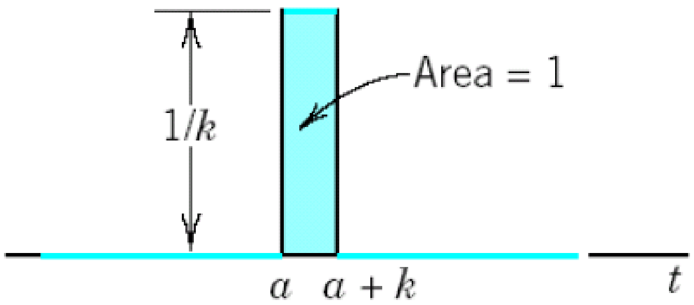
# Automatic Control

Hak-Tae Lee

# $2^{nd}$ Order System Responses

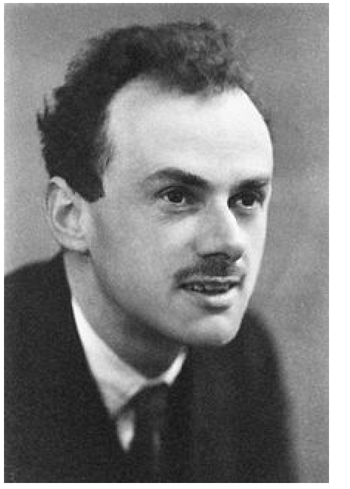
# Dirac Delta Function




$$f_k(t-a) = \begin{cases} \dfrac{1}{k} & (a \le t < a+k) \\ 0 & (otherwise) \end{cases} \qquad \delta(t-a) = \lim_{k \to 0} f_k(t-a)$$

$$\int_0^\infty f_k(t-a)dt = \int_a^{a+k} \frac{1}{k} dt = 1 \quad \Rightarrow \quad \int_0^\infty \delta(t-a)dt = 1$$

