Gambling-Based Confidence Sequences for Bounded Random Vectors

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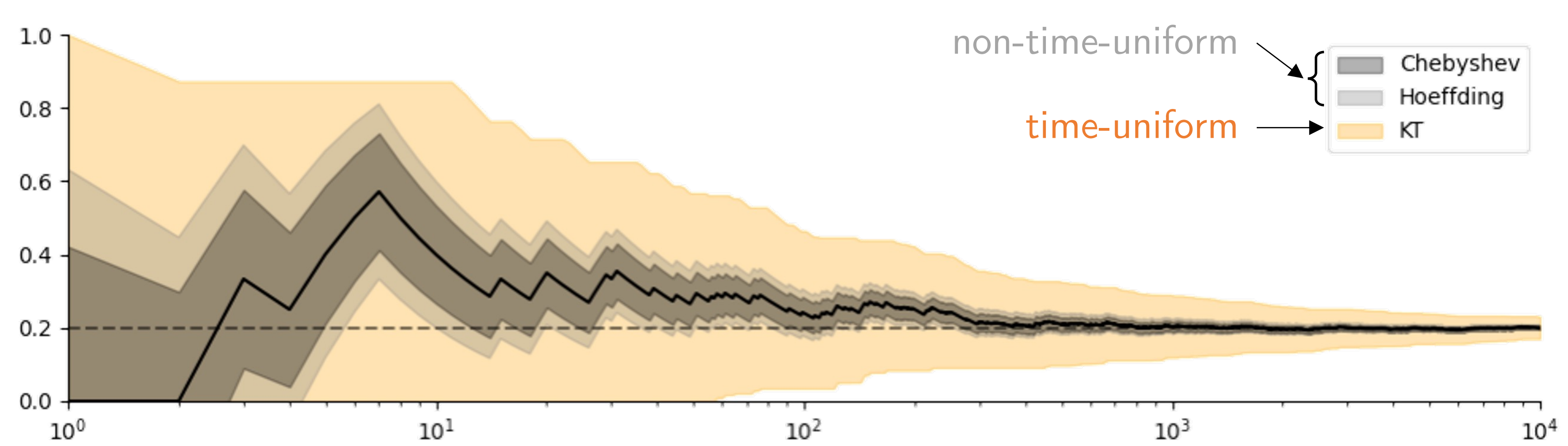
Time-Uniform Confidence Sets (a.k.a. Confidence Sequence)

- Suppose we observe $Y_1, Y_2, \ldots$ such that $\mathbb{E}[Y_t | Y^{t-1}] = \mu$
- **Confidence set**: for $\delta \in (0,1)$ w/ samples Y^t, construct $C_t(\delta)$ s.t. $\forall t \geq 1,\ \Pr(\mu \in C_t(\delta)) \geq 1-\delta$
  - standard notion in statistics
  - only valid for each time instance, not uniformly over time
- **Confidence sequence (CS)**: for $\delta \in (0,1)$

$$\Pr(\forall t \geq 1,\ \mu \in C_t(\delta)) \geq 1-\delta$$

  - suitable for sequential decision making!
- **Example**: i.i.d. Bern(0.2) process



Standard Recipe: Ville's inequality + Super-martingale

A stochastic process $(W_t)_{t=0}^{\infty}$ is said to be supermartingale if $(W_t)_{t=0}^{\infty}$ is adapted to the canonical filtration and satisfy

$$\forall t \geq 1,\ \mathbb{E}[W_t | \mathcal{F}_{t-1}] \leq W_{t-1}$$

Ville's inequality: For a nonnegative supermartingale sequence $(W_t)_{t=0}^{\infty}$ with $W_0 > 0$, for any $\delta > 0$,

$$\Pr\Big\{\sup_{t\geq 1} \frac{W_t}{W_0} \geq \frac{1}{\delta}\Big\} \leq \delta$$

Recipe:

- Construct $(W_t(y^t; m))_{t=1}^{\infty}$ as a function of y^t for each m;
- $\mathcal{C}(y^t;\delta) \triangleq \Big\{m\colon \frac{W_t(y^t;m)}{W_0(\emptyset;m)} < \frac{1}{\delta}\Big\}$ is a $(1-\delta)$-CS

Super-martingale Construction from Gambling

- **Intuition**: If the casino is (statistically) unfair to you, the wealth sequence from the gambling is a super-martingale
- Standard coin-betting formalism (Waudby-Smith & Ramdas, 2024)
  - For a sequence $Y_1, Y_2, \ldots \in [0,1]$, construct a wealth sequence

$$\boxed{\mathrm{K}_t(y^t; m) \triangleq \prod_{i=1}^{t} (1 + \lambda_i(m)\cdot(y_i - m)),\ \text{where } \lambda_t(m) \in \Big[-\frac{1}{1-m}, \frac{1}{m}\Big]}$$

