Automatic Control

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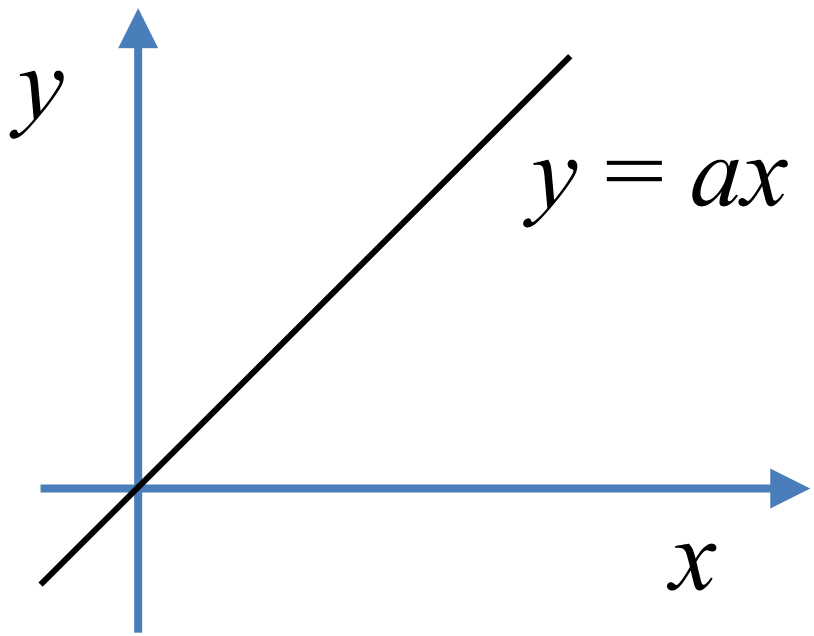
Mathematical Background

Linear System

Linear System – High School View



- A straight line going through (0, 0)
- Is a linear system that simple?

Linear System

- Input goes in, output comes out
- For the same input, output should be the same
- Input
  - Number
  - Function (signal)

Linear System

- Superposition

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

- Homogeneity

$$f(cx) = cf(x)$$

x can be a function, especially a function of time in most physical systems

Linear System Properties

- If x_1 and x_2 are solutions of a function f,

$$f(x_1) = 0$$
$$f(x_2) = 0$$

- $c_1x_1 + c_2x_2$ is also a solution

$$\begin{aligned} & f(c_1x_1 + c_2x_2) \\ & = f(c_1x_1) + f(c_2x_2) && \text{Superposition} \\ & = c_1 f(x_1) + c_2 f(x_2) && \text{Homogeneity} \\ & = 0 \end{aligned}$$

Examples of Non-linear Functions

- Constant $$f(x) = ax + b$$

$$f(x_1) + f(x_2) = (ax_1 + b) + (ax_2 + b) = a(x_1 + x_2) + 2b$$
$$f(x_1 + x_2) = a(x_1 + x_2) + b$$

- Higher order

$$f(x) = x^2$$

$$f(x_1) + f(x_2) = {x_1}^2 + {x_2}^2$$
$$f(x_1 + x_2) = (x_1 + x_2)^2 = {x_1}^2 + {x_2}^2 + 2x_1x_2$$

Linear Differential Equation

$$a_n \frac{d^n f}{dt^n} + a_{n\ \ 1} \frac{d^{n-1} f}{dt^{n-1}} + \cdots + a_2 \frac{d^2 f}{dt^2} + a_1 \frac{df}{dt} = 0$$

- Why is it important?
  - Many important physical system can be modeled or approximated using linear ODE
  - It can be solved!
  - Stability can easily be determined
- Is it really linear? Why?
  - Because differentiation is linear

$$\frac{d}{dt}\left(f+g\right) = \frac{df}{dt} + \frac{dg}{dt} \qquad\qquad \frac{d}{dt}\left(cf\right) = c\frac{df}{dt}$$