# Paul Dirac

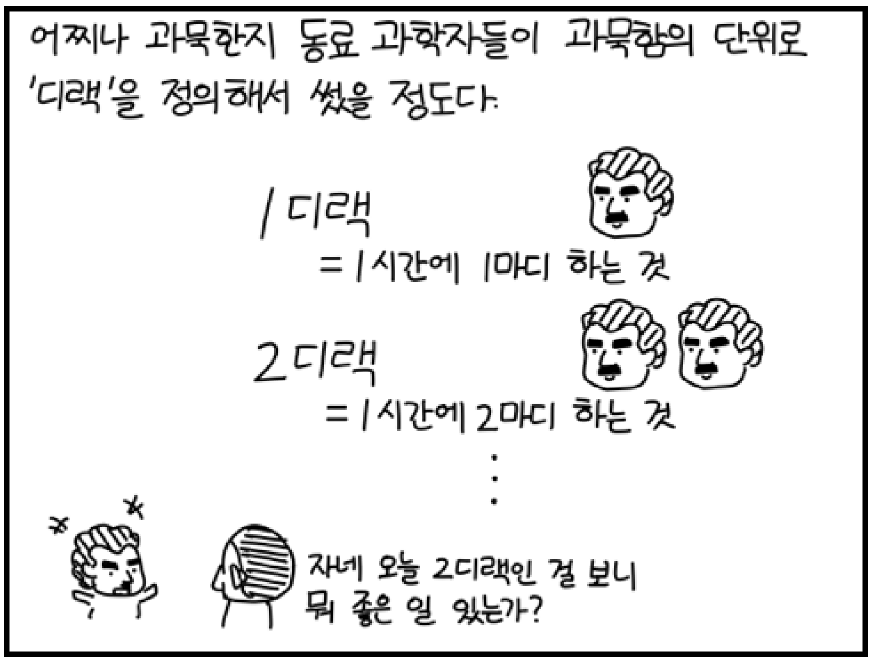


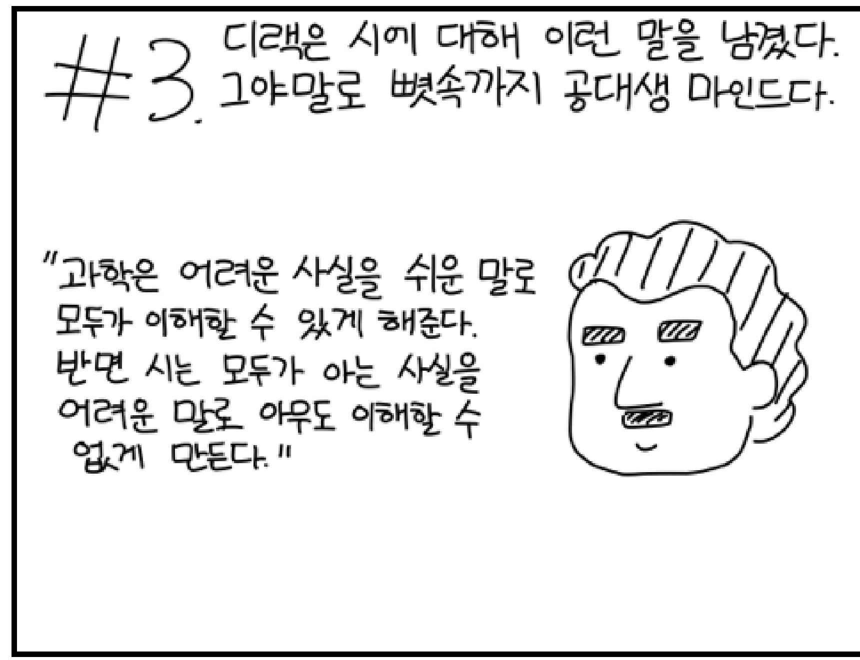
- August 1902 – October 1984
- English theoretical physicist

- Bio
  - 1926 Ph.D.
  - 1933 Nobel Prize in Physics
- Notable work
  - Quantum mechanics
  - Dirac equation – predicted the existence of antimattter

# Paul Dirac

https://goo.gl/c4gqq9



# Impulse Response

Solve the given ODE

$$\ddot{y} + 3\dot{y} + 2y = \delta(t)$$

$$\dot{y}(0) = 0$$
$$y(0) = 0$$

Laplace transform of Dirac delta function

$$L\{\delta(t-a)\} = \int_0^\infty \delta(t-a)e^{-st}dt = \lim_{k\to 0}\int_a^{a+k}\frac{1}{k}e^{-st}dt$$
$$= \lim_{k\to 0}\frac{e^{-as} - e^{-(a+k)s}}{ks} = e^{-as}$$

# Impulse Response

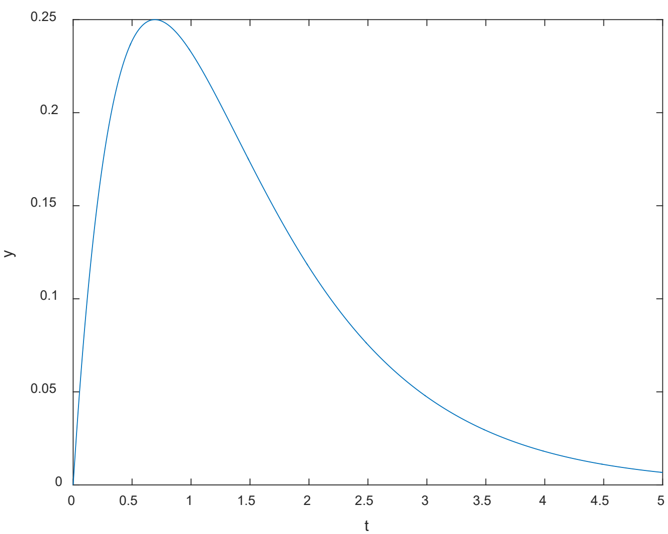
Laplace transform

$$L\{\ddot{y} + 3\dot{y} + 2y\} = (s^2 + 3s + 2)L\{y\} = L\{\delta(t)\} = 1$$

$$F(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$y(t) = e^{-t} - e^{-2t}$$

Homogenous solution!



