Automatic Control

Hak-Tae Lee

Frequency Response

Bode Plot 2

1^{st} Order System

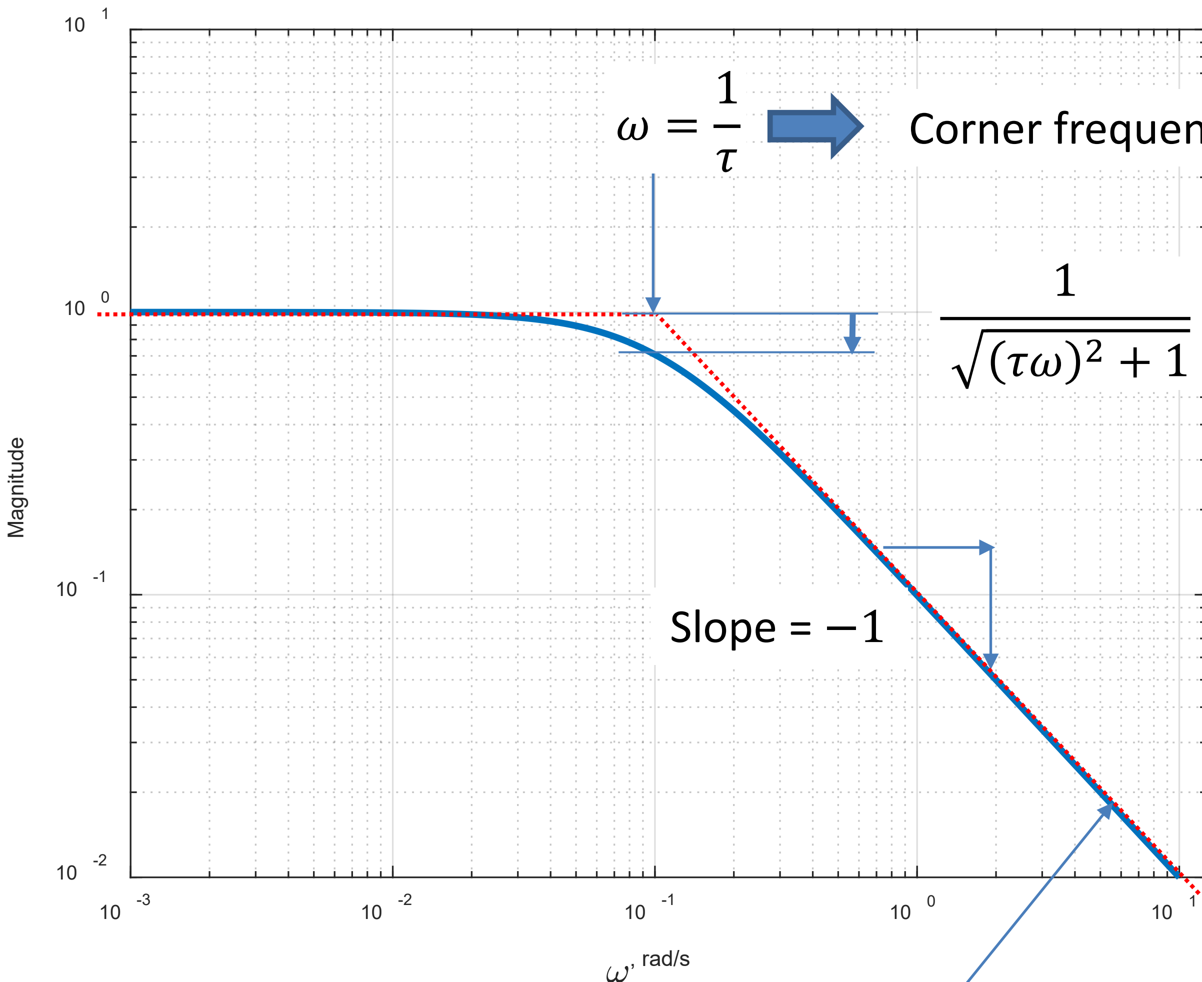
- $G(s) = \frac{1}{\tau s+1}$

- Magnitude
  - $|G(j\omega)| = \frac{1}{\sqrt{(\tau\omega)^2+1}}$
- Phase
  - $\angle G(j\omega) = -\angle \tan^{-1} \tau\omega$

- Corner frequency
  - $\tau\omega = 1 \rightarrow \omega = \frac{1}{\tau}$
  - Magnitude $\rightarrow \frac{1}{\sqrt{2}}$
  - Phase $\rightarrow -45°$

Magnitude Plot in Detail

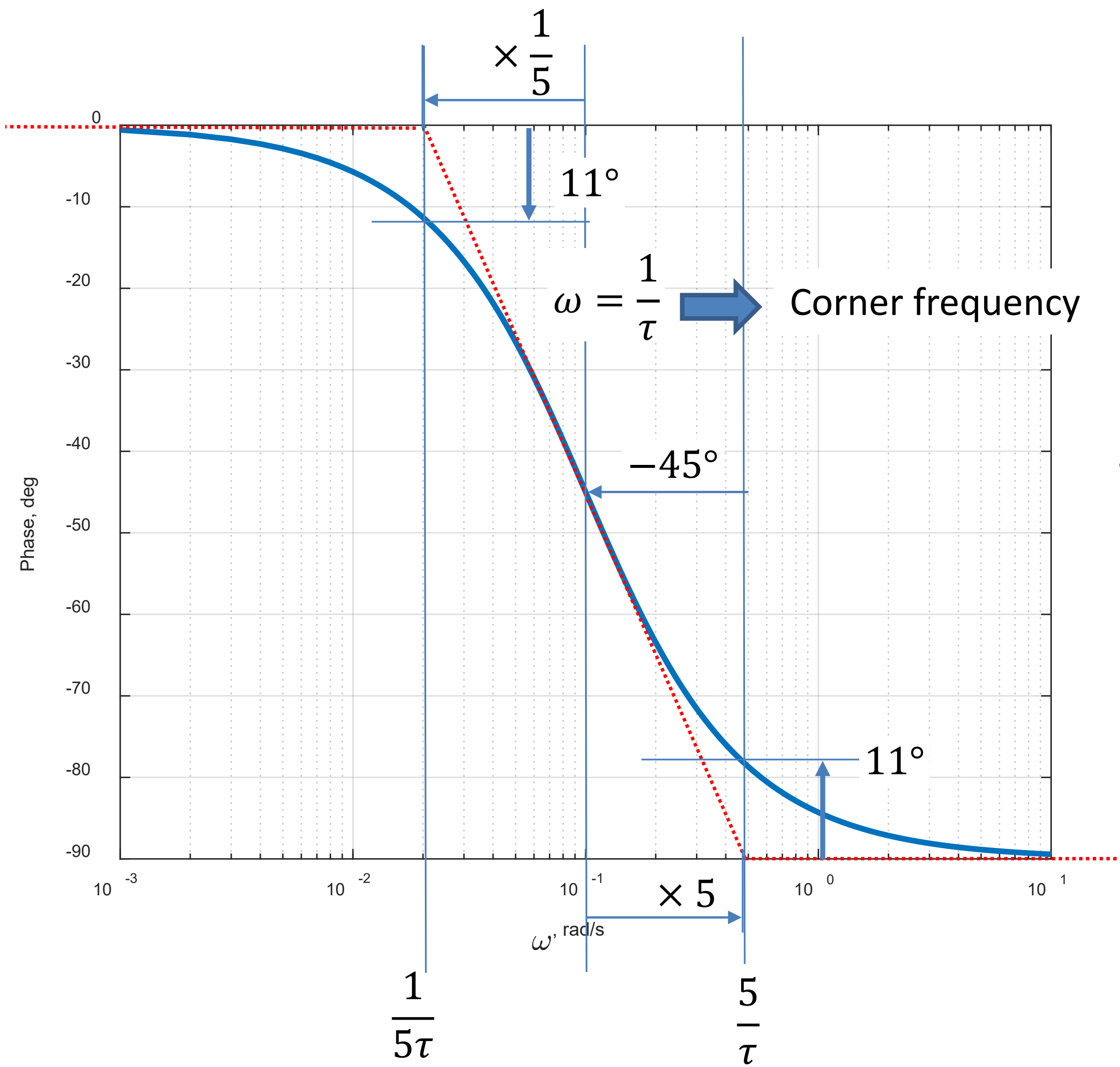
$$x = \frac{1}{\sqrt{(\tau\omega)^2 + 1}} \sin(\omega t + \phi)$$



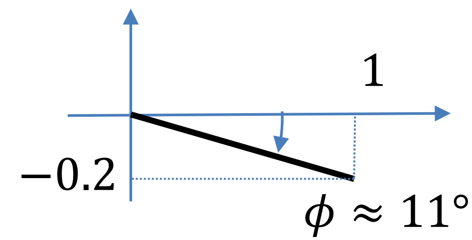
$$\frac{1}{\sqrt{(\tau\omega)^2 + 1}} = \frac{1}{\sqrt{2}} \approx 0.7$$

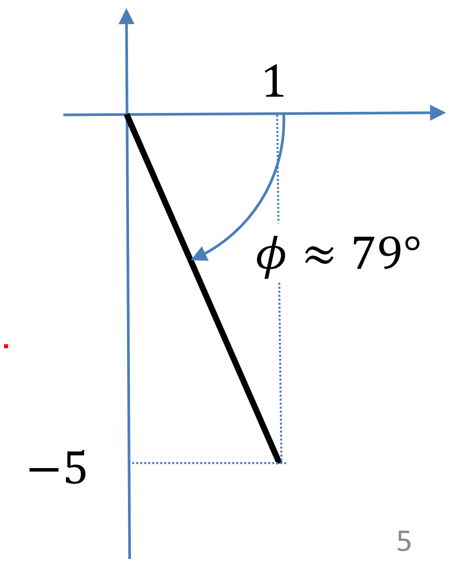
$$\log_{10}|G(j\omega)| = -\log_{10}\omega - \log_{10}\tau$$

