$

$$P[B_{-1} \mid A_1] = \frac{P[A_1 \mid B_{-1}]P[B_{-1}]}{\sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j]} = \frac{0 \cdot 0.2}{0.2} = 0$$

$$P[B_0 \mid A_1] = \frac{P[A_1 \mid B_0]P[B_0]}{\sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j]} = \frac{0.2 \cdot 0.2}{0.2} = 0.2$$

$$P[B_1 \mid A_1] = \frac{P[A_1 \mid B_1]P[B_1]}{\sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j]} = \frac{0.6 \cdot 0.2}{0.2} = 0.6$$

$$P[B_2 \mid A_1] = \frac{P[A_1 \mid B_2]P[B_2]}{\sum_{j=-2}^{2} P[A_1 \mid B_j]P[B_j]} = \frac{0.2 \cdot 0.2}{0.2} = 0.2$$

These conditional probabilities are *a posteriori* probabilities of the events B_i when the outcome of the measurement is A_1.

Bayes' Rule

❑ Ex) Estimation of an integer variable

Suppose that the first measurement is not A_1 but A_2. The conditional probability $P[B_n \mid A_2]$ can be calculated in a similar way. However, note that

$$P[A_2 \mid B_3] = P[z = 2 \mid x = 3] = P[v = -1] = 0.2 \neq 0$$

Hence, we need to include B_3 in this case:

$$\begin{aligned}
\text{Denominator} &= \sum_{j=-2}^{3} P[A_2 \mid B_j] P[B_j] \\
&= P[A_2 \mid B_{-2}]P[B_{-2}] + P[A_2 \mid B_{-1}]P[B_{-1}] + P[A_2 \mid B_0]P[B_0] \\
&\quad + P[A_2 \mid B_1]P[B_1] + P[A_2 \mid B_2]P[B_2] + P[A_2 \mid B_3]P[B_3] \\
&= P[v = 4]P[B_{-2}] + P[v = 3]P[B_{-1}] + P[v = 2]P[B_0] \\
&\quad + P[v = 1]P[B_1] + P[v = 0]P[B_2] + P[v = -1]P[B_3] \\
&= 0 \cdot 0.2 + 0 \cdot 0.2 + 0 \cdot 0.2 + 0.2 \cdot 0.2 + 0.6 \cdot 0.2 + 0.2 \cdot 0 \\
&= 0.16
\end{aligned}$$

Bayes' Rule

- Ex) Estimation of an integer variable

Hence, $$P[B_{-2} \mid A_2] = \frac{P[A_2 \mid B_{-2}]P[B_{-2}]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0 \cdot 0.2}{0.16} = 0$$

$$P[B_{-1} \mid A_2] = \frac{P[A_2 \mid B_{-1}]P[B_{-1}]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0 \cdot 0.2}{0.16} = 0$$

$$P[B_0 \mid A_2] = \frac{P[A_2 \mid B_0]P[B_0]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0 \cdot 0.2}{0.16} = 0$$

$$P[B_1 \mid A_2] = \frac{P[A_2 \mid B_1]P[B_1]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0.2 \cdot 0.2}{0.16} = 0.25$$

$$P[B_2 \mid A_2] = \frac{P[A_2 \mid B_2]P[B_2]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0.6 \cdot 0.2}{0.16} = 0.75$$

$$P[B_3 \mid A_2] = \frac{P[A_2 \mid B_3]P[B_3]}{\sum_{j=-2}^{3} P[A_2 \mid B_j]P[B_j]} = \frac{0.2 \cdot 0}{0.16} = 0$$

Bayes' Rule

❑ Ex) Estimation of an integer variable

Observe that $P[B_3 \mid A_2] = 0$ is against our intuition. The measurement A_2 is quite probable when $x = 3$. Then what is wrong?

The problem comes from that our *a priori* probability assignment is not adequate. If x has any chance of taking the value of 3, $P[B_3]$ should have some non-zero value. Otherwise, x can not be properly estimated.

Bayes' Rule

❑ Ex) Successive measurements

Let $A(i)$ denote the i-th measurement. Suppose that we have $A(1)$ and the associated *a posteriori* probabilities are calculated. In processing of $A(2)$, these *a posteriori* probabilities are then treated as *a priori* probabilities.

For simplicity, we use the following notation:

$$P[X, Y, Z] \triangleq P[X \cap Y \cap Z]$$

Then, note that

$$P[B_i \mid A(1), A(2)] = \frac{P[B_i, A(1), A(2)]}{P[A(1), A(2)]}$$

$$= \frac{P[B_i, A(1), A(2)]}{\sum_{j=1}^{m} P[B_j, A(1), A(2)]}$$

Bayes' Rule

❑ Ex) Successive measurements

but

$$P[B_j, A(1), A(2)] = \frac{n(B_j, A(1), A(2))}{n(B_j, A(1))} \frac{n(B_j, A(1))}{n(S)}$$

$$= P[A(2) \mid B_j, A(1)] \cdot P[B_j, A(1)]$$

Assume that each mesurement is independent, then we observe

$$P[A(2) \mid B_j, A(1)] = P[A(2) \mid B_j]$$

Also, $P[B_j, A(1)] = P[B_j \mid A(1)]P[A(1)]$

Hence, $P[B_j, A(1), A(2)] = P[A(2) \mid B_j] \cdot P[B_j \mid A(1)] \cdot P[A(1)]$

and

$$P[B_i \mid A(1), A(2)] = \frac{P[B_i, A(1), A(2)]}{\sum_{j=1}^{m} P[B_j, A(1), A(2)]}$$

$$= \frac{P[A(2) \mid B_j] \cdot P[B_j \mid A(1)] \cdot \cancel{P[A(1)]}}{\sum_{j=1}^{m} P[A(2) \mid B_j] \cdot P[B_j \mid A(1)] \cdot \cancel{P[A(1)]}}$$

Bayes' Rule

❑ Ex) Successive measurements

This expression provides a recursive algorithm for sequencial processing of the measurements;

$$P[B_j \mid A(1), A(2), A(3)] = \frac{P[A(3) \mid B_j] \cdot P[B_j \mid A(1), A(2)]}{\sum_{j=1}^{m} P[A(3) \mid B_j] \cdot P[B_j \mid A(1), A(2)]}$$

$$\vdots$$

$$P[B_j \mid A(1), A(2), \ldots, A(k+1)] = \frac{P[A(k+1) \mid B_j] \cdot P[B_j \mid A(1), A(2), \ldots, A(k)]}{\sum_{j=1}^{m} P[A(k+1) \mid B_j] \cdot P[B_j \mid A(1), A(2), \ldots, A(k)]}$$