# Unit Impulse Function

## Unit impulse function

$$\delta(t) = \lim_{\varepsilon \to 0} f_\epsilon(t) \quad \text{where} \quad f_\epsilon(t) = \begin{cases} 1/\epsilon & -\frac{\epsilon}{2} \leqq t \leqq \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases}$$

## properties of the unit impulse function

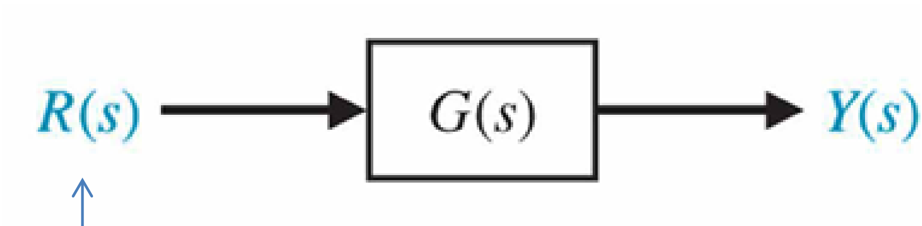
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a) g(t)\, dt = g(a)$$

## Laplace transform of the unit impulse function

$$\int_0^\infty \delta(t) e^{-st} dt = \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} \frac{1}{\varepsilon} e^{-st} dt = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( -\frac{1}{s} \right) e^{-st} \Big|_{-\varepsilon/2}^{\varepsilon/2}$$

$$= \left( -\frac{1}{s} \right) \lim_{\varepsilon \to 0} \frac{e^{-\varepsilon s/2} - e^{\varepsilon s/2}}{\varepsilon} = \left( -\frac{1}{s} \right) (-s) = 1$$

# Unit Impulse Response



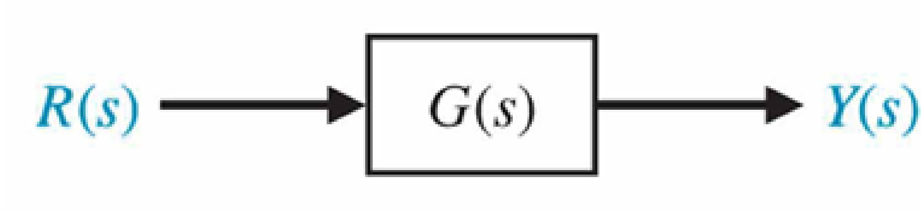

$$Y(s) = G(s)R(s) = G(s)$$

1

- Unit impulse response is equal to the transfer function itself
- Unit impulse response is the natural response of the system