Phase Plot in Detail



$\tan \phi = -\tau\omega$



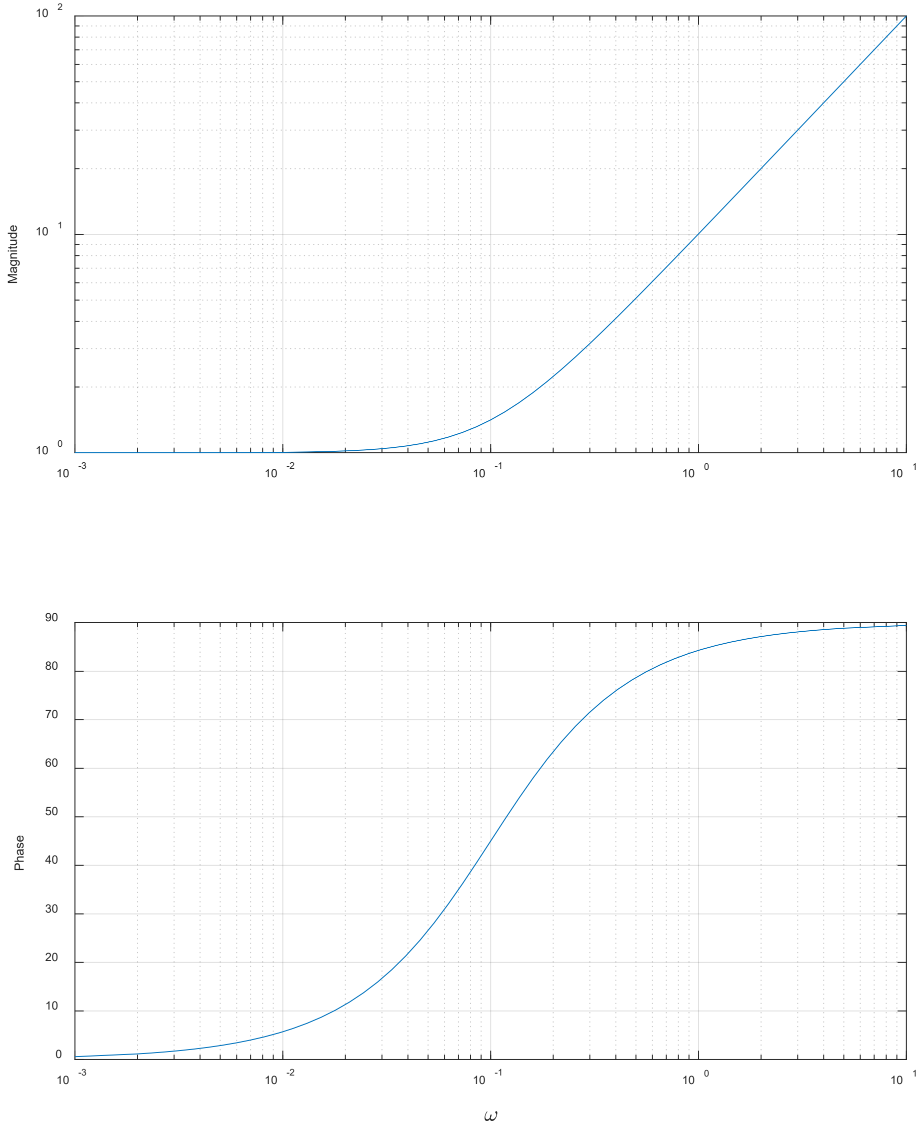


What about Zeros?

- $G(s) = \tau s + 1$
  - Magnitude – recall the characteristics of log
    - $\log_{10}|G(j\omega)| = \log_{10}|j\tau\omega + 1| = \log_{10}|j\tau\omega + 1|$
    - For small ω: $\log_{10}|G(j\omega)| \rightarrow 0$
    - For large ω: $\log_{10}|G(j\omega)| \rightarrow (+1)\log_{10}\omega + \log_{10}\tau$
  - Phase
    - $\angle G(j\omega) = \tan^{-1}\tau\omega$
    - For small ω → $\angle G(j\omega) \rightarrow 0$
    - For large ω → $\angle G(j\omega) \rightarrow 90°$

- In general $G(s) = (\tau s + 1)^n$
  - Magnitude → slope of n
  - Phase → from 0 to $n \cdot 90°$

Exact Opposite of $\dfrac{1}{\tau s+1}$



Frequency Response of s^n

- Magnitude
  - $|G(j\omega)| = |(j\omega)^n| = |\omega^n|$
    - $|G(j\omega)| = 1$ when $\omega = 1$
  - $\log_{10}|G(j\omega)| = \log_{10}|(j\omega)^n| = n\log_{10}|\omega|$

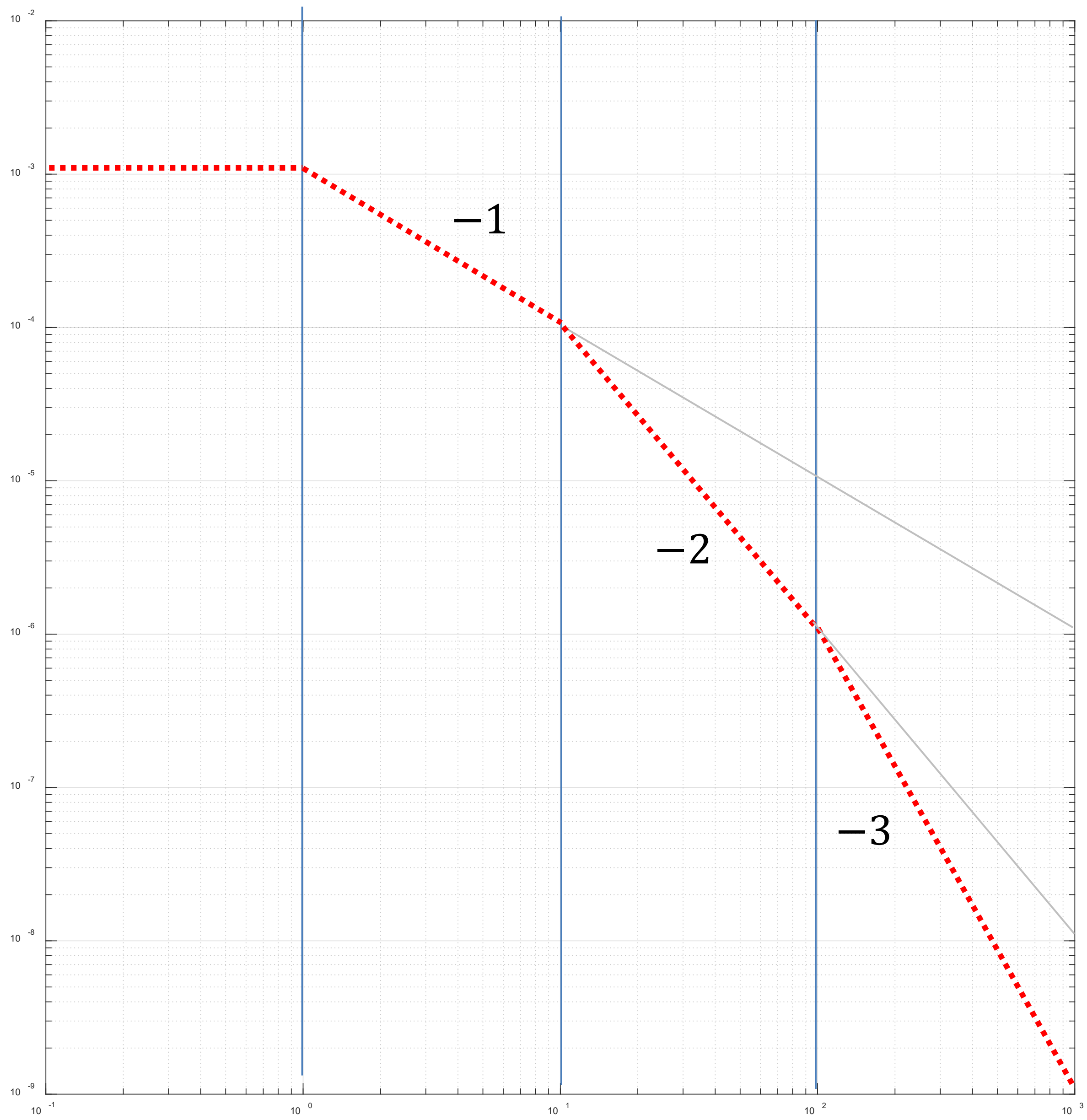
- Phase
  - $\angle G(j\omega) = \angle(j\omega)^n = n\angle j = 90°n$ (constant)
    - $n = \pm 1 \rightarrow \pm 90°$
    - $n = \pm 2 \rightarrow \pm 180°$
    - $n = \pm 3 \rightarrow \pm 270°$
    - $n = \pm 4 \rightarrow \pm 360°$