How to Solve Linear Differential Equations

$$\dot{x} + ax = 0$$

$$x(0) = x_0$$

Let

$$x = e^{kt}$$

$$\dot{x} = ke^{kt}$$

$$ke^{kt} + ae^{kt} = 0$$

$$(k+a)e^{kt} = 0$$

$$k = -a$$

$$\Rightarrow \quad x = e^{-at}$$

Because the equation is linear

$$x = Ce^{-at} = x_0 e^{-at}$$

How to Solve Linear Differential Equations

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(0) = x_0$$
$$\dot{x}(0) = v_0$$

Let

$$x = e^{at}$$
$$\dot{x} = ae^{at}$$
$$\ddot{x} = a^2 e^{at}$$

$$ma^2 e^{at} + bae^{at} + ke^{at} = 0$$
$$\left(ma^2 + ba + k\right)e^{at} = 0$$
$$a = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

$$\Rightarrow$$

$$x = e^{a_1 t}$$
$$x = e^{a_2 t}$$

How to Solve Linear Differential Equations

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(0) = x_0$$
$$\dot{x}(0) = v_0$$

Because the equation is linear

$$x = C_1 e^{a_1 t} + C_2 e^{a_2 t}$$
$$x(0) = C_1 + C_2 = x_0$$
$$\dot{x}(0) = a_1 C_1 + a_2 C_2 = v_0$$

How to Solve Linear Differential Equations

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$x(0) = x_0$$
$$\dot{x}(0) = v_0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = -\frac{b}{2m} \pm j\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Let

$$\omega = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{1}{2}\frac{b}{\sqrt{mk}}$$

$$-\frac{b}{2m} = -\zeta\omega \qquad \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \omega\sqrt{1-\zeta^2}$$

How to Solve Linear Differential Equations

$$e^{\left(-\zeta\omega \pm j\omega\sqrt{1-\zeta^2}\right)t} = e^{-\zeta\omega t}\left(\cos\omega\sqrt{1-\zeta^2}\,t \pm j\sin\omega\sqrt{1-\zeta^2}\,t\right)$$

$$x = Ae^{-\zeta\omega t}\sin\left(\omega\sqrt{1-\zeta^2}\,t + \varphi\right)$$

Exponential Function

- Basically, the exponential function is the solution of all the ‘homogeneous’ linear differential equation

- But what if the roots are complex?

Exponential of Imaginary Numbers

$$e^{j\omega} \equiv \cos\omega + j\sin\omega$$

This is a very clever definition!

$$f(a)f(b) = f(a+b)$$

Does the key exponentiation identity hold?

$$e^{ja}e^{jb} = (\cos a + j\sin a)(\cos b + j\sin b)$$
$$= \cos a\cos b + j\cos a\sin b + j\sin a\cos b - \sin a\sin b$$
$$= (\cos a\cos b - \sin a\sin b) + j(\sin a\cos b + \cos a\sin b)$$
$$= \cos(a+b) + j\sin(a+b)$$
$$= e^{j(a+b)}$$

Leonhard Euler

- 15 April 1707 – 18 September 1783
- Swiss mathematician and physicist

- Biography
  - Enrolled in University of Basel at the age of 13
  - Received Master of Philosophy in 1723 (dissertation that compared the philosophies of Descartes and Newton)
  - Receiving Saturday afternoon lessons from Johann Bernoulli (Father of Daniel Bernoulli)
  - 1727 – 1741, St. Petersburg Academy
  - 1741 – 1766, Berlin Academy (Frederick the Great of Prussia)
  - 1766 – 1783, St. Petersburg Academy (Catherine the Great)

Notable Work of Euler

- Mathematical notations $f(x), e, \Sigma, i$
- Analysis $e^{i\theta} = \cos\theta + i\sin\theta$
- Number theory $a^{\phi(n)} \equiv 1 \pmod{n}$
- Graph theory $v - e + f = 2$

Notable Work of Euler

- Engineering
  - Euler-Bernoulli beam equation

$$M = -EI \frac{d^2 w}{dx^2}$$

  - Euler equation for inviscid flow

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\mathbf{u} \otimes (\rho \mathbf{u})) + \nabla p = \mathbf{0}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}(E + p)) = 0,$$