# Unit Impulse Response Example

$r = \delta(t)$



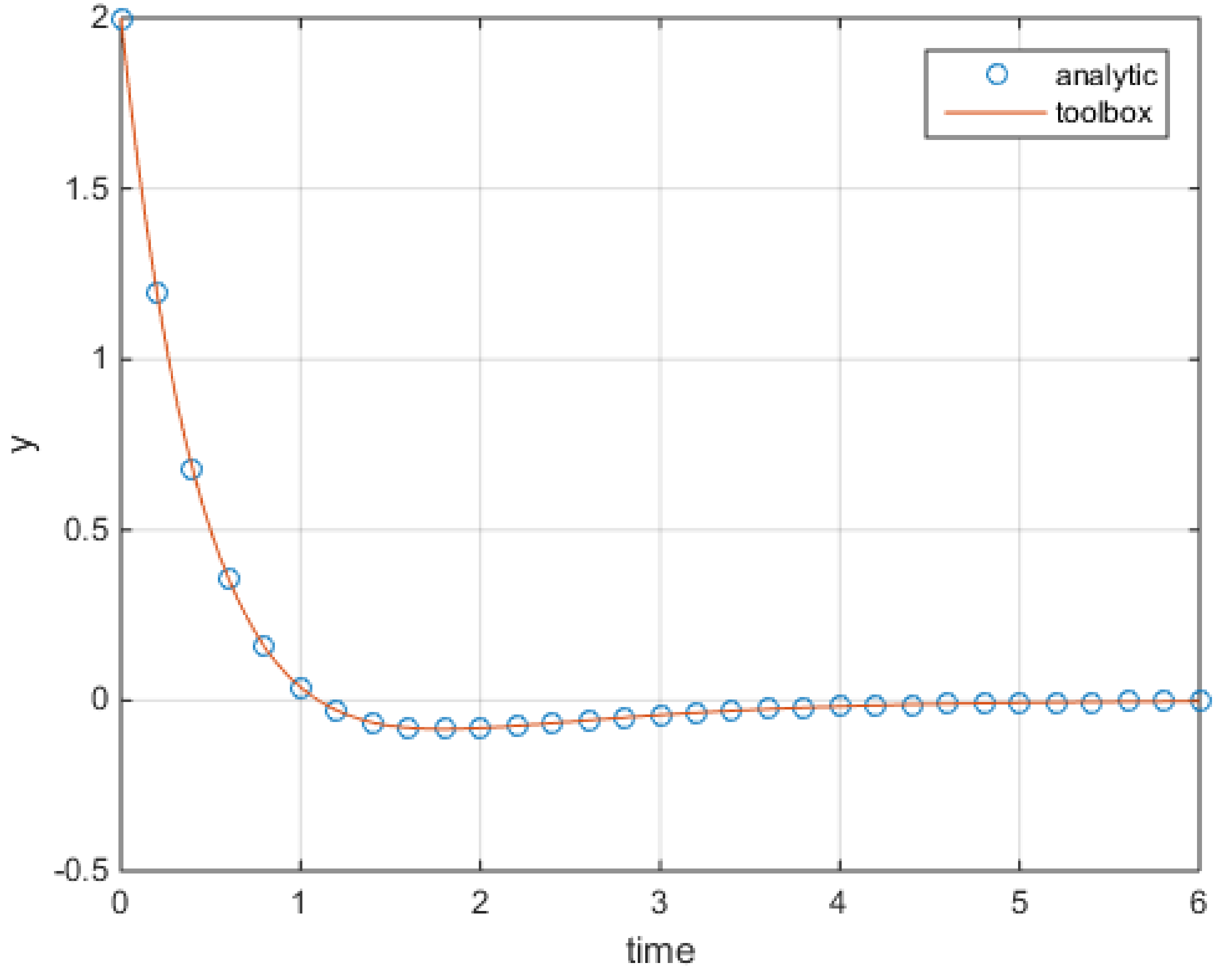

$y(t) = -e^{-t} + 3e^{-2t}$

$$G(s) = \frac{2s+1}{s^2+3s+2} = \frac{1}{s+1} + \frac{3}{s+2}$$

```
% Unit impulse response
clear all, close all

% Direct solution
t1 = 0:0.2:6;
y1 = -exp(-t1) + 3*exp(-2*t1);

% Using control system toolbox
num = [2 1];
den = [1 3 2];
sys = tf(num, den);

[y2, t2] = impulse(sys, 6);
```

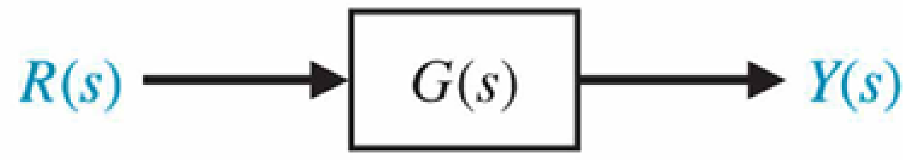
```
% plot
plot(t1, y1, 'o', t2, y2, '-')
xlabel('time');
ylabel('y');
legend('analytic', 'toolbox')
grid
```