Example 1

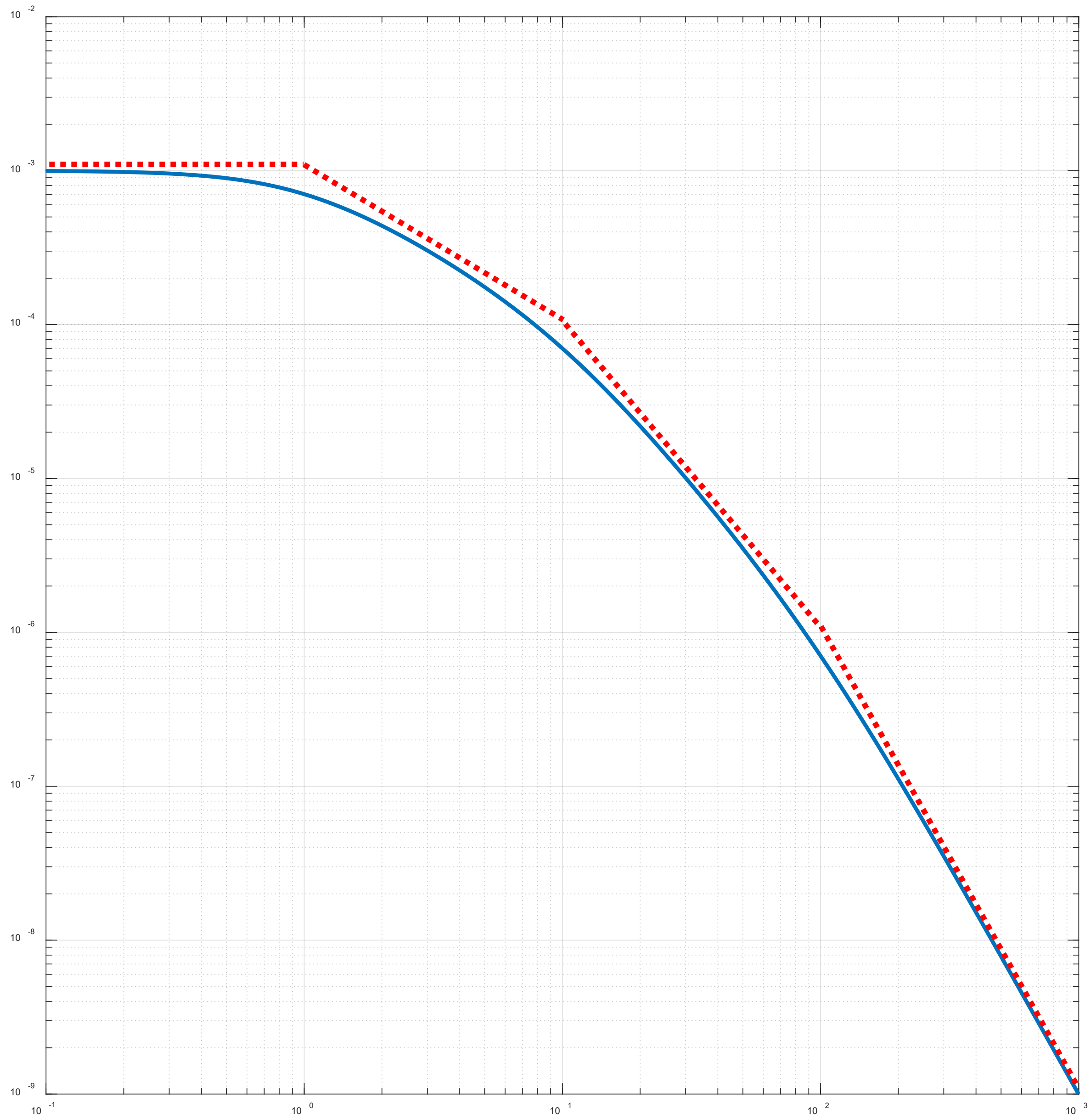
$$G(s) = \frac{1}{(s+1)(s+10)(s+100)}$$

- Step 1
  - Rewrite to a standard form
    - $G(s) = \frac{\frac{1}{10}\cdot\frac{1}{100}}{(s+1)\left(\frac{s}{10}+1\right)\left(\frac{s}{100}+1\right)}$
  - Magnitude at 0 frequency: 10^{-3}

- Step 2
  - Identify corner frequencies
    - 1
      - Magnitude slope: $-1 \rightarrow -1$
      - Phase: $-90° \rightarrow -90°$
    - 10
      - Magnitude slope: $-1 \rightarrow -2$
      - Phase: $-90° \rightarrow -180°$
    - 100
      - Magnitude slope: $-1 \rightarrow -3$
      - Phase: $-90° \rightarrow -270°$

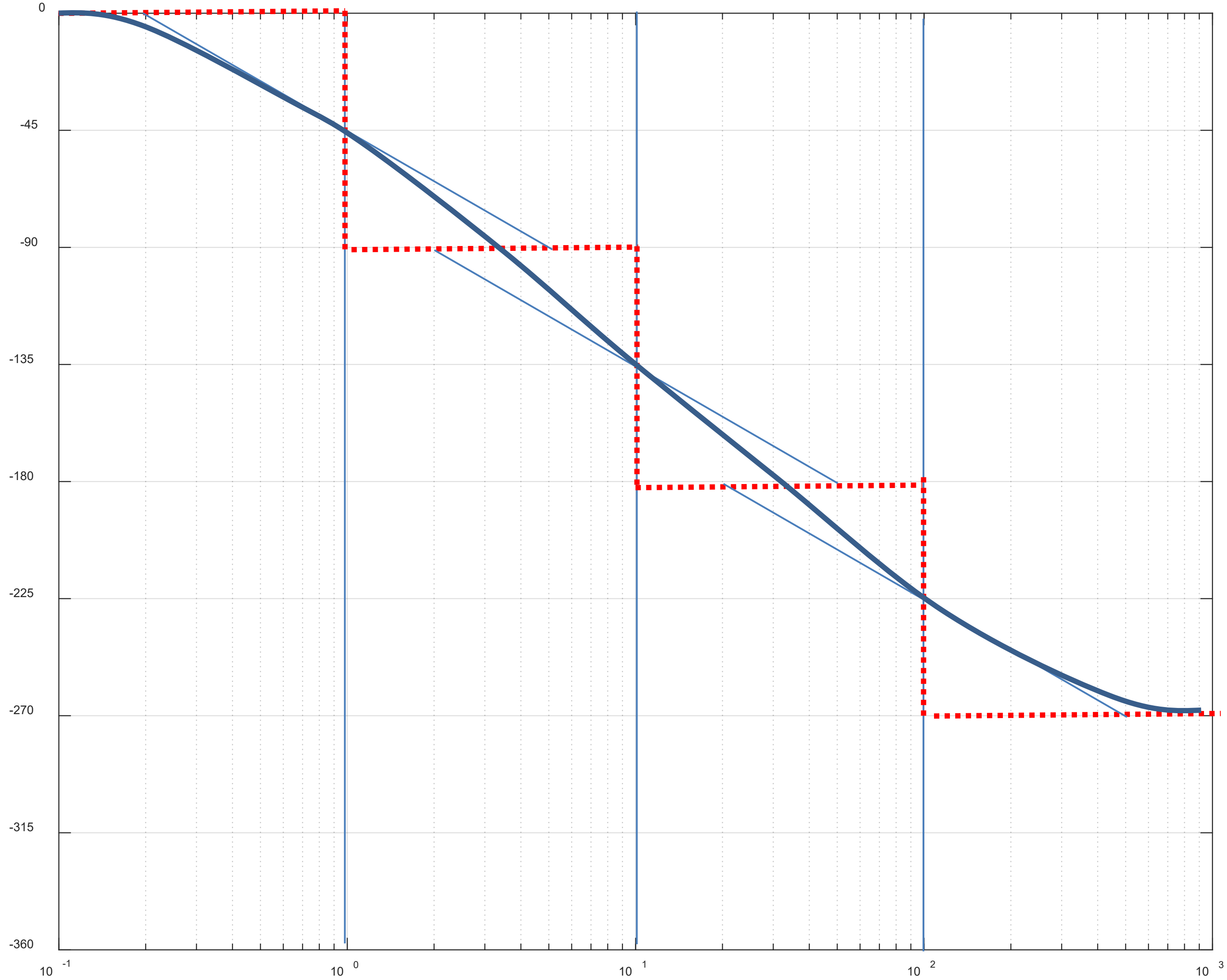


10 -2
10 -3
10 -4
10 -5
10 -6
10 -7
10 -8
10 -9
10 -1
10 0
10 1
10 2
10 3
−1
−2
−3

