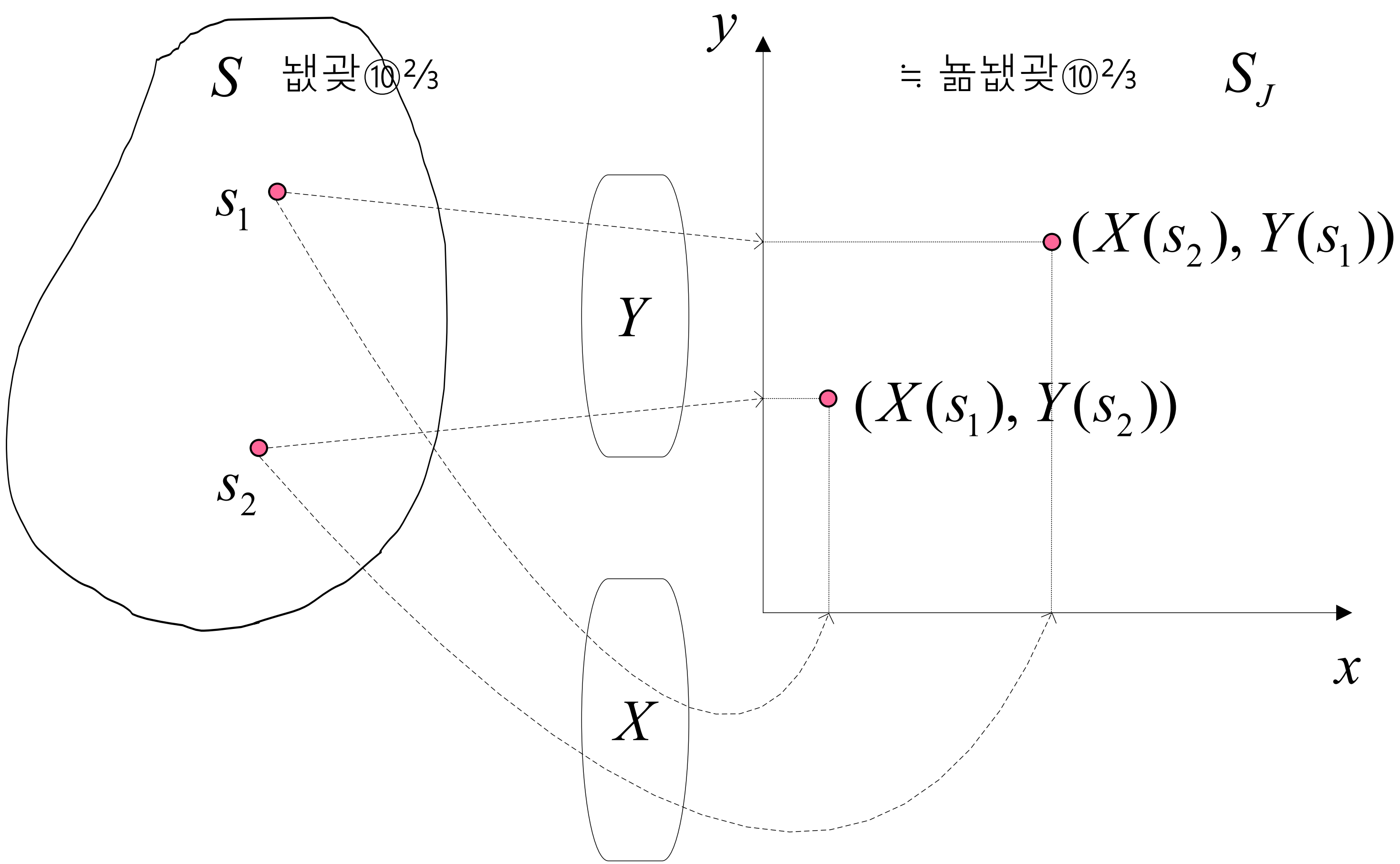
# Expectation and Estimation

# Multiple Random Variables

## ❑ 2차원 랜덤변수 사상



# Joint Probability Distribution Function

## ❑ Single Probability Distribution Function

$$F_X(x) = P\{X \le x\}$$
$$F_Y(y) = P\{Y \le y\}$$

## ❑ Joint Probability Distribution Function

$$F_{X,Y}(x,y) = P\{X \le x \text{ and } Y \le y\}$$

## ❑ Ex 4.2-1)

$$S_J = \{(1,1),(2,1),\ (3,3)\} \text{ and } P(1,1) = 0.2,\ P(2,1) = 0.3,\ P(3,3) = 0.5$$

$$\Rightarrow F_{X,Y}(0,1) = P\{X \le 0,\ Y \le 1\} = 0$$

$$F_{X,Y}(1,1) = P\{X \le 0,\ Y \le 1\} = P(1,1) = 0.2$$

$$F_{X,Y}(2,1) = P\{X \le 2,\ Y \le 1\} = P(1,1) + P(2,1) = 0.5$$

$$F_{X,Y}(2,2) = P\{X \le 2,\ Y \le 2\} = P(1,1) + P(2,1) = 0.5$$

$$F_{X,Y}(3,3) = P\{X \le 3,\ Y \le 3\} = P(1,1) + P(2,1) + P(3,3) = 1.0$$

# Joint Probability Distribution Function

## ❑ Properties of Joint Probability Distribution Function

- $F_{X,Y}(-\infty,-\infty) = F_{X,Y}(-\infty, y) = F_{X,Y}(x,-\infty) = 0$
- $F_{X,Y}(\infty,\infty) = 1$
- $0 \le F_{X,Y}(x, y) \le 1$
- $F_{X,Y}(x, y)$ is a non-decreasing function.
- $F_{X,Y}(x_2, y_2) + F_{X,Y}(x_1, y_1) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1)$

