Sequential First-Price Auctions Under Partial Disclosure: An Application to Korean Fruit Auction*

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November 7, 2024

Abstract

I consider a model in which a first-price auction sells one object at a time and repeats. During this repetition, only the winner and the winning bid are announced after each auction. A bidder uses this announcement to adjust his bidding strategies in order to win multiple objects across the repeated auctions. I narrow the repetition down to a two-period, so that I can nonparametrically identify a bidder's strategy and the complementarity between objects that motivates him to acquire multiple objects. I apply this model to the Korean Fruit Auction and suggest using an alternative auction design, Product-Mix Auction. This new design finds a uniform price for each variety and mitigates bidders' bid shading, thereby preventing the oscillatory winning bids observed in the current sequential auction and protecting farmers' interests, which aligns with the government's objectives.

Keywords: Sequential(repeated) Auction, First-price Auction, Market Design, Nonparametric Estimation

JEL Codes: C14, C51, C57, D47

*I thank Quang Vuong, Boyan Jovanovic, and Daniel Waldinger for their comments and guidance, and those who attended Student Econometrics Seminar; I thank a bidder at Seoul Auction House and an auctioneer at Joongang Auction House, both of whom I asked many questions. Lastly, I thank Seoul Agro-Fisheries & Food Corporation for providing the dataset in accordance with their internal regulations; all remaining errors are mine.

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1 Introduction

An auction is used when someone doubts market sentiments but must transact in large volume: well-known transactions include spectrum, timber rights, rough diamonds, highway paving contracts, and treasury bills[1].

Because of its large volume and bidders' demanding more than one unit[2], the stakes are high and the Revenue Equivalence Theorem fails[3], requiring the auctioneer to make decisions on the auction designs[4]. One of these decisions involves the extent of information disclosure between auctions, and papers (Bergemann and Hörner (2018) and Dufwenberg and Gneezy (2013)) suggest that revealing less information about the outcome of the previous auction to the bidders benefits the auctioneer.

Complementarity between objects also influences the auctioneer's decision in selecting the auction design. When objects are complements, the whole is worth more than the sum of its parts. Thus, in 2013, when the United Kingdom sold complementary spectrums, they selected a design that allowed a bidder to form packages (a combinatorial clock auction) rather than a design that did not (a simultaneous multiple round auction)[5].

The questions this paper tries to address are: (i) Under a model where the auctioneer sells multiple objects one at a time and discloses only the winning bids and the winner's identities, can the analyst separate complementarity and correlation across objects from the dataset? (ii) Given this separation, do the nonparametric estimators based on this model successfully estimate both the parameters of interest and also the bidders' bidding strategies? (iii) Can I use the model and its estimators to develop policy recommendations?

I answer these questions under the paradigm of Independent Private Value, focusing on first-price sealed-bid auctions. Each question is important, as answering the first question relates to distinguishing between structural state dependence and the persistent heterogeneity (Heckman (1981)), which is crucial for policy evaluation. Answering the second question also enhances the policy evaluation because the nonparametric estimator makes less assumptions on

[1] For spectrum, Myers (2023) elaborates on combinatorial clock auction and simultaneous multiple round auction used in Ofcom, UK. For other products, notable or recent papers are: timber rights for Athey et al. (2011); rough diamonds for Cramton et al. (2013); highway paving contracts for Jofre-Bonet and Pesendorfer (2003), Gentry et al. (2023), Silva and Rosa (2023), Kim (2024); treasury bills for Hortaçsu and McAdams (2010).

[2] Bidders' demanding only a single unit(known as unit demand) has been the main focus of theoretical literature. Milgrom and Weber (1982) incorporates the concept of affiliated values and ranks auction designs, Mcafee and Vincent (1993) explains the declining price in a repeated auction using risk-averse preference, Maskin and Riley (2000b) and Maskin and Riley (2003) focus on first-price sealed-bid and state condition for the existence of monotonic equilibrium strategy and condition for the unique equilibrium. These four papers are only a few among the voluminous notable papers, and chapters 1-11 of Krishna (2010a) introduce results from these notable theoretical papers. A recent empirical paper, Backus and Lewis (2024), discusses dynamic demand estimation under a unit demand assumption.

[3] The Revenue Equivalence Theorem assumes that a bidder desires only a single unit (Klemperer (2000)), which is why Ausubel et al. (2014) argues that the theorem is inapplicable when a bidder desires more than one unit. The inapplicability of the theorem in practice is detailed in Klemperer (2013b).

[4] Examples of auction designs can be found in Vulkan et al. (2013)(chapters 3, 10, 11, 12, 15, and 16), Hendricks and Porter (2007), Hortaçsu and Perrigne (2021), and Kaplan and Zamir (2015).

[5] Refer to Myers (2023). CCA(Combinatorial Clock Auction) was used in 2013 because the licenses that were auctioned were high-frequency(known as coverage spectrum) and low-frequency(known as capacity spectrum), which are complements. Figure A1.1 shows that 2013's CCA was not the first auction that used CCA and replaced SMRA(Simultaneous Multiple Round Auction); but, as page 228 denotes, the first two auctions(2007 and 2008) that used CCA were lower-stakes auctions prepared for the 2013's high-stakes auctions.

the parameters of the interest. Moreover, one could use the estimator to verify whether bidders engage in monotone strategies, thereby ensuring that the bidder who values the object most is awarded the object. Lastly, answering the last question demonstrates the policy implications of this paper.

I consider a two-period Perfect Bayesian Nash Equilibrium[6] model in Section 2 where the values of the first and second objects are correlated. Additionally, to separately account for correlation and complementarity, my model uses a function that takes as inputs the values of the first and second objects, and outputs an adjusted value for the second object that incorporates the complementarity effect of holding the first object. Under this function, bidders with different values for the objects experience different degrees of complementarity. This advances the structure of recent papers, where two bidders with different values for the objects experience the same degree of complementarity across those objects[7].

The bidder in my model wants both objects, leading the first auction losers and the first auction winner to compete in the second auction, creating an asymmetric auction[8]. Under this setting, I find that even with a dataset containing only the winning bids and the winner's identity, the analyst can nonparametrically identify the complementarity and correlation between objects, as demonstrated in Section 3. Moreover, the same section also demonstrates that the analyst can identify from the dataset the bidder's bidding strategy in the first auction, as well as his strategy in the second auction, depending on whether he won or lost the first auction. These demonstrations show that the *indirect approach* of Guerre et al. (2000), in which the bid distribution is first identified and the parameters of interest are subsequently identified, can be extended to a multi-period setting where bidders have multi-unit demand—an extension that has not yet been explored in the literature[9]. Given that this *indirect approach* is extensively used in real-world applications (see Hortaçsu and McAdams (2018)), my paper contributes to addressing real-world problems.

Building on the identification results in Section 3, I propose a multi-step estimator in Section 4 for these identified estimands and demonstrate, via Monte Carlo simulations, that the median estimates of the estimator are consistent with the true estimands. Policymakers may use the estimator to verify whether the bidders engage in a monotone bidding strategy or to assess the degree of complementarity between objects.

[6] Another concept used in empirical auction papers is Markov Perfect Equilibrium, as seen in notable or recent works such has Jofre-Bonet and Pesendorfer (2003), Altmann (2024b), and Kim (2024). Asker et al. (2020) also considers an infinite horizon game, but uses the new concept of Restricted Experience-Based Equilibrium; Aguirregabiria et al. (2021) in its chapter 4.1.3 introduces dynamic models used in empirical auction.

[7] For example, complementarity across objects, which are considered in the following papers, varies by auction-specific covariates not by a bidder's value for the objects: Arsenault-Morin et al. (2022) uses a function $K_i^{\mathcal{A}}$ for roof-maintenance contracts, Gentry et al. (2023) uses a function κ_i for highway procurement auctions, and Altmann (2024a) use a combinatorial pay-off function k for Feeding America's allocation mechanism. Another notable paper is Donna and Espín-Sánchez (2018) that estimates a complementarity parameter ρ in water auction. But, the estimated ρ and a bidder's value are linearly multiplied, which I do not assume.

[8] Maskin and Riley (2000a) also considers an asymmetric auction, but its focus is on a single period auction(known as a static auction).

[9] Kong (2021) also considers a two-period auction and discusses identification and estimation of the parameters, but it is a first-price auction followed by an English auction. For repeated first-price auctions, a well known paper is Milgrom and Weber (1999), which assumes a bidder with a unit demand; recent papers such as Kannan (2012), Bergemann and Hörner (2018), and Azacis (2020) assume a bidder with multi-unit demand and discuss the effect of changing disclosure policy, but they do not discuss identification or estimation strategies.

Section 5 shows the results of applying my model and its estimators to the Agricultural Produce Auction at Garak Market. Twenty-seven percent[10] of all vegetables and fruits in Korea are transferred from farmers to wholesalers through this auction, indicating that the stakes are high—to my knowledge, no paper has conducted a structural analysis of this auction.

Among the vegetables and fruits, I focus on apple auctions, as apples are one of top five fruits traded at Garak Market. By applying my model and its estimators to the last two auctions of each auction house at the Market, I estimate the substitutability between the second-last and last apples. This implies that a bidder who won the second-last auction becomes a weak type, while a bidder who lost becomes a strong type in the last auction. I estimate the bidding strategies of each type and find that each follows a monotone bidding strategy in the last auction, with the strong type shading more than the weak type; this differential shading algins with the predictions of Maskin and Riley (2000a).

Given the high stakes of this auction, various principles govern the auction. One such principle is Article 123 (4) of Constitution of The Republic of Korea, which mandates that the government protect the interest of farmers. Based on this article, the government has standardized the quality of agricultural products to ensure that each farmer receives higher winning bids in the auction. However, even with the assumption that the same product (e.g., high-quality Large Fuji apples) from different farmers is well standardized, selling them one by one would lead to oscillatory winning bids on any given day, as predict by papers suggesting a martingale path[11]. This would disadvantage farmers whose produce is auctioned at the trough of that oscillation.

Under the assumption that the same product from different farmers is well standardized, I propose using an alternative auction design: the Product-Mix Auction, as employed by the Bank of England. This design is a uniform-price auction in which there is only one winning bid for the same product, ensuring that all farmers of that product receive the same price. To mitigate the bid shading inherent in uniform-price auction, this design (i) first selects non-complementary products (e.g., high-quality Large Fuji apples and high-quality Medium Fuji apples) and declares the total quantities to be sold, (ii) then allows bidders to submit their demand schedules, which the auctioneer uses to construct a demand curve expressed in terms of the quantity ratio and price ratio of the two products, and (iii) enables the auctioneer to choose the quantity ratio to determine the competitive equilibrium price for each product. Under this design, a bidder shades his bid less because bid shading for one product is less likely to drive down the winning bid of that product compared to a traditional uniform-price auction. To demonstrate how the Product-Mix Auction works, I select one day from one of the auction houses and use my model and estimator to illustrate the demand curve and the equilibrium prices of the products.

[10]Thirty-three public wholesale markets account for 99.4% of the trade volume of vegetables and fruits in Korea. Garak Market, one of these thirty-three public wholesale markets, accounts for 34.5% of this trade volume, with auctions making up 79.6% of this volume. Multiplying these numbers yields twenty-seven percent.

[11]See Krishna (2010b) and Milgrom and Weber (1999).

2 Model

The auctioneer uses a first-price sealed-bid auction and sells one object[12] at a time. Between the first and the second auctions, he decides how much information to disclose about the result of the first auction. Less disclosure benefits the auctioneer[13], so the auctioneer in my model discloses only the winning bid and the winner's identity from the first auction[14], not all the bids.

Risk-neutral I bidders attend the auction. The set of I bidders remains the same across both auctions because a bidder who bids in the first auction also bids in the second auction; this repeated bidding occurs because both objects are valuable to each bidder. The value a bidder derives in apple auction (hereafter, Korean Fruit Auction) mostly comes from delivering them at an agreed-upon price to his customers. The customer base of each bidder differs, and bidders vary in their expertise, which aligns with the Independent Private Value paradigm[15].

Each v_1 and v_2 represents the private value a bidder derives from the first and second objects. If the objects are either complements or substitutes, the utility of having both should differ from the simple sum $v_1 + v_2$. I express this utility of having both as $v_1 + \delta(v_1, v_2)$, and one can interpret it as winning the v_1-valued first object modifying the value of the second object from v_2 to a $\delta(v_1, v_2)$.

Whether this modification moves upward or downward depends on whether both objects show complementarity or substitutability. In each case, we observe $\delta(v_1, v_2) > v_2$ or $\delta(v_1, v_2) < v_2$, indicating that the function δ is flexible enough to account for both directions. The assumption I impose on this flexible function δ is that it must be increasing in its second argument, v_2. This imposition reflects the idea that if the second object becomes more valuable, a bidder who already owns the first object also finds the second object more valuable, which aligns with common sense; from now on, the term 'increase' refers to 'strictly increase', and the same applies to 'monotone.'

The reason I impose that the function δ is monotone in v_2 is because it is necessary for nonparametric identification. Nonparametric identification of an unknown function, such as the function δ in my case and the function m in Matzkin (2003)'s, often requires monotonicity in unobserved heterogeneity — v_2 in my case and ϵ in Matzkin's. Imposing monotonicity on the function δ meets this requirement, which is also why I impose monotonicity on a bidder's strategy, as the strategy must also be nonparametrically identified.

Imposing monotonicity on a bidder's strategy means that a bidder places a higher bid when he values the object more; if the object is the first auctioned object, this implies that his bid in the first auction is monotone with respect to his v_1. This monotonicity makes sense if the

[12]For repeated first-price auction selling multiple objects at each period, see Altmann (2024a) and Altmann (2024b)

[13]Refer to both Bergemann and Hörner (2018) and Dufwenberg and Gneezy (2013). These studies compare allocative efficiency or revenue under three different disclosure policies. These policies include, after each round: disclosing privately to each bidder whether they have won or lost, revealing only the winning bid, and disclosing all bids. Both studies find that when all bids are disclosed, pooling equilibria arise, leading to lower revenue (Bergemann and Hörner (2018)) and higher procurement costs (Dufwenberg and Gneezy (2013)).

[14]For cases in which only the winner's identity is disclosed, see Choi (2024).

[15]As discussed in Perrigne and Vuong (2023), the Independent Private Value (IPV) model is the most commonly used framework in the empirical auction literature. A.9 discusses why the IPV model has gained popularity compared to other models.

bidder knows his v_1 but never his v_2 during the first auction: if he knew that both of his v_1 and v_2 were high, and given that the first auction's winning bid is disclosed, he might submit a low bid in the first auction to appear weak, manipulating others' belief about him, and win the second object at a lower price. This manipulative bidding often results in two bidders with different v_1s placing the same first auction bid, making the identification of the bidding strategy challenging[16].

To circumvent this challenge, the value of the second object remains a random variable V_2 during the first auction. Only after the first auction concludes is this randomness realized as v_2, drawn from the distribution $F_{2|1}(\cdot|v_1)$, implying that values of the first and second objects are dependent. This dependence assumption is reasonable because in most real-world applications, objects auctioned sequentially are correlated[17].

As the random variable and its distribution have been discussed, I specify what *symmetry* means in my model:

> *(Symmetry)* Every bidder is subject to the same parameters: $F_1(\cdot)$, $F_{2|1}(\cdot|\cdot)$, and $\delta(\cdot,\cdot)$. Each parameter represents $\Pr[V_1 \leq \cdot]$, $\Pr[V_2 \leq \cdot|V_1 = \cdot]$, and the function that takes v_1 and v_2 as arguments and outputs the adjusted value of v_2 from owning the v_1-valued object.

Symmetry shows how V_1s and V_2s are distributed across I bidders, and *independence* describes how these random variables are assumed to affect each other.

> *(Independence)* Under the same distribution F_1, each I bidder independently draws his value of the first object, V_1. Based on the drawn v_1, each bidder independently draws his value of the second object V_2 from the conditional distribution $F_{2|1}$, whose condition is set at $V_1 = v_1$.

Suppose a bidder values the first object at 10, i.e., $v_1 = 10$. If his values for the first and second objects are highly correlated, then $F_{2|1}$ predicts that his value for the second object will realize near 10. If this realization is $v_2 = 13$, then this value of 13 adjusts to $\delta(10, 13)$ if the bidder owns the first object, but remains as 13 otherwise. The adjustment by the function δ represents the causal effect of owning the first object, while the correlation between the objects is captured by $F_{2|1}$.

2.1 Equilibrium Strategies

Consider a situation where bidder i competes against $I - 1$ other bidders, who place Bayesian Nash equilibrium bids denoted by $\{(b_{2j}, b_{1j})_{j\neq i}\}$. In response to these equilibrium bids, bidder i maximizes his expected profit by submitting tilde bids $(\tilde{b}_{2i}, \tilde{b}_{1i})$. At the end of Section 2, I impose that bidder i's tilde bids are indeed Bayesian Nash equilibrium bids. Theorem 1 then specifies the testable restrictions that justify this imposition.

[16] This manipulative bidding can be understood as the ratchet effect; see Laffont and Tirole (1988). The phenomenon where bidders with different values place the same bid is known as pooling equilibria. Kong (2021), in her online appendix A.2.3, notes that the monotonicity of bidding strategies is crucial for structural analysis.

[17] This setup is the same as Kong (2021) and traces back to chapter 8 of Ortega-Reichert (1968); see A.10. Notable examples that justify the dependence between auctioned objects include wine auctions discussed in Ashenfelter (1989) and the transponder leases at Sotheby's in 1981 discussed in Milgrom and Weber (1999).

I begin with the second auction because I use the Perfect Bayesian Nash Equilibrium concept; details and proofs are included in the Appendix A.

2.1.1 Expected Profit Functions in the Second Auction

If bidder i wins the first auction, he acquires the first object, valued at v_{1i}. From the acquisition of the first object, his value for the second object changes from v_{2i} to $\delta(v_{1i}, v_{2i})$. To get the second object, now valued at $\delta(v_{1i}, v_{2i})$, he chooses the optimal bid $\tilde{b}_{2i}$ that maximizes the following expected profit function:

$$\big(\delta(v_{1i}, v_{2i}) - \tilde{b}_{2i}\big)\Pr[B^{\max}_{2,-i} \leq \tilde{b}_{2i} \mid B^{\max}_{1,-i} \leq \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}], \tag{1}$$

in which the random variable $B^{\max}_{2,-i}$ ($B^{\max}_{1,-i}$) represents the highest bid in the second (first) auction from the $I-1$ competitors, against which bidder i is bidding. The condition $\{B^{\max}_{1,-i} \leq \tilde{b}_{1i}\}$ inside the probability reflects bidder i is uncertain about other bidders' first auction bids. This uncertainty arises because the auctioneer does not disclose the losing bids from the first auction.

What the auctioneer discloses are the winning bid and the identity of the winner from the first auction. If bidder i loses and observes that the winning bid is b^w_1 and the winner is bidder j, then bidder i has the following expected profit function:

$$\big(v_{2i} - \tilde{b}_{2i}\big)\Pr[B^{\max}_{2,-i} \leq \tilde{b}_{2i} | B_{1j} = b^w_1, B_{1j} \geq B_{1k}, k \notin \{i, j\}, B_{1j} \geq \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]. \tag{2}$$

The condition $\{B_{1j} = b^w_1, B_{1j} \geq B_{1k}, k \notin \{i, j\}, B_{1j} \geq \tilde{b}_{1i}\}$ inside the probability expresses what bidder i knows after losing the first auction. Since he loses the first auction, he values the second object at v_{2i} instead of $\delta(v_{1i}, v_{2i})$ because he does not own the first object.

I introduce alternative expressions for both profit functions, (1) and (2).

A.1 shows the alternative expression for profit function (1) is the following equation,

$$\big(\delta(v_{1i}, v_{2i}) - \tilde{b}_{2i}\big)G_{B^l_2(\tilde{b}_{1i})}(\tilde{b}_{2i} | B_1 \leq \tilde{b}_{1i})^{I-1}. \tag{3}$$

A new notation $G_{B^l_2(\tilde{b}_{1i})}(\cdot | B_1 \leq \tilde{b}_{1i})$ represents a bid distribution $\Pr\big[B^l_2(\tilde{b}_{1i}) \leq \cdot | B_1 \leq \tilde{b}_{1i}\big]$, which is the distribution of second auction bid of a first auction loser, who lost to the winning bid $b^w_1 = \tilde{b}_{1i}$ in the first auction. A number $I-1$ is squared to this distribution; this squaring occurs because bidder i competes against the same first auction losers who bid independently in the second auction. They are the same as they decide their second auction bids according to the same equilibrium strategy s^l_2, and they bid independently by *independence*.

A.2 shows the alternative expression for profit function (2) is the following equation, in which the winning bid of b^w_1 is higher than bidder i's first auction bid $\tilde{b}_{1i}$:

$$\big(v_{2i} - \tilde{b}_{2i}\big)G^w_{2|1}(\tilde{b}_{2i} | b^w_1)G_{B^l_2(b^w_1)}(\tilde{b}_{2i} | B_1 \leq b^w_1)^{I-2}. \tag{4}$$

A new notation $G^w_{2|1}(\cdot | b^w_1)$ represents a bid distribution $\Pr[B^w_2 \leq \cdot | B_1 = b^w_1]$, which is the distribution of second auction bid of the first auction winner, who outbid bidder i with b^w_1. Equation (4) shows that bidder i competes against (i) $I-2$ first auction losers and (ii) the first auction winner who had bid b^w_1 in the first auction. This b^w_1 is known to every bidder by the disclosure policy.

Given bidder i's expected profit function (3), his optimal second auction bid $\tilde{b}_{2i}$ must satisfy the following equation (5); it is the first order condition coming from the derivative of equation (3) with respect to $\tilde{b}_{2i}$.

$$\delta(v_{1i}, v_{2i}) = \tilde{b}_{2i} + \frac{G_{B_2^l(\tilde{b}_{1i})}(\tilde{b}_{2i}|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{b}_{2i}|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \equiv \xi_2^w(\tilde{b}_{2i}, \tilde{b}_{1i}). \tag{5}$$

The left-hand side represents how much bidder i values the second object, while the right-hand side is a function of bids, denoted by a function ξ_2^w. This function is the sum of $\tilde{b}_{2i}$ and a certain fraction; this fraction only takes positive values, implying that it is optimal for bidder i, who won the first auction, to shade his bid in the second auction.

Similarly, given bidder i's another expected profit function (4), his optimal second auction bid $\tilde{b}_{2i}$ must satisfy the following equation (6), which is the first order condition coming from the derivative of equation (4) with respect to $\tilde{b}_{2i}$.

$$v_{2i} = \tilde{b}_{2i} + \frac{G^w_{2|1}(\tilde{b}_{2i}|b_1^w) G_{B_2^l(b_1^w)}(\tilde{b}_{2i}|B_1 \leq b_1^w)^{I-2}}{\partial (G^w_{2|1}(\tilde{b}_{2i}|b_1^w) G_{B_2^l(b_1^w)}(\tilde{b}_{2i}|B_1 \leq b_1^w)^{I-2})/\partial \tilde{b}_{2i}} \equiv \xi_2^l(\tilde{b}_{2i}, b_1^w), \tag{6}$$

in which bidder i's first auction bid $\tilde{b}_{1i}$ must be lower than the winning bid of b_1^w. Analogous to first-order condition (5), the left-hand side represents the value bidder i places on the second object, while the right-hand side is a function of bids, denoted by the function ξ_2^l. This function is the sum of $\tilde{b}_{2i}$ and a certain fraction; similar to equation (5), it is optimal for bidder i, who lost the first auction, to shade his bid in the second auction.

2.1.2 Expected Profit Function in the First Auction

First-order condition (5) expresses what bidder i must satisfy in choosing his optimal second auction bid had he won the first auction, while another first-order condition (6) expresses the optimal condition had he lost the first auction.

To choose the optimal first auction bid, $\tilde{b}_{1i}$, bidder i has to maximize the following expected profit function,

$$\left[v_{1i} - \tilde{b}_{1i} + \mathcal{V}^w(v_{1i}, \tilde{b}_{1i})\right] G_1(\tilde{b}_{1i})^{I-1} + \mathcal{V}^l(v_{1i}, \tilde{b}_{1i})\left[1 - G_1(\tilde{b}_{1i})^{I-1}\right]. \tag{7}$$

A new notation G_1 represents bid distribution $\Pr[B_1 \leq \cdot]$, which is the distribution of the first auction bid. $G_1(\tilde{b}_{1i})^{I-1}$ represents the probability of bidder i winning the first auction with a bid $\tilde{b}_{1i}$[18]. Because bidder i wins the first auction, he not only enjoys v_{1i} but also $\mathcal{V}^w(v_{1i}, \tilde{b}_{1i})$, which is the continuation value of being the first auction winner in the second auction. This concept of the continuation value also applies when bidder i loses the first auction with his first auction bid $\tilde{b}_{1i}$, which I denote as $\mathcal{V}^l(v_{1i}, \tilde{b}_{1i})$[19].

[18] A.3 shows the equivalence between $\Pr[B_{1,-i}^{\max} \leq \tilde{b}_{1i}]$, probability of bidder i winning the first auction with a bid $\tilde{b}_{1i}$, and $\Pr[B_1 \leq \tilde{b}_{1i}]^{I-1}$.

[19] A.4 and A.5 show analytical forms of both $\mathcal{V}^w$ and $\mathcal{V}^l$.