# Comparison between Results



# Unit Impulse Response Example 2

$r = \delta(t)$




$$G(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}}$$

Let

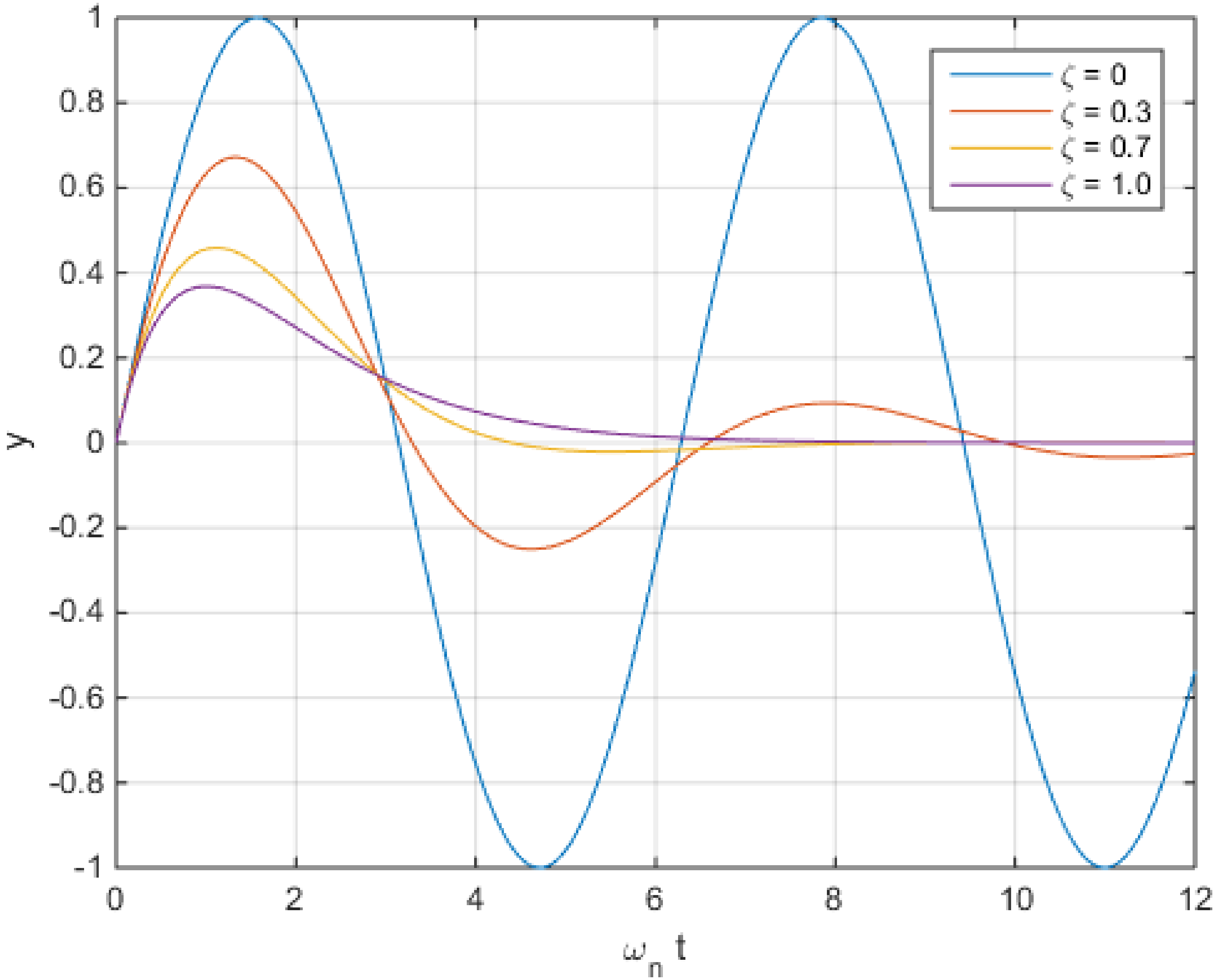
$$\sigma = \zeta\omega_n \;,\;\; \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Then

$$Y(s) = \frac{\omega_n^{\,2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} = \frac{\omega_n^{\,2}}{(s+\sigma)^2 + \omega_d^{\,2}}$$

$$y(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t)$$

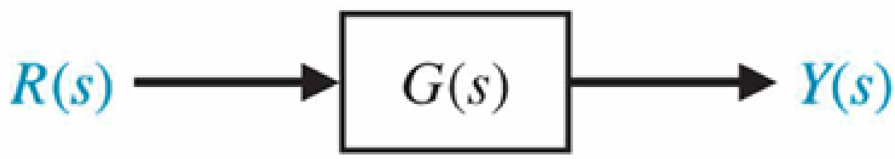
# Impulse Response of a Second Order System



# Unit Step Response

$$r = u(t)$$

$$R = \frac{1}{s}$$




$$y = L^{-1}\left(G(s)\frac{1}{s}\right) = L^{-1}\left(\frac{N(s)}{D(s)}\frac{1}{s}\right) = L^{-1}\left(\frac{1}{s} - \frac{N'(s)}{D(s)}\right)$$

