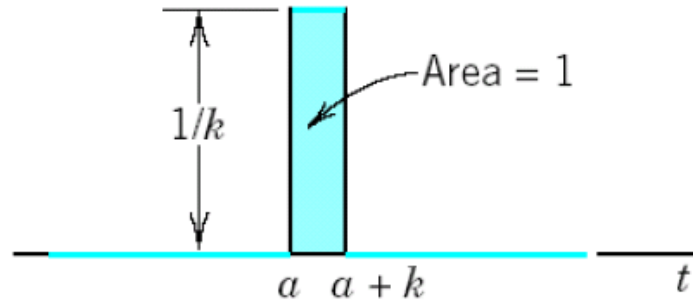


Automatic Control

Hak-Tae Lee

2nd Order System Responses

Dirac Delta Function

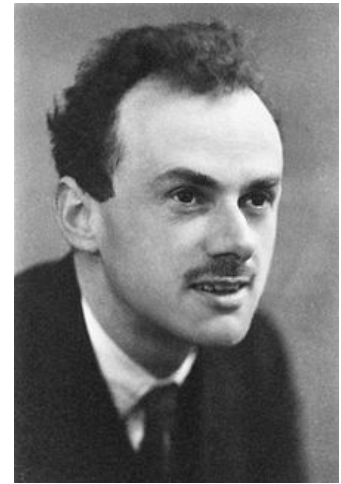


$$f_k(t-a) = \begin{cases} \frac{1}{k} & (a \leq t < a+k) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\delta(t-a) = \lim_{k \rightarrow 0} f_k(t-a)$$

$$\int_0^{\infty} f_k(t-a) dt = \int_a^{a+k} \frac{1}{k} dt = 1 \quad \Rightarrow \quad \int_0^{\infty} \delta(t-a) dt = 1$$

Paul Dirac



- August 1902 – October 1984
- English theoretical physicist
- Bio
 - 1926 Ph.D.
 - 1933 Nobel Prize in Physics
- Notable work
 - Quantum mechanics
 - Dirac equation – predicted the existence of antimatter

Paul Dirac

<https://goo.gl/c4gqq9>

어찌나 과묵한지 동료 과학자들이 과묵함의 단위로 '디랙'을 정의해서 썼을 정도다.

1 디랙

= 1시간에 1마디 하는 것



2 디랙

= 1시간에 2마디 하는 것



⋮



자네 오늘 2디랙인 걸 보니
뭐 좋은 일 있는가?

#3. 디랙은 시에 대해 이런 말을 남겼다.
그야말로 뱃속까지 공대생 마인드다.

"과학은 어려운 사실을 쉬운 말로
모두가 이해할 수 있게 해준다.
반면 시는 모두가 아는 사실을
어려운 말로 아무도 이해할 수
없게 만든다."



Impulse Response

Solve the given ODE

$$\ddot{y} + 3\dot{y} + 2y = \delta(t) \qquad \begin{aligned} \dot{y}(0) &= 0 \\ y(0) &= 0 \end{aligned}$$

Laplace transform of Dirac delta function

$$\begin{aligned} L\{\delta(t - a)\} &= \int_0^{\infty} \delta(t - a)e^{-st} dt = \lim_{k \rightarrow 0} \int_a^{a+k} \frac{1}{k} e^{-st} dt \\ &= \lim_{k \rightarrow 0} \frac{e^{-as} - e^{-(a+k)s}}{ks} = e^{-as} \end{aligned}$$