Given expected profit function (7), bidder i's optimal first auction bid $\tilde{b}_{1i}$ must satisfy the following equation (8)[20], which is the first order condition coming from the derivative of equation (7) with respect to $\tilde{b}_{1i}$.

$$\begin{aligned}
\tilde{b}_{1i} = v_{1i} &- \frac{1}{I-1}\frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})} \\
&+ \int_{\underline{b_2}}^{\overline{b_2}} \Bigg(\frac{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \frac{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2}}{g_1(\tilde{b}_{1i})} \\
&\quad \times \frac{\partial \{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i}) G_1(\tilde{b}_{1i})\}}{\partial \tilde{b}_{1i}} \Bigg) \underbrace{d\Pr[\tilde{B}_2^w \leq x|V_1 = v_{1i}]} \\
&- \int_{\underline{b_2}}^{\overline{b_2}} \frac{(G_{2|1}^w(x|\tilde{b}_{1i}) G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2})^2}{\partial (G_{2|1}^w(x|\tilde{b}_{1i}) G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2})/\partial \tilde{b}_{2i}} \underbrace{d\Pr[\tilde{B}_2^l(\tilde{b}_{1i}) \leq x|V_1 = v_{1i}]}. \qquad (8)
\end{aligned}$$

First-order condition (8) shows that bidder i's first auction bid $\tilde{b}_{1i}$, which is the left-hand side, equals his value for the first object v_{1i} with some adjustments. One of the adjustment terms, the fraction $\frac{1}{I-1}\frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})}$, represents how much bidder i shades his first auction bid if he were assumed to be interested in getting only the first object[21]. This assumption is not true because, in my model, every bidder demands both the first and second objects. This demand forces bidder i to consider the effect of adjusting his first auction bid $\tilde{b}_{1i}$ on his payoff in the second auction. The first integral represents how much bidder i needs to adjust his $\tilde{b}_{1i}$ if he were to enter the second auction as a first auction winner; the second integral is analogous to the first integral, except that the second integral represents bidder i entering the second auction as a first auction loser.

Since I define functions ξ_2^w and ξ_2^l from first-order conditions (5) and (6), I can define a new function ξ_1 from modifying first-order condition (8).

$$\begin{aligned}
v_{1i} = \tilde{b}_{1i} &+ \frac{1}{I-1}\frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})} \\
&- \int_{\underline{b_2}}^{\overline{b_2}} \Bigg(\frac{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \frac{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2}}{g_1(\tilde{b}_{1i})} \\
&\quad \times \frac{\partial \{G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i}) G_1(\tilde{b}_{1i})\}}{\partial \tilde{b}_{1i}} \Bigg) \underbrace{dG_{2|1}^w(x|\tilde{b}_{1i})} \\
&+ \int_{\underline{b_2}}^{\overline{b_2}} \frac{(G_{2|1}^w(x|\tilde{b}_{1i}) G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2})^2}{\partial (G_{2|1}^w(x|\tilde{b}_{1i}) G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})^{I-2})/\partial \tilde{b}_{2i}} \underbrace{dG_{B_2^l(\tilde{b}_{1i})|B_1}(x|\tilde{b}_{1i})} \\
&\equiv \xi_1(\tilde{b}_{1i}), \qquad (9)
\end{aligned}$$

in which the modification includes replacing the underbraced differential distributions in (8) with $dG_{2|1}^w$ and $dG_{B_2^l(\tilde{b}_{1i})|B_1}$. A new notation $G_{B_2^l(\tilde{b}_{1i})|B_1}(\cdot|\tilde{b}_{1i})$ represents bid distribution $\Pr[B_2^l(\tilde{b}_{1i}) \leq \cdot|B_1 = \tilde{b}_{1i}]$, which is the distribution of the second auction bid of the first auction loser whose first auction bid is $\tilde{b}_{1i}$ and the winning bid he observes is also $\tilde{b}_{1i}$. Equation (10) shows that

[20] A.6 shows the detailed derivation going from equation (7) to equation (8).

[21] Indeed, $\tilde{b}_{1i} = v_{1i} - \frac{1}{I-1}\frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})}$ appears in Guerre et al. (2000), which assumes a bidder with a unit demand.

this new bid distribution is the combination of other bid distributions that have already been defined[22].

$$G_{B_2^l(\tilde{b}_{1i})|B_1}(x|\tilde{b}_{1i}) = G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_1) + \frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})}\left\{\frac{\partial G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})}{\partial \tilde{b}_{1i}} - \frac{\partial \xi_2^l(x, \tilde{b}_{1i})/\partial \tilde{b}_{1i}}{\partial \xi_2^l(x, \tilde{b}_{1i})/\partial \tilde{b}_{2i}} \frac{\partial G_{B_2^l(\tilde{b}_{1i})}(x|B_1 \leq \tilde{b}_{1i})}{\partial \tilde{b}_{2i}}\right\}. \tag{10}$$

Because the combinations of bid distributions constitute the left-hand side of equation (10), one cannot guarantee that the resulting left-hand side is even a distribution. For this guaranteeing, I need to put some restrictions on my model, which leads me to discuss the equilibrium strategies.

2.1.3 Equilibrium Strategies

Up to now, I described bidder i's optimal bids, $\tilde{b}_{1i}$ and $\tilde{b}_{2i}$, which satisfy bidder i's first order conditions, (5), (6), and (9). Not only for bidder i but for other $I-1$ bidders, I can also derive their optimal bids $\{(\tilde{b}_{1j}, \tilde{b}_{2j})_{j\neq i}\}$, which satisfy their first order conditions. If all the bidders follow the optimal bids $\{(\tilde{b}_{1i}, \tilde{b}_{2i})_{i=1,\ldots,I}\}$, then equilibrium occurs when no bidder can gain by deviating from this set of optimal bids; this occurrence of equilibrium happens if the conditions in Theorem 1 are met.

Theorem 1. *(Equilibrium)* Conditions (i)-(iii) describe restrictions put on the bid distributions:
(i) Bid distributions are absolutely continuous, so that they have density.
(ii) A valid distribution must be formed from the right-hand side of equation (10).
(iii) Recall that the inputs for the functions ξ_2^w, ξ_2^l, and ξ_1 are bids and bid distributions; functions ξ_2^w and ξ_2^l must be increasing in the second auction bid for every first auction bid; a function ξ_1 must be increasing in the first auction bid.
Conditions (i)-(iii) are *necessary* if the bids $\{(\tilde{b}_{1i}, \tilde{b}_{2i})_{i=1,\ldots,I}\}$, which satisfy the first order conditions, were to become the bids coming from an increasing and differentiable Bayesian Nash Equilibrium strategy[23].

Theorem 1 ensures that my model is testable because three conditions restrict the bid distribution. One of these conditions, (iii), guarantees the second-order condition, thereby justifying the use of the first-order conditions. The monotonicity of the bidding strategy also follows from condition (iii): for instance, since $v_2 = \xi_2^l(b_2, b_1^w)$ holds by the first-order condition (6), the second auction bid b_2 of the first auction loser must increase with v_2, as condition (iii) stipulates that the function ξ_2^l is monotone. The monotonicity of ξ_2^l allows me to invert the function, yielding $\xi_2^{l,-1}(v_2; b_1^w) = b_2$.

This new inverse function $\xi_2^{l,-1}$, along with other inverse functions $\xi_2^{w,-1}$ and ξ_1^{-1}, are the increasing and differentiable Bayesian Nash Equilibrium strategies, as stated in Corollary 1.

Corollary 1. An inverse function $\xi_2^{l,-1}$, whose arguments are v_2 and b_1^w, is the equilibrium bidding strategy s_2^l in the second auction for the first auction loser; for the first auction winner in

[22] A.7 shows the detailed derivation of equation (10).
[23] I mention in A.8 that the proof for the Theorem is at the preliminary stage.

the second auction, his equilibrium bidding strategy s_2^w is an inverse function $\xi_2^{w,-1}$ whose inputs are v_1 and $\delta(v_1, v_2)$. Lastly, an inverse function ξ_1^{-1} taking v_1 as an input is the equilibrium first auction bidding strategy s_1.

Assume a bidder i who made a first auction bid of $s_1(v_{1i})$ but lost to a winning bid of b_1^w; Corollary 1 asserts that bidder i's second auction bid must be the amount of $s_2^l(v_{2i}, b_1^w)$, independent of bidder i's v_{1i}. This independence of v_{1i} arises because observing b_1^w informs bidder i that the V_1 values of remaining $I-1$ bidders are below $s_1^{-1}(b_1^w)$, and this information remains unaffected by what v_{1i} is.

But, if bidder i won the first auction, then his v_{1i} becomes relevant in the second auction because bidder i values the second object at $\delta(v_{1i}, v_{2i})$; this explains why the strategy s_2^w takes both v_{2i} and v_{1i} as inputs, and bidder i's second auction bid amounts to $s_2^w(v_{1i}, \delta(v_{1i}, v_{2i}))$, as stated in Corollary 1. Since Corollary 1 stems from Theorem 1, which links the bid distribution to my model, I discuss how to identify parameters of my model from the bid distributions in the next section.

3 Identification

I assume that I have access to the following dataset; this assumption holds because the dataset provided by the Seoul Agro-Fisheries & Food Corporation (hereafter, Corporation) also has a similar structure.

$$\{(B_{1\ell}^{\max}, W_{1\ell}, B_{2\ell}^{\max}, W_{2\ell}, Z_{1\ell}, Z_{2\ell}, I_{1\ell} = I_{2\ell})_{\ell=1,\dots,L}\},$$

in which the subscript ℓ denotes the ℓ-th auction pair. Then given any ℓ-th auction pair, $B_{1\ell}^{\max}$, $W_{1\ell}$, $Z_{1\ell}$, and $I_{1\ell}$ represent the winning bid, the winner's identity, the auction-specific covariate, and the set of bidders in the first auction, while $B_{2\ell}^{\max}$, $W_{2\ell}$, $Z_{2\ell}$, and $I_{2\ell}$ refer to those of the second auction.

Backbones of the model are the functions ξ_2^w, ξ_2^l, and ξ_1; these functions take as inputs four bid distributions, which are $G_{2|1}^w$, $G_{B_2^l(b_1^w)}$, G_1, and $G_{B_2^l(b_1^w)|B_1}$. Given that first three bid distributions constitute the last bid distribution as shown in equation (10)[24], expressing first three distributions as a function of the dataset, as in equations (11)-(13), suffices for the identification of the functions, ξ_2^w, ξ_2^l, and ξ_1.

$$\begin{aligned} &G_{2|1}^w(b_2|b_1) \\ &\quad = \exp\left\{-\int_{b_2}^{+\infty} \left(\Pr\left[B_2^{\max} \le b | B_1^{\max} = b_1\right]\right)^{-1} d\Pr\left[B_2^{\max} \le b, W_2 = W_1 | B_1^{\max} = b_1\right]\right\}, \quad (11) \\ &G_{B_2^l(b)}(b_2|B_1 \le b) \\ &\quad = \exp\left\{-\frac{1}{I-1}\int_{b_2}^{+\infty} \left(\Pr\left[B_2^{\max} \le b | B_1^{\max} = b_1\right]\right)^{-1} d\Pr\left[B_2^{\max} \le b, W_2 \ne W_1 | B_1^{\max} = b_1\right]\right\}, \quad (12) \\ &G_1(b_1) = \Pr[B_1^{\max} \le b_1]^{1/I}, \quad (13) \end{aligned}$$

[24](10) shows that it consists of G_1, $G_{B_2^l(b_1^w)}$, and the function ξ_2^l; this function uses $G_{2|1}^w$.

of which the right-hand sides consist of the probabilities that are directly identified from the dataset; for simplicity, I suppress the dependence of these bid distributions on auction covariate Z and the number of bidders I.

Since the dataset provided by the Corporation excludes the losing bids and the losers' identities, B.1 and B.2 show how to circumvent this exclusion in deriving equations (11)-(13): this circumvention relies on Theorem 7.3.1 of Rao (1992), which proves that one can uniquely identify the distribution of interest even when the dataset only contains the maximum values and the identity of the corresponding agents, a situation that aligns precisely with mine.

Given that equations (11)-(13) identify the functions ξ_2^w, ξ_2^l, and ξ_1, I use these functions to identify the parameters of interest $[F_1,\, F_{2|1},\, \delta]$. The first parameter F_1, value distribution of the first object, is identified because my model restricts the function ξ_1 to be monotone in the first auction bid. This monotonicity restriction, along with the first order condition $v_1 = \xi_1(b_1)$ from equation (9), implies that we must have $V_1^{\max} = \xi_1(B_1^{\max})$. Since random variable $B_1^{\max}$ and its distribution directly come from the dataset, it means that random variable $V_1^{\max}$ and its distribution are also identified. The distribution of $V_1^{\max}$ is F_1^I because of *independence*, which implies that I can recover F_1.

The second parameter $F_{2|1}$, value distribution of the second object given a certain value of the first object, is identified by the following equation (14).

$$\begin{aligned}\Pr[V_2 \le \cdot | V_1 = v_1] &= \Pr[V_2 \le \cdot | B_1 = b_1] \\ &= \Pr[V_2 \le \cdot | B_1 < b_1] + \frac{G_1(b_1)}{g_1(b_1)} \left(\frac{\partial}{\partial b_1} \Pr[V_2 \le \cdot | B_1 < b_1] \right) \\ &= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\xi_2^l(x, b_1) \le \cdot \right] dG_{B_2^l(b_1)}(x | B_1 \le b_1) \\ &\quad + \frac{G_1(b_1)}{g_1(b_1)} \left(\frac{\partial}{\partial b_1} \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\xi_2^l(x, b_1) \le \cdot \right] dG_{B_2^l(b_1)}(x | B_1 \le b_1) \right), \end{aligned} \tag{14}$$

in which the first equality holds because changing the condition from the event $\{V_1 = v_1\}$ to the event $\{\xi_1(V_1) = \xi_1(v_1)\} = \{B_1 = b_1\}$ keeps the original conditional distribution the same because function ξ_1 is monotone; the second and last equalities, established in B.3, prove that I can identify the value distribution $F_{2|1}$ from the bid distributions and the function ξ_2^l, which is also a function of bid distributions.

Since the function ξ_2^l is used in equation (14) to identify the value distribution $F_{2|1}$, I can analogously use the function ξ_2^w to identify another value distribution, i.e., the distribution of the adjusted value of the second object from having a v_1-valued object:

$$\Pr[\delta(v_1, V_2) \le \cdot | V_1 = v_1] = \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\xi_2^w(x, b_1) \le \cdot \right] dG_{2|1}^w(x | b_1), \tag{15}$$

whose detailed derivation is provided in B.4; the appendix uses the equality between $\delta(v_1, v_2)$ and $\xi_2^w(b_2, b_1)$, which I established in the first-order condition (5).

To identify the last parameter δ, a function that represents how much v_2 is adjusted from having v_1-valued object, we focus on the left-hand side of both equations (14) and (15). The left-hand side of (14) is a distribution of the random variable V_2 while that of (15) is also a distribution of the random variable, but a transformation of V_2; the transformation here is a function $\delta(v_1,\cdot)$. Because this function is monotone in its second argument, one can invoke the property of a random variable that a monotone function preserves the quantiles, leading to the following equality.

$$\alpha\text{-quantile of (15)} = \delta(v_1,\ \alpha\text{-quantile of (14)}).$$

This equality implies that varying α between 0 and 1 fills both the domain and the range of a function $\delta(v_1,\cdot)$, which finishes the identification of the function.

Computing quantile causes burden in practice, especially when we have to do it twice for each (14) and (15). B.5 shows that by starting from the grid on the second auction bids, instead of on the αs, we compute the quantile once for the bid distribution $G^w_{2|1}$, and still identify the function $\delta(v_1,\cdot)$. This alternative approach reduces the computational burden, which naturally leads to a discussion on parameter estimation.

4 Estimation and Monte Carlo

Previous section shows us that the parameters of the model are identified by three bid distributions $G^w_{2|1}$, $G_{B^l_2(b^w_1)}$ and G_1. These bid distributions are constructed from four random variables ($B^{\max}_2$, $B^{\max}_1$, W_2, W_1) and its distributions, such as $\Pr[B^{\max}_1 \leq \cdot]$, as shown in equations (11)-(13). I propose its kernel density estimators (16)-(19) as follows:

$$\widehat{\Pr}\left[B^{\max}_1 \leq b_1\right] = \int_{-\infty}^{b_1} \frac{1}{L_I}\frac{1}{h_{1,1}}\sum_{\ell\in\mathcal{L}_I} K\left(\frac{x - B^{max}_{1\ell}}{h_{1,1}}\right)dx, \tag{16}$$

$$\widehat{\Pr}\left[B^{\max}_2 \leq b | B^{\max}_1 = b_1\right] = \frac{\int_{-\infty}^{b}\frac{1}{h_{2,2}}\frac{1}{h_{1,2}}\sum_{\ell\in\mathcal{L}_I} K\left(\frac{x-B^{max}_{2\ell}}{h_{2,2}}\right)K\left(\frac{b_1-B^{max}_{1\ell}}{h_{1,2}}\right)dx}{\frac{1}{h_{1,1}}\sum_{\ell\in\mathcal{L}_I} K\left(\frac{b_1-B^{max}_{1\ell}}{h_{1,1}}\right)}, \tag{17}$$

in which the number of bidders is set at I. A set $\mathcal{L}_I$ includes all the auction pairs in which I bidders attended, and the number L_I denotes the size of the set $\mathcal{L}_I$, namely $L_I = |\mathcal{L}_I|$. Before describing the bandwidth $h^{superscript}_{subscript1,\, subscript2}$, which I choose as Silverman's rule of thumb, I introduce the remaining estimators (18) and (19).

$$\widehat{\Pr}\left[B^{\max}_2 \leq b, W_2 = W_1 | B^{\max}_1 = b_1\right]$$
$$= \int_{-\infty}^{b} \frac{\frac{1}{h^{1=2}_{2,2}}\frac{1}{h^{1=2}_{1,2}}\sum_{\ell\in\mathcal{L}^{1=2}_I} K\left(\frac{x-B^{\max}_{2\ell}}{h^{1=2}_{2,2}}\right)K\left(\frac{b_1-B^{\max}_{1\ell}}{h^{1=2}_{1,2}}\right)}{\frac{1}{h^{1=2}_{1,1}}\sum_{\ell\in\mathcal{L}^{1=2}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h^{1=2}_{1,1}}\right)} \frac{\sum_{\ell\in\mathcal{L}^{1=2}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h_{1,1}}\right)}{\sum_{\ell\in\mathcal{L}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h_{1,1}}\right)} dx, \tag{18}$$

$$\widehat{\Pr}\left[B^{\max}_2 \leq b, W_2 \neq W_1 | B^{\max}_1 = b_1\right]$$
$$= \int_{-\infty}^{b} \frac{\frac{1}{h^{1\neq2}_{2,2}}\frac{1}{h^{1\neq2}_{1,2}}\sum_{\ell\in\mathcal{L}^{1\neq2}_I} K\left(\frac{x-B^{\max}_{2\ell}}{h^{1\neq2}_{2,2}}\right)K\left(\frac{b_1-B^{\max}_{1\ell}}{h^{1\neq2}_{1,2}}\right)}{\frac{1}{h^{1\neq2}_{1,1}}\sum_{\ell\in\mathcal{L}^{1\neq2}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h^{1\neq2}_{1,1}}\right)} \frac{\sum_{\ell\in\mathcal{L}^{1\neq2}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h_{1,1}}\right)}{\sum_{\ell\in\mathcal{L}_I} K\left(\frac{b_1-B^{\max}_{1\ell}}{h_{1,1}}\right)} dx, \tag{19}$$

in which *subscript2* of the bandwidth[25] $h^{superscript}_{subscript1,\, subscript2}$ indicates the number of random variables, and *subscript1* indicates a random variable $B_1^{\max}$ if 1 and $B_2^{\max}$ if 2; *superscript* represents which auction pairs should be used: if $1 = 2$, it means I use auction pairs where the winners of both the first and second auctions are the same, i.e., auction pairs from a set $\{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\} \equiv \mathcal{L}_I^{1=2}$.

These estimators[26] (16)-(19) constitute the plug-in estimator for the bid distributions: for example, I can form the estimator of the bid distribution G_1 by replacing the estimand in equation (13) with the estimator (16) as follows,

$$\hat{G}_1(b_1) = \widehat{\Pr}[B_1^{\max} \leq b_1]^{1/I} = \left(\int_{-\infty}^{b_1} \frac{1}{L_I} \frac{1}{h_{1,1}} \sum_{\ell \in \mathcal{L}_I} K\left(\frac{x - B_{1\ell}^{max}}{h_{1,1}} \right) dx \right)^{1/I}.$$

Similarly, the estimands in equations (11)-(12) can be replaced by the estimators (17)-(19), meaning that I come up with plug-in estimators $\hat{G}^w_{2|1}$ and $\hat{G}_{B_2^l(b_1^w)}$; C.2-C.10 show the details of all the plug-in kernel density estimators.

Given these plug-in estimators of the bid distributions, I can estimate the parameters of the model $[\hat{F}_1, \hat{F}_{2|1}, \hat{\delta}]$ by following the steps in Identification section; whether following these steps accurately estimates the parameters or not is shown in the next Monte Carlo subsection.

4.1 Monte Carlo

I assume $I = 2$, namely two bidders. I independently draw their first auction bids from the common bid distribution G_1; the drawn bids decide the winner and the loser of the first auction. For the first auction winner whose bid is b_1^w, I draw his second auction bid from the bid distribution $G^w_{2|1}(\cdot|b_1^w)$; for the first auction loser who observes b_1^w, I draw his second auction bid from the bid distribution $G_{B_2^l(b_1^w)}(\cdot|B_1 \leq b_1^w)$.

This process concludes one auction pair ℓ. I repeat the process thousand times so that I generate the first set of thousand ℓs, denoted as $\mathcal{L}_{I,First}$. Then I repeat this generation two hundred times so that I get two hundred samples of $\mathcal{L}_{I,First}, ..., \mathcal{L}_{I,Two-hundredth}$.

Lastly, the bid distributions that I am considering are as follows,

$$\begin{aligned} G^w_{2|1}(b_2|b_1^w) &= b_2^{(b_1^w)^{1/70}+0.1} && b_2, b_1^w \in [0,1]^2, \\ G_{B_2^l(b_1^w)}(b_2|B_1 \leq b_1^w) &= b_2^{(b_1^w)^{1/70}+0.2} && b_2, b_1^w \in [0,1]^2, \\ G_1(b_1) &= b_1^{0.5} && b_1 \in [0,1]. \end{aligned}$$

C.11 shows that these bid distributions satisfy all the necessary conditions of the model, outlined in Theorem 1; a caution to this satisfaction is that the second condition is only nearly satisfied, though it comes very close to perfect satisfaction—a point I elaborate on when discussing Figure 5. Given the near-perfect satisfaction of this condition, I assume Theorem 1 holds for these bid distributions, which enables me to invoke Corollary 1 and plot the equilibrium bidding strategies as follows; dashed (median) and dotted (ninety percent confidence interval) lines come from

[25] C.1 shows the closed-form expression of the bandwidth.

[26] Derivations of (16) and (17)-(19) are introduced in C.2 and C.1; auction covariate Z is considered.

pointwise estimates of the two hundred bootstrapped samples, $\mathcal{L}_{I,First}, ..., \mathcal{L}_{I,Two-hundredth}$.

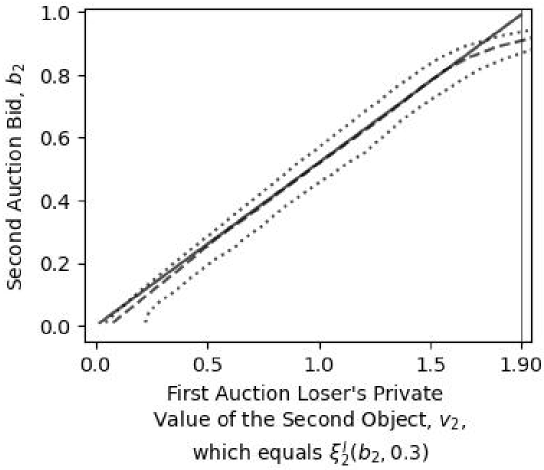

Fig. 1 – Inverse of $\hat{\xi}_2^l(\cdot, b_1^w)$ with $b_1^w = 0.3$ for dashed and dotted lines, and inverse of $\xi_2^l(\cdot, b_1^w)$ for solid line; by Corollary 1, these two inverses are equivalent to $\hat{s}_2^l(\cdot, \xi_1(b_1^w))$ and $s_2^l(\cdot, \xi_1(b_1^w))$, where $\xi_1(b_1^w) = 0.96$:

X-axis - $\hat{v}_2$ (dashed and dotted lines)
v_2 (solid line)

Y-axis - b_2

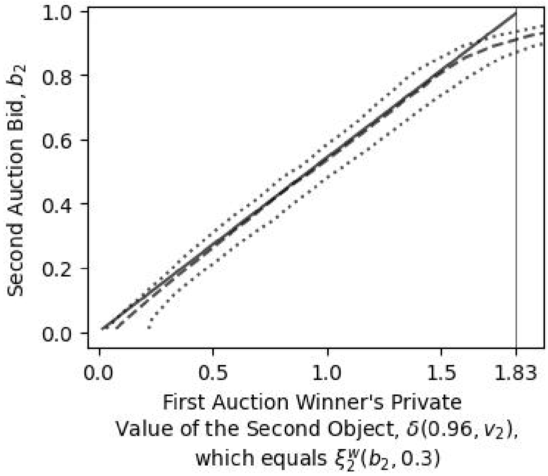

Fig. 2 – Inverse of $\hat{\xi}_2^w(\cdot, b_1^w)$ with $b_1^w = 0.3$ for dashed and dotted lines, and inverse of $\xi_2^w(\cdot, b_1^w)$ for solid line; by Corollary 1, these two inverses are equivalent to $\hat{s}_2^w(\xi_1(b_1^w), \cdot)$ and $s_2^w(\xi_1(b_1^w), \cdot)$, where $\xi_1(b_1^w) = 0.96$:

X-axis - $\widehat{\delta(\xi_1(b_1^w), v_2)}$ (dashed & dotted lines)
$\delta(\xi_1(b_1^w), v_2)$ (solid line)

Y-axis - b_2

Figure 1 shows the estimated (dashed and dotted) and the true (solid) second auction strategy of the first auction loser, who observed the winning bid[27] of $b_1^w = 0.3$. From this observation, the loser makes the second auction bid b_2 that fulfills the first-order condition (6); so, both the number $\xi_2^l(b_2, 0.3)$ and his value for the second object v_2 must equal. This equality pinpoints 0 and 1.90 on the horizontal axis, which are the lowest and the highest v_2s, because a function ξ_2^l is monotone in the second auction bid.