  $= P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} \ge 0$
- $F_{X,Y}(x,\infty) = F_X(x),\ \ F_{X,Y}(\infty, y) = F_Y(y)$

Let $A = \{X \le x\}$, $B = \{Y \le y\}$, then $F_{X,Y}(x, y) = P\{X \le x,\ Y \le y\} = P[A \cap B]$.

For $\mathrm{y} \to \infty$, $B = \{Y \le \infty\} = S \Rightarrow A \cap B = A$

Therefore, $F_{X,Y}(x,\infty) = P[A \cap S] = P[A] = P\{X \le x\} = F_X(x)$

# Joint Probability Density Function

## ❑ Joint Probability Density Function

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$

## ❑ Properties of Joint pdf

- $f_{X,Y}(x,y) \geq 0$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$
- $F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $F_X(x) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2 d\xi_1, \quad F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $P\{x_1 < X \leq x_2,\ y_1 < Y \leq y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(\xi_1,\xi_2)d\xi_1 d\xi_2$
- $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx \qquad (*)$

# Joint Probability Density Function

- **Proof of (*)**

$$f_X(x) = \frac{dF_X(x)}{dx} = \frac{d}{dx}\int_{-\infty}^{x} \underbrace{\int_{-\infty}^{\infty} f_{X,Y}(\xi_1,\xi_2)d\xi_2}_{\triangleq k(\xi_1)} d\xi_1$$

$$= \frac{d}{dx}\left[K(x) - K(-\infty)\right], \quad K = \int k(\xi_1)d\xi_1$$

$$= k(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,\xi_2)d\xi_2 = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

In similar way, $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$

# Conditional Joint Probability

## ❑ Conditional Probability of X on B

$$F_X(x \mid B) = P\{X \le x \mid B\} = \frac{P\{X \le x \bigcap B\}}{P[B]} \text{ for } P[B] \ne 0$$

$$\Rightarrow f_X(x \mid B) = \frac{dF_X(x \mid B)}{dx}$$

## ❑ Conditional Probability of X

$$\text{Let } B = \{y - \Delta y < Y \le y + \Delta y\}$$

$$\text{then } F_X(x \mid B) = F_X(x \mid y - \Delta y < Y \le y + \Delta y)$$

$$= \frac{P[X \le x \bigcap \{y - \Delta y < Y \le y + \Delta y\}]}{P\{y - \Delta y < Y \le y + \Delta y\}}.$$

# Conditional Joint Probability

## ❑ Conditional Probability of X

Recall that $P\{x_1 < X \le x_2,\ y_1 < Y \le y_2\} = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2.$

$$F_X(x \mid B) = \frac{\int_{y-\Delta y}^{y+\Delta y}\int_{-\infty}^{x} f_{X,Y}(\xi_1,\xi_2)\,d\xi_1 d\xi_2}{\int_{y-\Delta y}^{y+\Delta y} f_Y(\xi)\,d\xi}$$

$$\approx \frac{\int_{-\infty}^{x} f_{X,Y}(\xi_1, y)\,d\xi_1 \cdot 2\Delta y}{f_Y(y)\cdot 2\Delta y}$$

If $\Delta y \to 0,\ F_X(x \mid B) \to F_X(x \mid Y = y) = \dfrac{\int_{-\infty}^{x} f_{X,Y}(\xi_1, y)\,d\xi_1}{f_Y(y)}.$

$$\Rightarrow f_X(x \mid Y = y) = \frac{dF_X(x \mid Y = y)}{dx} = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Simply, $f_X(x \mid y) = f_X(x \mid Y = y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)},\ f_Y(y \mid x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

# Conditional Joint Probability

- **Statistically Independent**

It is said that $A$ is statistically independent on $B$,
if $P[A \cap B] = P[A]P[B]$.

Let $A = \{X \le x\}$, $B = \{Y \le y\}$, then $P\{X \le x, Y \le y\} = P\{X \le x\}P\{Y \le y\}$.
$\Rightarrow F_{X,Y}(x,y) = F_X(x)F_Y(y)$

$$F_X(x \mid Y \le y) = \frac{P\{X \le x \cap Y \le y\}}{P\{Y \le y\}} = \frac{F_{X,Y}(x,y)}{F_Y(y)} = \frac{F_X(x)F_Y(y)}{F_Y(y)}$$
$$= F_X(x)$$

In similar way, $F_Y(y \mid X \le x) = F_Y(y)$

Also, $f_X(x \mid Y \le y) = f_X(x)$, $\quad f_Y(y \mid X \le x) = f_Y(y)$

# Conditional Joint Probability

## ❑ Example

Let the joint density of two random variables $X$ and $Y$ be given by

$$f_{X,Y}(x,y)=\begin{cases}\frac{1}{6}(x+4y), & 0<x<2,\ 0<y<1\\ 0, & \text{otherwise}\end{cases}$$