  Here, the *signed bet* $\lambda_t(m)$ must be *causal*
- Stock market interpretation (Ryu & Bhatt, 2022)
  - Gambler chooses a betting $b_t \in [0,1]$ as a function of y^{t-1}
  - A *market vector* $\mathbf{x}_t(m) = [\frac{y_t}{m}, \frac{1-y_t}{1-m}]^\intercal$ is revealed;
  - Gambler's multiplicative gain at time t is

$$\boxed{\frac{\mathrm{W}_t(y^t; m)}{\mathrm{W}_{t-1}(y^{t-1}; m)} = \mathbf{x}_t(m)^\intercal \begin{bmatrix} b_t \\ 1-b_t \end{bmatrix} = b_t \frac{y_t}{m} + (1-b_t)\frac{1-y_t}{1-m}}$$

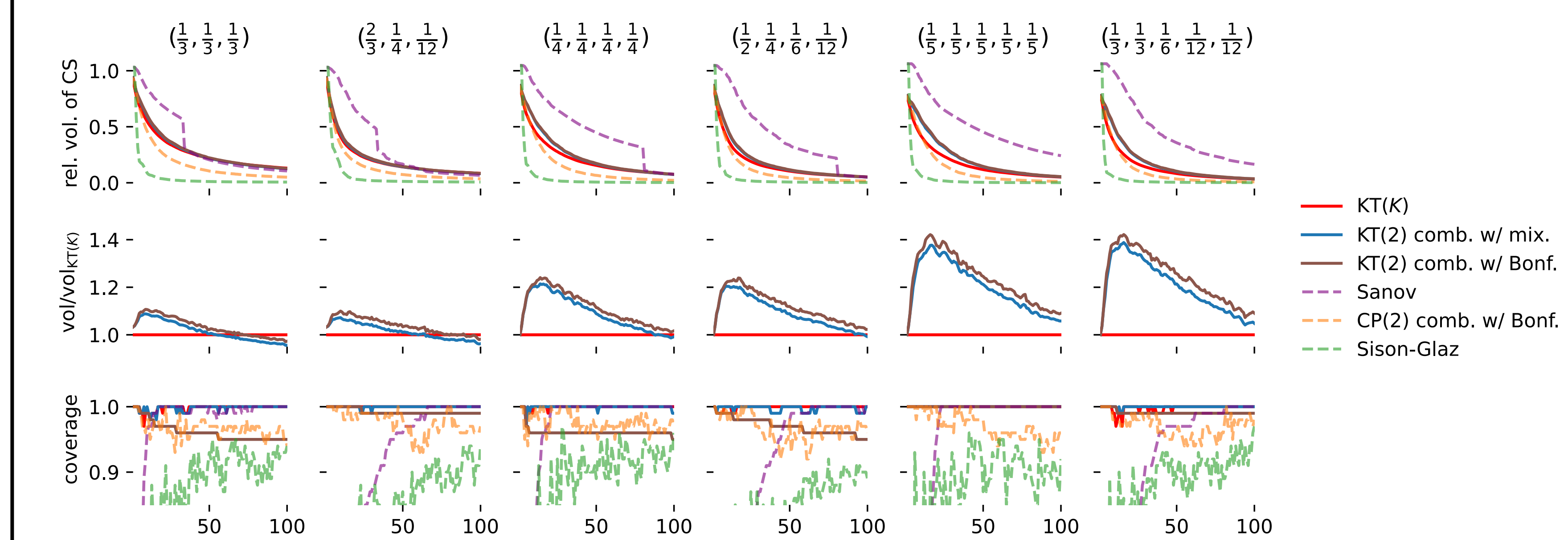
- Cover's universal portfolios (UP) works great (Jun & Orabona, 2024)
- Existing algorithms are designed for scalar-valued sequences

We extend this gambling framework to categorical/multivariate sequences!