Not only ξ_2^l, but also ξ_2^w is monotone in the second auction bid; Figure 2 shows the monotone second auction strategy of the first auction winner. Since he won with a bid of $b_1^w = 0.3$, first-order condition (5) predicts that his second auction bid b_2 must match a number $\xi_2^w(b_2, 0.3)$ with his value of the second object, not v_2 but the adjusted number $\delta(0.96, v_2)$. This adjustment occurs because of the first object he owns.

The value of the first object, $v_1 = 0.96$, comes from the first-order condition (9) of the first auction; the condition asserts that a bidder who bids 0.3 must have valued it at $0.96 = \xi_1(0.3)$, which is shown in Figure 3.

[27] I choose 0.3 because this number is close to the expectation of B_1, namely $\int_0^1 b\,(db^{0.5}/db)db = \frac{1}{3}$.

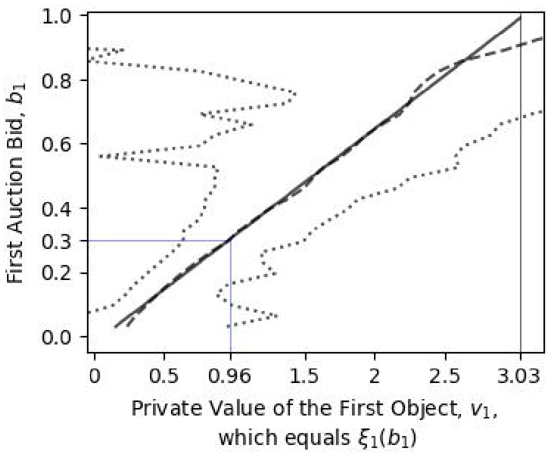

Fig. 3 – Inverse of $\hat{\xi}_1(\cdot)$ for dashed and dotted lines, and inverse of $\xi_1(\cdot)$ for solid line; by Corollary 1, these two inverses are equivalent to $\hat{s}_1(\cdot)$ and $s_1(\cdot)$:

X-axis - $\hat{v}_1$ (dashed and dotted lines)
v_1 (solid line)
Y-axis - b_1

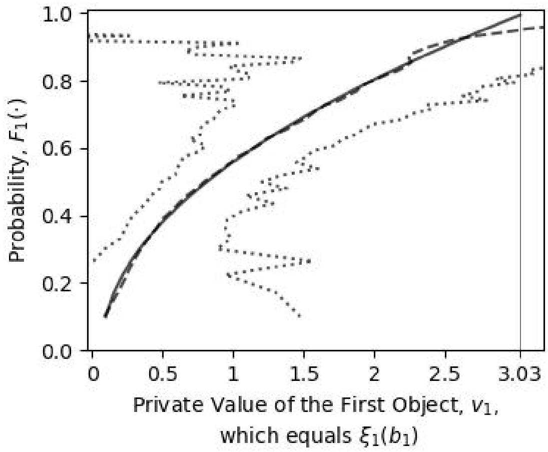

Fig. 4 – Value distribution of the first object: $\hat{F}_1(\cdot)$ for dashed and dotted lines, $F_1(\cdot)$ for solid line:

X-axis - $\hat{v}_1$ (dashed and dotted lines)
v_1 (solid line)
Y-axis - Cumulative Probability

A monotone function ξ_1 maps first auction bid 0.3 on the vertical axis to the value of the first object, $0.96 = \xi_1(0.3)$, on the horizontal axis. The estimator $\hat{\xi}_1$ also maps every first auction bid to its estimates, $\hat{v}_1$: median follows the true line, but the width of the confidence interval appears wide.

The wide interval occurs because various bid distributions and its integrations constitute the estimand ξ_1, shown in (9). Because the same estimand ξ_1 constitutes the value distribution of the first object F_1, the interval also gets widened in Figure 4, the estimates of $\hat{F}_1$.

Since F_1 is one of the parameters of my model $[F_1,\ F_{2|1},\ \delta]$, I show the estimates of the remaining parameters $\hat{F}_{2|1}$ and $\hat{\delta}$ in Figures 5 and 6: before discussing the findings from the figures, note that the upward-sloping solid line in Figure 5, representing the true value distribution $F_{2|1}$, appears to be valid: the line is strictly increasing, with probabilities of zero and one at the minimum and maximum of its support. This valid form of the distribution $F_{2|1}$ shows that the second condition of the model is nearly satisfied, as C.11.2 proves that $F_{2|1}$ and the bid distribution $G_{B_2^l(b_1)|B_1}$ are equivalent.

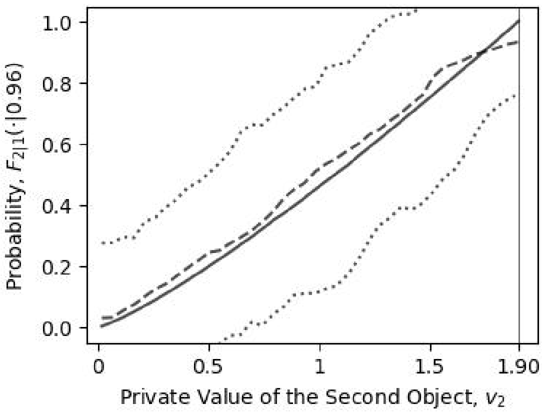

Fig. 5 – Value distribution of the second object given $v_1 = 0.96$: $\hat{F}_{2|1}(\cdot|0.96)$ for dashed and dotted lines, $F_{2|1}(\cdot|0.96)$ for solid line:

X-axis - $\hat{v}_2$ (dashed and dotted lines)
v_2 (solid line)

Y-axis - Cumulative Probability

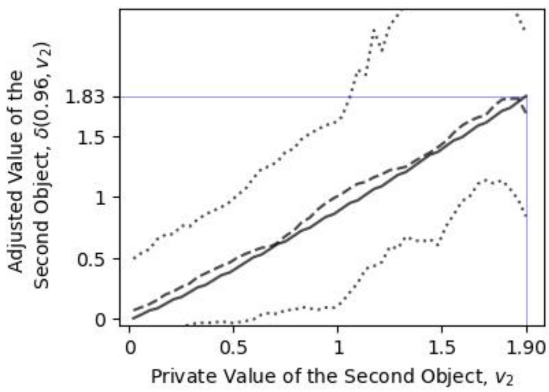

Fig. 6 – Complementarity function given $v_1 = 0.96$: $\hat{\delta}(0.96, \cdot)$ for dashed and dotted lines, $\delta(0.96, \cdot)$ for solid line:

X-axis - $\hat{v}_2$ (dashed and dotted lines)
v_2 (solid line)

Y-axis - $\widehat{\delta(0.96, v_2)}$ (dashed and dotted lines)
$\delta(0.96, v_2)$ (solid line)

Going back to the figures, Figure 5 shows the value distribution $F_{2|1}(\cdot|0.96)$, representing how likely some number from the random variable V_2 will be realized given that a bidder feels the first object at $v_1 = 0.96$. The solid line is almost linear, suggesting that the bidder expects any realization of V_2 between 0 and 1.90 has similar chance, as long as he values the first object at 0.96.

Testing with various values of the first object other than $0, 96$, which I do not show in this paper, still maintains the linearity, suggesting that both random variables V_1 and V_2 are nearly independent. This near-independence occurs because of the bid distributions $G^w_{2|1}$ and $G_{B^l_2(b^w_1)}$ that I started with; a common exponent $1/70$ weakens the first auction bid b^w_1's impact on the realization of the second auction bids, which my model rationalizes as random variables V_1 and V_2 being nearly independent.

From the bid distributions, my model also rationalizes that both V_1 and V_2 must be substitutes as shown in Figure 6: having a 0.96-valued first object decreases the value of the second object v_2 from 1.90 to 1.83. The decrease occurs because the bid distributions that I started with assume that a bidder bids less aggressive in the second auction if he owns the first object; a number 0.1, which is smaller than 0.2 from $G_{B^l_2(b^w_1)}$, is added to the exponent of $G^w_{2|1}$.

C.11 describes why I come up with such numbers: with these numbers, bid distributions pass the restrictions of my model. In the next section, I check whether the bid distributions of the Korean Fruit Auction also pass the model's restriction.

5 Application

My model applies to an auction where a single object is sold one at a time using a first-price sealed-bid format, with only the winning bid and the winner's identity disclosed from previous auctions. The Korean Fruit Auction aligns with this type of auction, as bidders submit their sealed bids using a gadget within the designated purple circle shown in Figure 7.

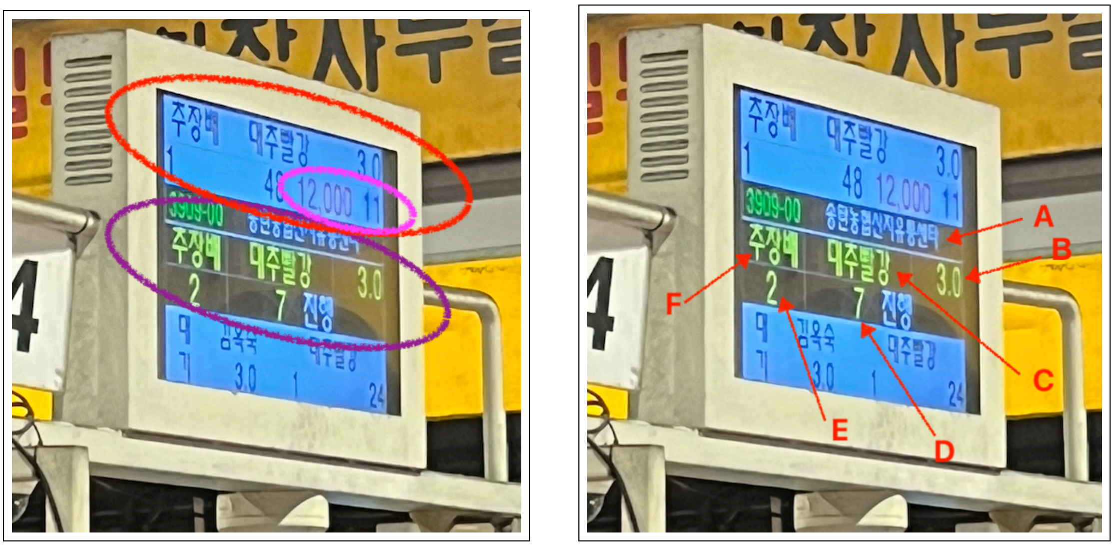

Fig. 7 — Auctioneer's Monitor at 3:02 AM on July 21, 2023, at Seoul auction house. Both the left and right panels display the same information. Information within the purple circle pertains to the ongoing auction, while information within the red circle denotes the previous auction. The pink circle inside the red circle displays the winning bid per box (12,000) and the winner's identity (11).

Figure 7 comes from the Seoul auction house, one of the five auction houses in the Korean Fruit Auction. Each auction house operates independently with its own monitor and conducts its own auctions.

The auction highlighted by the purple circle represents the ongoing auction, which typically lasts between three to ten seconds and involves the sale of a single item. This item has specific attributes displayed: the type of produce (Cell C), its origin (Cell A), and the producer's name (Cell F). Seven boxes of the produce are on sale (Cell D), each weighing three kilograms (Cell B) and classified as medium size (Cell E).

A bidder can place bids at only one of the auction houses, and his bid expresses how much, per box, he is willing to pay—hereafter, the term 'winning bid' refers to the winning bid per box.

The auction marked by the red circle in Figure 7 represents the previous auction, indicating that multiple auctions are conducted within a single day. Only the winning bid (12,000) and the winner's identity (11) from this previous auction are disclosed, as shown in the pink circle.

I provide below necessary context surrounding this auction.

5.1 Necessary Context for the Korean Fruit Auction

Garak Market, located in Seoul, is one of the largest of the thirty-three public wholesale markets that transfer vegetables and fruits from farmers to wholesalers. This market alone accounts for 34.3%[28] of Korea's total vegetable and fruit transaction volume. The primary transaction method at Garak Market is through auctions, which represent 79.6% of the total volume, trans-

[28] As of 2022, thirty-three public wholesale markets handle 99.4% of Korea's vegetable and fruit trade volume, with Garak Market representing 34.5% of this volume. The 34.3% figure results from multiplying these two percentages.

lating to approximately 27 percent[29] of all vegetables and fruits in Korea being transferred from farmers to wholesalers via auction—equating to a total volume of 1.8 million tons and a total value of $2.95 billion[30]. Auctions are held every day except Sundays and certain designated holidays. Further details about Garak Market and its auctions are provided in D.1.

Given the high stakes, a public corporation named Seoul Agro-Fisheries & Food Corporation (hereafter, the Corporation) regulates the market, including the auction. Six auction houses operate under the Corporation, each employing wholesalers who act as bidders in the auction. There are two types of bidders: 'Veggie' bidders, who specialize in purchasing vegetables, and 'Fruit' bidders, who focus on fruits. Veggie bidders typically concentrate on a single variety (e.g., a cabbage bidder), whereas Fruit bidders tend to buy across all fruit types, requiring a more complex bidding strategy than that of Veggie bidders. Notably, no bidder is known to participate in both categories.

In this paper, I focus on the Fruit auction (hereafter, 'auction') since the bidder I contacted is a Fruit bidder, allowing me to ask detailed questions that are not covered in government reports.

Of the six auction houses, five deal with fruits: Joongang, Nonghyup, Seoul, Donghwa, and Hankook. Each of these houses has approximately 90 fruit bidders and 10 fruit auctioneers (hereafter, 'bidders' and 'auctioneers'). Bidders enter contracts with one auction house, typically lasting between three to ten years, and are permitted to bid only at the house with which they are contracted. In 2022, the period covered by my dataset, no bidders were known to be expelled from any auction house, as the required monthly transaction value is easily met.

One of the regulations governing the auction requires each auction house to auction specific fruits within designated time periods[31]:

- 2:00 am - 5:00 am : perishable fruits such as grapes, tomatoes, and tangerines.
- 8:00 am[32] - 9:30 am: less perishable fruits such as pears, apples, and persimmons.

While the starting times are fixed, the ending times may vary depending on the number of fruit boxes arriving at the auction site.

My conversations with the auctioneer and the bidder align in that the exact number of boxes of a given variety from a particular farmer is known only the day before or on the day of its arrival. This uncertainty arises because farmers may adjust the agreed-upon number of boxes or the transportation date as harvesting continues. Acknowledging this uncertainty, the farmer and the auction house—where the auctioneer serves as a mediator—typically enter into a contract that specifies an estimated quantity of boxes and a general timeframe (e.g., the third week of July) for delivery. This uncertainty also explains why bidders inspect the items a few hours before each day's auction.

As boxes of fruit arrive at the auction site, the auctioneer has some discretion in determining the sequence of items to be auctioned. For example, an auctioneer responsible for selling apples

[29] This 27% figure is derived by multiplying 34.3% by 79.6%.
[30] 1,779,392 tons and ₩3,835,554 million, equivalent to $2,950 million, assuming an exchange rate of $1 = ₩1300.
[31] A similar rule applies to vegetables regarding the time period.
[32] Although the table in D.1.1 stipulates a start time of 8:30 am, the auction actually begins at 8:00 am.

can decide the order of apple auctions. Auctioneers often allocate high-quality products at the beginning of the auction to start with a high winning bid, aiming to set a positive tone for the remainder of the auction.

Among fruits, I focus specifically on apples, as they were one of the top five fruits traded in 2022—accounting for 38,155 tons and $89 million[33] in transaction volume and value. Additionally, the price surge of apples in the first half of 2024 was considered problematic, making policy recommendations regarding apples more impactful.

5.2 Apple Auctions

As mentioned in the previous subsection, each auction house is required to begin selling apples at 8:00 am. Typically, auction houses sell pears first, then move on to apples, and subsequently to other fruits, such as persimmons. On average, apple auctions begin at 8:05 am and conclude at 8:33 am, as shown in D.1.4.

The Corporation provided me with a dataset, an excerpt of which is shown below.

Table 1: Excerpt of a Dataset Showing the Auction Covariates Z for Apples

Ending Time	Day, Auction House, Order	Winning Bid Per Box	No. of Boxes	Kg per Box	Winner ID	Size	Type	Grade	Place of Origin	Group
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
8:37:55	July 19, Seoul, 6	$40.0	3	10	4044	40	Aori	High	Yesan	co-op.

Note: The dataset, from which this excerpt is taken, covers January, February, March, April, May, June, July, and December of 2022, showing that 87,349 apple auctions were conducted during this period. The first column shows the ending time of an auction, while the second column indicates that the auction was held on July 19 at the Seoul Auction House and was the sixth apple auction for that day-house pair. In this auction, three boxes (col. (4)) of high-grade (col. (9)) Aori apples (col. (8)) were auctioned, with each box containing 40 apples (col. (7)), sourced from Yesan (col. (10)) and grown by a farmer in a cooperative (col. (11)). A bidder with the anonymized ID 4044 (col. (6)) won the auction, bidding $40.0 per box (col. (3)), and each box weighed 10 kilograms (col. (5)).

The dataset includes 87,349 apple auctions that took place between January and July, and in December of 2022. Auction covariates, denoted as Z, indicate that the auction in Table 1 ended at 8:37:55 (column (1)), was held by the Seoul auction house on July 19, and was the sixth auction for that specific day-house pair (column (2)). The remaining columns indicate that three boxes of high-Grade Aori apples (columns (4), (9), and (8)) were sourced from a farmer in Yesan who belongs to a cooperative rather than operating individually (columns (10) and (11)). Each box weighed 10 kilograms and contained 40 apples (columns (5) and (7)). The auction was won by a bidder with anonymized ID 4044, who bid $40.0 per box (columns (6) and (3)).

As shown in Table 1, the dataset does not disclose the number of bids submitted in each auction, in accordance with the Corporation's internal regulations. However, various sources, including D.1.5, suggest that typically three to seven bids are submitted per auction, with the number of bids decreasing as the auction nears its end.

[33] ₩116 billion

The set of bidders submitting bids for each day-house pair is smaller than the total pool of approximately 90 potential bidders in that auction house. Only those bidders who wish to participate register before the auction begins and gather around the auctioneer's monitor to place their bids. The size of this group for any given day-house pair is not disclosed, but typically around 40 bidders[34] are observed surrounding the auctioneer's monitor.

The second column of the following table indicates that, within this group, the average number of unique winning bidders per day-house pair ranges from 15.5 to 37.0.

Table 2: Descriptive Statistics for the Number of Auctions and Unique Winner IDs per Day

Average (Std. Dev.) Number of Auctions per Day (1)		Average (Std. Dev.) Number of Unique Winner IDs per Day (2)	
- Joongang:	135.2 (93.4)	- Joongang:	37.0 (15.7)
- Nonghyup:	110.1 (68.4)	- Nonghyup:	29.5 (11.0)
- Seoul:	105.5 (79.0)	- Seoul:	31.8 (14.0)
- Donghwa:	58.8 (44.9)	- Donghwa:	18.9 (8.4)
- Hankook:	48.1 (39.3)	- Hankook:	15.5 (7.2)

Note: The figures in both columns are based on data from 87,349 auctions. Each column represents the average (with standard deviation) number of apple auctions per day and the number of unique winner IDs per day for each auction house.

The first column of Table 2 shows that, on average, 48.1 to 135.2 auctions occur in a single day, differing from the two periods covered by my model. Acknowledging this inherent difference, I use each house's last two auctions[35], with descriptive statistics provided below. From this point forward, I refer to the second-to-last auction as the first auction and the last auction as the second auction.

[34] Kim (2017) reports a range of fifty to eighty bidders, though the specific auction house and types of fruits or vegetables auctioned are unspecified. On the day I visited, July 21, 2023, thirty to fifty bidders were present at the Seoul Auction House.

[35] The approach of using the last two auctions in a sequential auction has also been adopted in Mcafee and Vincent (1993).

Table 3: Desc. Statistics for Covariates Z of the Last Two Auctions

A. Number of Auction Pairs per House

- Joongang (185), Seoul (195), Hankook (187), Donghwa (194), Nonghyup (192).
- Total pairs (953), i.e., 1,906 auctions.

B. Winning Bid per Box (Table 1, Col. 3)

- First auction
  (Mean, Std. Dev., 5th/50th/95th Percentiles)
  : 14.6, 7.7, 5.0/13.1/28.5.
- Second auction
  : 13.5, 7.7, 3.8/11.5/27.7.
- Correlation between first and second auctions: 0.768.

C. Number of Boxes (Table 1, Col. 4)

- First auction
  (Mean, Std. Dev., 5th/50th/95th Percentiles)
  : 24.8, 26.3, 1.0 / 16.0 / 71.4.
- Second auction
  : 22.8, 26.9, 1.0 / 11.0 / 69.4.

D. Winner ID (Table 1, Col. 6)

- Probability of first auction winner winning the second auction: 33.2% (316/953).
- Unique winning IDs appear 207 times out of 1,906 auctions.
- Top 20 IDs win 816 auctions
  : i.e., Top 10% (20/207) wins 42.8% (816/1,906) of auctions.

E. Size (Table 1, Col. 7)

- First auction; showing top five sizes
  (No. of apples inside a box (No. of auctions))
  : 50(207), 40(181), 30(83), 45(81), 60(57).
- Second Auction; showing top five sizes
  : 50(180), 60(127), 40(113), 30(78), 45(74).

F. Type (Table 1, Col. 8)

- Probability of same types being auctioned
  : 93.9% (895/953).
- Nine unique types exist
  : Mishima (606), Fuji (520), and Myanmar (400) represent over half of 1,906 auctions.

G. Place of Origin (Table 1, Col. 10)

- Probability of same origins being repeated: 94.2% (898/953).
- Cheongsong (443), Yeongju (261), Yeongcheon(234), and Andong(203) represent over half of 1,906 auctions.

H. Kg per Box, Grade, and Group (Table 1, Cols. 5, 9, and 11)

- 95% of the auctions deal with 10 kilograms (1,810/1,906).
- 87% of the apples are marked as the highest grade* (1,656/1,906).
- 68% of the apples come from farmers who are members of a cooperative (1,294/1,906).

Note: Of the 87,349 auctions, selecting the last two auctions from each auction house results in 1,906 auctions, or 953 auction pairs. There are eight items, labeled A through H, with all statistics for these items derived from the 953 auction pairs. Additionally, the covariates discussed for each of the eight items correspond to those in Table 1. Lastly, the asterisk (*) in Item H indicates that 87% of the apples are marked as the highest grade. Since farmers assign these grades to their produce, grading is often inflated, which explains why bidders inspect the boxes of fruit prior to the auction.

Of the 87,349 auctions, selecting the last two auctions from each auction house yields 1,906 auctions, or 953 auction pairs, for which Table 3 provides descriptive statistics. Eight items, labeled A through H, offer descriptive statistics for the corresponding columns in Table 1. Item B indicates that the mean winning bid of the second auction (13.5) is lower than that of the first auction (14.6) and that both winning bids are serially correlated—density plots for each winning bid, as well as their joint contour plot, are shown below.

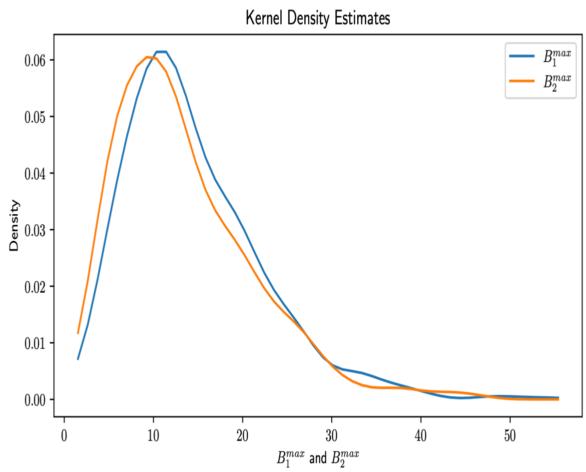

Fig. 8 – Marginal winning bid density of the first auction (blue) and the second auction (orange):

X-axis - Winning bid of the first (blue) and second (orange) auction

Y-axis - Density

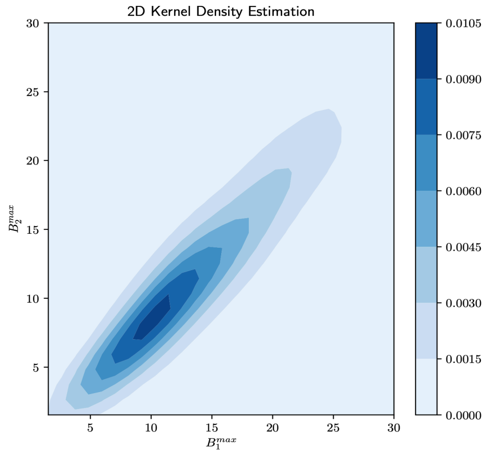

Fig. 9 – Joint winning bid contour plot of the first and second auctions:

X-axis - Winning bid of the first auction

Y-axis - Winning bid of the second auction

Figure 8, with an x-axis ranging from \$0 to \$60, displays the marginal winning bid densities for the first and second auctions. The winning bid density from the first auction is skewed further to the right than that of the second auction. Figure 9, with x- and y-axes representing the winning bids of the first and second auctions, each ranging from \$0 to \$30, illustrates that the two winning bids are indeed correlated.