1) $f_{X,Y}(x,y)\geq 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X,Y}(x,y)dxdy=\int_0^1\int_0^2\frac{1}{6}(x+4y)dxdy$$

$$=\int_0^1\left[\frac{1}{6}\left(\frac{1}{2}x^2+4xy\right)\right]_{x=0}^{x=2}dy=\int_0^1\frac{1}{6}(2+8y)dy$$

$$=\left[\frac{1}{6}(2y+4y^2)\right]_{y=0}^{y=1}=1$$

$\Rightarrow$ Probability Density function

# Conditional Joint Probability

- Example

$$2)\ f_X(x)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dy=\int_0^1 \frac{1}{6}(x+4y)dy=\frac{1}{6}(xy+2y^2)\Big]_0^1=\frac{1}{6}(x+2)$$

$$f_Y(y)=\int_{-\infty}^{\infty} f_{X,Y}(x,y)dx=\int_0^2 \frac{1}{6}(x+4y)dx=\frac{1}{6}(\frac{1}{2}x^2+4xy)\Big]_0^2=\frac{1}{6}(2+8y)=\frac{1}{3}(1+4y)$$

$$f_X(x\,|\,y)=f_X(x\,|\,Y=y)=\begin{cases}\dfrac{f_{X,Y}(x,y)}{f_Y(y)}=\dfrac{\frac{1}{6}(x+4y)}{\frac{1}{3}(1+4y)}=\dfrac{x+4y}{2(1+4y)}, & 0<x<2\\ 0, & \text{otherwise}\end{cases}$$

$$3)\ F_X(1\,|\,0.5)=F_X(x\,|\,Y=0.5)=\frac{\int_{-\infty}^{x} f_{X,Y}(\xi_1,y)d\xi_1}{f_Y(y)}=\frac{\int_{-\infty}^{1} f_{X,Y}(\xi_1,0.5)d\xi_1}{f_Y(0.5)}$$

$$=\frac{\int_0^1 \frac{1}{6}(x+2)dx}{1}=\frac{1}{6}(\frac{1}{2}x^2+2x)\Big]_0^2=\frac{5}{12}$$

# Expected Value

❑ **Expected value or Mean of a discrete random variable X**

$$m_X = E[X] = \sum_{x \in S_X} x p_X(x) = \sum_k x_k p_X(x_k)$$

- The expected value (or expectation) refers, intuitively, to the value of a random variable one would "expect" to find if one could repeat the random variable process an infinite number of times and take the average of the values obtained.

- The expected value is a weighted average of all possible values.

❑ **Ex) 주사위를 1회 던졌을 때 나타나는 눈의 기대값**

$$X = \{1, 2, 3, 4, 5, 6\},\ p_X(x_i) = 1/6,\ i = 1,2,\ldots,6$$

$$m_X = E[X] = \sum_k x_k p_X(x_k) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

# Expected Value

- **Expected value a random variable X(General meaning)**

$$m_X = E\left[X\right] = \int_{-\infty}^{\infty} x f_X(x)dx$$

- **Ex 3.1-2)**

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-(x-a)/b}, & x > a \\ 0, \ x < a \end{cases}$$

$$E[X] = \frac{1}{b}\int_a^{\infty} x e^{-\frac{(x-a)}{b}} dx = \frac{1}{b} e^{\frac{a}{b}} \int_a^{\infty} x e^{-\frac{x}{b}} dx = \left(\frac{1}{b} e^{\frac{a}{b}}\right) \underbrace{\left[ e^{-\frac{x}{b}} \left( \frac{x}{-\frac{1}{b}} - \frac{1}{\frac{1}{b^2}} \right) \right]_a^{\infty}}_{\text{from C-46}}$$

$$= \left(\frac{1}{b} e^{\frac{a}{b}}\right) e^{-\frac{a}{b}} \left[ab + b^2\right] = a + b$$

# Expected Value

- Ex) Mean of normal pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

$$E[X] = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{\infty} [x e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} - a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} + a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}] dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_X^2}} [\underbrace{\int_{-\infty}^{\infty} (x - a_X) e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx}_{=0 \because odd\ \ function} + \int_{-\infty}^{\infty} a_X e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx]$$

$$= a_X \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}} dx = a_X \underbrace{\int_{-\infty}^{\infty} f_X(x) dx}_{=1} = a_X$$

# Moments

## ❑ Expected value functions of a random variable X

$$E\left[g(X)\right]=\int_{-\infty}^{\infty} g(x)f_X(x)dx$$

## ❑ Moments about the origin

Let $g(X)=X^n, \quad n=0,1,2,\ldots$

then the n-th moment of $X$ is given by $m_n=E[X^n]=\int_{-\infty}^{\infty} x^n f_X(x)dx$

## ❑ Central Moments

Let $g(X)=(X-\overline{X})^n, \quad n=0,1,2,\ldots$

where $\overline{X}$ is the mean of $X$, then the n-th central moment of $X$

is given by $\mu_n=E[(X-\overline{X})^n]=\int_{-\infty}^{\infty}(x-\overline{X})^n f_X(x)dx.$