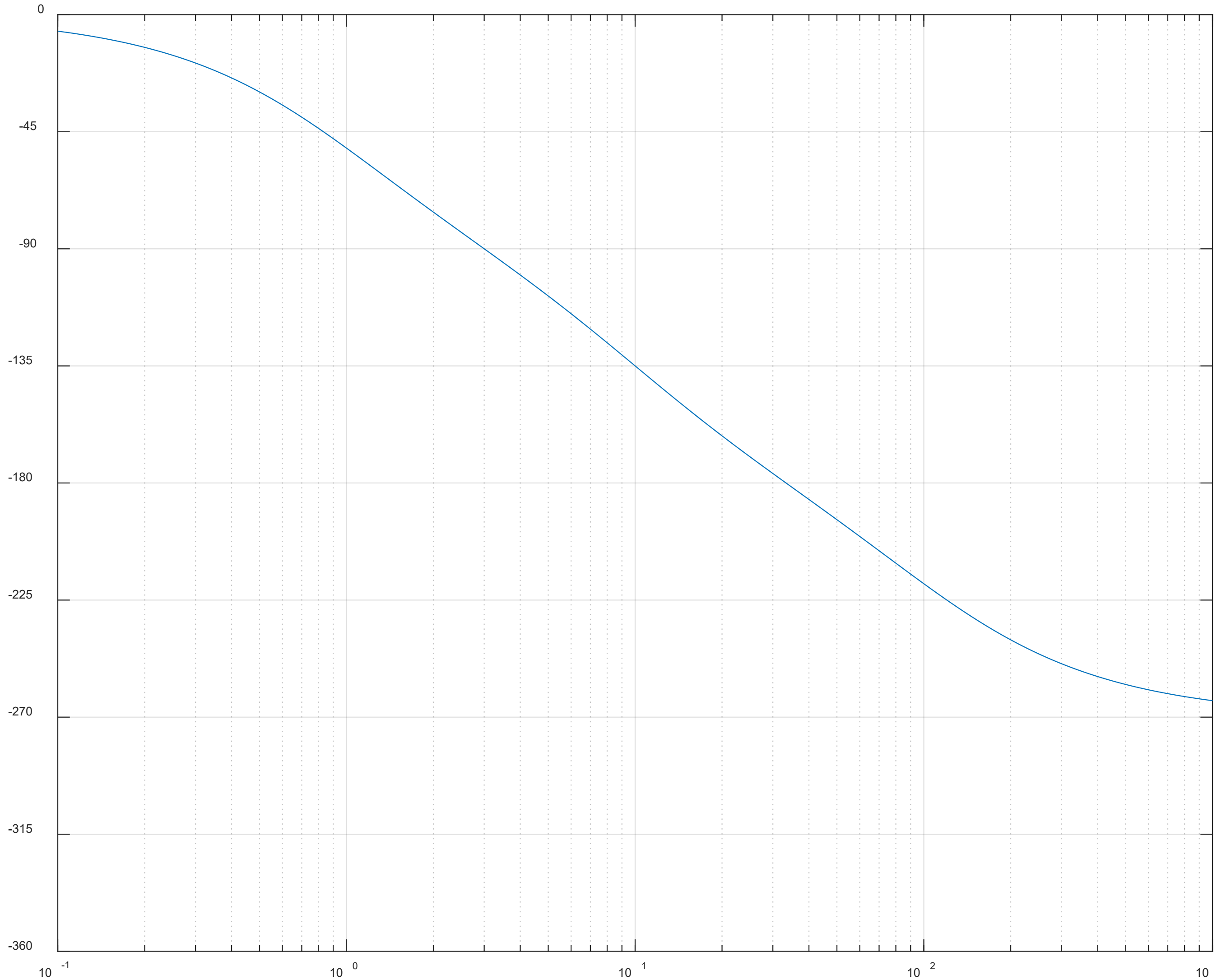


10 -2
10 -3
10 -4
10 -5
10 -6
10 -7
10 -8
10 -9
10 -1
10 0
10 1
10 2
10 3

Phase



Phase - Actual

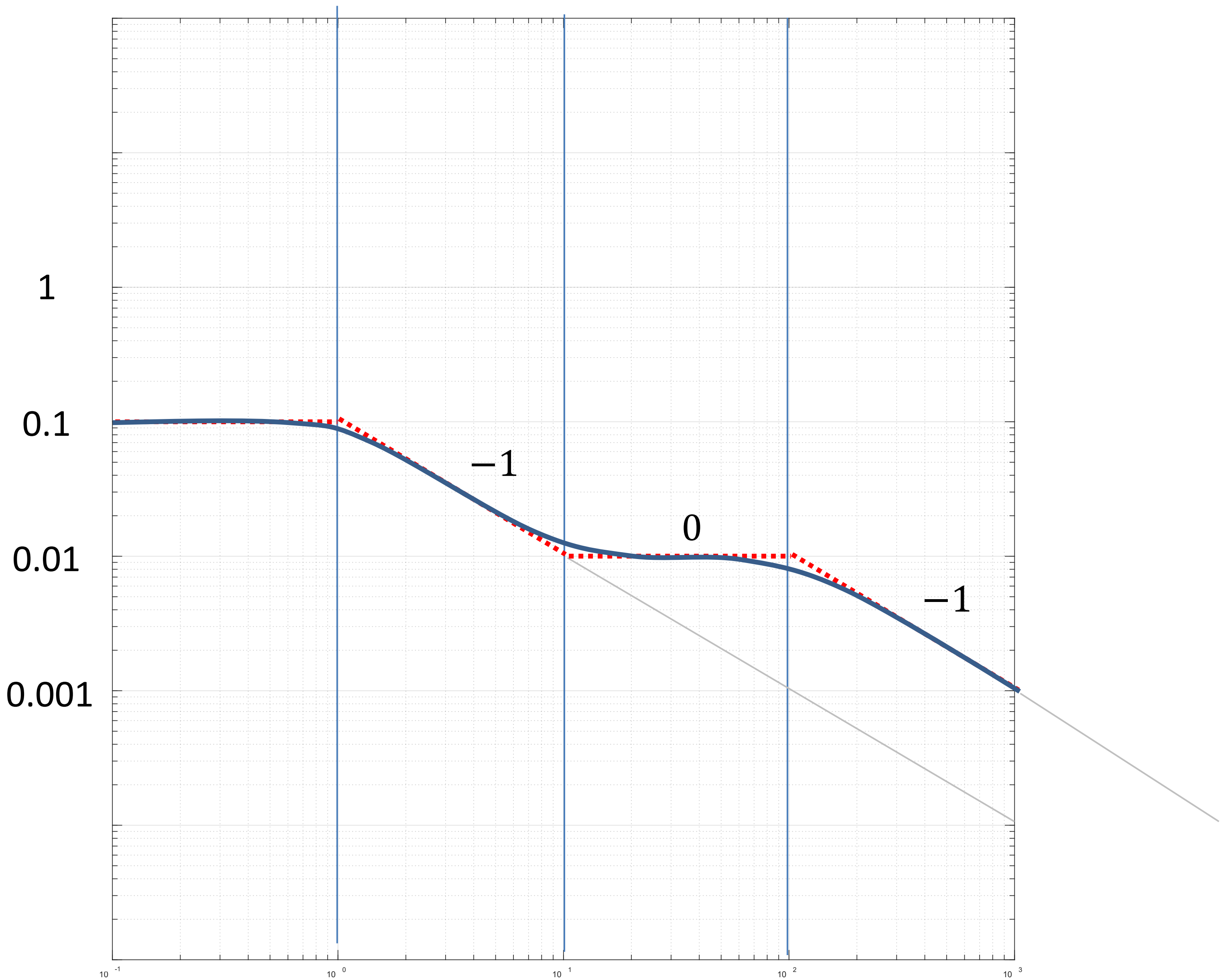


Example 2

$$G(s) = \frac{(s+10)}{(s+1)(s+100)}$$

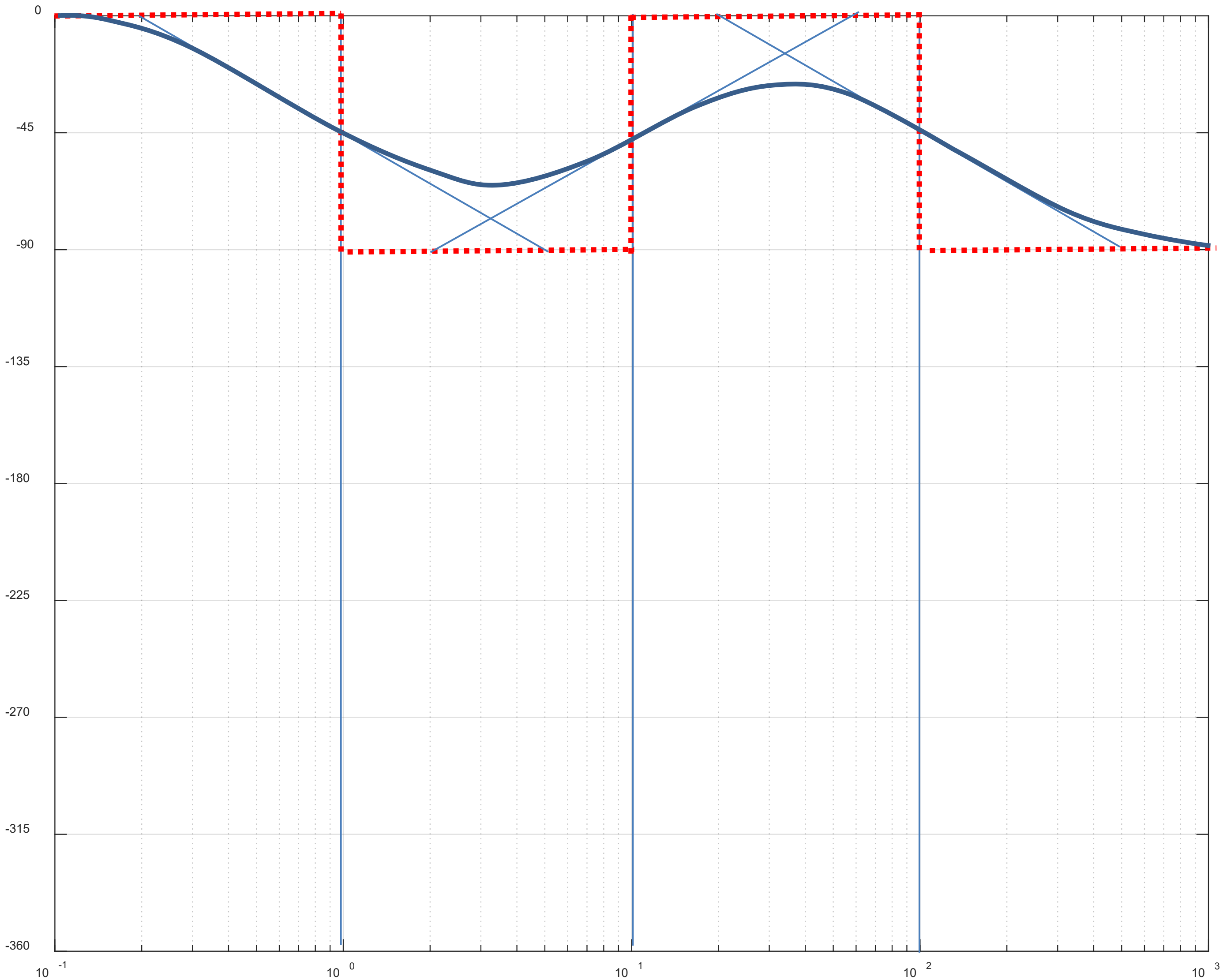
- Step 1
  - Rewrite to a standard form
    - $G(s) = \frac{\left(\frac{s}{10}+1\right)\cdot\frac{1}{100}}{(s+1)\frac{1}{10}\left(\frac{s}{100}+1\right)} = \frac{1}{10}\frac{\left(\frac{s}{10}+1\right)}{(s+1)\left(\frac{s}{100}+1\right)}$
  - Magnitude at 0 frequency: 10^{-1}

- Step 2
  - Identify corner frequencies
    - 1
      - Magnitude slope: $-1 \rightarrow -1$
      - Phase: $-90° \rightarrow -90°$
    - 10
      - Magnitude slope: +1 $\rightarrow 0$
      - Phase: $+90° \rightarrow 0°$
    - 100
      - Magnitude: $-1 \rightarrow -1$
      - Phase: $-90° \rightarrow -90°$