Regarding Item D of Table 3, bidders are indeed asymmetric, with the top 10% of bidders winning 42.8% of the auctions. One approach to addressing this asymmetry is to use a model in which each bidder draws their value for the first object from their own individual value distribution, rather than from a common value distribution F_1 as in the current model. However, allowing for these initial differences increases the number of testable restrictions that the bid distribution must satisfy, as detailed in D.2. To limit these restrictions, I continue to use the original model and present the estimation results based on this model in the next subsection.

5.3 Estimation Results for Apple Auctions

As mentioned previously when introducing Table 1, the number of bidders in each auction is not disclosed in the dataset. Nonetheless, various sources, including D.1.5, indicate that three to seven bidders typically participate in each auction, with the number of bids decreasing as the auction progresses. Given this decline, I set the number of bidders to the lowest possible number[36], $I = 3$, meaning that each of the 953 pairs was attended exogenously[37] by three

[36] Guerre and Luo (2022) discusses methods for identifying the distribution of the number of bidders, N (which corresponds to I in my case), when the analyst has access only to winning bids.

[37] Two notable papers that incorporate a bidder's entry decision (known as endogenous participation) into the model are Samuelson (1985) and Levin and Smith (1994). These papers differ in whether a bidder knows his private value before entry (Samuelson) or after (Levin and Smith). To determine which model should be used, Li and Zheng (2012) develops a Bayesian model selection method, while Marmer et al. (2013) proposes a nonparametric test. Among the many studies that incorporate endogenous participation, both Athey et al. (2011) and Li and Zhang (2015) assume that a bidder knows his private value after paying the entry cost, aligning with the approach of Levin and Smith (1994). A recent notable paper, Gentry and Li (2014), proposes an Affiliated-

bidders: for robustness, D.4 presents the estimation results when $I = 5$.

In the estimation, I treat each of the 953 auction pairs as independent, implicitly meaning that (i) pairs from different auction houses are considered independent (e.g., two auction pairs from Feb 22, 2022—one from the Seoul house and one from the Joongang house—are treated as independent), and (ii) any two pairs from the same auction house are also independent (e.g., within the Seoul house, a pair from Feb 19 and a pair from Feb 22 are independent).

The context of the Garak market partially supports these implicit assumptions because, as discussed in subsection 5.1, (i) each auction house is allowed to begin selling apples only after 8 am, and no bidder can participate in auctions at multiple houses, and (ii) bidders can only accurately predict the exact number of boxes of fruit from a specific farmer, either on the day of arrival or the day before. Furthermore, a conversation with a bidder indicates that bidders do not engage in fruit hoarding, as 90% of their daily purchases are promptly delivered to their customers, and regulations are in place to curb secondary markets among bidders after the auction.

While items F and G in Table 3 indicate that the types and places of origin of the apples in the first and second auctions are alike, items C and E reveal heterogeneity between the auctions: (i) the number of boxes sold decreases in the second auction, and (ii) size 60 becomes the second most frequent in the second auction.

To allow the estimates from my model to incorporate observed heterogeneity in covariates Z, I use the bid homogenization method, with details provided in D.5.

The homogenization method assumes that, given a specific value z of the covariate Z, any bidder i's value for the object at the k-th auction, denoted v_{ik}, is given by $v_{ik} = \exp(z'\beta) \times u_{ik}\epsilon_k$. In this formulation, u_{ik} represents bidder i's private value for the object at the k-th auction, independent of the covariate, and ϵ_k captures auction-level unobserved heterogeneity. This multiplicatively separable structure, as adopted in Asker (2010) and Sant'Anna (2018), allows both the mean and variance of a bidder's value distribution to be influenced by the covariate Z.

For this paper, I set $\epsilon_k = 1$, thereby disregarding unobserved heterogeneity in the analysis[38]. Sant'Anna (2018) establishes that, under this assumption, the analyst proceeds with estimation by following three steps: (i) regressing the log winning bids on the covariates to obtain $\hat{\beta}$ and the residuals, denoted as b^o, which are called homogenized bids and are independent of the covariate Z; (ii) using b^o to obtain the covariate-free estimate $\hat{v}^o$ from the estimators; and (iii) multiplying both the covariate-free estimate, $\hat{v}^o$, and the homogenized bid, b^o, by $\exp(z'\hat{\beta})$ to recover the value estimate, $\hat{v}$, and the unhomogenized bid, b, which incorporate the effect of a specific covariate value, z.

Upon completing the first step of the estimation (namely, (i)), I obtain 953 pairs of homogenized bids, denoted by $\{(b^o_{1\ell}, b^o_{2\ell}) : \ell \in 1, \ldots, 953\}$. The regression results for this initial step are provided in D.5.1. Below, I summarize the mean, standard deviation, and 5th/50th/95th

Signal model that encompasses both the Samuelson and Levin-Smith approaches and discusses identification results.

[38]Unobserved heterogeneities, which refer to factors observed by the bidder but not captured in the dataset, include details such as the exact identity of a farmer (Cell F of Figure 7). A notable paper that discusses addressing unobserved heterogeneity in auctions is Krasnokutskaya (2011).

percentiles for each b_1^o and b_2^o.

- Homogenized winning bid of the first auction (b_1^o) : 1.02, 0.40, 0.42/1.00/1.73,
- Homogenized winning bid of the second auction (b_2^o): 1.00, 0.42, 0.36/0.97/1.72.

These descriptive statistics are comparable to Item B of Table 3, and one can observe that after homogenization, the mean and the 50th percentile have become much closer.

Given the homogenized pairs, I create 200 bootstrap samples by sampling with replacement to come up with the following figures — I chose a Gaussian kernel and set the bandwidth based on Silverman's rule of thumb[39]; I use a dashed line to denote the median of the estimates and a dotted line to represent the 90% nonparametric bootstrap confidence interval.

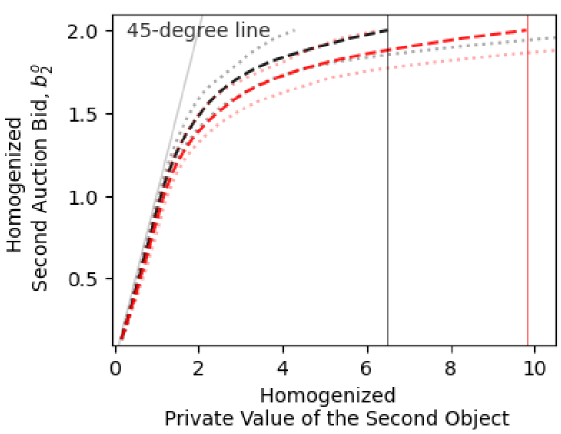

Fig. 10 – Inverse of two estimators, $\hat{\xi}_2^w$(black) and $\hat{\xi}_2^l$(red), given $b_1^o = 0.40$:

X-axis - $\widehat{\delta^o(\xi_1(b_1^o), v_2^o)}$ for black, $\hat{v}_2^o$ for red
Y-axis - b_2^o

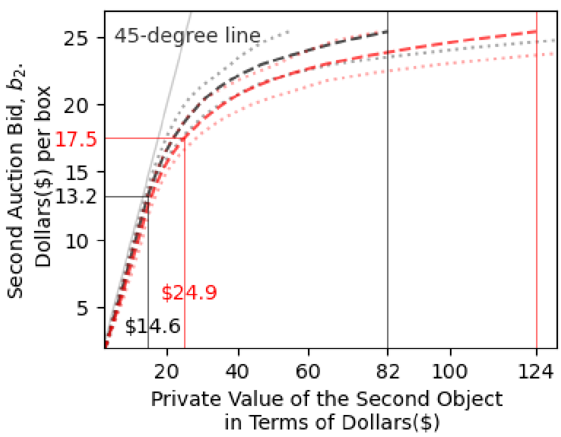

Fig. 11 – Bidding strategies, $\hat{s}_2^w$(black) and $\hat{s}_2^l$(red), given $B_1^{\max} = \$5.4$, along with median values of $\exp(Z_1'\hat{\beta})$ and $\exp(Z_2'\hat{\beta})$:

X-axis - $\widehat{\delta(\xi_1(B_1^{\max}), v_2)}$ for black, $\hat{v}_2$ for red
Y-axis - b_2

Note the y-axis of Figure 10, which represents the homogenized second auction bid b_2^o. By fixing the homogenized first auction bid at $b_1^o = 0.40$, which is slightly below the fifth percentile of b_1^o, I derive each red-colored and black-colored estimate by plugging the b_2^o values from the y-axis into $\hat{\xi}_2^l(\cdot; b_1^o)$ and $\hat{\xi}_2^w(\cdot; b_1^o)$. This produces covariate-free private values on the x-axis, represented as $\hat{v}_2^o$ and $\widehat{\delta^o(\xi_1(b_1^o), v_2^o)}$.

Figure 10 displays the result following the second step of the estimation (namely, (ii)), where the estimators here are $\hat{\xi}_2^l$ and $\hat{\xi}_2^w$. To implement the third step (namely, (iii)), I need to multiply the homogenized values and bids by $\exp(z'\hat{\beta})$ to recover values and bids that reflect the influence of the covariate z. To this end, I present the mean, standard deviation, and 5th/50th/95th percentiles for each $\exp(Z_1'\hat{\beta})$ and $\exp(Z_2'\hat{\beta})$.

- Effect of covariates in the first auction ($\exp(Z_1'\hat{\beta})$) : \$14.5, \$5.7, \$7.1/13.6/25.4,
- Effect of covariates in the second auction ($\exp(Z_2'\hat{\beta})$): \$13.7, \$5.7, \$6.5/12.7/24.6.

[39] Refer to Silverman (1986) for details on the rule of thumb. In the estimation, I adjusted the bandwidth from the value recommended by the rule of thumb, as my estimators are multi-step, which differs from the one-step estimator assumed by Silverman (1986). The purpose of the adjustment was to narrow the width of the confidence interval.

To construct Figure 11 from Figure 10, I assume that the effects of covariates in both the first and second auction are at their median values. I then multiply both x- and y-axes of Figure 10 by the median of $\exp(Z_2'\hat{\beta})$ to obtain Figure 11 and multiply $b_1^o = 0.40$ by the median of $\exp(Z_1'\hat{\beta})$ to calculate $B_1^{\max} = \$5.4$, as noted in the caption of Figure 11.

I discuss three key points from Figures 10 and 11.

First, both estimators, $\hat{\xi}_2^l(\cdot, b_1^o)$ and $\hat{\xi}_2^w(\cdot, b_1^o)$, in Figure 10 are monotonic in the homogenized second auction bid b_2^o, and D.4 confirms that this monotonicity holds even when $b_1^o \neq 0.40$. Therefore, two of the four testable restrictions of the model, as stated in Theorem 1, are met. The two remaining testable restrictions are: (a) whether the function ξ_1 increases with the first auction bid, which I do not examine in this paper[40], and (b) whether a valid distribution is formed from the right-hand side of equation (10), which holds when b_1^o is set at 0.40, as discussed in D.4.

Second, since three out of the four testable restrictions of the model are met when b_1^o is set to 0.40, I proceed under the assumption that all four restrictions in Theorem 1 are satisfied. As discussed in Corollary 1, if all the restrictions in Theorem 1 are satisfied, then the inverses of both estimators, $\hat{\xi}_2^l(\cdot, b_1^o)$ and $\hat{\xi}_2^w(\cdot, b_1^o)$, correspond to the estimators of the second auction equilibrium bidding strategies, $s_2^l(\cdot, \xi_1(b_1^o))$ and $s_2^w(\xi_1(b_1^o), \cdot)$. Each strategy represents, respectively, the first auction loser who observed the winning bid of b_1^o and the winner whose first auction bid is b_1^o.

Lastly, Figure 11 illustrates the second auction bidding strategies, incorporating the effect of covariates, with the covariate effect in both auctions set to its median value. This setup can be interpreted as both the first and second objects being of median quality. Accordingly, $B_1^{\max} = \$5.4$ reflects that the winning bid for a median-quality first object was \$5.4. The red-colored and black-colored estimates in the figure represent the bidding strategies for the median-quality second object: $\hat{s}_2^l(\cdot, \xi_1(\$5.4); \$12.7)$ and $\hat{s}_2^w(\xi_1(\$5.4), \cdot; \$12.7)$, where \$12.7 is the median of $\exp(Z_2'\hat{\beta})$. The support for each bidding strategies is given by $\hat{v}_2 \in [\$0, \$124]$ and $\widehat{\delta(\xi_1(\$5.4), v_2)} \in [\$0, \$82]$.

All the estimation results presented below also follow steps (i)-(iii) of bid homogenization: namely, I use 953 homogenized auction pairs to estimate the homogenized parameters, then dehomogenize either the bids or the estimated parameters by multiplying by the median of $\exp(Z_1'\hat{\beta})$ or $\exp(Z_2'\hat{\beta})$—for simplicity, I omit these two notations in the following results.

[40] Given that the bidding densities I am using are unimodal, as shown in Figure 8, I expect this restriction to be satisfied.

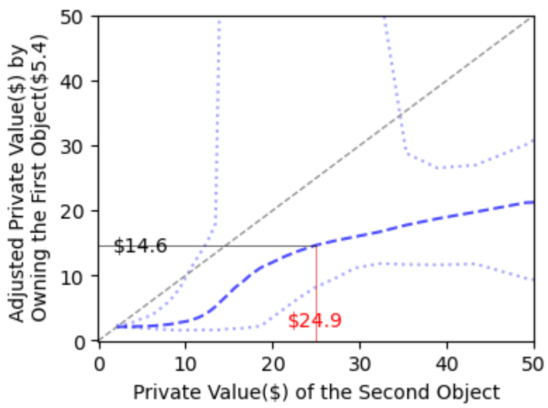

Fig. 12 – Estimated complementarity function given $B_1 = \$5.4$, namely $\hat{\delta}(\xi_1(B_1), \cdot)$:

X-axis - $\hat{v}_2$
Y-axis - $\widehat{\delta(\xi_1(B_1), v_2)}$

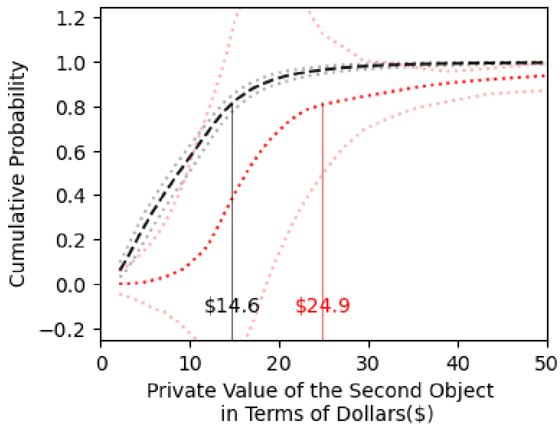

Fig. 13 – Value distribution of the second object for the first auction winner (black) and loser (red), given $B_1 = \$5.4$:

X-axis - $\widehat{\delta(\xi_1(B_1), v_2)}$ for black, $\hat{v}_2$ for red
Y-axis - Cumulative Probability

Figure 12 displays the estimated complementarity function, $\hat{\delta}(\xi_1(5.4), \cdot)$. As discussed in Sections 3 and 4, this function is identified and estimated by comparing the percentiles of the two estimators, $\widehat{\Pr}[V_2 \leq \cdot | V_1 = \xi_1(\$5.4)]$ and $\widehat{\Pr}[\delta(V_1, V_2) \leq \cdot | V_1 = \xi_1(\$5.4)]$, which correspond to the red-colored and black-colored estimates in Figure 13. The values \$24.9 and \$14.6 correspond to the 80th percentiles of each estimator, respectively, with the estimands of both estimators given by equations (14) and (15)—the closed form expressions for each estimator can be found in C.8 and C.7.

We observe a negative complementarity between the first and second objects in Figure 12, as the median estimate falls below the 45-degree line. For example, if the standalone value of the second object is \$24.9, then winning the first object valued at $\xi_1(\$5.4)$ reduces the second object's value to \$14.6. The extent of this reduction, or adjustment, varies by where v_2 lies. This variation shows that any two bidders with different values for the objects experience different degree of complementarity between the objects, as emphasized in Introduction.

Figure 13 shows that the median of $\widehat{\Pr}[V_2 \leq \cdot | V_1 = \xi_1(\$5.4)]$ first-order stochastically dominates that of $\widehat{\Pr}[\delta(V_1, V_2) \leq \cdot | V_1 = \xi_1(\$5.4)]$. This indicates that, given both the first auction loser and the winner bid $B_1 = \$5.4$, the value of the second object for the first auction loser stochastically dominates that for the first auction winner.

According to one of the definitions[41] introduced in Maskin and Riley (2000a), the stochastic dominance observed in Figure 13 implies that, in the second auction, the first auction loser is a strong bidder, while the first auction winner is a weak bidder. Equation (3.11) of their paper predicts that the strong bidder shades more than the weak bidder, aligning with the findings in Figure 11, where the first auction loser's bidding strategy lies below that of the first auction winner. Furthermore, Proposition 3.3 in their paper predicts that the equilibrium bid distribution of the strong bidder first-order stochastically dominates that of the weak bidder. In our context, this implies that the first auction loser's bidding density in the second auction first-order stochastically dominates that of the first auction winner—Figure 14 confirms that

[41] Refer to equation (3.1) of Maskin and Riley (2000a).

this prediction also holds in our context.

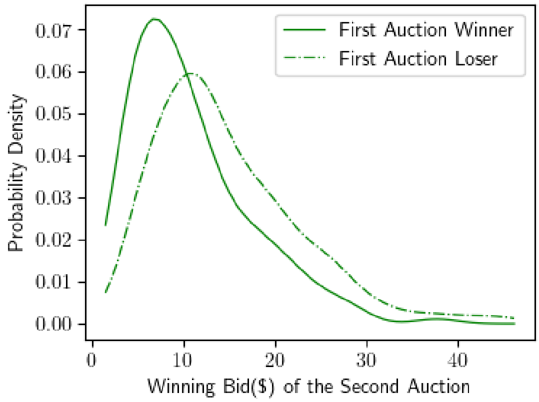

Fig. 14 – Unconditional density of second auction winning bid for first auction winner and loser from 953 auction pairs:

X-axis - Winning bid of the second auction for the first auction winner (solid) and loser (dash-dotted)

Y-axis - Density

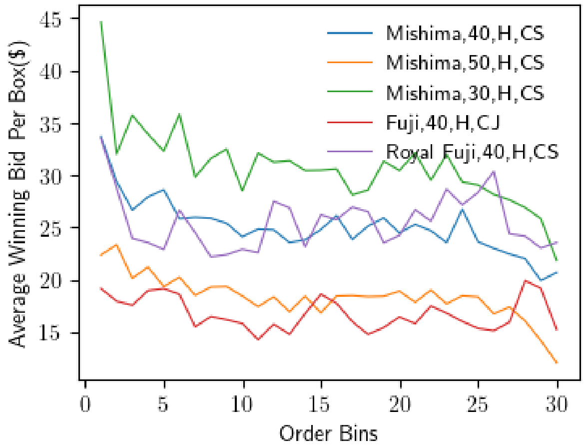

Fig. 15 – Average winning bid per order bin for the five most-auctioned type-size-grade-origin combinations; 'H' denotes highest grade, and 'CS' and 'CJ' represent Cheongsong and Chungju:

X-axis - Auction order bins (sequentially categorized into 30 bins)

Y-axis - Average winning bid by order bin

To summarize, the estimates shown in Figures 11 and 13, along with the descriptive statistics in Figure 14, align with the theoretical predictions made by Maskin and Riley (2000a) for an asymmetric auction. In our case, the second auction represents the asymmetric auction, with the first auction loser and winner corresponding to the strong and weak types, respectively.

In the next subsection, I discuss the problems associated with the current auction design and propose an alternative auction design, the Product-Mix Auction, to address these issues. I will use my estimator to demonstrate the outcomes under this new design.

5.4 Product-Mix Auction

The current auction design used in Garak Market is a sequential first-price sealed-bid auction with partial disclosure. For example, if one of the auction houses were to sell 100 objects, it would conduct 100 first-price auctions. A single object being sold in each auction might be described as 'three boxes of high-graded Aori apples, with each box containing 40 apples sourced from a farmer in Yesan,' as illustrated in Table 1. Furthermore, each of the 100 first-price auctions only discloses the winning bid and the identities of the winners to the bidders, as shown in Figure 7.

As shown in Figure 15, in the current design, the winning bids for a certain category oscillate and decline as the auction progresses throughout a given day. Each category in the figure's legend represents one of the five most frequently[42] auctioned type-size-grade-origin combinations out of a total of 87,349 auctions: these categories account for 1,949, 1,555, 1,453, 1,221, and

[42]These five categories are actually the second, fourth, fifth, seventh, and eighth most frequent combinations. The remaining combinations (first, third, and sixth) do not include size information, which is why they were excluded.

1,177 auctions, respectively. Each auction within a category was assigned to one of thirty order bins based on its auction sequence on a given day[43]. The average winning bid was then calculated for each order bin.

D.1.6 shows the variants of Figure 15 by changining the number of orders bins from 30 to 10 or 50, and also using the weighted average winning bid where the weight is number of boxes of each auction; still, the oscillatory and decreasing patterns are found.

An oscillatory winning bid implies that the same product would receive a higher winning bid if auctioned at a peak and a lower winning bid if auctioned at a trough. This situation disadvantages farmers whose products are auctioned during trough periods, which conflicts with the spirit of Article 123 (4) of the Constitution of the Republic of Korea, mandating that the government endeavor to stabilize agricultural product prices to protect farmers' interests.

To this end, one of the government's agendas has been to enhance and standardize product quality, ensuring that farmers receive higher winning bids at auction and that winning bids are stabilized across different farmers and locations, thereby reducing oscillation.

However, in the current auction design—a sequential first-price auction with partial disclosure—both Milgrom and Weber (1999) and Krishna (2010b) predict that even with standardization, the path of winning bids will follow a martingale process, meaning that the declining trend will disappear[44], though some degree of oscillation will persist. While the assumptions in these papers do not perfectly align with the context of the apple auctions—since they assume perfectly homogeneous goods and bidders with unit demand, whereas apples within a certain category in Figure 15 are not exactly identical, and bidders in apple auctions often participate in multiple auctions—the auction designs are indeed identical. Intuitively, it is reasonable to expect that auctioning homogeneous goods one item at a time would produce oscillatory[45] winning bids, such that a good auctioned off at the trough is at a disadvantage.

Among the recommendations made in various reports (D.1.5), changing the sequential nature of the current design has not been considered. To this end, I propose that policymakers consider an auction design called the Product-Mix Auction (hereafter referred to as the alternative auction), which is used by the Bank of England and the Icelandic government. The details of this alternative auction design are available in Klemperer (2013a) and Klemperer (2018), while Baldwin et al. (2023) discusses the computational aspects of the design, such as the optimal tie-breaking rule.

Using the example from the first paragraph of this subsection, this alternative auction is a simultaneous single-round auction, as it conducts auction once to sell 100 objects, and a uniform-price auction, meaning that if 60 of them belong to a category named A and 40 belong to a category named B, it determines a single clearing price for each category. Therefore, a farmer selling products in category A need not worry about whether his products will be auctioned at a peak or trough. Furthermore, the alternative auction determines the clearing price in a way that reduces bid shading, thereby protecting farmers' interests; bid shading or demand

[43]For example, 1,949 auctions in the category 'Mishima,40,H,CS' were distributed across 30 order bins as follows: 70/64/70/70/64/78/79/92/82/78/65/74/57/63/59/49/63/54/53/66/78/72/61/66/59/68/59/51/49/36.

[44]D.1.6 provides an explanation for the declining prices observed in Figure 15 and how standardization can address this decline.

[45]The same phenomenon is predicted to be observed even when the auction format changes to a sequential English auction (Van Den Berg et al. (2001)).

reduction by bidders in uniform-price auctions are well-known practices, and for details, refer to Ausubel et al. (2014) and Kaplan and Zamir (2015)[46].

I first outline the rules of this alternative auction, then select apple auctions held on February 12, 2022, at the Seoul auction house to demonstrate its outcome, and lastly provide additional details about this alternative auction.

The rules of this alternative auction are as follows, in the context of selling apples in two categories, A and B —D.3 provides additional details of this rule using another example.

- Step 1. Before the auction, the auctioneer announces the total number of boxes of apples to be supplied across categories A and B.

- Step 2. Bidders submit their demand schedules for categories A and B, as they would in a uniform-price auction.

- Step 3. Given the submitted demand schedules for categories A and B, the auctioneer forms a single demand curve, where the x-axis represents the ratio of the number of boxes in category A to category B, and the y-axis represents the ratio of bids per box in category A to category B.