$cf)\ \ \mu_0=E[(X-\overline{X})^0]=\int_{-\infty}^{\infty} f_X(x)dx=1$

$\mu_1=E[(X-\overline{X})]=\int_{-\infty}^{\infty}(x-\overline{X})f_X(x)dx=0$

# Moments

## ❑ Properties of Mean

- $E\left[g(X)+h(X)\right]=E\left[g(X)\right]+E\left[h(X)\right]$
- $E\left[ag(X)\right]=aE\left[g(X)\right]$
- $E\left[g(X)+c\right]=E\left[g(X)\right]+c$
- $E\left[c\right]=c$
- $E\left[\sum g_k(X)\right]=\sum E\left[g_k(X)\right]$

# Moments

- Ex) 주사위를 1회 던졌을 때 나타나는 눈의 모멘트

$$X = \{1, 2, 3, 4, 5, 6\},\ p_X(x_i) = 1/6,\ i = 1, 2, \ldots, 6$$

$$\text{Mean (the 1}^{st}\text{ moment)}:\ \bar{X} = E[X] = \sum_k x_k p_X(x_k) = 1\cdot\frac{1}{6} + \ldots + 6\cdot\frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

$$\text{The 2}^{nd}\text{ moment}:\ E[X^2] = \sum_k k^2 p_X(x_k) = 1^2\cdot\frac{1}{6} + 2^2\cdot\frac{1}{6} + \ldots + 6^2\cdot\frac{1}{6} = \frac{91}{6}$$

$$\text{Variance(the 2}^{nd}\text{ central moment)}:$$

$$E[(X-\bar{X})^2] = \sum_k \left(k - \frac{7}{2}\right)^2 p_X(x_k) \Leftarrow \text{Too tedious}$$

$$= E[X^2 - 2X\bar{X} + \bar{X}^2]$$

$$= E[X^2] - 2\bar{X}\,E[X] + \bar{X}^2 = E[X^2] - \bar{X}^2$$

$$= \frac{91}{6} - \left(\frac{7}{2}\right)^2$$

# Moments

- **Ex) Gaussian Random Variable**

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-a_X)^2}{2\sigma_X^2}}$$

$$\text{Mean (the } 1^{st} \text{ moment)}: \ \bar{X} = E[X] = a_X$$

$$\text{Variance(the } 2^{nd} \text{ central moment)}: \ E[(X-\bar{X})^2] = \sigma_X^2$$

$\Rightarrow$ The pdf of the gaussian RV is represented by its mean and variance.

# Moments

## ❑ Joint moments about the origin

For two random variables $X$ and $Y$,

the joint moment is given by $m_{nk} = E\left[X^n Y^k\right] = \int_{-\infty}^{\infty} x^n y^k f_{X,Y}(x,y)dxdy$,

where $n+k$ is the order of the joint moment.

## ❑ Correlation

$$R_{XY} = m_{11} = E\left[XY\right] = \int_{-\infty}^{\infty} x\, y\, f_{X,Y}(x,y)dxdy$$

If $R_{XY} = E[X]E[Y]$ is satisfied, we say that there is no correlation between $X$ and $Y$. Or, it is said that $X$ is statistically indpendent on $Y$.

If $R_{XY} = 0$, two random variables $X$ and $Y$ are orthogonal each other.

# Moments

## ❑ Joint central moments

For two random variables $X$ and $Y$,

$$\mu_{nk} = E\left[ (X-\bar{X})^n (Y-\bar{Y})^k \right] = \int_{-\infty}^{\infty} (x-\bar{X})^n (y-\bar{Y})^k f_{X,Y}(x,y)dxdy$$

is called the joint central moment.

# Moments

- **Covariance**

$$C_{XY} = \mu_{11} = E\left[(X-\bar{X})(Y-\bar{Y})\right] = \int_{-\infty}^{\infty} (x-\bar{X})(y-\bar{Y}) f_{X,Y}(x,y)dxdy$$

$$= \underbrace{\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy}_{=E[XY]} - \underbrace{\bar{X}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dxdy}_{=E[\bar{X}Y]=\bar{X}E[Y]=\bar{X}\bar{Y}} - \underbrace{\bar{Y}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)dxdy}_{=E[X\bar{Y}]=\bar{Y}E[X]=\bar{X}\bar{Y}}$$

$$+ \bar{X}\bar{Y}\underbrace{\int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy}_{=1} = R_{XY} - \bar{X}\bar{Y}$$

If $C_{XY} = 0 (\Rightarrow R_{XY} = E[X]E[Y])$, there is no correlation or independent each other between $X$ and $Y$.

If $X$ and $Y$ are orthogonal($R_{XY} = 0$), $C_{XY} = -E[X]E[Y]$.

# Moments

- Correlation coefficient

$$\rho = E\left[\frac{(X-\bar{X})}{\sigma_X}\frac{(Y-\bar{Y})}{\sigma_Y}\right]$$

where $\sigma_X^2 = E[(X-\bar{X})^2],\ \sigma_Y^2 = E[(Y-\bar{Y})^2].$

HW) Prove $-1 \le \rho \le 1.$

# Variance Estimates[Wiki]

## ❑ Population Variance

In general, the population variance of a finite population of size N with values $x_i$ is given by

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 = \frac{1}{N}\sum_{i=1}^{N}x_i^2 - m^2, \quad m = \frac{1}{N}\sum_{i=1}^{N}x_i$$

## ❑ Sample Variance

In many practical situations, the true variance of a population is not known a priori and must be computed somehow. When dealing with extremely large populations, it is not possible to count every object in the population, so the computation must be performed on a sample of the population. Sample variance can also be applied to the estimation of the variance of a continuous distribution from a sample of that distribution.