1
0.1
0.01
0.001
−1
0
−1
10 -1
10 0
10 1
10 2
10 3

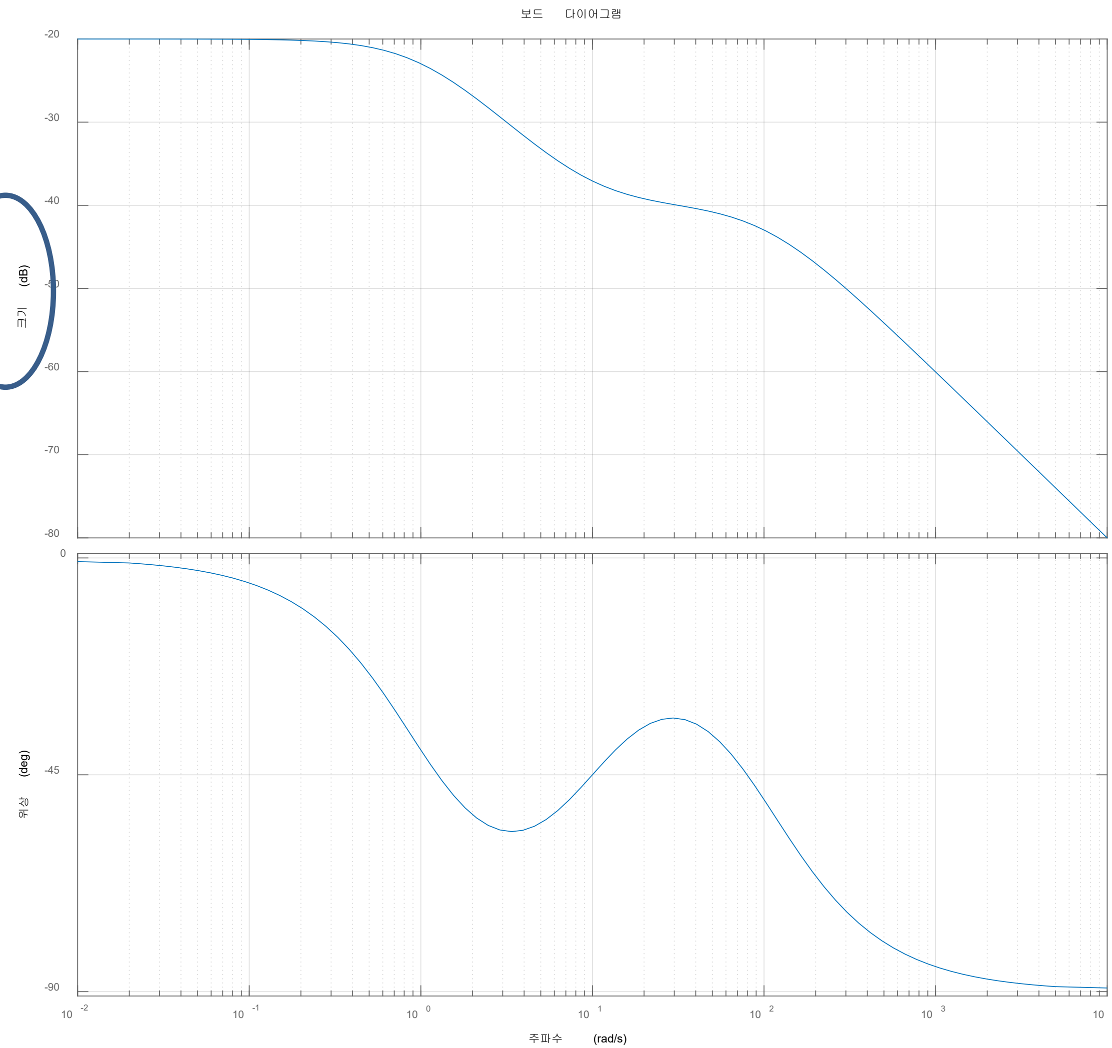
Phase



Bode Plot Using Matlab

$$dB = 20\log_{10}|G(j\omega)|$$

```
>> num = [1 10]

num =

    1    10

>> den = conv([1 1], [1 100])

den =

    1   101   100

>> bode(num, den)
>> grid on
```



dB Scale

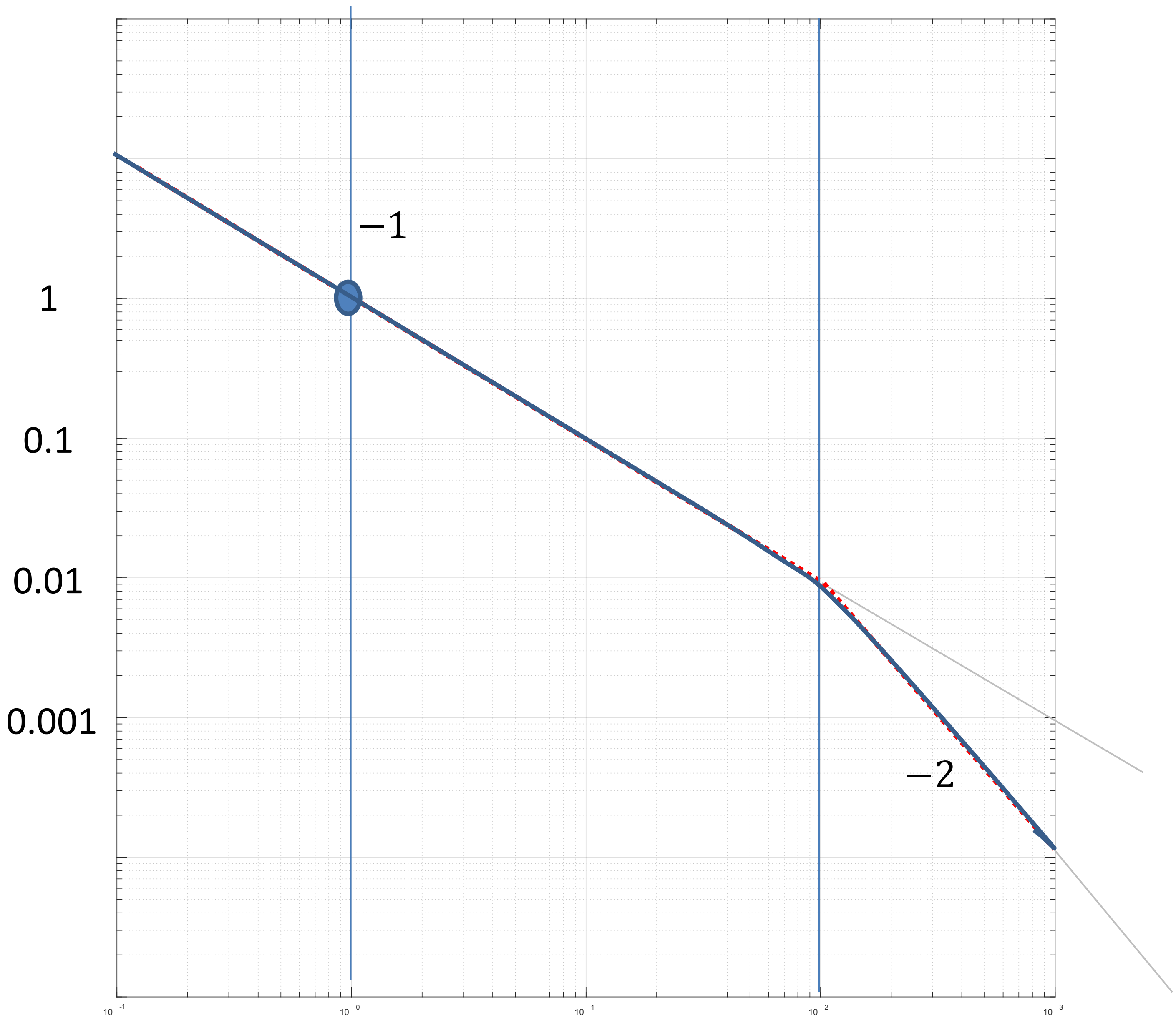
- Definition
  - $dB = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$
  - Scale for power or energy ratio
  - $-10\ dB$ means output is $\frac{1}{10}$ of input

- For Bode plots
  - $dB = 20\log_{10}\left(\frac{y}{u}\right) = 20\log_{10}|G(j\omega)|$
  - In most cases, input and output are ‘signals’ that is in ‘voltage’
  - $P \propto V^2$ → So 20 is used for ratios that represent square root of power

Example 3

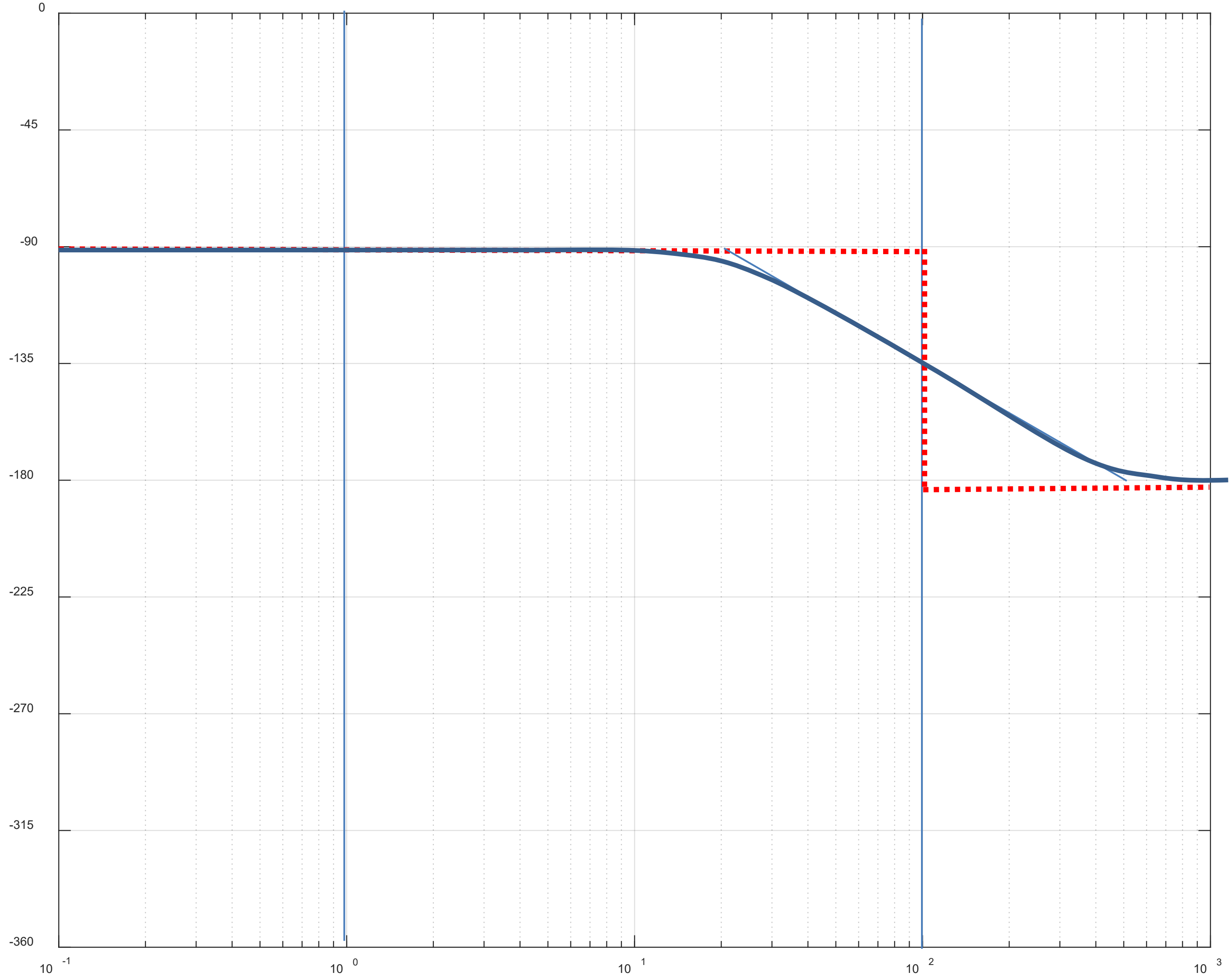
$$G(s) = \frac{1}{s(s/100 + 1)}$$

- Step 1
  - Already in standard form
- Step 2
  - Corner frequencies
    - $0??$
      - $|G(j\omega)| = 1$ when $\omega = 1$
      - Slope → -1
      - Phase → $-90°$
    - 100
      - Magnitude slope: $-1 \rightarrow -2$
      - Phase: $-90° \rightarrow -180°$