- Step 4. After observing the demand curve, the auctioneer determines the ratio of the number of boxes in category A to category B to achieve the competitive equilibrium clearing price for each category. In making this decision, the auctioneer adheres to the total number of boxes announced in Step 1.

These four steps constitute the implementation of the alternative auction design. The reason that bidders in this auction engage in less bid-shading, to the extent that they are known to bid approximately truthfully (Grace (2024)), relates to Steps 1 and 4. In Step 1, bidders cannot perfectly forecast how many boxes of apples will be supplied for each category, and in Step 4, the auctioneer decides the supply of boxes for each category after observing bidders' demands. Additionally, the clearing price for each category is determined simultaneously, so that any reduction in a bidder's demand for one category may not drive down the clearing price of that category.

Step 3 mentions a single demand curve, represented as a blue-dotted curve in Figure 16.

[46] Ausubel et al. (2014) finds evidence of demand reduction and the possible presence of inefficient allocation in uniform-price auctions, though these findings vary slightly across settings and assumptions. Chapter 7.3 of Kaplan and Zamir (2015) points out that, typically, the uniform-price auction is subject to inefficient allocation and manipulative bidding.

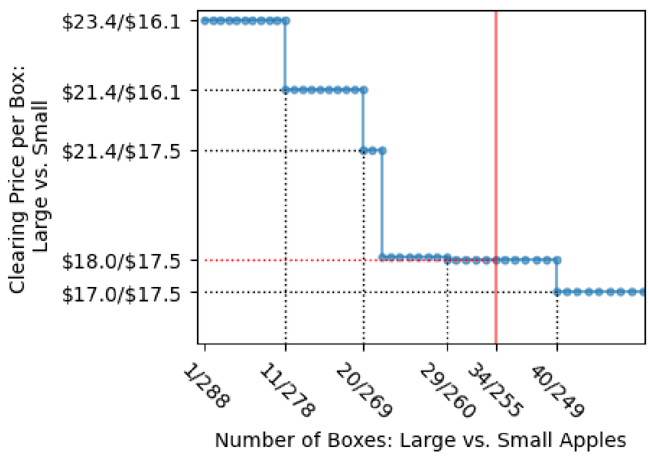

Fig. 16 – Predicted outcome of apple auction at Seoul auction house on February 12, 2022, using Product-Mix Auction:

Categories - Myanmar Large apples and Myanmar Small apples from Cheongsong

X-axis - Ratio of number of boxes

Y-axis - Ratio of uniform clearing price per box

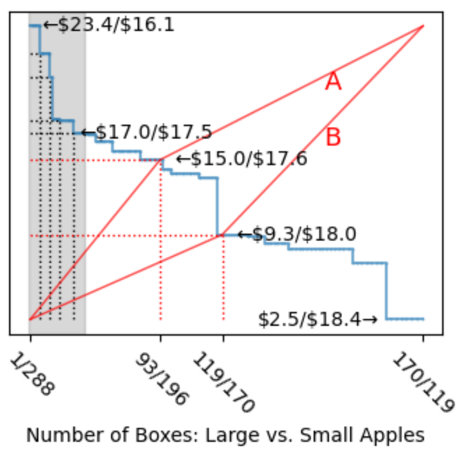

Fig. 17 – Predicted outcome of apple auction at Seoul auction house on February 12, 2022, using Product-Mix Auction with a non-vertical supply curve; the shaded region corresponds to the demand curve in Figure 16:

The categories, X-axis, and Y-axis match those in Figure 16.

The demand curve represents what would have happened if the Seoul auction house had implemented the Product-Mix Auction for selling apples on February 12, 2022. As shown in Table 4 below, twelve apple auctions were sequentially conducted, with apples in each auction being of the Myanmar type and originating from the same place. The Size column (col. (4)) shows the number of apples inside a single box; as this number decreases, the size of each apple inside the box increases. Therefore, if this number was less than or equal to 30, I categorized the corresponding apples as Large (col. (7)) and as Small otherwise, thereby yielding two categories. Thus, categories A and B in the previous example correspond to Large apples and Small apples, respectively.

The total number of boxes supplied (col. (2)) was 289[47]. If the auctioneer had declared in Step 1 that he would supply this total, then, as stated in Step 4, the boxes allocated to each category must add up to 289. This is why the ratio on the x-axis of Figure 16 always sums to 289 (e.g., 34/255 and 40/249). If the auctioneer determined, after forming and observing this demand curve, that it was optimal to supply 34 boxes of Large apples and 255 boxes of Small apples, then the equilibrium clearing price per box would be $18.0 for Large apples and $17.5 for Small apples. Thus, a farmer selling Large apples need not worry about whether his produce will be sold at a peak or trough, as occurs in a sequential auction; all farmers of Large apples would receive $18.0 per box. Furthermore, because the alternative auction is known to be robust against demand reduction, the clearing prices for each category can be thought of as the maximum price that can be attained in a uniform-price auction.

The reason the supply curve (red solid line) in Figure 16 is vertical is that, in the current institutional setting of Garak Market, the auctioneer cannot adjust the ratio of boxes between Large apples and Small apples. The numbers 34 and 255 represent the total number of boxes (col. (2)) for Large apples and Small apples, respectively. If the institutional setting were to

[47] As noted in Table 4, I excluded the 11 boxes from the first auction in the summation.

change, allowing the auctioneer to adjust the ratio of boxes, the supply curve would no longer be vertical. This scenario is illustrated in Figure 17, where curve A and curve B represent supply curves under different settings, each leading to different equilibrium price vectors; the shaded region in Figure 17 corresponds to the demand curve in Figure 16.

Below, I describe how I obtained the demand curve, which is derived from the sixth column of the table.

Table 4: Apple Auctions on February 12, 2022 at Seoul Auction House

	Winning Bid Per box($) (1)	No.of Boxes (2)	Winner ID (3)	Size (4)	$\exp(Z'\hat{\beta})$ ($) (5)	Estimated Private Values ($) (6)	Binary Size (7)
1	28.5	11	4728	25	23.3	-	-
2	20.0	11	4507	30	19.5	*23.4*, 21.4, 18.0, 16.5, 15.0	Large
3	19.2	37	4765	40	17.6	*26.3*, 14.1, 13.9, 12.4, 5.4	Small
4	16.9	30	4303	40	17.0	*21.1*, 18.0, 17.5, 16.1, 12.6	Small
5	13.8	46	4266	50	13.5	*18.4*, 15.6, 12.2, 6.1, 6.0	Small
6	10.0	12	4266	50	11.7	*12.1*, 10.4, 7.9, 7.1, 6.5	Small
7	13.8	16	4266	25	22.0	*17.0*, 15.7, 14.2, 9.3, 8.3	Large
8	13.8	7	4306	30	17.8	*18.1*, 13.9, 8.8, 7.2, 2.5	Large
9	13.8	48	4781	40	17.0	*18.5*, 17.6, 15.7, 11.3, 10.4	Small
10	10.0	22	4306	40	15.2	*12.6*, 11.2, 9.5, 5.4, 4.2	Small
11	8.5	46	4266	50	12.5	*11.0*, 5.1, 4.5, 2.6, 1.8	Small
12	7.7	14	4266	50	10.9	*10.0*, 9.0, 6.2, 3.4, 1.5	Small

Note: The Seoul auction house conducted twelve auctions on February 12, 2022, which is why there are twelve rows. *The apples sold in each auction were all of the Myanmar variety and came from Cheongsong.* As discussed in Table 1, the fourth column denotes the number of apples inside a box. The larger this number, the smaller each individual apple; therefore, when the value in the fourth column was less than or equal to 30, I defined the apples as Large in the seventh column, and as Small otherwise. The fifth column represents the covariate-explained value, as introduced in bid homogenization. The sixth column displays the estimated private values of the bidders, which incorporate the effects of covariates, as all estimates were obtained through steps (i)-(iii) of the bid homogenization process. The sixth column of the first auction remains vacant because the required estimators, $\hat{\xi}_1$ and $\hat{F}_1$, are still being evaluated for their performance. Thus, I discard the first row when constructing the demand curve in Figure 16, which is also why I leave the seventh column of the first row vacant.

Recalling that the number of bidders in each auction is not specified in the dataset, and that three to seven bidders are known to participate in each auction, I assumed that five bidders—the median number—attended each auction. This assumption explains why there are five values for each auction in the sixth column. Focusing on the last row, the values—*10.0*, 9.0, 6.2, 3.4, 1.5—were estimated as follows: the italicized value (10.0) was obtained by applying $\hat{\xi}_2^l(b_2, b_1^{\max})$, where b_2 is 7.7 (last row, col. (1)) and $b_1^{\max}$ is 8.5 (eleventh row, col. (1)). The remaining four values were independently drawn from $\widehat{\Pr}[V_2 \leq \cdot | B_1 \leq b_1^{\max}]$[48], whose estimand is one of the components of another value distribution (14). I repeated this backward approach for each pair of auctions from the eleventh to the second, with all estimates incorporating the effect of covariates (col. (5)), as each estimate underwent steps (i)-(iii) of the bid homogenization process.

The sixth column of the first auction remains vacant because the required estimators, $\hat{\xi}_1$

[48]The closed-form expression is provided in C.8

and $\hat{F}_1$, are still being evaluated for their performance. Therefore, excluding the first auction, the demand curve shown in Figures 16 and 17 was derived by assuming that the values in the sixth column represent the submitted demand schedules across bidders in the alternative auction design. This assumption is based on the fining that, as discussed previously, bidders are known to bid approximately truthfully in this alternative auction design. The demand schedule suggests that, focusing on the last row of the sixth column, one bidder expressed a willingness to pay \$10.0 per box for up to 14 boxes of Small apples, while another bidder expressed a willingness to pay \$9.0 per box for up to 14 boxes of Small apples.

Lastly, I discuss two key details of the Product-Mix Auction.

First, replacing Large and Small apples with Strong and Weak collaterals results in the Product-Mix Auction used by the Bank of England. In its Indexed Long-Term Repo operations, the Bank provides reserves to counterparties by purchasing either Strong collateral (e.g., highly liquid sovereign securities) or Weak collateral (e.g., mortgage-backed bonds). The aim of the Product-Mix Auction is to find a competitive equilibrium for two non-complementary products (in our case, Large apples and Small apples) simultaneously. Extending this to more than two products, such as three, involves grouping two of the products together, comparing that group with the third product, and then comparing the two products within the group[49].

Lastly, as Palacios-Huerta et al. (2024) points out, a sizable number of auction designs exist, but only a few are used in practice[50]. The Product-Mix Auction is one of these practical designs, and because it is not a multi-round auction, it concludes quickly, allowing bidders to continue engaging in daily transactions with their customers; as discussed in subsection 5.3, 90% of the fruits they win are delivered to customers on the same day. However, changing or introducing a new auction design should be approached with caution. This is exemplified by the Federal Communications Commission when it devised the Simultaneous Multiple-Round Auction[51].

6 Conclusion

I focus on sequential first-price auctions with partial disclosure, where each auction sells a single object at a time, and only the winning bid and the winner's identity are disclosed for that object.

In my model, this disclosure influences a bidder's decision when placing a bid. His decision also considers the complementarity between objects, captured by a function δ. Unlike in other studies, the output of this function varies not only by auction covariates but also by the bidder's own value for the objects.

Under the Independent Private Value paradigm with a two-period model, I demonstrate that, even with a dataset containing only winning bids and winner identities, an analyst can nonparametrically identify and separately estimate the complementarity and correlation across objects, as well as bidders' bidding strategies. This result shows that the *indirect approach* of Guerre et al. (2000) can be extended beyond a single period to cases where bidders have multi-

[49] Refer to Frost et al. (2015)

[50] As Palacios-Huerta et al. (2024) notes, more than 8,000 papers discuss combinatorial auctions, with most proposing a variation. However, only three variations are commonly used in practice: Sealed-bid, Multi-round, and Combinatorial Clock.

[51] Federal Communications Commission (1994) notes that 222 comments and 169 reply comments were received.

unit demand. Extending the model to more than two periods while ensuring it remains testable through the bid distribution, and developing estimators for the parameters of this extended model, are focuses of my future work.

In the Application section, I use my model and estimator to propose using an alternative auction design, the Product-Mix Auction, to help the government achieve its objectives. Other approaches could be explored to assess the effects of alternative designs: adopting a model of asymmetric bidders (D.2), devising a model[52] that incorporates risk-averse or budget-constrained bidders, applying machine learning techniques[53], or conducting field or lab experiments, as in List (2000), Ausubel et al. (2013) or Katok (2013) are a few examples.

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[52]Such models would impose additional testable restrictions on the bid distribution if one were to continue using the approach of Guerre et al. (2000).

[53]Refer to the conference titled *New Directions in Market Design*, held from May 11-12, 2023, at the National Bureau of Economic Research.

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Appendices

A Section 2, Model

A.1 Equivalence between $\Pr[B_{2,-i}^{\max} \le \cdot | B_{1,-i}^{\max} \le \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]$ and $G_{B_2^l(\tilde{b}_{1i})}(\cdot | B_1 \le \tilde{b}_{1i})^{I-1}$

[Back to ToC] Refer to the following equalities; $\tilde{b}_{1i}$ is equivalent to $\tilde{s}_1(v_{1i})$ where the strategy $\tilde{s}_1$ need not be the equilibrium strategy s_1.

$$\begin{aligned}
&\Pr\left[B_{2,-i}^{\max} \le \cdot \mid B_{1,-i}^{\max} \le \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\
&= \Pr\left[s_2^l(V_{2j}, \tilde{s}_1(V_{1i})) \le \cdot, \forall j \ne i \mid s_1(V_{1j}) \le \tilde{s}_1(V_{1i}), \forall j \ne i, V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\
&= \frac{\Pr[s_2^l(V_{2j}, \tilde{s}_1(V_{1i})) \le \cdot, s_1(V_{1j}) \le \tilde{s}_1(V_{1i}), \forall j \ne i, V_{1i} = v_{1i}, V_{2i} = v_{2i}]}{\Pr[s_1(V_{1j}) \le \tilde{s}_1(V_{1i}), \forall j \ne i, V_{1i} = v_{1i}, V_{2i} = v_{2i}]} \\
&= \frac{\Pr[s_2^l(V_{2j}, \tilde{b}_{1i}) \le \cdot, s_1(V_{1j}) \le \tilde{b}_{1i}, \forall j \ne i]\ \Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}]}{\Pr[s_1(V_{1j}) \le \tilde{b}_{1i}, \forall j \ne i]\ \Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}]} \\
&= \Pr\left[s_2^l(V_{2j}, \tilde{b}_{1i}) \le \cdot, \forall j \ne i \mid s_1(V_{1j}) \le \tilde{b}_{1i}, \forall j \ne i\right] \\
&= \prod_{j \ne i} \Pr\left[s_2^l(V_{2j}, \tilde{b}_{1i}) \le \cdot \mid s_1(V_{1j}) \le \tilde{b}_{1i}\right] = \Pr\left[s_2^l(V_2, \tilde{b}_{1i}) \le \cdot \mid s_1(V_1) \le \tilde{b}_{1i}\right]^{I-1} \\
&= \Pr\left[B_2^l(\tilde{b}_{1i}) \le \cdot \mid B_1 \le \tilde{b}_{1i}\right]^{I-1} \equiv G_{B_2^l(\tilde{b}_{1i})}(\cdot \mid B_1 \le \tilde{b}_{1i})^{I-1},
\end{aligned}$$

in which the equalities hold by *independence*, *symmetry*, and by the assumption that the competitors' bids originate from equilibrium strategies; especially, the sixth equality uses *symmetry*.

A.2 Equivalence between $\Pr[B_{2,-i}^{\max} \le \cdot | B_{1j} = b_1^w, B_{1j} \ge B_{1k}, k \notin \{i,j\}, B_{1j} \ge \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]$ and $G_{2|1}^w(\cdot|b_1^w) G_{B_2^l(b_1^w)}(\cdot|B_1 \le b_1^w)^{I-2}$

[Back to ToC] The following equalities also use *independence*, *symmetry*, and the notion that the competitors' bids come from equilibrium strategies.

$$\begin{aligned}
&\Pr[B_{2,-i}^{\max} \le \cdot \mid B_{1j} = b_1^w, B_{1j} \ge B_{1k}, k \notin \{i,j\}, B_{1j} \ge \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}] \\
&= \Pr\left[s_2^w(V_{1j}, V_{2j}) \le \cdot, s_2^l(V_{2k}, s_1(V_{1j})) \le \cdot, k \notin \{i,j\} \mid \right. \\
&\qquad \left. s_1(V_{1j}) = b_1^w, s_1(V_{1j}) \ge s_1(V_{1k}), k \notin \{i,j\}, s_1(V_{1j}) \ge \tilde{s}_1(V_{1i}), V_{1i} = v_{1i}, V_{2i} = v_{2i}\right] \\
&= \frac{\Pr[s_2^w(V_{1j}, V_{2j}) \le \cdot, s_1(V_{1j}) = b_1^w, s_2^l(V_{2k}, b_1^w) \le \cdot, b_1^w \ge s_1(V_{1k}), k \notin \{i,j\}, b_1^w \ge \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]}{\Pr[s_1(V_{1j}) = b_1^w, b_1^w \ge s_1(V_{1k}), k \notin \{i,j\}, b_1^w \ge \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]} \\
&= \frac{\Pr[s_2^w(V_{1j}, V_{2j}) \le \cdot, s_1(V_{1j}) = b_1^w, s_2^l(V_{2k}, b_1^w) \le \cdot, b_1^w \ge s_1(V_{1k}), k \notin \{i,j\}]}{\Pr[s_1(V_{1j}) = b_1^w, b_1^w \ge s_1(V_{1k}), k \notin \{i,j\}]} \\
&\qquad \times \frac{\Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}]}{\Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}]} \\
&= \frac{\Pr[s_2^w(V_{1j}, V_{2j}) \le \cdot, s_1(V_{1j}) = b_1^w] \prod_{k \notin \{i,j\}} \Pr[s_2^l(V_{2k}, b_1^w) \le \cdot, b_1^w \ge s_1(V_{1k})]}{\Pr[s_1(V_{1j}) = b_1^w] \prod_{k \notin \{i,j\}} \Pr[b_1^w \ge s_1(V_{1k})]} \\
&= \Pr\left[s_2^w(V_{1j}, V_{2j}) \le \cdot \mid s_1(V_{1j}) = b_1^w\right] \prod_{k \notin \{i,j\}} \Pr[s_2^l(V_{2k}, b_1^w) \le \cdot \mid s_1(V_{1k}) \le b_1^w] \\
&= \Pr\left[s_2^w(V_1, V_2) \le \cdot \mid s_1(V_1) = b_1^w\right]\ \Pr[s_2^l(V_2, b_1^w) \le \cdot \mid s_1(V_1) \le b_1^w]^{I-2}
\end{aligned}$$

$$= \Pr\left[B_2^w \leq \cdot \mid B_1 = b_1^w\right] \Pr[B_2^l(b_1^w) \leq \cdot \mid B_1 \leq b_1^w]^{I-2} \equiv G_{2|1}^w(\cdot \mid b_1^w) G_{B_2^l(b_1^w)}(\cdot \mid B_1 \leq b_1^w)^{I-2},$$

in which the third holds as long as $b_1^w \geq \tilde{b}_{1i}$; the sixth equality equality holds by *symmetry*, and it is this *symmetry* that makes the identity of the winner to be irrelevant.

A.3 Equivalence between $\Pr[B_{1,-i}^{\max} \leq \tilde{b}_{1i}]$ and $\Pr[B_1 \leq \tilde{b}_{1i}]^{I-1}$

[Back to ToC] Refer to the following equalities.

$$\begin{aligned}
\Pr[B_{1,-i}^{\max} \leq \tilde{b}_{1i}] &= \Pr[s_1(V_{1j}) \leq \tilde{b}_{1i}, \forall j \neq i] = \prod_{j \neq i} \Pr[s_1(V_{1j}) \leq \tilde{b}_{1i}] \\
&= \Pr[s_1(V_1) \leq \tilde{b}_{1i}]^{I-1} = \Pr[B_1 \leq \tilde{b}_{1i}]^{I-1},
\end{aligned}$$

in which the second equality holds by *independence*; the third holds by *symmetry*.

A.4 Closed form expression for the continuation value of being the first auction winner in the second auction, i.e., $\mathcal{V}^w(v_{1i}, \tilde{b}_{1i})$

[Back to ToC] Given that the optimal second auction bid $\tilde{b}_{2i}$ comes from equation (5), I can plug this $\tilde{b}_{2i}$ into bidder i's expected profit equation (3), which yields the following number,

$$\frac{(G_{B_2^l(\tilde{b}_{1i})}(\tilde{b}_{2i}|B_1 \leq \tilde{b}_{1i})^{I-1})^2}{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{b}_{2i}|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}}.$$

This number expresses the maximum profit bidder i can enjoy in the second auction assuming that he knows his v_{1i} and v_{2i}.

If bidder i knows his v_{1i} but not v_{2i}, then the maximum profit number becomes a random variable where v_{2i} is replaced with V_{2i} — the strategy $\tilde{s}_2^w$ chosen by bidder i need not be equilibrium strategy,

$$\frac{(G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, V_{2i})|B_1 \leq \tilde{b}_{1i})^{I-1})^2}{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, V_{2i})|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}}.$$

This new random variable is what bidder i faces in the first auction. Since bidder i draws his value of the second object from the distribution $F_{2|1}(\cdot|v_{1i})$, his continuation value $\mathcal{V}^w(v_{1i}, \tilde{b}_{1i})$ is calculated as follows.

$$\begin{aligned}
&\int_{\underline{v_2}}^{\overline{v_2}} \frac{(G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, x)|B_1 \leq \tilde{b}_{1i})^{I-1})^2}{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, x)|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} dF_{2|1}(x|v_{1i}) \\
&= \int_{\underline{b_2}}^{\overline{b_2}} \frac{(G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1})^2}{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} d\Pr[\tilde{B}_2^w \leq b_2|V_1 = v_{1i}] \\
&= \mathrm{E}_{\tilde{B}_2^w|V_1}\left[\frac{(G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1})^2}{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \,\middle|\, V_1 = v_{1i}\right] \equiv \mathcal{V}^w(v_{1i}, \tilde{b}_{1i}),
\end{aligned} \tag{20}$$

in which the first equality holds because of $F_{2|1}(x|v_{1i}) \equiv \Pr[V_2 \leq x|V_1 = v_{1i}] = \Pr[\tilde{s}_2^w(v_{1i}, V_2) \leq \tilde{s}_2^w(v_{1i}, x)|V_1 = v_{1i}] \equiv \Pr[\tilde{B}_2^w \leq b_2|V_1 = v_{1i}]$ where I assume that the strategy $\tilde{s}_2^w$, which bidder i plays, is monotone — at the end of Section 2, I show that bidder i's strategy $\tilde{s}_2^w$ must equal monotone equilibrium strategy s_2^w, so the assumption that $\tilde{s}_2^w$ being monotone makes no harm.

A.5 Closed form expression for the continuation value of being the first auction loser in the second auction, i.e., $\mathcal{V}^l(v_{1i}, \tilde{b}_{1i})$

[Back to ToC] Given that the optimal second auction bid $\tilde{b}_{2i}$ comes from equation (6), I can plug this $\tilde{b}_{2i}$ into bidder i's expected profit equation (4), which yields the following number,

$$\frac{(G^w_{2|1}(\tilde{b}_{2i}|b_1^w)G_{B_2^l(b_1^w)}(\tilde{b}_{2i}|B_1 \leq b_1^w)^{I-2})^2}{\partial(G^w_{2|1}(\tilde{b}_{2i}|b_1^w)G_{B_2^l(b_1^w)}(\tilde{b}_{2i}|B_1 \leq b_1^w)^{I-2})/\partial\tilde{b}_{2i}}.$$

This number expresses the maximum profit bidder i can enjoy in the second auction assuming that he knows his v_{1i}, v_{2i}, and observes the winning bid of b_1^w.