Mathematical Background

Laplace Transform

Laplace Transform

Dynamic System Model: Differential Equations $\xrightarrow{\text{Laplace Transform}}$ Control System Model

$$F(s) = L\{f\} = \int_0^\infty f(t)e^{-st}\,dt$$

Inverse Laplace Transform

$$f(t) = L^{-1}\{F\}$$

Pierre-Simon Laplace

- 1749 – 1827
- French scientist and politician

- Bio
  - Studied under d'Alembert
  - 1799, appointed to as the Minister of the Interior by Napoleon
  - 1817, rewarded with the title of marquis by Bourbons

Pierre-Simon Laplace

- Stability of the Solar System
  - Solved a problem left by Euler and Lagrange
- Potential Theory
  - Laplace equation $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$
- Celestial mechanics
- Black hole
- Theory of probabilities
- Laplace transform
- Speed of sound

Examples

$$L\{e^{-at}\} = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$= \frac{1}{s+a}$$

Unit step function

$$u(t) = \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

$$L\{u(t)\} = \int_0^\infty u(t) e^{-st} dt = \int_0^\infty e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_0^\infty = 0 - \left(-\frac{1}{s} \right)$$

$$= \frac{1}{s}$$

Important Properties

Superposition

$$L\{f(t)+g(t)\}=L\{f(t)\}+L\{g(t)\}$$

$\Rightarrow$ Linear!

Homogeneity

$$L\{cf(t)\}=cL\{f(t)\}$$

The Laplace Transform

Table 2.3 Important Laplace Transform Pairs

$f(t)$	$F(s)$
Step function, $u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$	$s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$

Exercise

Find the Laplace transform of

$$\sin \omega t$$

$$t^2$$

Laplace Transform of Differentiation

Differential and Integral Operators

$$\frac{d}{dt} f \Leftrightarrow \mathrm{sF}(\mathrm{s})$$

$$\int_0^t f\,dt \Leftrightarrow \frac{1}{\mathrm{s}}\mathrm{F}(\mathrm{s})$$

Differentiation
- Multiply by s

Integration
- Divide by s

$$L\left\{\frac{df}{dt}\right\} = \int_0^\infty \frac{df}{dt} e^{-st} dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st} dt = sF(s) - f(0)$$

$$L\left\{\frac{d^2 f}{dt^2}\right\} = sL\left\{\frac{df}{dt}\right\} - \frac{df}{dt}(0) = s\big(sF(s) - f(0)\big) - \frac{df}{dt}(0) = s^2 F(s) - sf(0) - \frac{df}{dt}(0)$$

Solution with Laplace Transform

2nd order free vibration system

$$m\ddot{x}+b\dot{x}+kx=0$$

$$x(0)=x_0$$
$$\dot{x}(0)=v_0$$

Laplace Transform

$$L(m\ddot{x}+b\dot{x}+kx)$$
$$=m\left(s^2X(s)-sx(0)-\dot{x}(0)\right)+b\left(sX(s)-x(0)\right)+kX(s)=0$$

$$\left(ms^2+bs+k\right)X(s)=msx_0+mv_0+bx_0$$

$$X(s)=\frac{mx_0s+mv_0+bx_0}{ms^2+bs+k}=\frac{p(s)}{q(s)}$$

Solution with Laplace Transform

2nd order free vibration system example

$$m\ddot{x} + b\dot{x} + kx = 0 \quad \underset{\longleftrightarrow}{\xrightarrow{\text{Laplace Transform}}} \quad X(s) = \frac{mx_0 s + mv_0 + bx_0}{ms^2 + bs + k} = \frac{p(s)}{q(s)}$$

For example, $m = 1, b = 3, k = 2, v_0 = 0$

$$X(s) = \frac{x_0 s + 3x_0}{s^2 + 3s + 2} = \frac{(s+3)x_0}{(s+1)(s+2)} = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