Impulse Response

Laplace transform

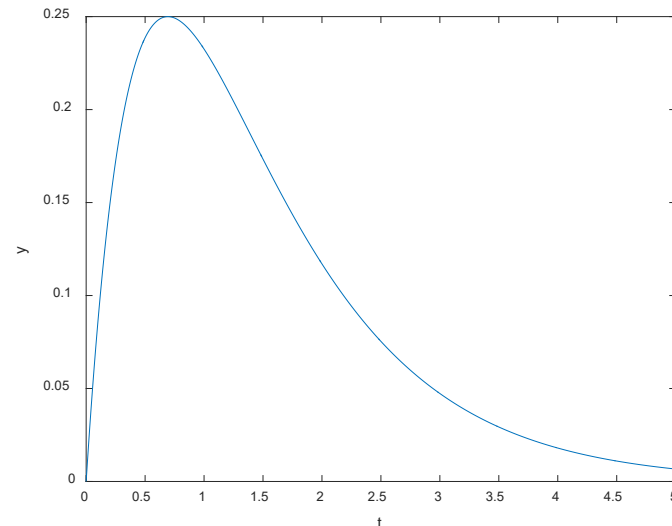
$$L\{\ddot{y} + 3\dot{y} + 2y\} = (s^2 + 3s + 2)L\{y\} = L\{\delta(t)\} = 1$$

$$F(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 2)(s + 1)} = \frac{1}{s + 1} - \frac{1}{s + 2}$$

$$y(t) = e^{-t} - e^{-2t}$$



Homogenous solution!



Unit Impulse Function

Unit impulse function

$$\delta(t) = \lim_{\epsilon \rightarrow 0} f_{\epsilon}(t) \quad \text{where} \quad f_{\epsilon}(t) = \begin{cases} 1/\epsilon - \frac{\epsilon}{2} \leq t \leq \frac{\epsilon}{2} \\ 0 \quad \text{otherwise} \end{cases}$$

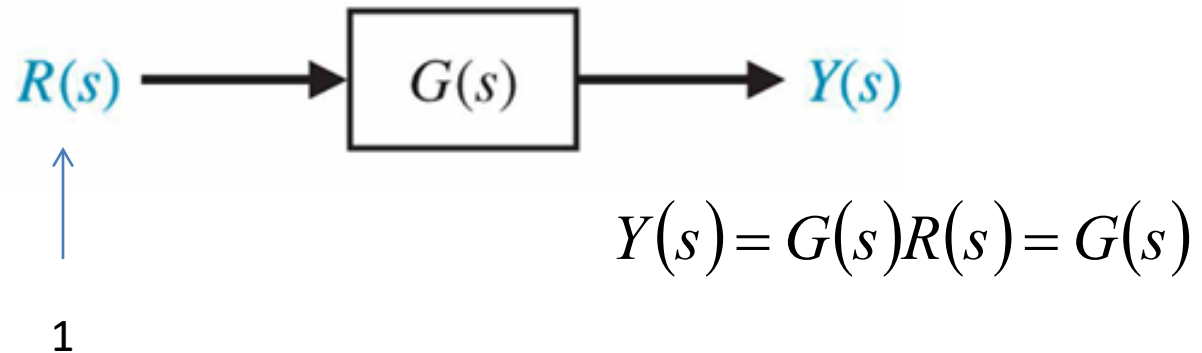
properties of the unit impulse function

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a)g(t) dt = g(a)$$

Laplace transform of the unit impulse function

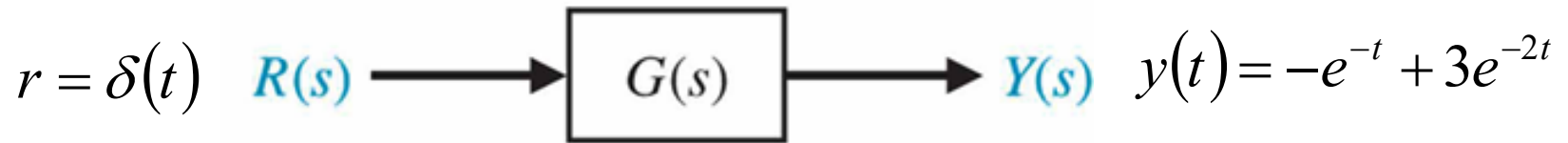
$$\begin{aligned} \int_0^{\infty} \delta(t) e^{-st} dt &= \lim_{\epsilon \rightarrow 0} \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon} e^{-st} dt = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left(-\frac{1}{s} \right) e^{-st} \Bigg|_{-\epsilon/2}^{\epsilon/2} \\ &= \left(-\frac{1}{s} \right) \lim_{\epsilon \rightarrow 0} \frac{e^{-\epsilon s/2} - e^{\epsilon s/2}}{\epsilon} = \left(-\frac{1}{s} \right) (-s) = 1 \end{aligned}$$