If bidder i knows his v_{1i} but neither v_{2i} nor b_1^w, then the maximum profit number becomes a function that takes V_{2i} and B_1^w as random variables — the strategy $\tilde{s}_2^l$ chosen by bidder i need not be equilibrium strategy.

$$\frac{(G^w_{2|1}(\tilde{s}_2^l(v_{1i}, V_{2i}, B_1^w)|B_1^w)\, G_{B_2^l(B_1^w)}(\tilde{s}_2^l(v_{1i}, V_{2i}, B_1^w)|B_1 \leq B_1^w)^{I-2})^2}{\partial(G^w_{2|1}(\tilde{s}_2^l(v_{1i}, V_{2i}, B_1^w)|B_1^w)G_{B_2^l(B_1^w)}(\tilde{s}_2^l(v_{1i}, V_{2i}, B_1^w)|B_1 \leq B_1^w)^{I-2})/\partial\tilde{b}_{2i}}.$$

This function of random variables, which are V_{2i} and B_1^w, is what bidder i faces in the first auction. The distribution of both random variables given that bidder i's situation is $\{B_{1,-i}^{\max} \geq \tilde{b}_1, V_{1i} = v_{1i}\}$ as follows:

$$\begin{aligned}
&\Pr\left[V_{2i} \leq v_2, B_{1,-i}^{\max} \leq b \mid B_{1,-i}^{\max} \geq \tilde{b}_{1i}, V_{1i} = v_{1i}\right] \\
&= \frac{\Pr\left[V_{2i} \leq v_2, V_{1i} = v_{1i}, B_{1,-i}^{\max} \leq b, B_{1,-i}^{\max} \geq \tilde{b}_{1i}\right]}{\Pr\left[V_{1i} = v_{1i}, B_{1,-i}^{\max} \geq \tilde{b}_{1i}\right]} \\
&= \frac{\Pr\left[V_{2i} \leq v_2, V_{1i} = v_{1i}, \max_{j\neq i} s_1(V_{1j}) \leq b, \max_{j\neq i} s_1(V_{1j}) \geq \tilde{b}_{1i}\right]}{\Pr\left[V_{1i} = v_{1i}, \max_{j\neq i} s_1(V_{1j}) \geq \tilde{b}_{1i}\right]} \\
&= F_{2|1}(v_2 \mid v_{1i}) \Pr[\max_{j\neq i} s_1(V_{1j}) \leq b \mid \max_{j\neq i} s_1(V_{1j}) \geq \tilde{b}_{1i}] \\
&= \frac{1}{1 - \Pr[\max_{j\neq i} s_1(V_{1j}) \leq \tilde{b}_{1i}]} F_{2|1}(v_2|v_{1i}) \Pr[\tilde{b}_{1i} \leq \max_{j\neq i} s_1(V_{1j}) \leq b] \\
&= \frac{1}{1 - G_1(\tilde{b}_{1i})^{I-1}} F_{2|1}(v_2|v_{1i}) \Pr[B_{1,-i}^{\max} \leq b]\mathbb{1}[\tilde{b}_{1i} \leq B_{1,-i}^{\max}] \\
&= \frac{1}{1 - G_1(\tilde{b}_{1i})^{I-1}} F_{2|1}(v_2|v_{1i}) G_1(b)^{I-1}\mathbb{1}[\tilde{b}_{1i} \leq B_{1,-i}^{\max}]
\end{aligned}$$

in which the second equality holds by monotone strategy s_1 played by $I-1$ bidders; the third holds by *independence* and *symmetry*; the fifth and the last hold by the equivalence between $\Pr[B_{1,-i}^{\max} \leq b]$ and $\Pr[B_1 \leq b]^{I-1}$.

Given the conditional distribution of V_{2i} and B_1^w, I can calculate bidder i's continuation value $\mathcal{V}^l(v_{1i}, \tilde{b}_{1i})$ as follows.

$$\begin{aligned}
&\frac{1}{1-G_1(\tilde{b}_{1i})^{I-1}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \Bigg[\\
&\frac{(G_{2|1}^w(\tilde{s}_2^l(v_{1i}, v, b)|b)\, G_{B_2^l(b)}(\tilde{s}_2^l(v_{1i}, v, b)|B_1 \leq b)^{I-2})^2}{\partial (G_{2|1}^w(\tilde{s}_2^l(v_{1i}, v, b)|b) G_{B_2^l(b)}(\tilde{s}_2^l(v_{1i}, v, b)|B_1 \leq b)^{I-2})/\partial \tilde{b}_{2i}}\Bigg] dF_{2|1}(v|v_{1i}) dG_1(b)^{I-1} \\
&= \frac{1}{1-G_1(\tilde{b}_{1i})^{I-1}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{b_2}}^{\overline{b_2}} \Bigg[\\
&\frac{(G_{2|1}^w(b_2|b)\, G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})^2}{\partial (G_{2|1}^w(b_2|b) G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})/\partial \tilde{b}_{2i}}\Bigg] d \Pr[\tilde{B}_2^l(b) \leq b_2 | V_1 = v_{1i}] dG_1(b)^{I-1} \\
&\equiv \mathcal{V}^l(v_{1i}, \tilde{b}_{1i}), \qquad (21)
\end{aligned}$$

in which the first equality holds because of $F_{2|1}(v|v_{1i}) \equiv \Pr[V_2 \leq v | V_1 = v_{1i}] = \Pr[\tilde{s}_2^l(v_{1i}, V_2, b) \leq \tilde{s}_2^l(v_{1i}, v, b) | V_1 = v_{1i}] \equiv \Pr[\tilde{B}_2^l(b) \leq b_2 | V_1 = v_{1i}]$ where I assume that the strategy $\tilde{s}_2^l$, which bidder i plays, is monotone in V_2 — at the end of Section 2, I show that bidder i's strategy $\tilde{s}_2^l$ must equal monotone equilibrium strategy s_2^l, so the assumption that $\tilde{s}_2^l$ being monotone makes no harm.

A.6 Detailed derivation of going from equation (7) to equation (8)

[Back to ToC] Taking the derivative of equation (7) with respect to $\tilde{b}_{1i}$ yields the following:

$$\begin{aligned}
v_{1i} = \tilde{b}_{1i} &+ \frac{1}{I-1} \frac{G_1(\tilde{b}_{1i})}{g_1(\tilde{b}_{1i})} - \mathcal{V}^w(v_{1i}, \tilde{b}_{1i}) + \mathcal{V}^l(v_{1i}, \tilde{b}_{1i}) \\
&- \frac{\partial \mathcal{V}^w(v_{1i}, \tilde{b}_{1i})}{\partial \tilde{b}_{1i}} \frac{G_1(\tilde{b}_{1i})^{I-1}}{dG_1(\tilde{b}_{1i})^{I-1}/d\tilde{b}_{1i}} - \frac{\partial \mathcal{V}^l(v_{1i}, \tilde{b}_{1i})}{\partial \tilde{b}_{1i}} \frac{[1-G_1(\tilde{b}_{1i})^{I-1}]}{dG_1(\tilde{b}_{1i})^{I-1}/d\tilde{b}_{1i}},
\end{aligned}$$

in which we need derivatives of $\mathcal{V}^w$ (i.e., (20)) and $\mathcal{V}^l$ (i.e., (21)) with respect to the first auction bid $\tilde{b}_{1i}$. These derivatives exploit the following equivalence the distribution $G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})$ has.

$$\begin{aligned}
G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i}) &\equiv \Pr[B_2^l(\tilde{b}_{1i}) \leq b_2 | B_1 \leq \tilde{b}_{1i}] \\
&= \Pr[s_2^l(V_2, \tilde{b}_{1i}) \leq b_2 | B_1 \leq \tilde{b}_{1i}] \\
&= \frac{1}{G_1(\tilde{b}_{1i})} \int_{\underline{b_1}}^{\tilde{b}_{1i}} \Pr[s_2^l(V_2, \tilde{b}_{1i}) \leq b_2, B_1 = x] dx \\
&= \frac{1}{G_1(\tilde{b}_{1i})} \int_{\underline{b_1}}^{\tilde{b}_{1i}} \Pr[s_2^l(V_2, \tilde{b}_{1i}) \leq b_2 | B_1 = x] dG_1(x) \\
&\equiv \frac{1}{G_1(\tilde{b}_{1i})} \int_{\underline{b_1}}^{\tilde{b}_{1i}} G_{B_2^l(\tilde{b}_{1i})|B_1}(b_2|x) dG_1(x), \qquad (22)
\end{aligned}$$

in which I use equilibrium strategy s_2^l because a bidder other than bidder i plays equilibrium strategy; the newly defined term, $G_{B_2^l(\tilde{b}_{1i})|B_1}(b_2|x)$, represents "the bid distribution of the first auction loser who had bid $B_1 = x$ but lost to the winning bid of $\tilde{b}_{1i}$" — given this equivalence, I introduce the following derivation.

$$\frac{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial \tilde{b}_{1i}} = \frac{dG_1(\tilde{b}_{1i})^{I-1}/d\tilde{b}_{1i}}{G_1(\tilde{b}_{1i})^{I-1}} \times$$
$$\left[\frac{G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-2}}{g_1(\tilde{b}_{1i})} \frac{\partial \left\{ G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i}) G_1(\tilde{b}_{1i}) \right\}}{\partial \tilde{b}_{1i}} - G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1} \right],$$

which is used in the derivative of $\mathcal{V}^w(v_{1i}, \tilde{b}_{1i})$ with respect to $\tilde{b}_{1i}$ as follows.

$$\frac{\partial \mathcal{V}^w(v_{1i}, \tilde{b}_{1i})}{\partial \tilde{b}_{1i}}$$
$$= \int_{\underline{v_2}}^{\overline{v_2}} \frac{G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, x)|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, x)|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \frac{\partial G_{B_2^l(\tilde{b}_{1i})}(\tilde{s}_2^w(v_{1i}, x)|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial \tilde{b}_{1i}} dF_{2|1}(x|v_{1i})$$
$$= \int_{\underline{b_2}}^{\overline{b_2}} \frac{G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \frac{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial \tilde{b}_{1i}} d\Pr[\tilde{B}_2^w \leq b_2|V_1 = v_{1i}]$$
$$= \frac{dG_1(\tilde{b}_{1i})^{I-1}/d\tilde{b}_{1i}}{G_1(\tilde{b}_{1i})^{I-1}} \times \int_{\underline{b_2}}^{\overline{b_2}} \left\{ \frac{G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}}{\partial G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1}/\partial \tilde{b}_{2i}} \left[\frac{G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-2}}{g_1(\tilde{b}_{1i})} \right.\right.$$
$$\left.\left. \frac{\partial \left\{ G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i}) G_1(\tilde{b}_{1i}) \right\}}{\partial \tilde{b}_{1i}} - G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-1} \right] \right\} d\Pr[\tilde{B}_2^w \leq b_2|V_1 = v_{1i}],$$

in which the first equality holds by (i) taking the derivative of equation (3) with respect to $\tilde{b}_{1i}$ and replacing $\tilde{b}_{2i}$ with the $\tilde{b}_{2i}$ from equation (5) (i.e., using Envelope Theorem), and by (ii) noting that $\tilde{b}_{2i}$ is equivalent to $\tilde{s}_2^w(v_{1i}, v_{2i})$ so that we have to take an expectation; the second equality holds by the same logic used in equation (20).

The derivative of $\mathcal{V}^l$(i.e., (21)) with respect to the first auction bid $\tilde{b}_{1i}$ is as follows.

$$\frac{\partial \mathcal{V}^l(v_{1i}, \tilde{b}_{1i})}{\partial \tilde{b}_{1i}}$$
$$= \frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{1}{1 - G_1(\tilde{b}_{1i})^{I-1}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{b_2}}^{\overline{b_2}} \left[\right.\right.$$
$$\left.\left. \frac{(G_{2|1}^w(b_2|b)\, G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})^2}{\partial (G_{2|1}^w(b_2|b) G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})/\partial \tilde{b}_{2i}} \right] d\Pr[\tilde{B}_2^l(b) \leq b_2|V_1 = v_{1i}] dG_1(b)^{I-1} \right)$$
$$= \left(\frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{1}{1 - G_1(\tilde{b}_{1i})^{I-1}} \right) \right) \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{b_2}}^{\overline{b_2}} \left[\right.$$
$$\left. \frac{(G_{2|1}^w(b_2|b)\, G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})^2}{\partial (G_{2|1}^w(b_2|b) G_{B_2^l(b)}(b_2|B_1 \leq b)^{I-2})/\partial \tilde{b}_{2i}} \right] d\Pr[\tilde{B}_2^l(b) \leq b_2|V_1 = v_{1i}] dG_1(b)^{I-1}$$
$$- \frac{dG_1(\tilde{b}_{1i})^{I-1}/d\tilde{b}_{1i}}{1 - G_1(\tilde{b}_{1i})^{I-1}} \int_{\underline{b_2}}^{\overline{b_2}} \left[\frac{(G_{2|1}^w(b_2|\tilde{b}_{1i})\, G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-2})^2}{\partial (G_{2|1}^w(b_2|\tilde{b}_{1i}) G_{B_2^l(\tilde{b}_{1i})}(b_2|B_1 \leq \tilde{b}_{1i})^{I-2})/\partial \tilde{b}_{2i}} \right] d\Pr[\tilde{B}_2^l(\tilde{b}_{1i}) \leq b_2|V_1 = v_{1i}].$$

A.7 Detailed derivation of equation (10)

[Back to ToC] I use equation (22), which asserts the following equality,

$$G_{B_2^l(b)}(\cdot|B_1 \leq b) = \frac{1}{G_1(b)} \int_{\underline{b}}^{b} G_{B_2^l(b)|B_1}(\cdot|u) dG_1(u).$$

Taking the derivative of both sides with respect to b yields the following result.

$$\begin{aligned}
\frac{\partial}{\partial b} G_{B_2^l(b)}(\cdot|B_1 \leq b) &= \frac{\partial}{\partial b}\left(\frac{1}{G_1(b)} \int_{\underline{b}}^{b} G_{B_2^l(b)|B_1}(\cdot|u) dG_1(u)\right) \\
&= -\frac{g_1(b)}{G_1(b)} G_{B_2^l(b)}(\cdot|B_1 \leq b) + \frac{1}{G_1(b)} \frac{\partial}{\partial b}\left\{\int_{\underline{b}}^{b} G_{B_2^l(b)|B_1}(\cdot|u) dG_1(u)\right\} \\
&= -\frac{g_1(b)}{G_1(b)} G_{B_2^l(b)}(\cdot|B_1 \leq b) + \frac{1}{G_1(b)}\left\{G_{B_2^l(b)|B_1}(\cdot|b) g_1(b) + \int_{\underline{b}}^{b} \frac{\partial G_{B_2^l(b)|B_1}(\cdot|u)}{\partial b} dG_1(u)\right\}.
\end{aligned}$$

in which the last equality holds by Leibniz Integral Rule. Moreover, the integrand, $\partial G_{B_2^l(b)|B_1}(\cdot|u)/\partial b$, has the following equality.

$$\begin{aligned}
\frac{\partial G_{B_2^l(b)|B_1}(\cdot|u)}{\partial b} &= \frac{\partial \Pr[s_2^l(V_2, b) \leq \cdot|s_1(V_1) = u]}{\partial b} = \frac{\partial \Pr[V_2 \leq \xi_2^l(\cdot, b)|V_1 = \xi_1(u)]}{\partial b} \\
&\equiv \frac{\partial F_{2|1}[\xi_2^l(\cdot, b)|\xi_1(u)]}{\partial b} = f_{2|1}[\xi_2^l(\cdot, b)|\xi_1(u)] \frac{\partial \xi_2^l(\cdot, b)}{\partial b} \\
&= g_{B_2^l(b)|B_1}(\cdot|u) \frac{\partial \xi_2^l(\cdot, b)/\partial b}{\partial \xi_2^l(\cdot, b)/\partial b_2^l},
\end{aligned}$$

in which the first line uses the equivalence $G_{B_2^l(b)|B_1}(\cdot|u) \equiv \Pr[B_2^l(b) \leq \cdot|B_1 = b]$ and that the random variables $[B_2^l(b), B_1]$ come from the equilibrium strategies $[s_2^l, s_1]$; the last line uses the equality $g_{B_2^l(b)|B_1}(\cdot|u) = f_{2|1}[\xi_2^l(\cdot, b)|\xi_1(u)] \frac{\partial \xi_2^l(\cdot, b)}{\partial b_2^l}$, which is the density of $B_2^l(b) = s_2^l(V_2, b)$ given $B_1 = u$.

Hence, we can extend further the derivation of $\frac{\partial}{\partial b} G_{B_2^l(b)}(\cdot|B_1 \leq b)$ as follows.

$$\begin{aligned}
\frac{\partial}{\partial b} G_{B_2^l(b)}(\cdot|B_1 \leq b) &= -\frac{g_1(b)}{G_1(b)} G_{B_2^l(b)}(\cdot|B_1 \leq b) + \frac{1}{G_1(b)}\left\{G_{B_2^l(b)|B_1}(\cdot|b) g_1(b) + \int_{\underline{b}}^{b} \frac{\partial G_{B_2^l(b)|B_1}(\cdot|u)}{\partial b} dG_1(u)\right\} \\
&= -\frac{g_1(b)}{G_1(b)} G_{B_2^l(b)}(\cdot|B_1 \leq b) \\
&\quad + \frac{1}{G_1(b)}\left\{G_{B_2^l(b)|B_1}(\cdot|b) g_1(b) + \frac{\partial \xi_2^l(\cdot, b)/\partial b}{\partial \xi_2^l(\cdot, b)/\partial b_2^l} \int_{\underline{b}}^{b} g_{B_2^l(b)|B_1}(\cdot|u) dG_1(u)\right\} \\
&= \frac{g_1(b)}{G_1(b)}\left\{G_{B_2^l(b)|B_1}(\cdot|b) - G_{B_2^l(b)}(\cdot|B_1 \leq b)\right\} + \frac{\partial \xi_2^l(\cdot, b)/\partial b}{\partial \xi_2^l(\cdot, b)/\partial b_2^l} g_{B_2^l(b)}(\cdot|B_1 \leq b).
\end{aligned}$$

This gives,

$$G_{B_2^l(b)|B_1}(\cdot|b) = G_{B_2^l(b)}(\cdot|B_1 \leq b) + \frac{G_1(b)}{g_1(b)}\left\{\frac{\partial G_{B_2^l(b)}(\cdot|B_1 \leq b)}{\partial b} - \frac{\partial \xi_2^l(\cdot, b)/\partial b}{\partial \xi_2^l(\cdot, b)/\partial b_2^l} g_{B_2^l(b)}(\cdot|B_1 \leq b)\right\},$$

and if we change the notations from $[\cdot, b, b_2^l]$ to $[x, \tilde{b}_{1i}, \tilde{b}_{2i}]$, we get equation (10).

A.8 Proof of Theorem 1

[Back to ToC] The preliminary proof is available upon request. The complete proof follows a structure similar to Proposition 1 in Li et al. (2002), Proposition 1 in Campo et al. (2003), and Theorem 1 in Guerre et al. (2000).

A.9 IPV and other Models

[Back to ToC] As discussed in Perrigne and Vuong (2023), the Independent Private Value (IPV) model is the most commonly used framework in the empirical auction literature. Other models, such as the Affiliated Value (AV) or Pure Common Value (PCV) models, are susceptible to non-identification issues because they are observationally equivalent to some Affiliated Private Value (APV) models (see Section 8 of Perrigne and Vuong (2023) and Laffont and Vuong (1996): Laffont and Vuong's paper focuses on first-price sealed-bid auctions where bidders are symmetric and desire a single object). As noted by Laffont and Vuong (1996), both symmetric IPV and symmetric APV models can be identified from the bid distribution, which aligns with the needs of empirical auction literature (For the details of identification in APV model, see Li et al. (2002)). The reason IPV has gained more popularity than APV can be though of that (i) the affiliation across bids in APV can be addressed by using a suitable conditioning variable under the IPV framework, and (ii) various techniques to handle unobserved heterogeneity in IPV models can also account for the observed affiliation in bids.

A.10 Ortega-Reichert (1968)

[Back to ToC] Chapter 8 of Ortega-Reichert (1968) considers a two-period, two-player procurement auction. In this auction, all bids are disclosed, and both bidders desire both units. Ortega-Reichart assumes a pure-strategy monotone equilibrium (Section 8.2) and establishes its existence; as pointed out in Klemperer (1999) and in section 3.1 of Weber (1983), the values across both periods of a bidder in Ortega-Reichart are correlated, and the bidder does not know his value for the second item until the first item is sold; no complementarity is considered, so a bidder's value from acquiring the objects is $V_1 + V_2$. Ortega-Reichart finds that, in this model, a bidder shades more in the first auction (compared to a single-object first-price auction) so that, when his bid is revealed, he can mitigate the fierce competition in the second auction. As pointed out in footnote 26 of Klemperer (1999) and in footnote 1 of working paper, Chapter 8 of Ortega-Reichart was influential in guiding Milgrom and Roberts (1982).

B Section 3, Identification

B.1 Derivation of equations (11)-(12) from the dataset, and Lemmas 1 and 2

[Back to ToC] *Independence* implies that the pairs in the set $\{(V_{1i}, V_{2i})_{i=1,...,I}\}$ are independent, i.e., the pairs come from the joint density $f(v_{11}, v_{21}, ..., v_{1I}, v_{2I}) = \prod_{j=1}^{I} f(v_{1j}, v_{2j})$ which has the following equivalences.

$$f(v_{11}, v_{21}, ..., v_{1I}, v_{2I}) \equiv \Pr[V_{11} = v_{11}, V_{21} = v_{21}, ..., V_{1I} = v_{1I}, V_{2I} = v_{2I}],$$
$$f(v_{1j}, v_{2j}) \equiv \Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}].$$

The independence among pairs in the set $\{(V_{1i}, V_{2i})_{i=1,...,I}\}$, which I refer to as *pair independence*, is used in the proof of following Lemma 1.

Lemma 1. *Let the first auction winner be any bidder i, i.e., $W_1 = i$, and his winning bid be b_1, i.e., $B_1^{\max} = b_1$. Then I second-auction bids are independent conditional on the event $\{W_1 = i, B_1^{\max} = b_1\} = \{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$. In this case, the distribution of the first auction winner's B_{2i} given the event is $G^w_{2|B_1^{\max}}(\cdot|b_1) = G^w_{2|1}(\cdot|b_1)$, whereas for the first auction losers $j \neq i$, the distribution of B_{2j} given the event is $G^l_{2|B_1^{\max}}(\cdot|b_1) = G_{B_2^l(b_1)}(\cdot|B_1 \leq b_1)$.*

Proof: *Let $V_{1,-i}^{\max} \equiv \max\{V_{1j}, j \neq i\}$. Using independence and pair independence, the joint density of V_{2i} and $(V_{1j}, V_{2j})_{j\neq i}$ given the event $\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$ is equivalent to (23).*

$$\begin{aligned}
&\Pr[V_{2i} = v_{2i}, (V_{1j} = v_{1j}, V_{2j} = v_{2j}), j \neq i \mid V_{1i} = v_{1i}, V_{1j} \leq V_{1i}, j \neq i] \\
&= \frac{\Pr[(V_{1i} = v_{1i}, V_{2i} = v_{2i}), (V_{1j} = v_{1j}, V_{2j} = v_{2j}, V_{1j} \leq V_{1i}), j \neq i]}{\Pr[V_{1i} = v_{1i}, V_{1j} \leq V_{1i}, j \neq i]} \\
&= \frac{\Pr[V_{1i} = v_{1i}, V_{2i} = v_{2i}] \prod_{j\neq i} \Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}]}{f_{1i}(v_{1i}) \prod_{j\neq i} F_{1j}(v_{1i})} \\
&= \Pr[V_{2i} = v_{2i}|V_{1i} = v_{1i}] \prod_{j\neq i} \Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}|V_{1j} \leq v_{1i}] \\
&= \Pr[V_{2i} = v_{2i}|V_{1i} = v_{1i}, V_{1j} \leq V_{1i}, j \neq i] \times \\
&\quad \prod_{j\neq i} \Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}|V_{1i} = v_{1i}, V_{1j} \leq V_{1i}, j \neq i], \qquad (23)
\end{aligned}$$

in which the second equality holds if $v_{1j} \leq v_{1i}, j \neq i$ holds; the third equality holds by noting that $\Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}]/F_{1j}(v_{1i}) = \Pr[V_{1j} = v_{1j}, V_{2j} = v_{2j}|V_{1j} \leq v_{1i}]$ holds as long as $v_{1j} \leq v_{1i}, j \neq i$ holds; the last equality holds by independence. As a result, equation (23) shows that V_{2i} and $(V_{1j}, V_{2j})_{j\neq i}$ are independent given the condition $\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$. Next, we translate this conditional independence in terms of bids.

Because $B_{1j} = s_1(V_{1j})$ with the monotonicity of s_1, the conditioning event $\{V_{1,-i}^{\max} \leq V_{1i} = v_{1i}\}$ is the same as $\{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$ with $b_1 = s_1(v_{1i})$. Given $\{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$, we have $B_{2i} = s_2^w(v_{1i}, V_{2i})$ for i and $B_{2j} = s_2^l(V_{2j}, b_1) = s_2^l(V_{2j}, s_1(v_{1i}))$ for $j \neq i$. Since strategies s_2^w and s_2^l are continuous and thus measurable, the elements in the set $\{s_2^w(v_{1i}, V_{2i}), s_2^l(V_{2j}, s_1(v_{1i})), j \neq i\}$ are independent given the event $\{V_{1,-i}^{\max} \leq V_{1i} = v_1\}$. This is equivalent to elements in the set $\{B_{2i}, B_{2j}, j \neq i\}$ being independent given $\{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$, which proves that the I second-auction bids are independent conditional on $\{W_1 = i, B_1^{\max} = b_1\} = \{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$.