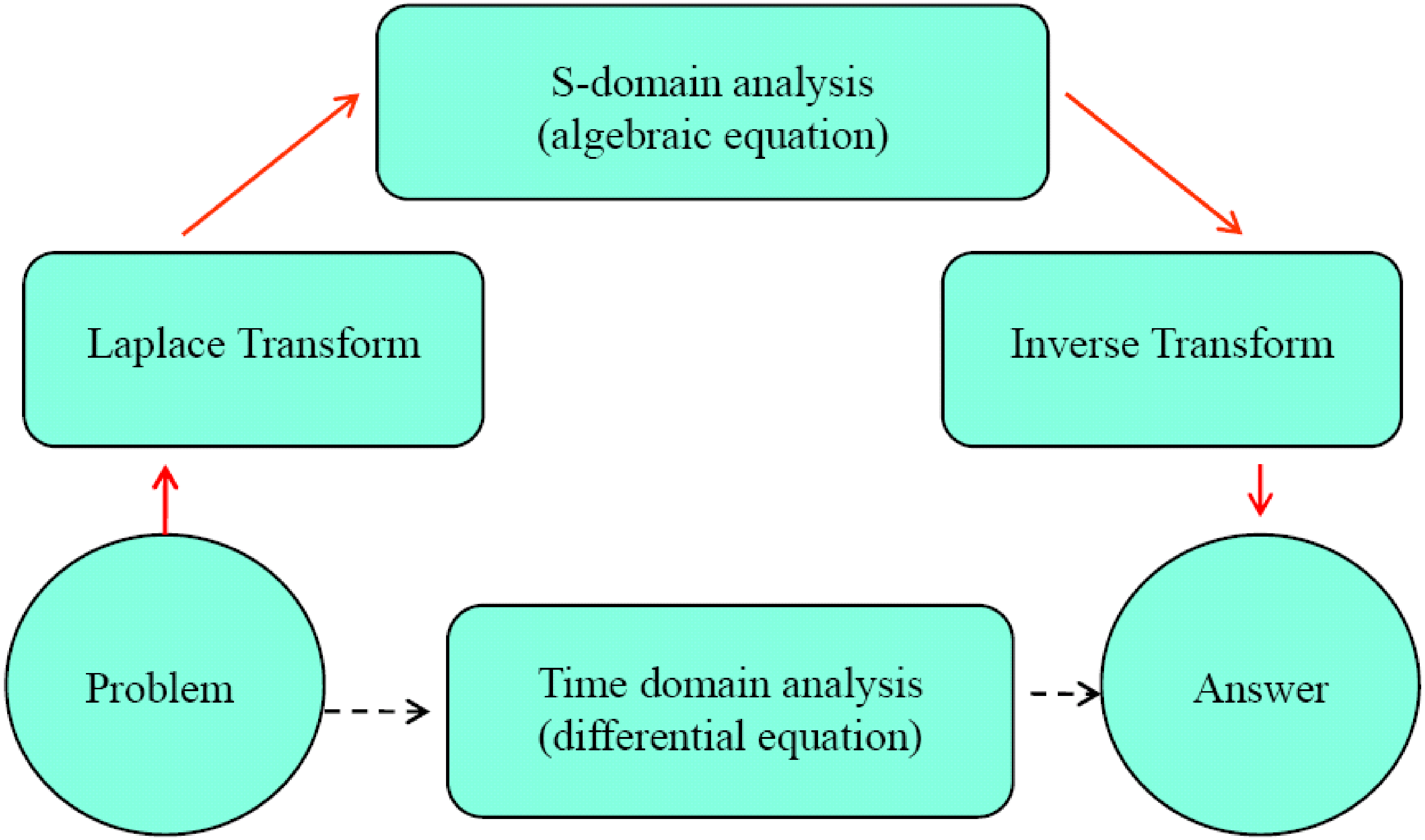
$$k_1 = (s+1)\frac{p(s)}{q(s)}\bigg|_{s=-1} = 2x_0 \qquad k_2 = (s+2)\frac{p(s)}{q(s)}\bigg|_{s=-2} = -x_0$$

Inverse Laplace Transform

$$\Longrightarrow \quad x(t) = L^{-1}\{X(s)\} = L^{-1}\left\{\frac{2x_0}{s+1}\right\} + L^{-1}\left\{\frac{-x_0}{s+2}\right\}$$

$$\Longrightarrow \quad = x_0\left(2e^{-t} - e^{-2t}\right)$$

Solution Process



< Relation of Time domain and s-Domain >

Distinct Real Root Exercise

$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Find y(t)