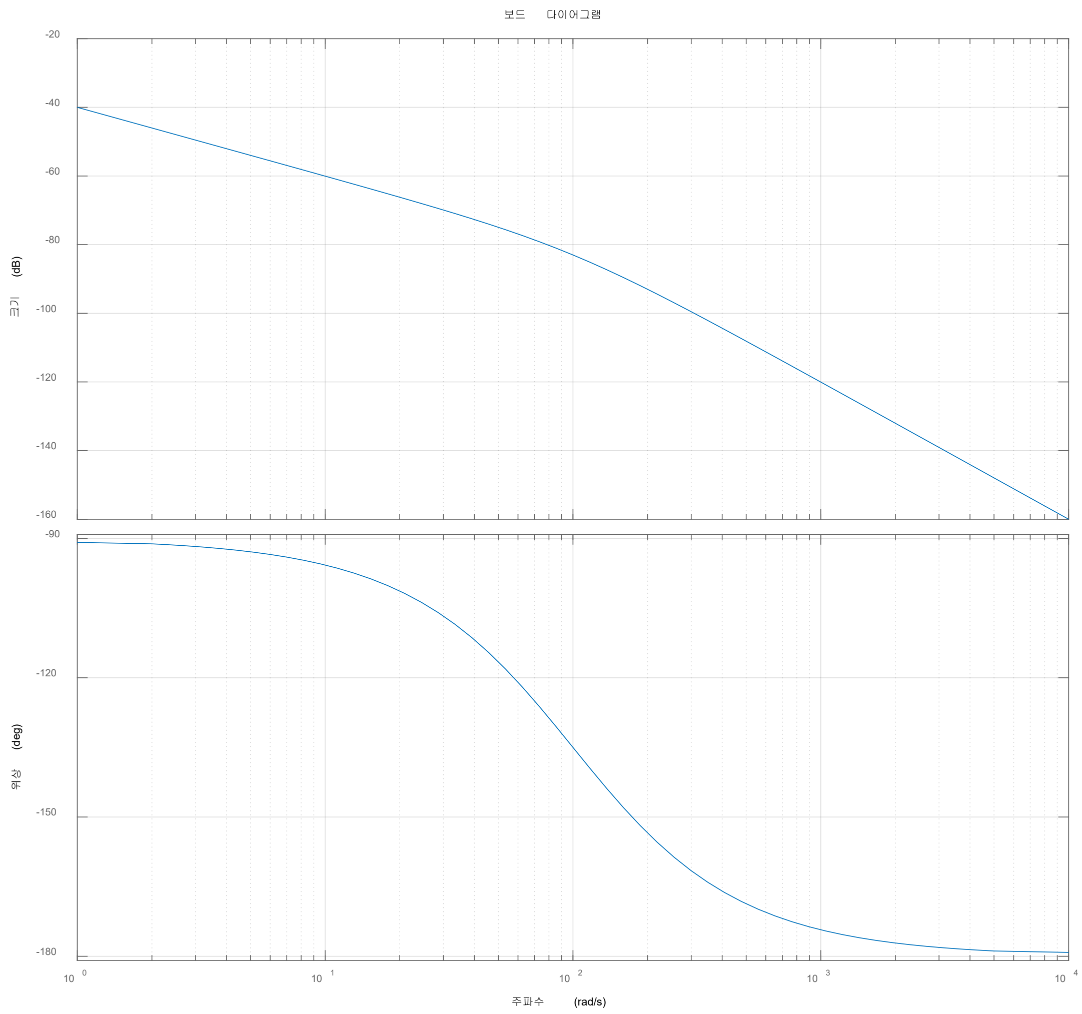


−1
1
0.1
0.01
0.001
−2
10 -1
10 0
10 1
10 2
10 3

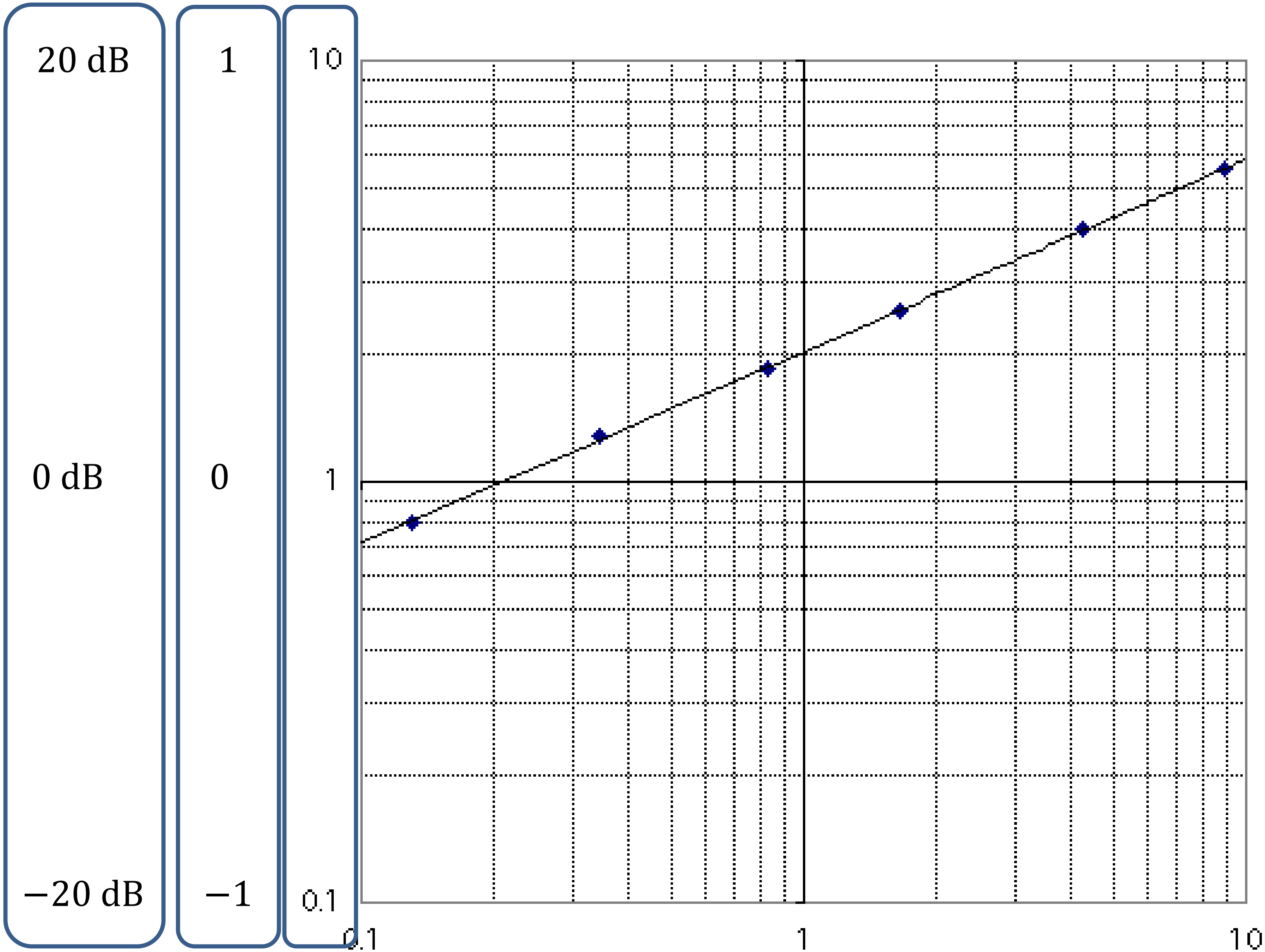
Phase



Verification



Log Scale



2^{nd} Order System

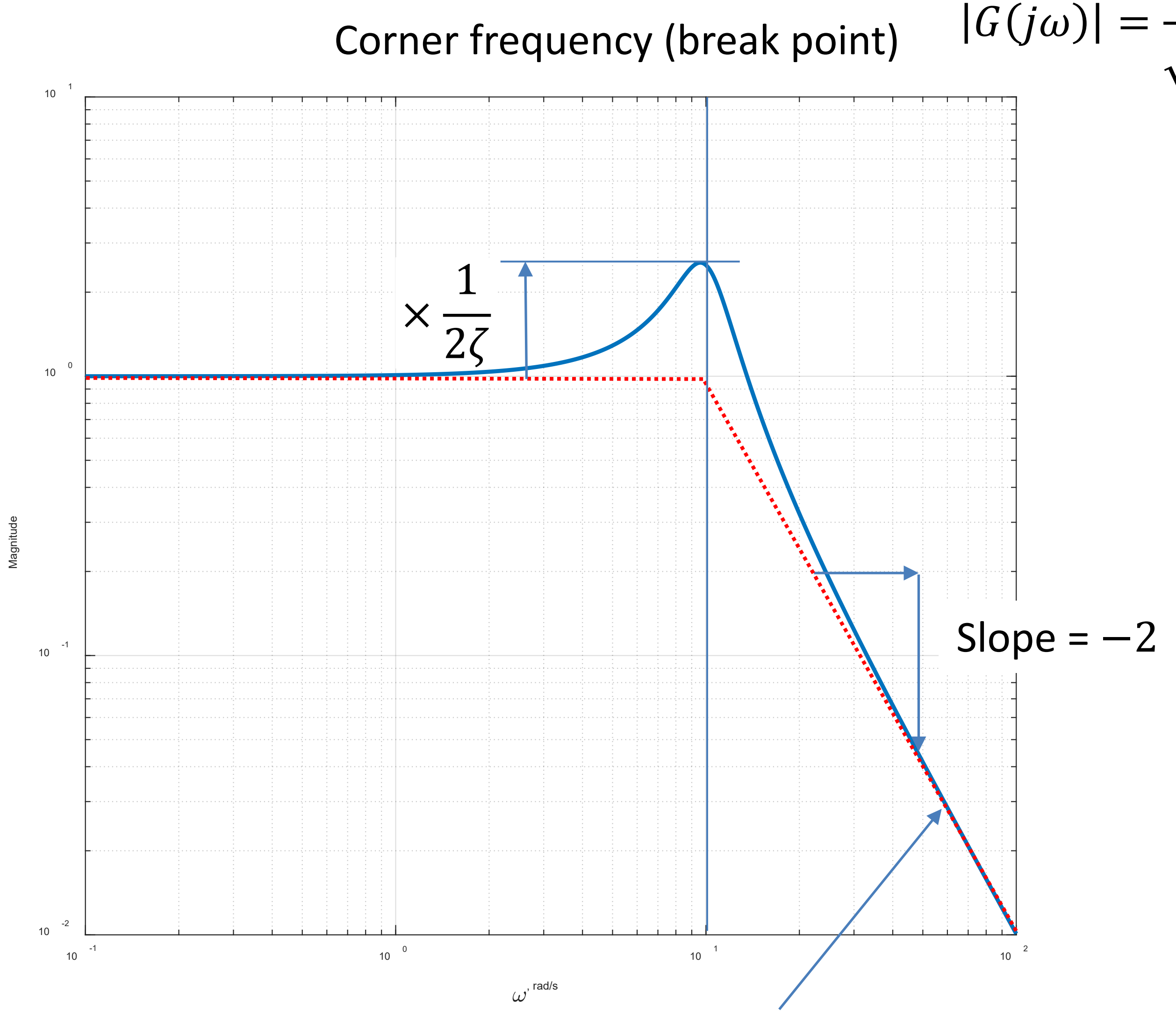
- $G(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$