# Variance Estimates

## ❑ Sample Variance

Let the sample variance given by

$$\sigma_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2, \text{where the sample mean is } \bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

Since $x_i$ are selected randomly, $\bar{x}$ and $\sigma^2$ are random variables.

The expected value of sample variance is

$$E[\sigma_x^2] = E\left[\frac{1}{N}\sum_{i=1}^{N}\left(x_i - \frac{1}{N}\sum_{j=1}^{N} x_j\right)^2\right] = \frac{1}{N}\sum_{i=1}^{N} E\left[x_i^2 - \frac{2}{N}x_i\sum_{j=1}^{N}x_j + \frac{1}{N^2}\sum_{j=1}^{N}x_j\sum_{k=1}^{N}x_k\right]$$

$$= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{N-2}{N}\underbrace{E[x_i^2]}_{=\sigma^2+m^2} - \frac{2}{N}\underbrace{\sum_{j\neq i}\underbrace{E[x_i x_j]}_{\substack{=E[x_i]E[x_j]\\ =m^2}}}_{=(N-1)m^2} + \frac{1}{N^2}\underbrace{\sum_{j=1}^{N}\underbrace{\sum_{k\neq j}\underbrace{E[x_j x_k]}_{\substack{=E[x_j]E[x_k]\\ =m^2}}}_{=(N-1)m^2}}_{=N(N-1)m^2} + \frac{1}{N^2}\underbrace{\sum_{j=1}^{N}\underbrace{E[x_j^2]}_{=\sigma^2+m^2}}_{=N(\sigma^2+m^2)}\right]$$

# Variance Estimates

## ❑ Sample Variance

$$=\frac{1}{N}\sum_{i=1}^{N}\left[\frac{N-2}{N}(\sigma^2+m^2)-\frac{2(N-1)}{N}m^2+\frac{N(N-1)}{N^2}m^2+\frac{1}{N}(\sigma^2+m^2)\right]$$

$$=\frac{1}{N}\sum_{i=1}^{N}\left[(\sigma^2+m^2)-\frac{1}{N}(\sigma^2+m^2)-\frac{(N-1)}{N}m^2\right]$$

$$=\frac{1}{N}\sum_{i=1}^{N}\left[\frac{N-1}{N}\sigma^2\right]$$

$$=\frac{N-1}{N}\sigma^2 \text{ : Biased estimate}$$

$$\Rightarrow s^2=\frac{1}{N-1}\sum_{i=1}^{N}(x_i-\bar{x})^2\text{:Unbiased estimate}$$

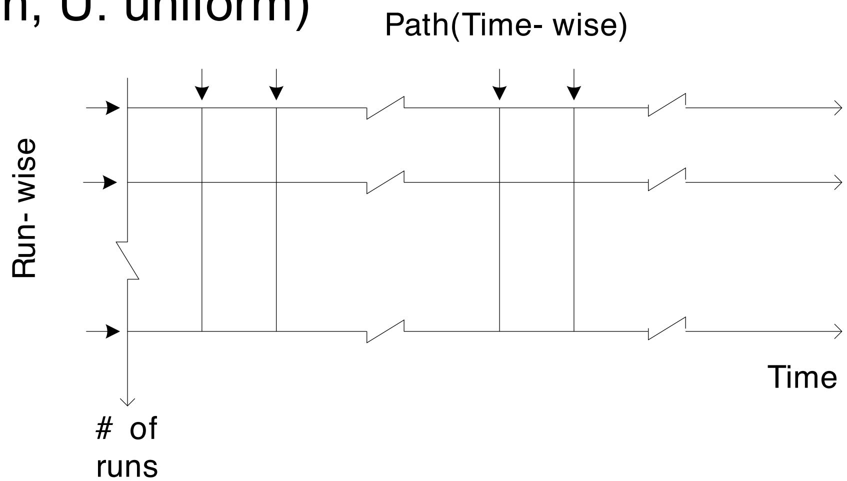
# Monte-Carlo Method

## ❑ Type of uncertainties

- Run-wise uncertainties

  - Fixed value for each run

  - Aerodynamic coefficients(N)., Thrust misalignment(U), Launch angle(G), wind(N,U), etc. (N: Gaussian, U: uniform)




- Path(Time)-wise uncertainties

  - Measurement noise, wind, etc.

  - Process noise(Stochastic Integ.)

- Considerations on process noise

  - Calibrated s.d. must be used to generate random numbers to avoid averaging effect for each step of integration.

$$\dot{x} = f(x,t) + g(x,t)w(t),\ E[w(t)w^T(t)] = Q(t)\delta(t-\tau)$$

$$\text{- r.n. for each step: } w_k \sim N(m,\sigma^2),\ \sigma^2 \approx Q(k)/\Delta t$$

$$\text{- r.n. for sub call in RK4: } \sigma^2 \approx {}^{18}\!/_{5}\, Q(k)\Big/\Delta t$$

# Monte-Carlo Method

## ❑ Mean & Variance Estimate[Lin]

- Representative statistics of M-C simulation results

M-C run results: $x_i,\ i=1,\ldots,N$

$$\hat{m}_x = \frac{1}{N}\sum_{i=1}^{N} x_i,\ \ \hat{\sigma}_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i-\hat{m}_x)^2$$

- Confidence interval

Confidence: $100(1-\alpha)\%$

$\alpha$: Confidence coeff.