Distinct Real Root Exercise

$$Y(s)=\frac{(s+2)(s+4)}{s(s+1)(s+3)}$$

Find y(t)

$$Y(s)=\frac{(s+2)(s+4)}{s(s+1)(s+3)}=\frac{C_1}{s}+\frac{C_2}{s+1}+\frac{C_3}{s+3}$$

$$C_1=sY(s)\Big|_{s=0}=\frac{(s+2)(s+4)}{(s+1)(s+3)}\Bigg|_{s=0}=\frac{8}{3}$$

$$C_2=(s+1)Y(s)\Big|_{s=-1}=\frac{(s+2)(s+4)}{s(s+3)}\Bigg|_{s=-1}=-\frac{3}{2}$$

$$C_3=(s+3)Y(s)\Big|_{s=-3}=\frac{(s+2)(s+4)}{s(s+1)}\Bigg|_{s=-3}=-\frac{1}{6}$$

$$Y(s)=\left(\frac{8}{3}\right)\frac{1}{s}-\left(\frac{3}{2}\right)\frac{1}{s+1}-\left(\frac{1}{6}\right)\frac{1}{s+3}$$

$$y(t)=\frac{8}{3}-\frac{3}{2}e^{-t}-\frac{1}{6}e^{-3t}$$

Forced Vibration Example

$$\ddot{x} - x = t \qquad \begin{aligned} x(0) &= 1 \\ \dot{x}(0) &= 1 \end{aligned}$$

Laplace Transform

$$L(\ddot{x} - x) = L(t) \qquad \left(s^2 X(s) - sx(0) - \dot{x}(0)\right) - X(s) = \left(s^2 - 1\right)X(s) - s - 1 = \frac{1}{s^2}$$

$$X(s) = \frac{s+1}{s^2-1} + \frac{1}{s^2\left(s^2-1\right)} = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

$$= \frac{1}{s-1} + \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) - \frac{1}{s^2} = \frac{3}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} - \frac{1}{s^2}$$

$$x(t) = \frac{3}{2}L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s^2}\right) = \frac{3}{2}e^{t} + \frac{1}{2}e^{-t} - t$$

One more Laplace Transform Formula

$$L\left\{e^{-at}\cos bt\right\} = \frac{s+a}{\left(s+a\right)^2 + b^2}$$

$$L\left\{e^{-at}\sin bt\right\} = \frac{b}{\left(s+a\right)^2 + b^2}$$

Distinct Complex Root Exercise

$\ddot{y} + \dot{y} + y = u(t)$ $\qquad$ $y(0) = 0, \dot{y}(0) = 0$ $\qquad$ Find y(t)

Distinct Complex Root Exercise

$\ddot{y} + \dot{y} + y = u(t)$ $\quad$ $y(0) = 0, \dot{y}(0) = 0$ $\quad$ Find y(t)

$$Y(s) = \frac{1}{s(s^2 + s + 1)} \qquad Y(s) = \frac{1}{s(s^2 + s + 1)} = \frac{C_1}{s} + \frac{C_2 s + C_3}{s^2 + s + 1}$$

Find residual

$$C_1 = sY(s)\Big|_{s=0} = \left.\frac{1}{(s^2 + s + 1)}\right|_{s=0} = 1$$

Sometimes it is better to just compare the coefficients

$$s^2 + s + 1 + s(C_2 s + C_3) = 1$$

$$(1 + C_2)s^2 + (1 + C_3)s = 0$$

$$C_2 = C_3 = -1$$

Distinct Complex Root Exercise

$$Y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{s} - \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)} \frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= 1 - e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

Final Value Theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

$$\lim_{s\to 0}\left(\int_0^\infty \frac{dx}{dt}(t)e^{-st}dt\right) = \lim_{s\to 0}(sX(s) - x(0)) = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty \lim_{s\to 0}\frac{dx}{dt}(t)e^{-st}dt = \int_0^\infty \frac{dx}{dt}(t)dt = x(\infty) - x(0)$$

For the previous system

$$X(s) = \frac{(s+3)x_0}{(s+1)(s+2)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{s(s+3)x_0}{(s+1)(s+2)} = 0$$

Verification with the solution

$$x(t=\infty) = x_0\left(2e^{-t} - e^{-2t}\right)\Big|_{t=\infty} = 0$$