Unit Impulse Response



- Unit impulse response is equal to the transfer function itself
- Unit impulse response is the natural response of the system

Unit Impulse Response Example



$$G(s) = \frac{2s+1}{s^2+3s+2} = \frac{1}{s+1} + \frac{3}{s+2}$$

```
% Unit impulse response  
clear all, close all
```

```
% Direct solution
```

```
t1 = 0:0.2:6;  
y1 = -exp(-t1) + 3*exp(-2*t1);
```

```
% Using control system toolbox
```

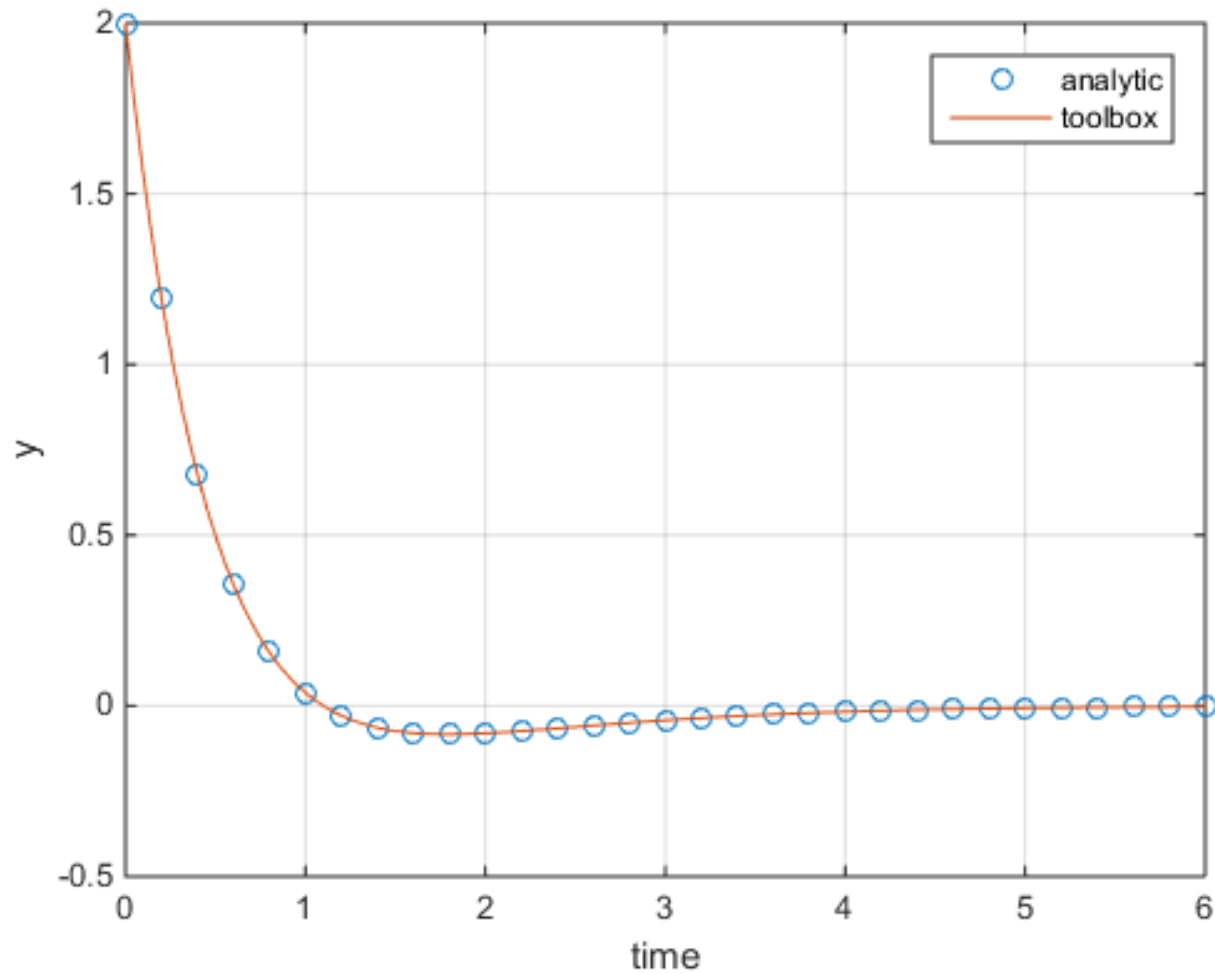
```
num = [2 1];  
den = [1 3 2];  
sys = tf(num, den);
```

```
[y2, t2] = impulse(sys, 6);
```