- Magnitude
  - $|G(j\omega)| = \frac{1}{\sqrt{(1-\lambda^2)^2 + (2\zeta\lambda)^2}}$ where $\lambda = \frac{\omega}{\omega_n}$
- Phase
  - $\angle G(j\omega) = -\tan^{-1}\frac{2\zeta\lambda}{1-\lambda^2}$

- Corner frequency (break point)
  - $\lambda = 1 \rightarrow \omega = \omega_n$
  - Magnitude $\rightarrow \frac{1}{2\zeta}$
  - Phase $\rightarrow -90°$

Magnitude Plot in Detail

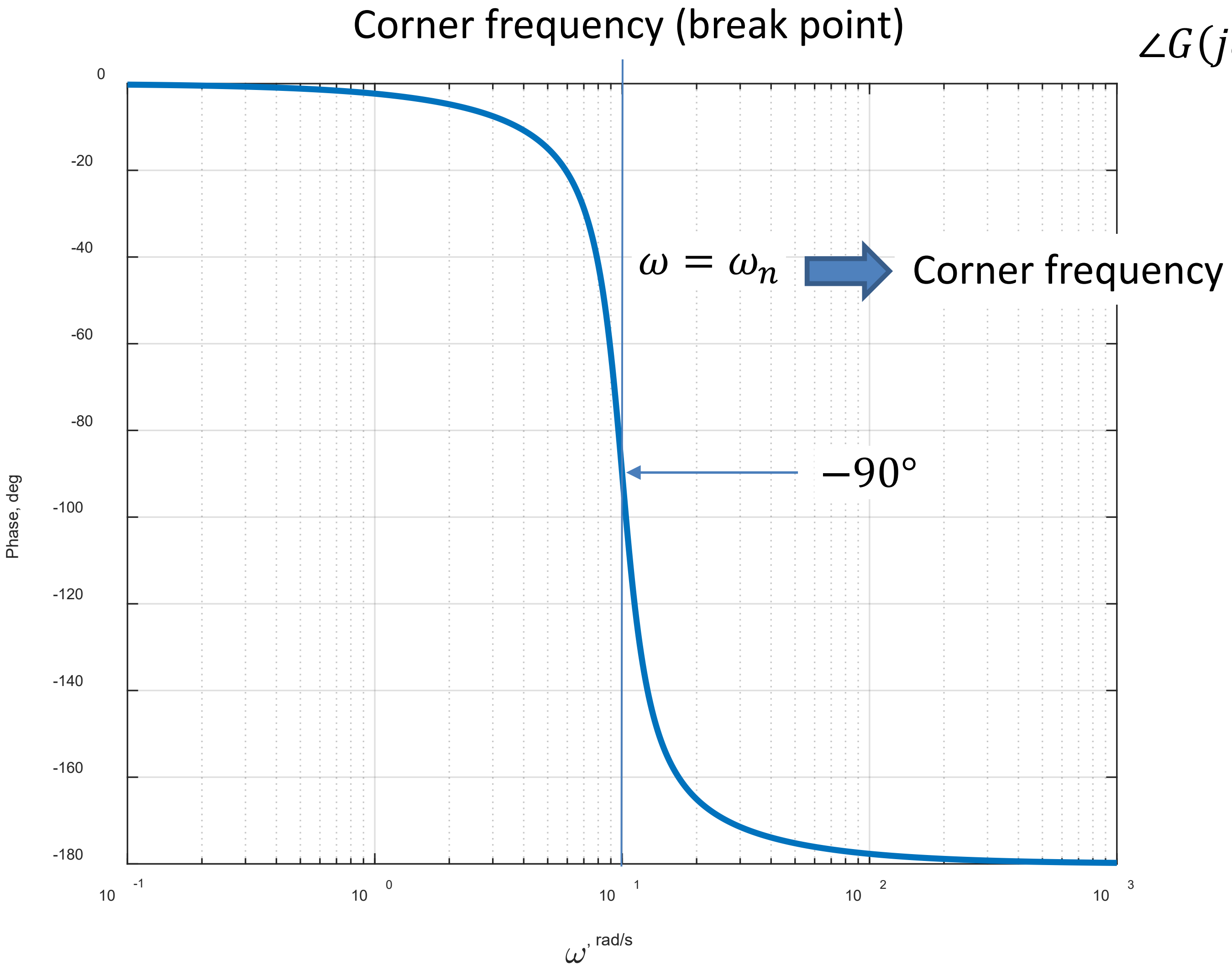
$$|G(j\omega)| = \frac{1}{\sqrt{(1-\lambda^2)^2 + (2\zeta\lambda)^2}}$$



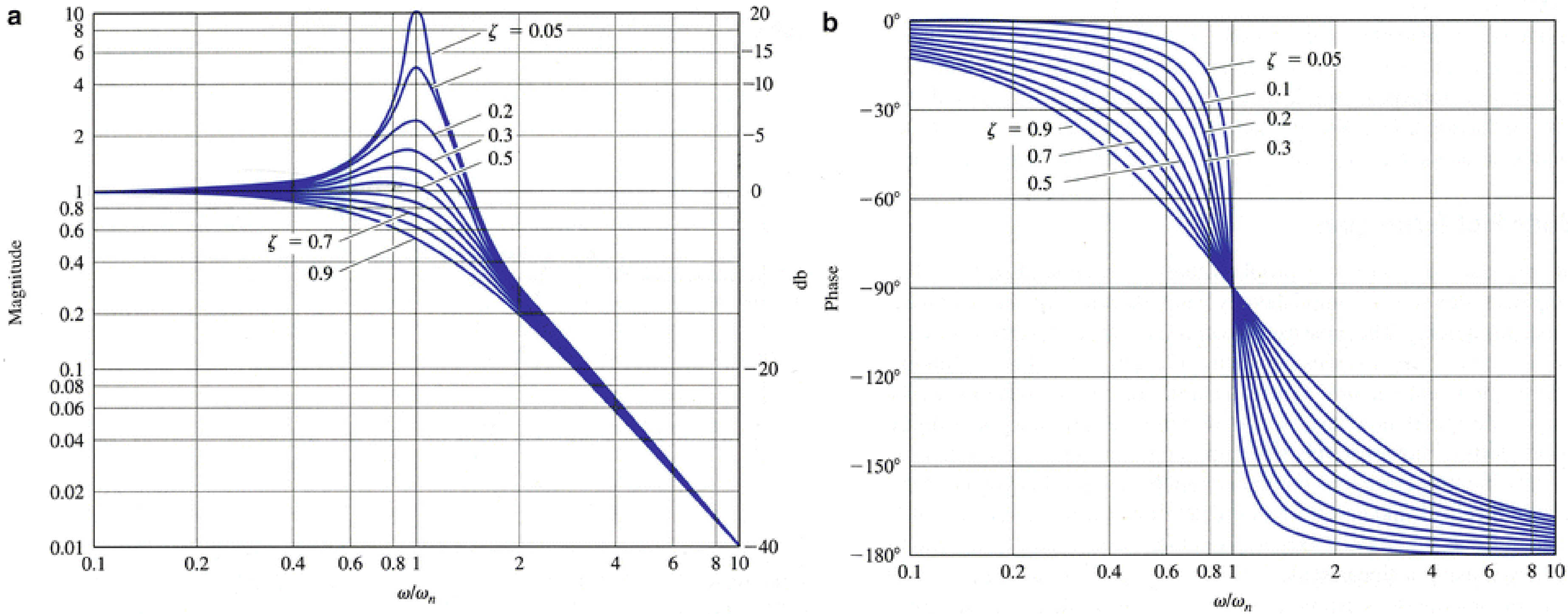
$$\log_{10}|G(j\omega)| \approx -2\log_{10}\omega + 2\log_{10}\omega_n$$