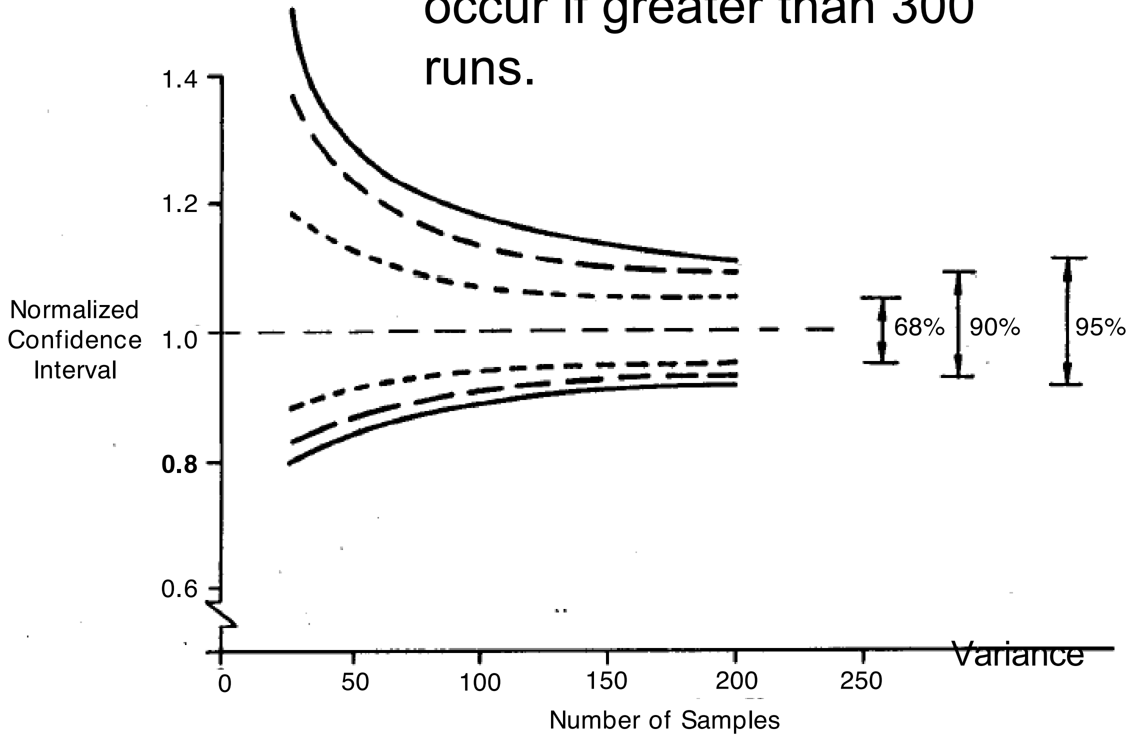
Mean:

$$-\frac{t_{N-1;\alpha/2}}{\sqrt{N}} \le \frac{m_x-\hat{m}_x}{\sigma_x} \le \frac{t_{N-1;\alpha/2}}{\sqrt{N}}$$

Variance:

$$\frac{N-1}{\chi^2_{N-1;\alpha/2}} \le \frac{\sigma_x}{\hat{\sigma}_x} \le \frac{N-1}{\chi^2_{N-1;1-\alpha/2}}$$

- Confidence interval depends only on # of runs.
- No more significant changes occur if greater than 300 runs.




[Lin] C.-F. Lin, Modern Navigation, Guidance, and Control Processing, Prentice-Hall, 1991.

# Joint Gaussian Random Variables

## ❑ Two Jointly Gaussian Probability Density Function

- Single Gaussian pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\bar{X})^2}{2\sigma_X^2}}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}}$$

- Two random variables are jointly Gaussian if their joint pdf has the form (or called bivariate Gaussian)

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\bar{X})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{X})(y-\bar{Y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]}$$