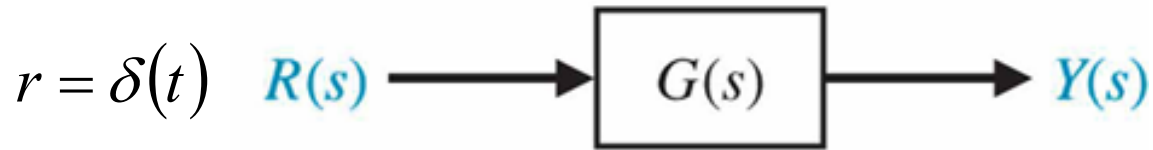
```
% plot
```

```
plot(t1, y1, 'o', t2, y2, '-')  
xlabel('time');  
ylabel('y');  
legend('analytic', 'toolbox')  
grid
```

Comparison between Results



Unit Impulse Response Example 2



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Let

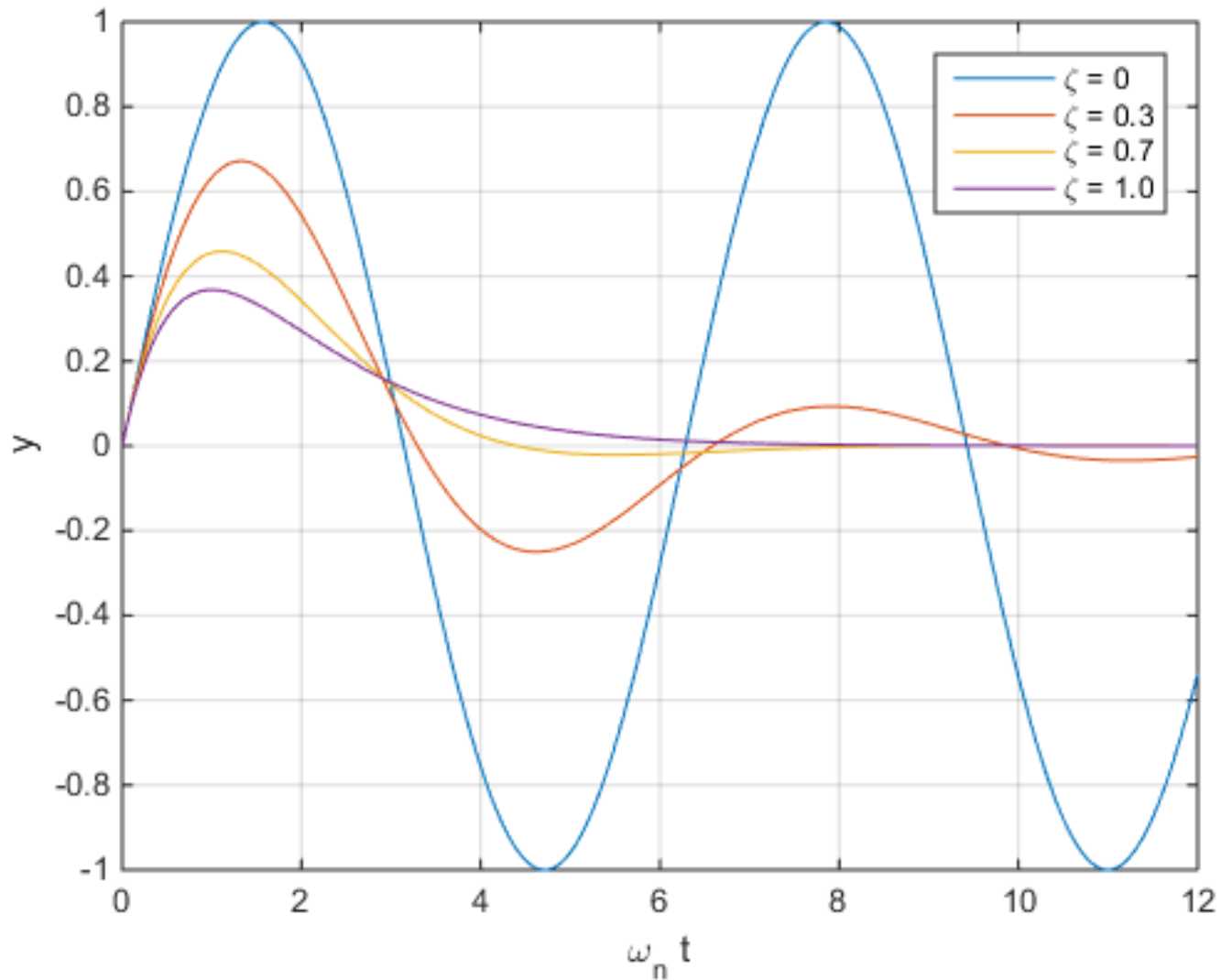
$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Then