Phase Plot in Detail

$$\angle G(j\omega) = \tan^{-1}\frac{-2\zeta\lambda}{1-\lambda^2}$$



Effects of the Damping Ratio, ζ



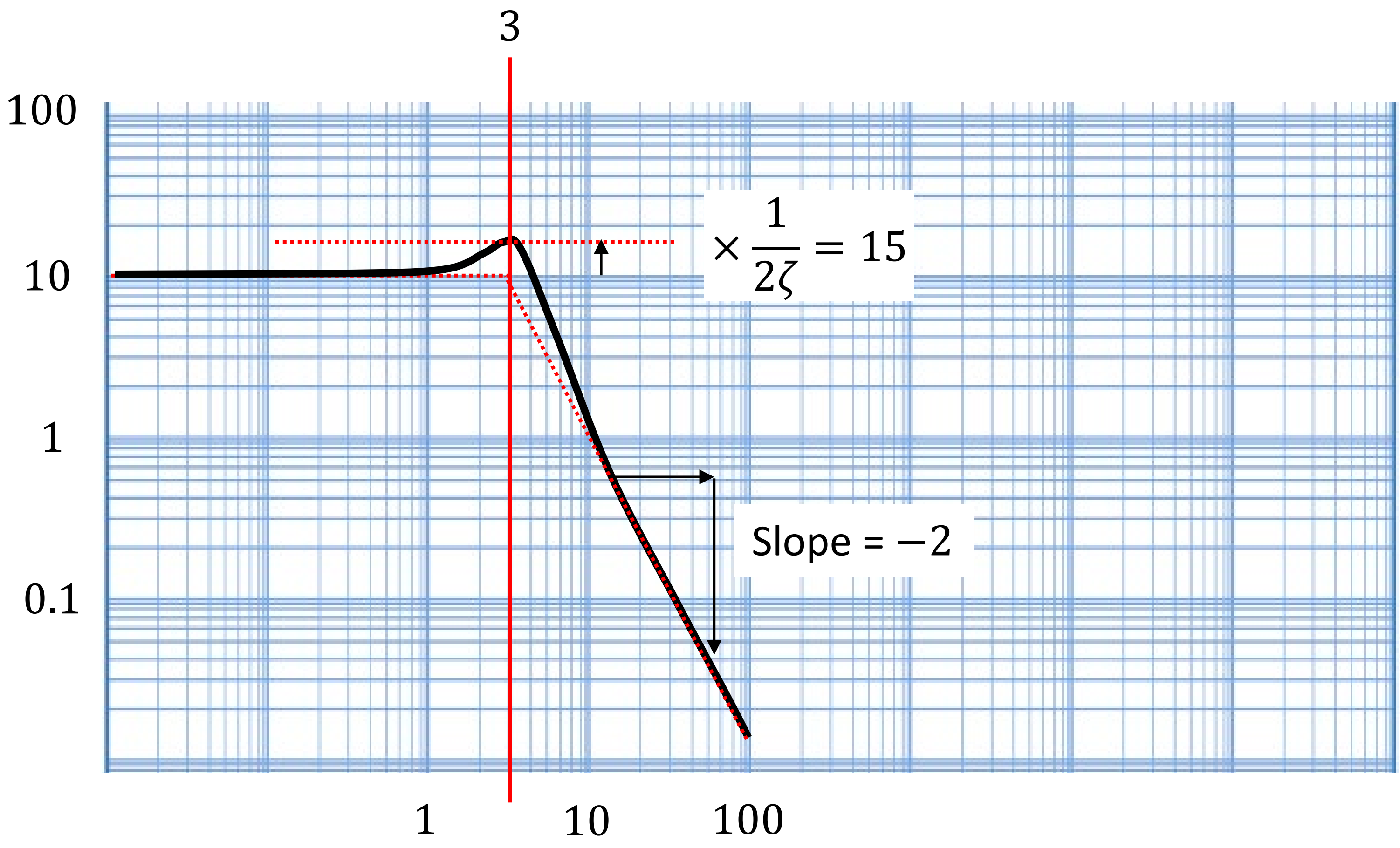
Example

- $G(s) = \frac{90}{s^2+2s+9}$

- Step 1
  - Rewrite to a standard form
    - $G(s) = 10\frac{9}{s^2+2s+9} = 10\frac{1}{\left(\frac{s}{3}\right)^2+2\left(\frac{1}{3}\right)\left(\frac{s}{3}\right)+1}$
  - Magnitude at 0 frequency: 10^1

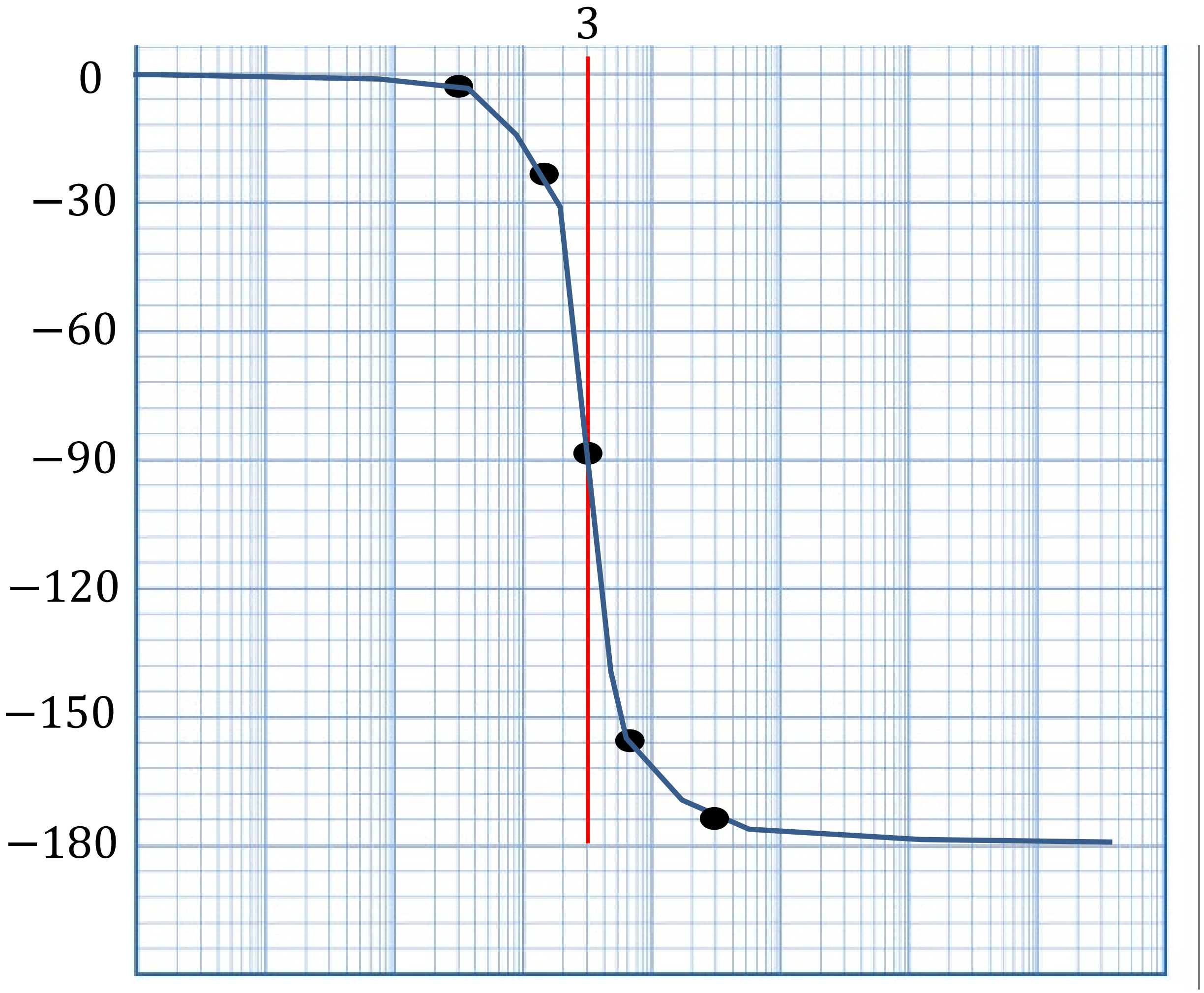
- Step 2
  - Identify corner frequencies (break points)
    - $\omega_n = 3$
  - Identify damping ratio
    - $\zeta = 0.33 \rightarrow \frac{1}{2\zeta} = 1.5$
    - Magnitude at $\omega_n \rightarrow 10 \times 1.5 = 15$
  - Phase
    - $0 \rightarrow -90 \ \rightarrow -180$

Magnitude Plot

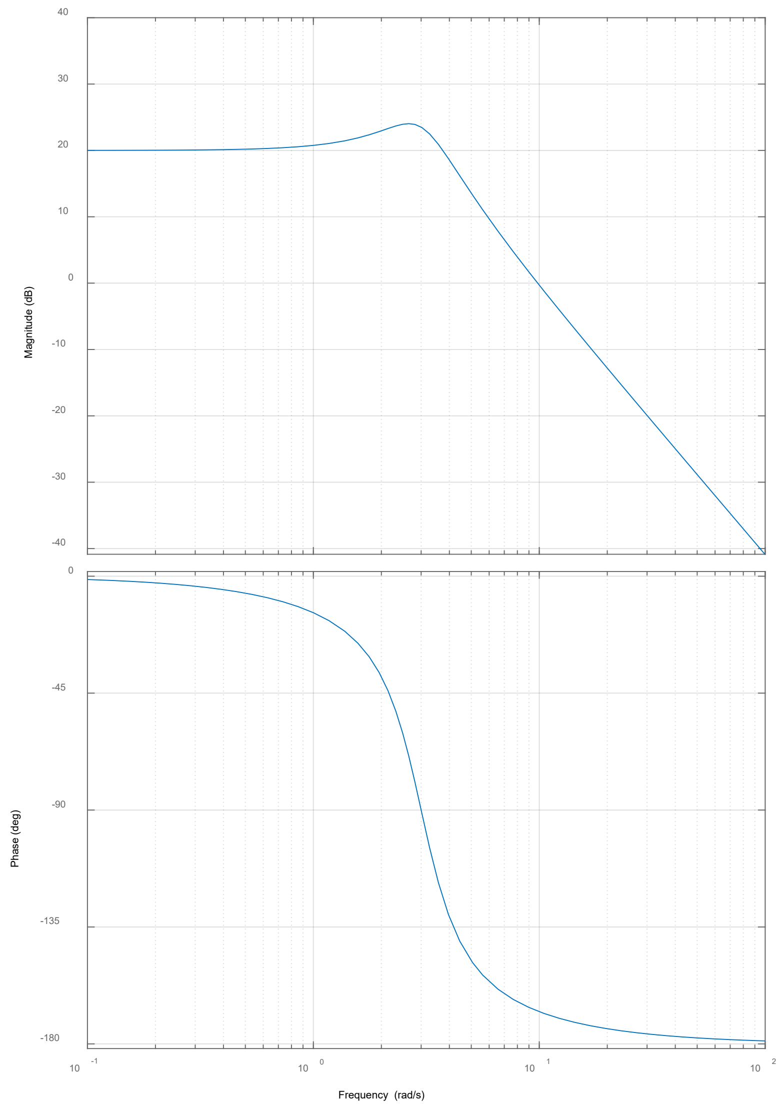


Phase Plot

ω	$3/10$	$3/2$	3	3×2	3×10
ϕ	-3.9	-24	-90	-156	-176.1



Verification with Matlab




```
>> num = 90

num =

   90

>> den = [1 2 9]

den =

    1    2    9

>> bode(num, den)
```

$$dB = 20\log_{10}|G(j\omega)|$$