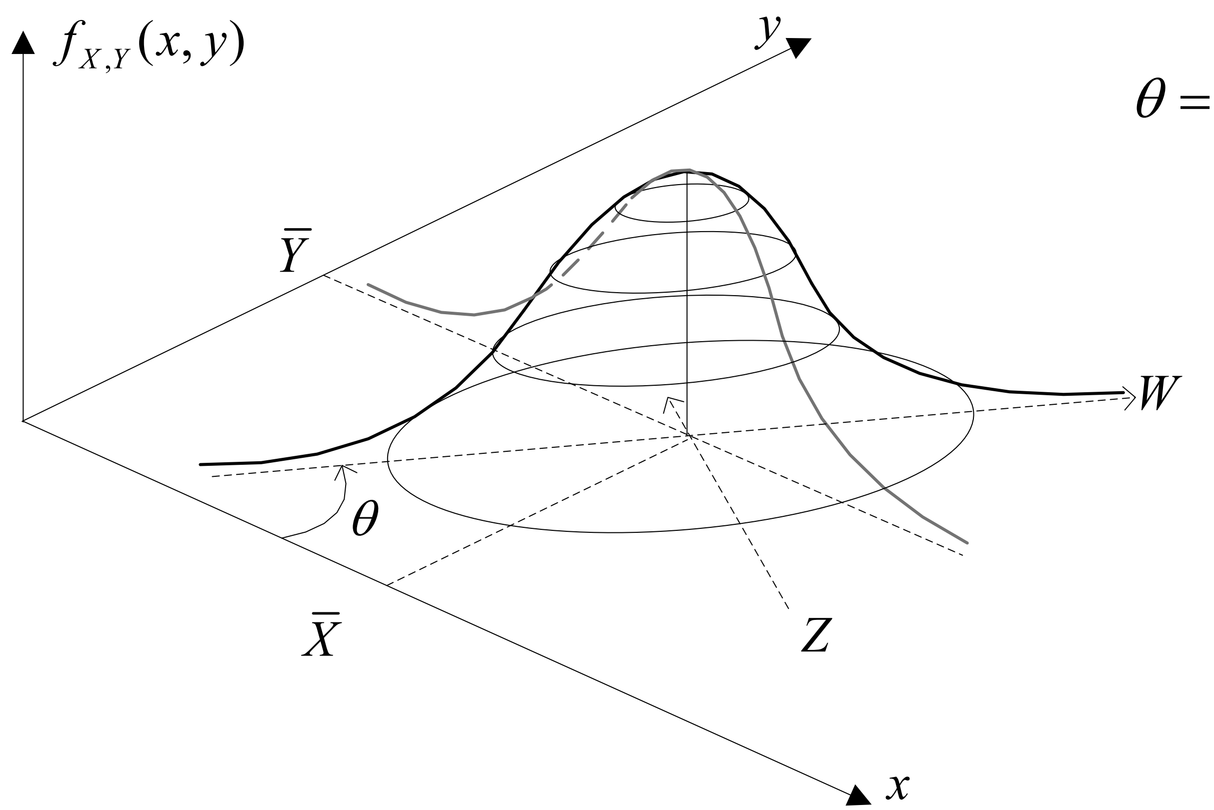
where $\bar{X} = E[X],\ \sigma_X^2 = E[(X-\bar{X}^2)],\ \ \bar{Y} = E[Y],\ \sigma_X^2 = E[(Y-\bar{Y}^2)]$

$$\rho = \frac{E[(X-\bar{X})(Y-\bar{Y})]}{\sigma_X\sigma_Y}$$

# Joint Gaussian Random Variables

❑ **Two Jointly Gaussian Probability Density Function**

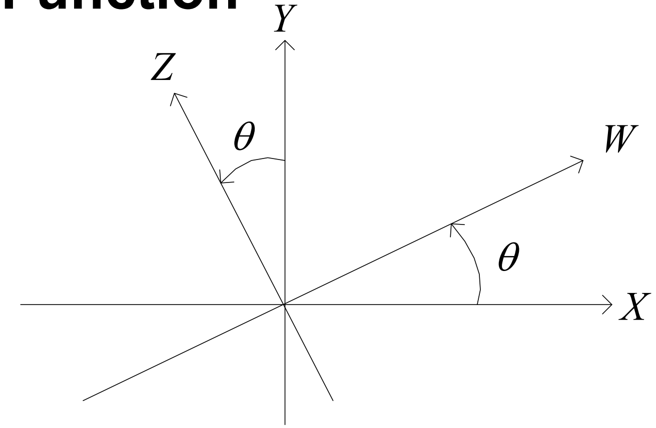
$$\max\ f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \quad \text{at } (\bar{X},\bar{Y})$$




$$\theta = \frac{1}{2}\tan^{-1}\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}$$

# Joint Gaussian Random Variables

## ❑ Two Jointly Gaussian Probability Density Function




- Proof of $\theta$ equation

$$\begin{bmatrix} W = X\cos\theta + Y\sin\theta \\ Z = -X\sin\theta + Y\cos\theta \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{W} = \bar{X}\cos\theta + \bar{Y}\sin\theta \\ \bar{Z} = -\bar{X}\sin\theta + \bar{Y}\cos\theta \end{bmatrix}$$

$$\begin{aligned} C_{WZ} &= E\left[(W-\bar{W})(Z-\bar{Z})\right] \\ &= E\left[[(X-\bar{X})\cos\theta + (Y-\bar{Y})\sin\theta][-(X-\bar{X})\sin\theta + (Y-\bar{Y})\cos\theta]\right] \\ &= E\left[[(Y-\bar{Y})^2 - (X-\bar{X})^2]\sin\theta\cos\theta + (X-\bar{X})(Y-\bar{Y})(\cos^2\theta - \sin^2\theta)\right] \\ &= (\sigma_Y^2 - \sigma_X^2)\sin\theta\cos\theta + C_{XY}(\cos^2\theta - \sin^2\theta) \\ &= \frac{1}{2}(\sigma_Y^2 - \sigma_X^2)\sin 2\theta + C_{XY}\cos 2\theta \end{aligned}$$

Since $W$ and $Z$ are uncorrelated, $C_{WZ} = 0$. Therefore

$$\tan 2\theta = \frac{2C_{XY}}{\sigma_X^2 - \sigma_Y^2} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2\rho\sigma_X\sigma_Y}{\sigma_X^2 - \sigma_Y^2}$$

# Joint Gaussian Random Variables

## ❑ Two Jointly Gaussian Probability Density Function

- If the two Gaussian random variables are uncorrelated, $\rho$=0.