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \sigma)^2 + \omega_d^2}$$

$$y(t) = \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\sigma t} \sin(\omega_d t)$$

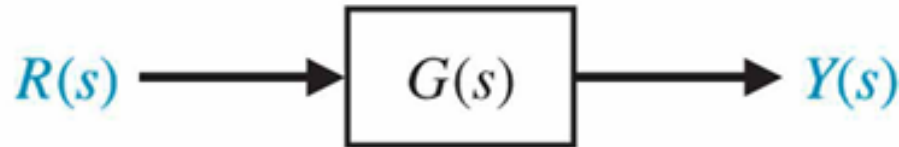
Impulse Response of a Second Order System



Unit Step Response

$$r = u(t)$$

$$R = \frac{1}{s}$$



$$y = L^{-1}\left(G(s)\frac{1}{s}\right) = L^{-1}\left(\frac{N(s)}{D(s)}\frac{1}{s}\right) = L^{-1}\left(\frac{1}{s} - \frac{N'(s)}{D(s)}\right)$$

- Basic behavior is still governed by the roots of

$$D(s) = 0$$

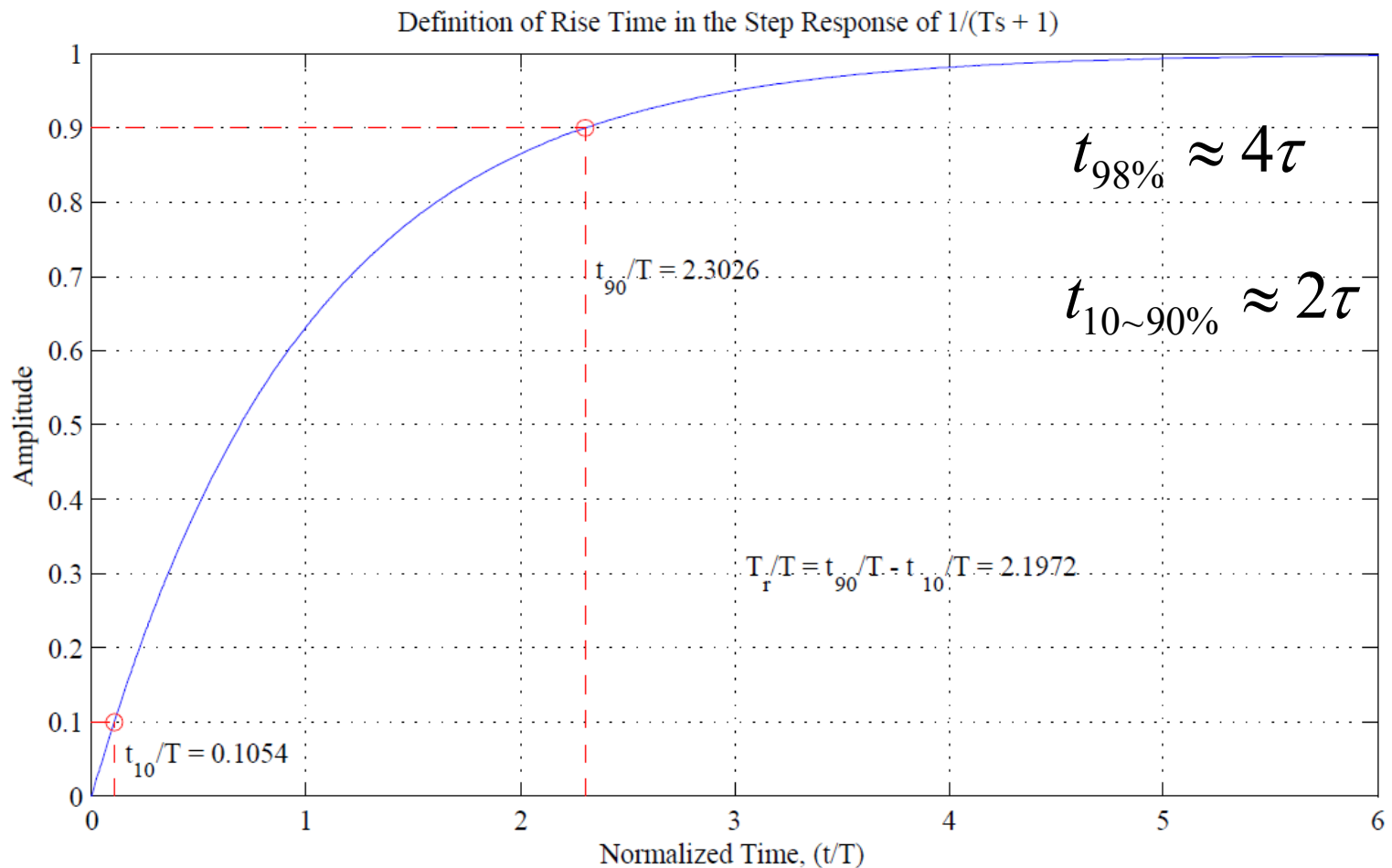
- These roots are called poles

Step Response of First-Order Systems

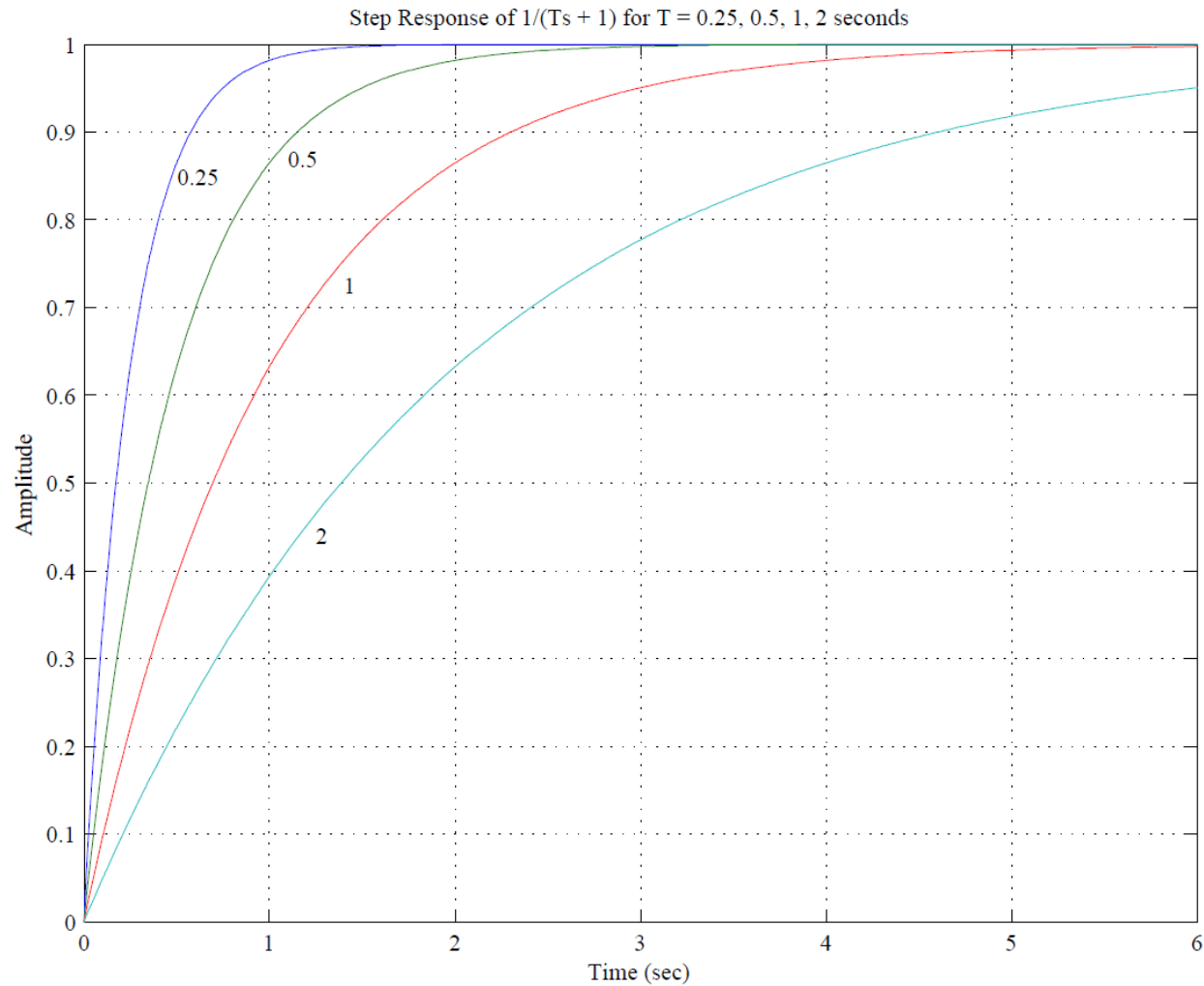
$$\frac{Y(s)}{U(s)} = \frac{b}{s+a} = \frac{b/a}{(1/a)s+1} = \frac{c}{\tau s+1}$$

Steady-state response: $c=b/a$

Quickness: $\tau = 1/a$



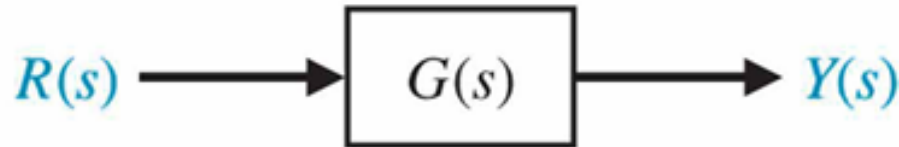
Step Response of First-Order Systems



Step Response of Second-Order Systems

$$r = u(t)$$

$$R = \frac{1}{s}$$



$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = 1 - e^{-\sigma t} \left(\cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \quad \left(\sigma = \zeta\omega_n, \omega_d = \omega_n \sqrt{1 - \zeta^2} \right)$$

$$\text{Poles of } G(s) : s = -\sigma \pm j\omega_d$$

Geometric Representation of Second-Order Poles