- Basic behavior is still governed by the roots of

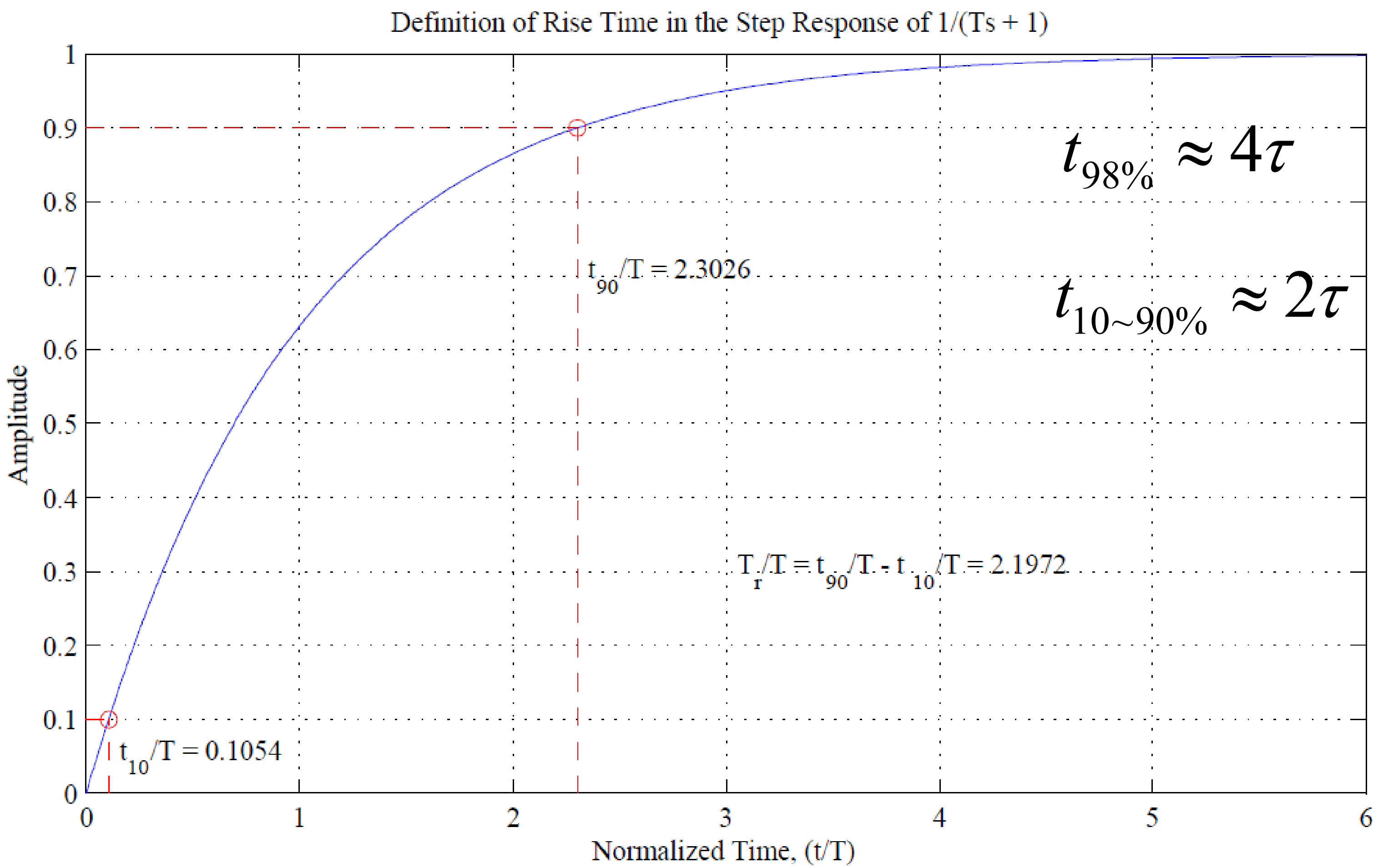
$$D(s) = 0$$

- These roots are called poles

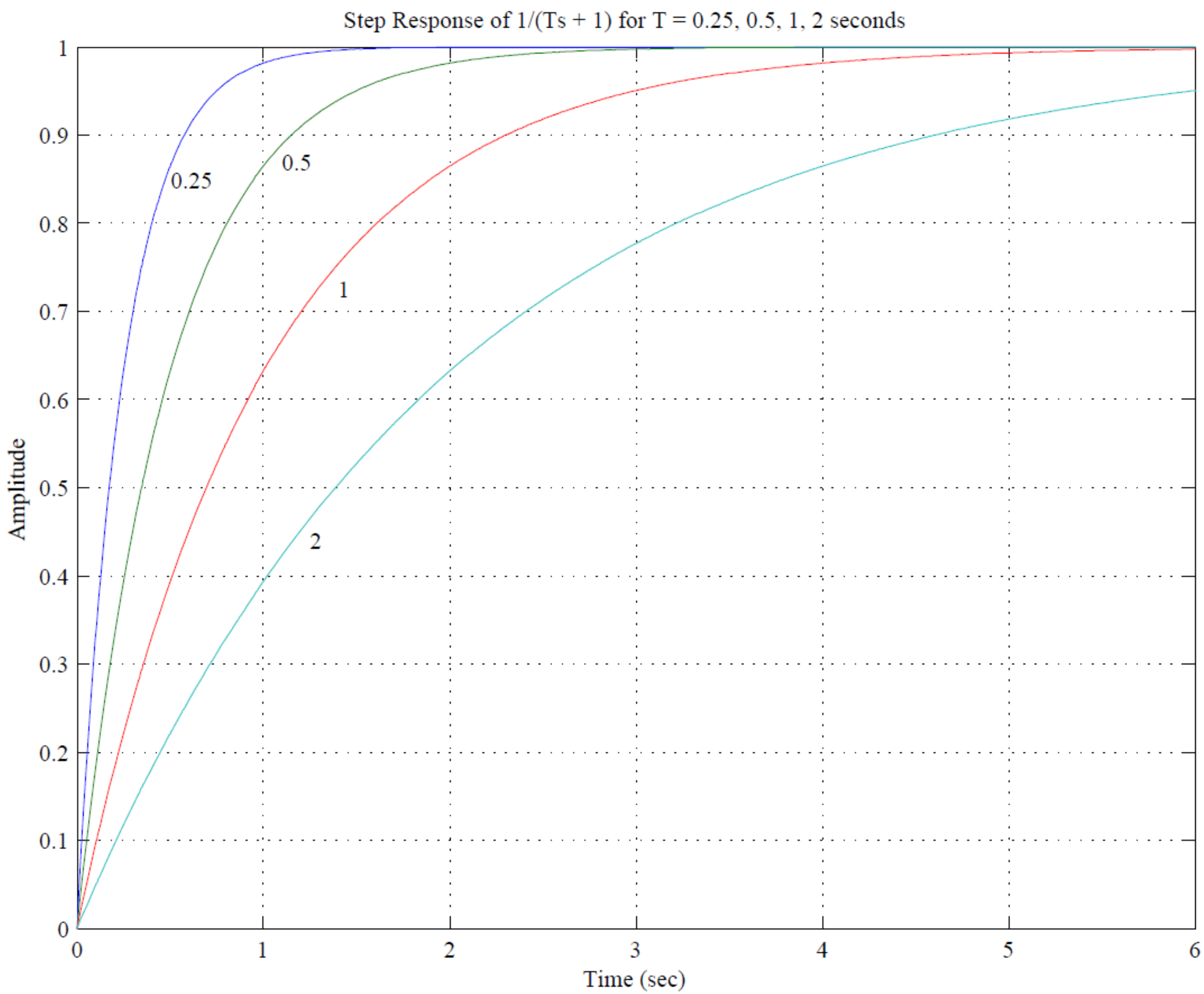
# Step Response of First-Order Systems

$$\frac{Y(s)}{U(s)} = \frac{b}{s+a} = \frac{b/a}{(1/a)s+1} = \frac{c}{\tau s+1}$$

Steady-state response: *c=b/a*
Quickness: $\tau = 1/a$



# Step Response of First-Order Systems



# Step Response of Second-Order Systems

$$r = u(t)$$

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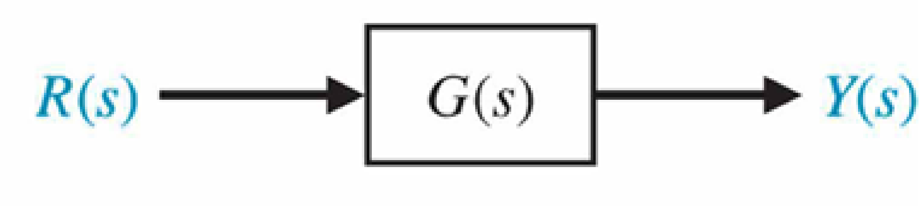


$$G(s) = \frac{\omega_n^{\;2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\;2}}$$

$$y(t) = 1 - e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right) \qquad \left(\sigma = \zeta\omega_n\ ,\ \ \omega_d = \omega_n\sqrt{1-\zeta^2}\right)$$

Poles of $G(s)$ : $\quad s = -\sigma \pm j\omega_d$

# Geometric Representation of Second-Order Poles