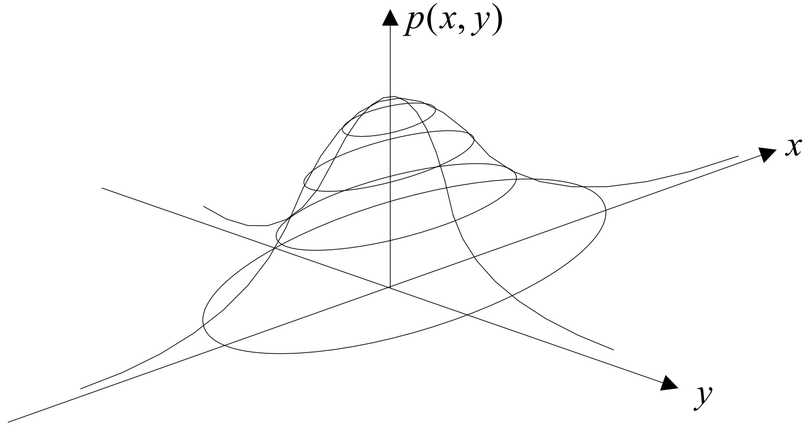
$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\bar{X})^2}{\sigma_X^2}+\frac{(y-\bar{Y})^2}{\sigma_Y^2}\right]} = f_X(x)f_Y(y)$$

where $f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-\frac{(x-\bar{X})^2}{2\sigma_X^2}}$, $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\bar{Y})^2}{2\sigma_Y^2}}$

- Uncorrelated Gaussian random variables are also statistically independent.
- Gaussian r.v.s are completely defined through their means, variances, and covariances.
- Random variables produced by a linear transformation of jointly Gaussian r.v.s are also Gaussian.
- The conditional density functions defined over jointly Gaussian r.v.s is also Gaussian.

# Circular Error Provable

- CEP is an intuitive measure of accuracy defined as the radius of a circle, centered about the mean, whose boundary is expected to include 50% of the hits within it. [Sio]




If there's no correlation between $x$ and $y$.

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} Exp\left[ -\frac{(x-m_x)^2}{2\sigma_x^2} - \frac{(y-m_y)^2}{2\sigma_y^2} \right]$$

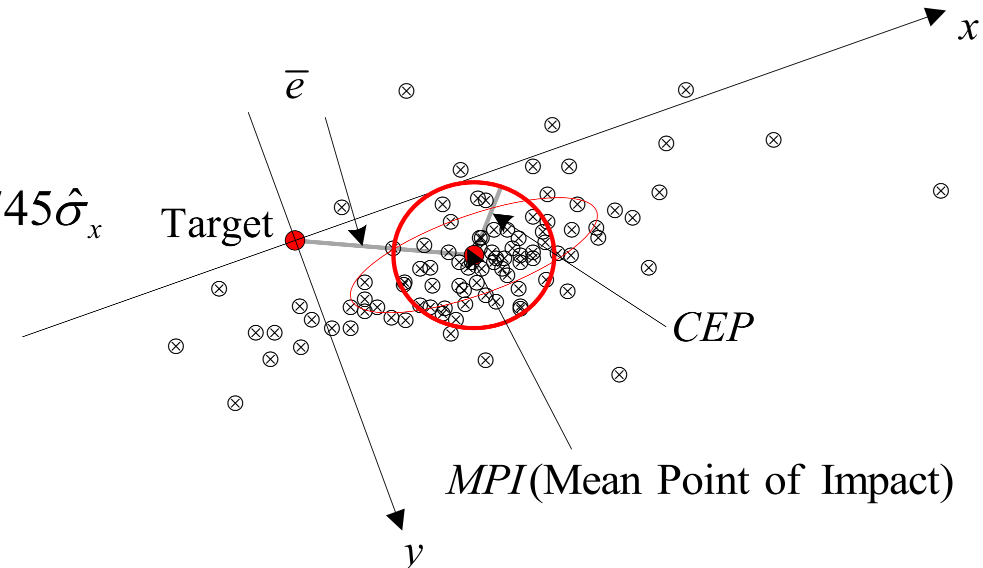
Let $\hat{\sigma}_x > \hat{\sigma}_y$, then $\nu = \dfrac{\hat{\sigma}_y}{\hat{\sigma}_x}$

$i)\ \nu \geq 0.28 \Rightarrow CEP \approx 0.589(\hat{\sigma}_x + \hat{\sigma}_y)$

$ii)\ \nu < 0.28 \Rightarrow CEP \approx 0.9263\left(\dfrac{\hat{\sigma}_y}{\hat{\sigma}_x}\right)^{2.09} + 0.6745\hat{\sigma}_x$

If $\bar{e} > 0.25\sqrt{\hat{\sigma}_x^2 + \hat{\sigma}_y^2}$, then

$\hat{\sigma}_x \Leftarrow \sqrt{\hat{\sigma}_x^2 + \bar{e}_x^2},\ \ \hat{\sigma}_y \Leftarrow \sqrt{\hat{\sigma}_y^2 + \bar{e}_y^2}$





[Sio] G. M. Siouris, Missile Guidance and Control Systems, Springer-Verlag, 2004.