For i, the distribution of B_{2i} given $\{W_1 = i, B_1^{\max} = b_1\} = \{B_{1,-i}^{\max} \leq B_{1i} = b_1\}$ is

$$\begin{aligned}
\Pr[B_{2i} \leq \cdot \mid W_1 = i, B_1^{\max} = b_1] &= \Pr[s_2^w(V_{1i}, V_{2i}) \leq \cdot \mid s_1(V_{1j}) \leq b_1, \forall j \neq i, s_1(V_{1i}) = b_1] \\
&= \Pr[s_2^w(V_{1i}, V_{2i}) \leq \cdot \mid s_1(V_{1i}) = b_1], \qquad (24)
\end{aligned}$$

in which the second equality holds by independence. Since the left-hand side probability of (24) does not depend on the first-auction winner's identity i by symmetry, we have $\Pr[B_{2i} \le \cdot \mid W_1 = i, B_1^{\max} = b_1] = \Pr[B_2^w \le \cdot | B_1^{\max} = b_1] \equiv G^w_{2|B_1^{\max}}(\cdot|b_1)$*. And, the right-hand side probability of (24) also does not depend on winner's identity i by symmetry, so we have* $\Pr[s_2^w(V_{1i}, V_{2i}) \le \cdot \mid s_1(V_{1i}) = b_1] = \Pr[s_2^w(V_1, V_2) \le \cdot \mid s_1(V_1) = b_1] \equiv G^w_{2|1}(\cdot|b_1)$*.*

For $j \neq i$, the distribution of B_{2j} given $\{W_1 = i, B_1^{\max} = b_1\} = \{B_{1,-i}^{\max} \le B_{1i} = b_1\}$ is

$$\begin{aligned}
\Pr\left[B_{2j} \le \cdot \mid W_1 = i, B_1^{\max} = b_1\right] &= \Pr[s_2^l(V_{2j}, s_1(V_{1i})) \le \cdot | s_1(V_{1k}) \le s_1(V_{1i}), \forall k \neq i, s_1(V_{1i}) = b_1] \\
&= \Pr[s_2^l(V_{2j}, s_1(V_{1i})) \le \cdot | s_1(V_{1j}) \le s_1(V_{1i}), s_1(V_{1i}) = b_1] \\
&= \Pr[s_2^l(V_{2j}, b_1) \le \cdot \mid s_1(V_{1j}) \le b_1], \qquad (25)
\end{aligned}$$

in which the first equality holds by B_{2j} being $s_2^l(V_{2j}, B_1^{\max})$, and the rest of the equalities hold by independence. Since the left-hand side probability of (25) depends neither on the first auction winner's identity i nor on a first auction loser's identity j because of symmetry, we have $\Pr\left[B_{2j} \le \cdot \mid W_1 = i, B_1^{\max} = b_1\right] = \Pr[B_2^l \le \cdot | B_1^{\max} = b_1] \equiv G^l_{2|B_1^{\max}}(\cdot|b_1)$ *holds. And, the right-hand side probability of (25) also does not depend on bidder's identity by symmetry, so we have* $\Pr[s_2^l(V_{2j}, b_1) \le \cdot \mid s_1(V_{1j}) \le b_1] = \Pr[s_2^l(V_2, b_1) \le \cdot \mid s_1(V_1) \le b_1] \equiv G_{B_2^l(b_1)}(\cdot|B_1 \le b_1)$*. Q.E.D.*

Lemma 2 below, along with Lemma 1, is also needed in the derivation of equations (11)-(12); Lemma 2 closely relates to Remarks 7.3.1 and Theorem 7.3.1 in Rao (1992).

Lemma 2. *Let the tuple (Z, J) be the 'identified maximum tuple' where $Z = \max\{X_1, ..., X_k\}$ and $X_J = Z$. If $X_1, ..., X_k$ are mutually independent, then their distributions $F_1(\cdot), ..., F_k(\cdot)$ are identified by a tuple (Z, J) such that the following holds.*

$$F_j(x) = \exp\left\{-\int_x^{+\infty}\left[\sum_{i=1}^k H_i(t)\right]^{-1} dH_j(t)\right\} = \exp\left\{-\int_x^{+\infty}(\Pr[Z \le t])^{-1} dH_j(t)\right\}, \qquad (26)$$

in which $H_j(x) \equiv \Pr[Z \le x, J = j]$ for $j = 1, \ldots, k$.
Proof: *Since $H_j(x) = \Pr\left[X_j \text{ is the maximum among } X_1, \ldots, X_k, \text{ and } X_j \le x\right]$, we have*

$$H_j(x) = \int_{-\infty}^x \prod_{i \neq j} F_i(t) dF_j(t) = \int_{-\infty}^x \frac{\prod_{i=1}^k F_i(t)}{F_j(t)} dF_j(t) = \int_{-\infty}^x \prod_{i=1}^k F_i(t) d\log F_j(t). \qquad (27)$$

But, $\sum_{i=1}^k H_i(t) = \sum_{i=1}^k \Pr[Z \le t, J = i] = \Pr[Z \le t] = \prod_{i=1}^k F_i(t)$*. Thus, equation (27) becomes*

$$H_j(x) = \int_{-\infty}^x \sum_{i=1}^k H_i(t) d\log F_j(t). \qquad (28)$$

Differentiating equation (28) with respect to x gives the following equation.

$$d\log F_j(x) = \left[\sum_{i=1}^k H_i(x)\right]^{-1} dH_j(x) \qquad (29)$$

Integrating both sides of equation (29) from x to $+\infty$, while noting that $\log F_j(+\infty) = 0$, gives

$$-\log F_j(x) = \int_x^{+\infty} \left[\sum_{i=1}^{k} H_i(t)\right]^{-1} dH_j(t),$$

which implies equation (26) since $\sum_{i=1}^{k} H_i(t) = \Pr[Z \leq t]$. Q.E.D.

Given Lemmas 1 and 2, I prove in words that the distributions $G^w_{2|B_1^{max}}(\cdot|b_1)$ and $G^l_{2|B_1^{max}}(\cdot|b_1)$ are identified.

As an analyst, I observe $(B_1^{\max}, W_1, B_2^{\max}, W_2)$, where $B_t^{\max} \equiv \max\{B_{t1}, \ldots, B_{tI}\}$ and W_t are the winning bid and the random winner's index in the t-th auction, respectively, for $t = 1, 2$. From Lemma 1, we know that the I second auction bids $\{B_{2j}, j = 1\ldots, I\}$ are independent with distributions $G^w_{2|B_1^{max}}(\cdot|b_1)$ when $j = i$ and $G^l_{2|B_1^{max}}(\cdot|b_1)$ for $j \neq i$ conditional on the event $\{B_{1,-i}^{\max} \leq B_{1i} = b_{1i}\} = \{W_1 = i, B_1^{\max} = b_1\}$ where $b_1 = b_{1i}$. It follows from Lemma 2 with $F_i(\cdot) = G^w_{2|B_1^{max}}(\cdot|b_1)$ for i, $F_j(\cdot) = G^l_{2|B_1^{max}}(\cdot|b_1)$ for $j \neq i$, and $H_j(\cdot|b_1) = \Pr[B_2^{\max} \leq \cdot, W_2 = j | W_1 = i, B_1^{\max} = b_1]$ for $j \in \{1, ..., I\}$ that $G^w_{2|B_1^{max}}(\cdot|b_1)$ and $G^l_{2|B_1^{max}}(\cdot|b_1)$ are identified.

To derive equations (11)-(12), note the alternative expression of $H_j(\cdot|b_1)$ for $j \neq i$.

$$\begin{aligned}
H_j(\cdot|b_1) &= \frac{1}{I-1} \Pr[B_2^{\max} \leq \cdot, W_2 \neq i \mid W_1 = i, B_1^{\max} = b_1] \\
&= [1/(I-1)] \Pr[B_2^{\max} \leq \cdot \mid W_2 \neq i, W_1 = i, B_1^{\max} = b_1] \times \Pr[W_2 \neq i \mid W_1 = i, B_1^{\max} = b_1] \\
&= [1/(I-1)] \Pr[B_2^{\max} \leq \cdot \mid W_2 \neq W_1, W_1 = i, B_1^{\max} = b_1] \times \Pr[W_2 \neq W_1 \mid W_1 = i, B_1^{\max} = b_1] \\
&= [1/(I-1)] \Pr[B_2^{\max} \leq \cdot \mid W_2 \neq W_1, B_1^{\max} = b_1] \times \Pr[W_2 \neq W_1 \mid B_1^{\max} = b_1] \\
&= [1/(I-1)] \Pr[B_2^{\max} \leq \cdot, W_2 \neq W_1 \mid B_1^{\max} = b_1],
\end{aligned}$$

in which the first equality uses the symmetry of the first auction losers in the second auction, which holds by *symmetry*. Alternative expression for $H_i(\cdot|b_1)$ is,

$$\begin{aligned}
H_i(\cdot|b_1) &= \Pr[B_2^{\max} \leq \cdot, W_2 = i \mid W_1 = i, B_1^{\max} = b_1] \\
&= \Pr[B_2^{\max} \leq \cdot \mid W_2 = i, W_1 = i, B_1^{\max} = b_1] \times \Pr[W_2 = i \mid W_1 = i, B_1^{\max} = b_1] \\
&= \Pr[B_2^{\max} \leq \cdot \mid W_2 = W_1, W_1 = i, B_1^{\max} = b_1] \times \Pr[W_2 = W_1 \mid W_1 = i, B_1^{\max} = b_1] \\
&= \Pr[B_2^{\max} \leq \cdot \mid W_2 = W_1, B_1^{\max} = b_1] \times \Pr[W_2 = W_1 \mid B_1^{\max} = b_1] \\
&= \Pr[B_2^{\max} \leq \cdot, W_2 = W_1 \mid B_1^{\max} = b_1],
\end{aligned}$$

in which the fourth equality holds because the bidders are symmetric in the first auction. Hence, Lemma 2 and Lemma 1 give us,

$$\begin{aligned}
&G^w_{2|1}(b_2|b_1) = G^w_{2|B_1^{\max}}(b_2|b_1) \\
&\quad = \exp\left\{-\int_{b_2}^{+\infty} \left(\Pr[B_2^{\max} \leq b \mid B_1^{\max} = b_1]\right)^{-1} d\Pr[B_2^{\max} \leq b, W_2 = W_1 | B_1^{\max} = b_1]\right\}, \\
&G^l_{2|B_1^{max}}(b_2|b_1) = G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)
\end{aligned}$$

$$= \exp\left\{-\frac{1}{I-1}\int_{b_2}^{+\infty}\left(\Pr\left[B_2^{\max} \le b \mid B_1^{\max} = b_1\right]\right)^{-1} d\Pr[B_2^{\max} \le b, W_2 \ne W_1 | B_1^{\max} = b_1]\right\}.$$

B.2 Derivation of equation (13) from the dataset

[Back to ToC] Refer to the following equalities.

$$\begin{aligned}\Pr[B_1^{\max} \le b_1] = \Pr[s_1(V_{1j}) \le b_1, \forall j] &= \prod_{j\in\{1,\dots,I\}} \Pr[s_1(V_{1j}) \le b_1] \\ &= \Pr[s_1(V_1) \le b_1]^I = \Pr[B_1 \le b_1]^I \equiv G_1(b_1)^I,\end{aligned}$$

in which the second equality holds by *independence*; the third holds by *symmetry*.

B.3 Derivation of equation (14)

[Back to ToC] I progress in three steps.

✓ *Step 1*: Assume an arbitrary bidder i who had bid any bid b_{1i} below the winning bid of b_1 in the first auction. I want to first identify a distribution $\Pr[V_{2i} \le \cdot | B_{1i} < B_1^{\max} = b_1]$. From first-order condition (6), we know the following holds.

$$V_{2i} = B_{2i}^l + \frac{G_{2|1}^w(B_{2i}^l|b_1)G_{B_2^l(b_1)}(B_{2i}^l|B_1 \le b_1)^{I-2}}{\partial(G_{2|1}^w(B_{2i}^l|b_1)G_{B_2^l(b_1)}(B_{2i}^l|B_1 \le b_1)^{I-2})/\partial b_{2i}^l} \equiv \xi_2^l(B_{2i}^l, b_1),$$

in which $B_{2i}^l = s_2^l(V_{2i}, b_1)$ and $B_{1i} = s_1(V_{1i})$ for any $B_{1i} < b_1$. However, since I only have access to $(B_1^{\max}, W_1, B_2^{\max}, W_2)$, I can't observe B_{2i}^l if i never wins the second auction.

To circumvent this situation, I rely on the identified bid distribution $G_{2|B_1^{\max}}^l(\cdot|b_1) = G_{B_2^l(b_1)}(\cdot|B_1 \le b_1)$ to identify a distribution $\Pr[V_2 \le \cdot|B_1 < B_1^{\max} = b_1]$. Since the bid distribution $G_{B_2^l(b_1)}(\cdot|B_1 \le b_1)$ is the distribution of B_2^l given that this loser saw the winning bid of b_1, I can rewrite $\Pr[V_2 \le \cdot|B_1 < B_1^{\max} = b_1]$ as follows.

$$\begin{aligned}\Pr[V_2 \le \cdot|B_1 < B_1^{\max} = b_1] &= \Pr[\xi_2^l(B_2^l, b_1) \le \cdot|B_1 < B_1^{\max} = b_1] \\ &= \mathbb{E}[\mathbb{1}[\xi_2^l(B_2^l, b_1) \le \cdot]|B_1 < B_1^{\max} = b_1] \\ &= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}[\xi_2^l(x, b_1) \le \cdot]dG_{B_2^l(b)}(x|B_1 \le b_1), \qquad (30)\end{aligned}$$

✓ *Step 2*: Note that the identified distribution $\Pr[V_2 \le \cdot|B_1 \le B_1^{max} = b_1]$ is equivalent to another distribution $\Pr[V_2 \le \cdot|B_1 \le b_1]$, as I show below. Choose an arbitrary bidder i and assume $W_1 = j$, then a distribution $\Pr[V_{2i} \le \cdot|B_{1i} \le B_1^{\max} = b_1]$ is as follows.

$$\begin{aligned}&\Pr[V_{2i} \le \cdot|B_{1i} \le B_1^{\max} = b_1] \text{ where } B_{1i} = s_1(V_{1i}) \\ &= \frac{\Pr[B_{11} < b_1, \dots, (V_{2i} \le \cdot,\ B_{1i} < b_1), B_{1j} = b_1, \dots, B_{1I} < b_1]}{\Pr[B_{11} < b_1, \dots, B_{1i} < b_1, B_{1j} = b_1, \dots, B_{1I} < b_1]} \\ &= \frac{\Pr[B_{11} < b_1]\dots\Pr[V_{2i} \le \cdot, B_{1i} < b_1]\ \Pr[B_{1j} = b_1]\dots\Pr[B_{1I} < b_1]}{\Pr[B_{11} < b_1]\dots\Pr[B_{1i} < b_1]\ \Pr[B_{1j} = b_1]\dots\Pr[B_{1I} < b_1]}\end{aligned}$$

$$= \frac{\Pr[V_{2i} \leq \cdot, B_{1i} < b_1]}{\Pr[B_{1i} < b_1]}$$
$$= \Pr[V_{2i} \leq \cdot | B_{1i} < b_1], \tag{31}$$

in which the second equality holds by *independence*. Also by *symmetry*, the result above is equivalent to $\Pr[V_2 \leq \cdot | B_1 \leq B_1^{\max} = b_1] = \Pr[V_2 \leq \cdot | B_1 < b_1]$.

✓ *Step 3*: Note that the following equation holds for $\Pr\left[V_2 \leq \cdot | B_1 < b_1\right]$.

$$\Pr[V_2 \leq \cdot \mid B_1 < b_1] = \frac{1}{G_1(b_1)} \int_{\underline{b_1}}^{b_1} \Pr[V_2 \leq \cdot,\ B_1 = x] dx.$$

If we take the derivative of both sides with respect to b_1, we have the following.

$$\frac{\partial}{\partial b_1} \Pr[V_2 \leq \cdot \mid B_1 < b_1]$$
$$= -\frac{g_1(b_1)}{G_1(b_1)^2} \int_{\underline{b_1}}^{b_1} \Pr[V_2 \leq \cdot,\ B_1 = x] dx + \frac{1}{G_1(b_1)} \Pr[V_2 \leq \cdot, B_1 = b_1]$$
$$= -\frac{g_1(b_1)}{G_1(b_1)} \Pr[V_2 \leq \cdot \mid B_1 < b_1] + \frac{g_1(b_1)}{G_1(b_1)} \Pr[V_2 \leq \cdot \mid B_1 = b_1]$$

Rearranging the equation above yields the following equation,

$$\Pr[V_2 \leq \cdot \mid B_1 = b_1] = \Pr[V_2 \leq \cdot \mid B_1 < b_1] + \frac{G_1(b_1)}{g_1(b_1)} \left(\frac{\partial}{\partial b_1} \Pr[V_2 \leq \cdot \mid B_1 < b_1] \right).$$

We already identified a distribution $\Pr[V_2 \leq \cdot \mid B_1 < b_1]$ in *Steps 1* and *2*.

B.4 Derivation of equation (15)

[Back to ToC] I start with a distribution $\Pr[\tilde{\delta}\left(B_1, V_2\right) \leq \cdot | B_1 = b_1]$ where $\tilde{\delta}\left(B_1, V_2\right) \equiv \delta(s_1^{-1}\left(B_1\right), V_2) = \delta\left(V_1, V_2\right)$. Pick an arbitrary bidder i and assume $W_1 = i$. Then for any $B_{2i}^w \in [\underline{b_2}, \overline{b_2}]$, the following must hold by first order condition (5):

$$\delta(v_{1i}, V_{2i}) = \tilde{\delta}(b_{1i}, V_{2i}) = B_{2i}^w + \frac{G_{B_2^l(b_{1i})}(B_{2i}^w | B_1 \leq b_{1i})^{I-1}}{\partial G_{B_2^l(b_{1i})}(B_{2i}^w | B_1 \leq b_{1i})^{I-1} / \partial b_{2i}^w} \equiv \xi_2^w(B_{2i}^w, b_{1i}),$$

in which $B_{2i}^w = s_2^w\left(v_{1i}, V_{2i}\right)$ and $b_{1i} = s_1\left(v_{1i}\right)$. However, since I only have access to $(B_1^{\max}, W_1, B_2^{\max}, W_2)$, I observe B_{2i}^w only if bidder i had won both the first and the second auction, i.e., $W_1 = W_2 = i$.

To circumvent this situation, I rely on the identified bid distribution $G^w_{2|B_1^{\max}}(\cdot | b_1) = G^w_{2|1}\left(\cdot | b_1\right)$ to identify a distribution $\Pr[\tilde{\delta}\left(B_1, V_2\right) \leq \cdot | B_1 = b_1]$. Since the bid distribution $G^w_{2|1}\left(\cdot | b_1\right)$ is the distribution of B_2^w given $B_1 = b_1$, I can rewrite $\Pr[\tilde{\delta}\left(B_1, V_2\right) \leq \cdot | B_1 = b_1]$ as follows.

$$\Pr[\tilde{\delta}\left(B_1, V_2\right) \leq \cdot | B_1 = b_1] = \Pr[\xi_2^w(B_2^w, B_1) \leq \cdot | B_1 = b_1]$$
$$= \mathbb{E}[\mathbb{1}[\xi_2^w(B_2^w, B_1) \leq \cdot] | B_1 = b_1]$$
$$= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}[\xi_2^w(x, b_1) \leq \cdot] dG^w_{2|1}\left(x | b_1\right), \tag{32}$$

in which the first equality holds by the first-order condition. Note that equation (32) is equivalent to $\Pr[\tilde{\delta}\left(B_1^{\max}, V_2\right) \le \cdot | B_1^{\max} = b_1]$ because B_1 in (32) is the winning bid and both distributions $G_{2|1}^w$ and $G_{2|B_1^{\max}}^w$ are the same.

Since $\tilde{\delta}\left(B_1, V_2\right) \equiv \delta(s_1^{-1}\left(B_1\right), V_2) = \delta\left(V_1, V_2\right)$ holds and $s_1 = \xi_1^{-1}$ is a monotone strategy, we have the following result,

$$\Pr[\tilde{\delta}\left(B_1, V_2\right) \le \cdot | B_1 = b_1] = \Pr[\delta\left(V_1, V_2\right) \le \cdot | V_1 = v_1].$$

B.5 Alternative Identification Strategy for the function δ

[Back to ToC] For the ease of exposition in B.5, I introduce a new notation below, namely $\tilde{F}_{2|1}$; I omit the dependence of the auction covariate Z and the number of bidders I in B.5 whenever possible.

$$F_{2|1}(v_2|v_1) \equiv \Pr[V_2 \le v_2 | V_1 = v_1] = \Pr[V_2 \le v_2 | B_1 = b_1] \equiv \tilde{F}_{2|1}(v_2|b_1),$$

where the equality holds by the reason described in equation (14). Analogously, I introduce two new notations, $D_{2|1}$ and $\tilde{D}_{2|1}$, as follows,

$$D_{2|1}(d|v_1) \equiv \Pr[\delta(v_1, V_2) \le d | V_1 = v_1] = \Pr[\tilde{\delta}(b_1, V_2) \le d | B_1 = b_1] \equiv \tilde{D}_{2|1}(d|b_1),$$

where the equality holds from the result in B.4.

The original approach, namely "α-quantile of (15) $= \delta(v_1,\ \alpha$-quantile of (14))", asserts that I can identify a function $\delta(v_1, \cdot)$ by comparing the quantiles of two distributions $D_{2|1}(\cdot|v_1)$ and $F_{2|1}(\cdot|v_1)$. In detail, the comparison of the quantiles follows the steps below, which I refer to as Original Identification Strategy.

1. Make a grid of α, say $[0.01, ..., 0.99]$.
2. For each grid point of α, we can calculate the quantiles of $\tilde{F}_{2|1}(\cdot|b_1)$ and $\tilde{D}_{2|1}(\cdot|b_1)$. Denote those quantiles as $\tilde{v}_{2|1}(\alpha|b_1)$ and $\tilde{d}_{2|1}(\alpha|b_1)$.
3. Compare $\tilde{v}_{2|1}(\alpha|b_1)$ and $\tilde{d}_{2|1}(\alpha|b_1)$ for every grid point in α, then we can identify a function $\tilde{\delta}(b_1, \cdot)$.

As a result, both the domain and the range of the function $\tilde{\delta}(b_1, \cdot) = \delta(v_1, \cdot)$ are $[\tilde{v}_{2|1}(0|b_1),\ \tilde{v}_{2|1}(1|b_1)]$ and $[\tilde{d}_{2|1}(0|b_1),\ \tilde{d}_{2|1}(1|b_1)]$. But, Original Identification Strategy causes huge computational burden in Monte Carlo simulation: calculating the quantile of a distribution is taxing, and it becomes more taxing when the distribution is complicated as is our case in (14) and (15).

To circumvent the computational burden in Monte Carlo simulation, I describe Alternative Identification Strategy to identify a function δ as follows:

1. Pick arbitrary value for the first auction max bid $B_1^{\max}$, say $B_1^{\max} = 0.3$. Then make a grid of b_2 such that it has fifty points, $\left[b^1_{2,\{B_1^{\max}=0.3\}}, ..., b^{50}_{2,\{B_1^{\max}=0.3\}}\right]$.

2. Do $\xi_2^l(b_2, 0.3)$ for every b_2 in the grid. Then by the first-order condition (6), this outputs the grid of v_2 given $B_1^{\max} = 0.3$, which means we will have the following grid:

$$\left[\xi_2^l(b^1_{2,\{B_1^{\max}=0.3\}}, 0.3), ..., \xi_2^l(b^{50}_{2,\{B_1^{\max}=0.3\}}, 0.3)\right] = \left[v^1_{2,\{B_1^{\max}=0.3\}}, ..., v^{50}_{2,\{B_1^{\max}=0.3\}}\right].$$

Since a function $\xi_2^l(\cdot, 0.3)$ is strictly increasing in $\cdot$, we know that $v^1_{2,\{B_1^{\max}=0.3\}}$ and $v^{50}_{2,\{B_1^{\max}=0.3\}}$ are the smallest and the largest v_2 given $B_1^{\max} = 0.3$.

3. Similarly, define $v^1_{2,\{B_1=0.3\}}$ and $v^{50}_{2,\{B_1=0.3\}}$ as the smallest and the largest v_2 given the first auction bid $B_1 = 0.3$. I claim that these two values must be the same as $v^1_{2,\{B_1^{\max}=0.3\}}$ and $v^{50}_{2,\{B_1^{\max}=0.3\}}$ by the following reason:

   "*Step 1* from B.3 implies that the interval $[\xi_2^l(b^1_{2,\{B_1^{\max}=0.3\}}, 0.3), \xi_2^l(b^{50}_{2,\{B_1^{\max}=0.3\}}, 0.3)]$ is the support of the distribution $\Pr[V_2 \leq \cdot | B_1 < B_1^{\max} = 0.3]$. And, (31) from B.3 implies that $\Pr[V_2 \leq \cdot | B_1 \leq B_1^{\max} = 0.3]$ and $\Pr[V_2 \leq \cdot | B_1 \leq 0.3]$ are the same. And, equation (14) implies that the support of $\Pr[V_2 \leq \cdot | B_1 \leq 0.3]$ and the support of $\Pr[V_2 \leq \cdot | B_1 = 0.3] \equiv \tilde{F}_{2|1}(\cdot|0.3)$ must be the same by noting the right-hand side of (14)."