Our Contributions

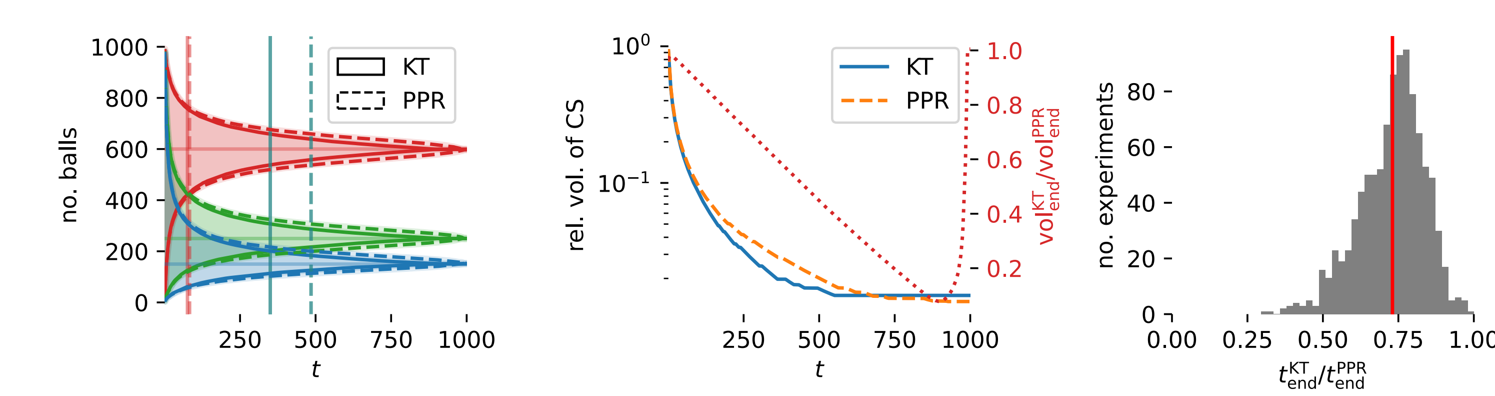
- Observation: For $\mathbb{R}^1$, map into a prob. simplex Δ^1, and use 2-stock market
- Generalization: For Δ^{K-1} (K-dim. prob. vector), use K-stock market
- Special case: For K-categorical data, use K-horse race
- Extension: For $[0,1]^{K-1}$, embed it into Δ^{K-1} and apply K-stock market
- We consider Cover's UP for the gambling strategy
  - For sampling without replacement with categorical observations, our construction with Cover's UP is provably tighter than (Waudby-Smith & Ramdas, 2020)

Categorical Random Variables



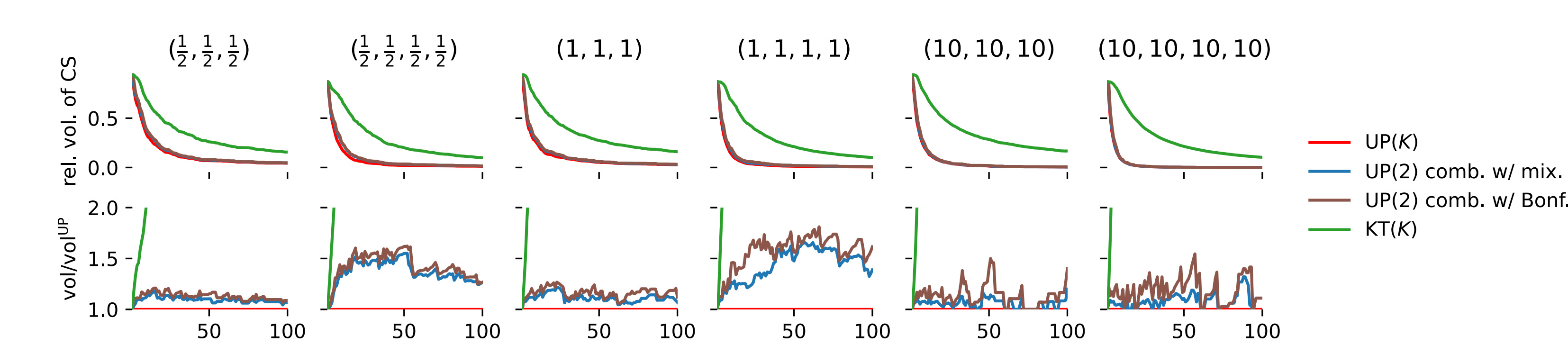
- Tighter than Sanov's bound / valid at all times (cf. Sison-Glaz)
- Tighter than baselines of combining "scalar" techniques
- Tight especially in small-sample regime

Sampling Without Replacement



- Tighter than PPR (WS&R, 2020), leading to faster decision for auditing

Probability-Vector-Valued Random Vectors



- Tighter than baselines of combining "scalar" techniques
- UP is not scalable due to the time complexity $O(T^K)$

References

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