4. Because the claim asserts $v^1_{2,\{B_1=0.3\}} = v^1_{2,\{B_1^{\max}=0.3\}}$ and $v^{50}_{2,\{B_1=0.3\}} = v^{50}_{2,\{B_1^{\max}=0.3\}}$, I can get the set of probabilities, $\{0, \alpha_1, ..., \alpha_{49}, 1\}$, by using the right-hand side of (14) as follows.

$$\begin{aligned}
&\Pr\left[V_2 \leq v^1_{2,\{B_1^{max}=0.3\}} = v^1_{2,\{B_1=0.3\}} \mid B_1 = 0.3\right] = 0,\\
&\Pr\left[V_2 \leq v^2_{2,\{B_1^{max}=0.3\}} \mid B_1 = 0.3\right] \qquad\qquad\quad = \alpha_2,\\
&...\\
&...\\
&\Pr\left[V_2 \leq v^{49}_{2,\{B_1^{max}=0.3\}} \mid B_1 = 0.3\right] \qquad\qquad\quad = \alpha_{49},\\
&\Pr\left[V_2 \leq v^{50}_{2,\{B_1^{max}=0.3\}} = v^{50}_{2,\{B_1=0.3\}} \mid B_1 = 0.3\right] = 1.0.
\end{aligned}$$

   Given the set $\{0, \alpha_2, ..., \alpha_{49}, 1.0\}$, we can get the α-quantiles of $\tilde{D}_{2|1}(\cdot|b_1)$, namely $\tilde{d}_{2|1}(\alpha|b_1)$. But, computing the quantiles directly from the distribution $\tilde{D}_{2|1}$ is taxing because the distribution itself is already complicated as can be seen from (15). To circumvent this computational burden, I claim that we can get $\tilde{d}_{2|1}(\alpha|b_1)$ from the α-quantile of the bid distribution $G^w_{2|1}$ by the following reason:

   "B.4 asserts that $\tilde{D}_{2|1}(\cdot|b_1)$ and $\Pr[\xi_2^w(B_2^w, B_1) \leq \cdot | B_1 = b_1]$ are the same. The random variable of the distribution $\Pr[\xi_2^w(B_2^w, B_1) \leq \cdot | B_1 = b_1]$ is B_2^w transformed by a monotone function $\xi_2^w(\cdot, B_1 = b_1)$. Note that random variable B_2^w given $B_1 = b_1$ comes from the distribution $G^w_{2|1}(\cdot|b_1)$. Thus, the following must hold by the quantile's invariance to monotone transformation:

$$\alpha\text{-quantile of } \tilde{D}_{2|1}(\cdot|b_1) = \xi_2^w(\alpha\text{-quantile of } G^w_{2|1}(\cdot|b_1),\, b_1),$$

which I can alternatively express as follows

$$\tilde{d}_{2|1}(\alpha|b_1) = \xi_2^w(b_{2|1}^w(\alpha|b_1),\, b_1), \tag{33}$$

where $b_{2|1}^w(\alpha|b_1)$ stands for the α-quantile of the bid distribution $G_{2|1}^w(\cdot|b_1)$."

5. As a result, we have fifty sets as follows.

$$\{\ \underbrace{v^1_{2,\{B_1^{max}=0.3\}}}_{=v^1_{2,\{B_1=0.3\}}},\ \alpha=0,\ \tilde{d}_{2|1}(0|B_1=0.3)\}$$
$$\{\ v^2_{2,\{B_1^{max}=0.3\}},\ \alpha=\alpha_2,\ \tilde{d}_{2|1}(\alpha_2|B_1=0.3)\}$$
$$...$$
$$...$$
$$\{\ v^{49}_{2,\{B_1^{max}=0.3\}},\ \alpha=\alpha_{49},\ \tilde{d}_{2|1}(\alpha_{49}|B_1=0.3)\}$$
$$\{\ \underbrace{v^{50}_{2,\{B_1^{max}=0.3\}}}_{=v^{50}_{2,\{B_1=0.3\}}},\ \alpha=1,\ \tilde{d}_{2|1}(1|B_1=0.3)\}$$

Compare the first and the third elements of each set, which identifies a function $\tilde{\delta}(0.3,\cdot) = \delta(s_1^{-1}(0.3),\cdot)$.

The difference between ORIGINAL and ALTERNATIVE lies where they start: the original starts from the grid $\alpha \in [0.01, ..., 0.99]$ while the alternative stats from the grid $[\underline{b_2}, \overline{b_2}]$. Because the alternative starts from the grid of bids, it avoids the direct calculation of the quantiles $\tilde{v}_2(\alpha|b_1)$ from the distribution $\tilde{F}_{2|1}(\cdot|b_1)$. This avoidance is what makes the alternative much faster than the original in Monte Carlo simulation.

C Section 4, Estimation and Monte Carlo

C.1 Bandwidth and the derivations of (17)-(19)

[Back to ToC] Note the following equality where a random variable Z stands for the auction covariate.

$$\begin{aligned}&\Pr\left[B_2^{\max}=b_2, D=1 \mid B_1^{\max}=b_1, z, I\right]\\ &= \frac{\Pr\left[B_2^{\max}=b_2, B_1^{\max}=b_1, z \mid D=1, I\right]}{\Pr\left[B_1^{\max}=b_1, z \mid D=1, I\right]}\Pr[D=1 \mid B_1^{max}=b_1, z, I]\end{aligned} \tag{34}$$

where D is $\mathbb{1}[W_2 = W_1]$ so that d can be either 1 or 0; $d = 1$ outputs $m_2^w(b_2|b_1, z, I)$ while $d = 0$ outputs $m_2^l(b_2|b_1, z, I)$. Two new notations m_2^w and m_2^l are the densities of the following probabilities M_2^w and M_2^l:

$$\begin{aligned} M_2^w(b_2|b_1, z, I) &\equiv \Pr[B_2^{\max} \le b_2, W_2 = W_1 | B_1^{\max} = b_1, z, I],\\ M_2^l(b_2|b_1, z, I) &\equiv \Pr[B_2^{\max} \le b_2, W_2 \neq W_1 | B_1^{\max} = b_1, z, I],\end{aligned}$$

where these M_2^w and M_2^l are analogous to the estimands of (18) and (19).

Regarding the probabilities inside the right-hand side of (34), we can come up with the following estimators; throughout this document, I will assume a random variable Z to be a unit dimension.

$$\begin{aligned}
&\widehat{\Pr}\left[B_2^{\max} = b_2, B_1^{\max} = b_1, z \mid D = 1, I\right] \\
&\quad = \frac{1}{L_I^{1=2} h_{2,3}^{1=2} h_{1,3}^{1=2} h_{Z,3}^{1=2}} \sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_2 - B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right) K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,3}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,3}^{1=2}}\right) && (35) \\
&\widehat{\Pr}\left[B_1^{\max} = b_1, z \mid D = 1, I\right] \\
&\quad = \frac{1}{L_I^{1=2} h_{1,2}^{1=2} h_{Z,2}^{1=2}} \sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}^{1=2}}\right) && (36) \\
&\widehat{\Pr}\left[D = 1 \mid B_1^{\max} = b_1, z, I\right] \\
&\quad = \frac{\sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)} && (37)
\end{aligned}$$

where (37) holds by employing kernel regression on $\mathbb{E}[\mathbb{1}[D = d] | B_1^{max} = b_1, z, I] = \Pr[D = d | B_1^{max} = b_1, z, I]$. Some description about the notations used in (35)-(37) follows.

✓ Let $L_I^{1=2}$ and $L_I^{1\neq 2}$ be the numbers of auction pairs with I bidders such that $W_{1\ell} = W_{2\ell}$ and $W_{1\ell} \neq W_{2\ell}$. Thus, $L_I^{1=2} + L_I^{1\neq 2} = L_I$ where L_I is the number of auction pairs with I bidders.

✓ Let $\mathcal{L}_I \equiv \{\ell : I_\ell = I\}$ be the index set corresponding to auction pairs with I bidders such that we have $\mathcal{L}_I^{1=2} \cup \mathcal{L}_I^{1\neq 2} = \mathcal{L}_I$. Here $\mathcal{L}_I^{1=2}$(resp., $\mathcal{L}_I^{1\neq 2}$) denotes the subset of $\mathcal{L}_I$ that satisfies $W_{1\ell} = W_{2\ell}$(resp., $W_{1\ell} \neq W_{2\ell}$). That is, $\mathcal{L}_I^{1=2} \equiv \{\ell \in \mathcal{L}_I : W_{1\ell} = W_{2\ell}\}$ and $\mathcal{L}_I^{1\neq 2} \equiv \{\ell \in \mathcal{L}_I : W_{1\ell} \neq W_{2\ell}\}$.

✓ There are three types of bandwidths, $(h_{var,dim}, h_{var,dim}^{1=2}, h_{var,dim}^{1\neq 2})$. $h_{var,dim}$ uses the entire auction pairs $\mathcal{L}_I$ while $h_{var,dim}^{1=2}$ and $h_{var,dim}^{1\neq 2}$ use $\mathcal{L}_I^{1=2}$ and $\mathcal{L}_I^{1\neq 2}$. The subscripts (var, dim) are as follows:

  – var is one of $\{1, 2, Z\}$ where each element represents B_1^{max}, B_2^{max}, and Z.
  – dim stands for the dimension of the probability density. In (35)-(37), dim is either 2 or 3.

For example, assume that I choose Silverman's rule of thumb(Silverman (1986)) for the bandwidths — then, we can come up with the following bandwidths.

$$h_{var,dim}^{event} = \left(\frac{4}{dim + 2}\right)^{1/(dim+4)} (L_I^{event})^{-1/(dim+4)} \hat{\sigma}_{\widetilde{var},I}^{event} \tag{38}$$

where if I were to use a Triweight kernel, I would multiply 2.978: this number comes from Table 6.3 of Scott (2015). Here, the subscript $(var, \widetilde{var})$ can be one of $(1, B_1^{max})$, $(2, B_2^{max})$, and (Z, Z). The superscript $event$ can be one of $1 = 2$ and $1 \neq 2$. Thus, one example where

$(var,\widetilde{var})$=$(2, B_2^{max})$ and $event = 1 \neq 2$, and $dim = 3$ will output the following bandwidth.

$$h_{2,3}^{1\neq 2} = \left(\frac{4}{3+2}\right)^{1/(3+4)} (L_I^{1\neq 2})^{-1/(3+4)} \hat{\sigma}_{B_2^{max},I}^{1\neq 2}$$

where $\hat{\sigma}_{B_2^{max},I}^{1\neq 2}$ is calculated from $\{B_{2\ell}^{max} : \ell \in \mathcal{L}_I^{1\neq 2}\}$. Going back to (38), there are bandwidths that do not have the superscript *event* such as $h_{1,2}$ inside (37). In this case, the right-hand side of (38) uses L_I and $\hat{\sigma}_{\widetilde{var},I}$ instead of L_I^{event} and $\hat{\sigma}_{\widetilde{var},I}^{event}$.

Now, we go back to making the estimator of (34). Given (35)-(37), the estimator $\widehat{\Pr}[B_2^{\max} = b_2, D = 1 \mid B_1^{\max} = b_1, z, I]$ is as follows.

$$\begin{aligned}
\widehat{\Pr}\left[B_2^{\max} = b_2, D = 1 \mid B_1^{\max} = b_1, z, I\right] &= \frac{(35)}{(36)}(37) \\
&= \frac{h_{1,2}^{1=2} h_{Z,2}^{1=2}}{h_{2,3}^{1=2} h_{1,3}^{1=2} h_{Z,3}^{1=2}} \frac{\sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_2 - B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right) K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,3}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,3}^{1=2}}\right)}{\sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}^{1=2}}\right)} \\
&\quad \times \frac{\sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}.
\end{aligned}$$

Since it is cumbersome, Let's define a new notation $\lambda_\ell^{1=2}(b_1)$ as follows.

$$\lambda_\ell^{1=2}(b_1) \equiv \frac{\frac{1}{h_{1,3}^{1=2}} \frac{1}{h_{Z,3}^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,3}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,3}^{1=2}}\right)}{\frac{1}{h_{1,2}^{1=2}} \frac{1}{h_{Z,2}^{1=2}} \sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}^{1=2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}^{1=2}}\right)} \frac{\sum_{\ell \in \mathcal{L}_I^{1=2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}. \tag{39}$$

For a future use, I also define a new notation $\lambda_\ell^{1\neq 2}(b_1)$ as follows.

$$\lambda_\ell^{1\neq 2}(b_1) \equiv \frac{\frac{1}{h_{1,3}^{1\neq 2}} \frac{1}{h_{Z,3}^{1\neq 2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,3}^{1\neq 2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,3}^{1\neq 2}}\right)}{\frac{1}{h_{1,2}^{1\neq 2}} \frac{1}{h_{Z,2}^{1\neq 2}} \sum_{\ell \in \mathcal{L}_I^{1\neq 2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}^{1\neq 2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}^{1\neq 2}}\right)} \frac{\sum_{\ell \in \mathcal{L}_I^{1\neq 2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}. \tag{40}$$

If I use (39), we have a simpler form of the estimator as follows.

$$\begin{aligned}
\widehat{\Pr}\left[B_2^{\max} = b_2, D = 1 \mid B_1^{\max} = b_1, z, I\right] &= \frac{(35)}{(36)}(37) \\
&= \sum_{\ell \in \mathcal{L}_I^{1=2}} \lambda_\ell^{1=2}(b_1) \frac{1}{h_{2,3}^{1=2}} K\left(\frac{b_2 - B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right).
\end{aligned} \tag{41}$$

Up to now, I've gone through a process to make the estimator of (34), namely (41). Employing the same process, I can come up with the following estimators.

$$\hat{m}_2^w(b_2|b_1,z,I) \equiv \widehat{\Pr}[B_2^{\max}=b_2, W_2=W_1|B_1^{\max}=b_1,z,I]$$

$$= \sum_{\ell\in\mathcal{L}_I^{1=2}} \lambda_\ell^{1=2}(b_1)\frac{1}{h_{2,3}^{1=2}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right), \tag{42}$$

$$\hat{M}_2^w(b_2|b_1,z,I) \equiv \widehat{\Pr}[B_2^{\max}\le b_2, W_2=W_1|B_1^{\max}=b_1,z,I]$$

$$= \sum_{\ell\in\mathcal{L}_I^{1=2}} \lambda_\ell^{1=2}(b_1)\underbrace{\int_{-\infty}^{b_2}\frac{1}{h_{2,3}^{1=2}}K\left(\frac{x-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right)dx}_{\overline{K}_{2\ell}^{1=2}(b_2)}, \tag{43}$$

$$\hat{m}_2^l(b_2|b_1,z,I) \equiv \widehat{\Pr}[B_2^{\max}=b_2, W_2\neq W_1|B_1^{\max}=b_1,z,I]$$

$$= \sum_{\ell\in\mathcal{L}_I^{1\neq 2}} \lambda_\ell^{1\neq 2}(b_1)\frac{1}{h_{2,3}^{1\neq 2}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1\neq 2}}\right), \tag{44}$$

$$\hat{M}_2^l(b_2|b_1,z,I) \equiv \widehat{\Pr}[B_2^{\max}\le b_2, W_2\neq W_1|B_1^{\max}=b_1,z,I]$$

$$= \sum_{\ell\in\mathcal{L}_I^{1\neq 2}} \lambda_\ell^{1\neq 2}(b_1)\underbrace{\int_{-\infty}^{b_2}\frac{1}{h_{2,3}^{1\neq 2}}K\left(\frac{x-B_{2\ell}^{max}}{h_{2,3}^{1\neq 2}}\right)dx}_{\overline{K}_{2\ell}^{1\neq 2}(b_2)}, \tag{45}$$

where (42)-(45) constitute other estimators in the following appendices C.2-C.10. Equations (43) and (45) are comparable to the estimators (18) and (19).

The following (46)-(48) are also used in the following subsections — before I describe about (46)-(48), I mention the following equality,

$$\Pr[B_2^{max}=b_2|B_1^{max}=b_1,z,I] = \frac{\Pr[B_2^{max}=b_2, B_1^{max}=b_1,z,I]}{\Pr[B_1^{max}=b_1,z,I]}$$
$$= \frac{\Pr[B_2^{max}=b_2, B_1^{max}=b_1,z|I]}{\Pr[B_1^{max}=b_1,z|I]}.$$

(46) employs the equality above as shown below.

$$\hat{g}_{B_2^{\max}|B_1^{\max}}(b_2\mid b_1,z,I) \equiv \widehat{\Pr}[B_2^{\max}=b_2\mid B_1^{\max}=b_1,z,I]$$

$$= \frac{\frac{1}{L_I h_{2,3}h_{1,3}h_{Z,3}}\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}}\right)K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,3}}\right)K\left(\frac{z-Z_\ell}{h_{Z,3}}\right)}{\frac{1}{L_I h_{1,2}h_{Z,2}}\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}$$

$$= \sum_{\ell\in\mathcal{L}_I}\lambda_\ell(b_1)\frac{1}{h_{2,3}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}}\right), \tag{46}$$

where $\lambda_\ell(b_1)$ inside (46) denotes the following.

$$\lambda_\ell(b_1) = \frac{\frac{1}{h_{1,3}}\frac{1}{h_{Z,3}}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,3}}\right)K\left(\frac{z-Z_\ell}{h_{Z,3}}\right)}{\frac{1}{h_{1,2}}\frac{1}{h_{Z,2}}\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}. \tag{47}$$

Note that if I erase all the superscripts '1 = 2' in (39), we get (47). Given (46) we can define

(48) as follows.

$$\begin{aligned}\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I) &\equiv \widehat{\Pr}[B_2^{\max} \le b_2 | B_1^{\max} = b_1, z, I] \\ &= \sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \underbrace{\int_{-\infty}^{b_2} \frac{1}{h_{2,3}} K\left(\frac{x - B_{2\ell}^{max}}{h_{2,3}}\right) dx}_{\overline{K}_{2\ell}(b_2)},\end{aligned} \tag{48}$$

which compares to the estimator (17).

C.2 Derivations of (16), $\hat{G}_1(\cdot|z, I)$, and $\hat{g}_1(\cdot|z, I)$

[Back to ToC] Note the following equality.

$$\Pr[B_1^{max} = b_1 | z, I] = \frac{\Pr[B_1^{max} = b_1, Z = z, I]}{\Pr[Z = z, I]} = \frac{\Pr[B_1^{max} = b_1, Z = z | I]}{\Pr[Z = z | I]}.$$

Given the quality, I can come up with the following estimators.

$$\begin{aligned}\hat{g}_1^{max}(b_1|z, I) &\equiv \widehat{\Pr}[B_1^{\max} = b_1 | z, I] = \frac{\frac{1}{h_{1,2} h_{Z,2}} \sum_{\ell \in \mathcal{L}_I} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right) K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\frac{1}{h_{Z,1}} \sum_{\ell \in \mathcal{L}_I} K\left(\frac{z - Z_\ell}{h_{Z,1}}\right)} \\ &= \sum_{\ell \in \mathcal{L}_I} \omega_\ell \frac{1}{h_{1,2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right),\end{aligned} \tag{49}$$

$$\begin{aligned}\hat{G}_1^{max}(b_1 \mid z, I) &\equiv \widehat{\Pr}[B_1^{\max} \le b_1 | z, I] \\ &= \sum_{\ell \in \mathcal{L}_I} \omega_\ell \underbrace{\int_{-\infty}^{b_1} \frac{1}{h_{1,2}} K\left(\frac{x - B_{1\ell}^{max}}{h_{1,2}}\right) dx}_{\overline{K}_{1\ell}(b_1)},\end{aligned} \tag{50}$$

$$\hat{G}_1(b_1 \mid z, I) = \left(\hat{G}_1^{max}(b_1 \mid z, I)\right)^{1/I} = \left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(b_1)\right)^{1/I}, \tag{51}$$

$$\hat{g}_1(b_1 \mid z, I) = \frac{1}{I}\left(\sum_{\ell \in \mathcal{L}_I} \omega_\ell \overline{K}_{1\ell}(b_1)\right)^{(1-I)/I} \sum_{\ell \in \mathcal{L}_I} \omega_\ell \frac{1}{h_{1,2}} K\left(\frac{b_1 - B_{1\ell}^{max}}{h_{1,2}}\right), \tag{52}$$

where ω_ℓ that first appeared in (49) is defined as follows.

$$\omega_\ell \equiv \frac{\frac{1}{h_{Z,2}} K\left(\frac{z - Z_\ell}{h_{Z,2}}\right)}{\frac{1}{h_{Z,1}} \sum_{\ell \in \mathcal{L}_I} K\left(\frac{z - Z_\ell}{h_{Z,1}}\right)}$$

C.3 Derivations of $\hat{G}_{2|1}^w(\cdot|b_1, z, I)$ and $\hat{G}_{B_2^l(b_1)}(\cdot|B_1 \le b_1, z, I)$

[Back to ToC] Using equation (11), the plug-in estimator of $G_{2|1}^w(b_2|b_1, z, I)$ is as follows.

$$\hat{G}_{2|1}^w(b_2 \mid b_1, z, I) = \exp\left\{-\int_{-\infty}^{+\infty} \frac{\mathbb{1}[b_2 \le b]}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1, z, I)} \hat{m}_2^w(b \mid b_1, z, I)\, db\right\}$$

$$= \exp\left\{-\int_{b_2}^{+\infty}\frac{\sum_{\ell\in\mathcal{L}_I^{1=2}}\lambda_\ell^{1=2}(b_1)\frac{1}{h_{2,3}^{1=2}}K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}db\right\}$$

$$= \prod_{\ell\in\mathcal{L}_I^{1=2}}\exp\left\{-\frac{\lambda_\ell^{1=2}(b_1)}{h_{2,3}^{1=2}}\int_{b_2}^{+\infty}\frac{K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}db\right\}. \tag{53}$$

Using equation (12), the plug-in estimator of $G_{B_2^l(b_1)}(b_2|B_1\le b_1,z,I)$ is as follows.

$$\hat{G}_{B_2^l(b_1)}(b_2|B_1\le b_1,z,I) = \exp\left\{-\frac{1}{I-1}\int_{-\infty}^{+\infty}\frac{\mathbb{1}\left[b_2\le b\right]}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}\hat{m}_2^l\left(b\mid b_1,z,I\right)db\right\}$$

$$= \exp\left\{-\frac{1}{I-1}\int_{b_2}^{+\infty}\frac{\sum_{\ell\in\mathcal{L}_I^{1\neq2}}\lambda_\ell^{1\neq2}(b_1)\frac{1}{h_{2,3}^{1\neq2}}K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}db\right\}$$

$$= \prod_{\ell\in\mathcal{L}_I^{1\neq2}}\exp\left\{-\frac{\lambda_\ell^{1\neq2}(b_1)}{h_{2,3}^{1\neq2}(I-1)}\int_{b_2}^{+\infty}\frac{K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}db\right\}. \tag{54}$$

C.4 Derivation of $\partial\hat{G}_{B_2^l(b_1)}(\cdot|B_1\le b_1,z,I)/\partial b_1$

[Back to ToC] We will need the following estimator, namely the derivative of (54) with respect to the first auction winning bid, in the following subsections:

$$\frac{\partial\hat{G}_{B_2^l(b_1)}\left(b_2\mid B_1\le b_1,z,I\right)}{\partial b_1}$$

$$= \frac{\partial}{\partial b_1}\exp\left\{-\frac{1}{I-1}\int_{b_2}^{+\infty}\frac{\sum_{\ell\in\mathcal{L}_I^{1\neq2}}\lambda_\ell^{1\neq2}(b_1)\frac{1}{h_{2,3}^{1\neq2}}K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}db\right\}$$

$$= -\hat{G}^l_{2|B_1^{max}}\left(b_2\mid b_1,z,I\right)\frac{1}{h_{2,3}^{1\neq2}(I-1)}\left(\int_{b_2}^{+\infty}\frac{\partial}{\partial b_1}\left(\frac{\sum_{\ell\in\mathcal{L}_I^{1\neq2}}\lambda_\ell^{1\neq2}(b_1)K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)}{\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}\right)db\right)$$

$$= -\hat{G}^l_{2|B_1^{max}}\left(b_2\mid b_1,z,I\right)\frac{1}{h_{2,3}^{1\neq2}(I-1)}\times$$

$$\int_{b_2}^{+\infty}\left(\frac{\sum_{\ell\in\mathcal{L}_I^{1\neq2}}\frac{\partial\lambda_\ell^{1\neq2}(b_1)}{\partial b_1}K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)}{\left(\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)\right)^2}-\right.$$

$$\left.\frac{\sum_{\ell\in\mathcal{L}_I^{1\neq2}}\lambda_\ell^{1\neq2}(b_1)K\left(\frac{b-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right)\left(\sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell(b_1)}{\partial b_1}\overline{K}_{2\ell}(b)\right)}{\left(\hat{G}_{B_2^{max}|B_1^{max}}(b|b_1,z,I)\right)^2}\right)db \tag{55}$$

As you can see from (55), there are two terms, $\partial\lambda_\ell^{1\neq 2}(b_1)/\partial b_1$ and $\partial\lambda_\ell(b_1)/\partial b_1$. First, $\partial\lambda_\ell(b_1)/\partial b_1$ is as follows by noting (47).

$$\begin{aligned}
\frac{\partial\lambda_\ell(b_1)}{\partial b_1} &= \frac{\partial}{\partial b_1}\left(\frac{\frac{1}{h_{1,3}}\frac{1}{h_{Z,3}}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,3}}\right)K\left(\frac{z-Z_\ell}{h_{Z,3}}\right)}{\frac{1}{h_{1,2}}\frac{1}{h_{Z,2}}\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}\right) \\
&= \frac{\frac{1}{h_{1,3}}\frac{1}{h_{Z,3}}}{\frac{1}{h_{1,2}}\frac{1}{h_{Z,2}}}\left(\frac{\frac{1}{h_{1,3}}k\left(\frac{b_1-B_{1\ell}^{\max}}{h_{1,3}}\right)K\left(\frac{z-Z_\ell}{h_{Z,3}}\right)\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{\max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}{\left(\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)\right)^2}\right. \\
&\qquad\left.-\frac{K\left(\frac{b_1-B_{1\ell}^{\max}}{h_{1,3}}\right)K\left(\frac{z-Z_\ell}{h_{Z,3}}\right)\frac{1}{h_{1,2}}\sum_{\ell\in\mathcal{L}_I}k\left(\frac{b_1-B_{1\ell}^{\max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}{\left(\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)\right)^2}\right) \\
&= \lambda_\ell(b_1)\left(\frac{\frac{1}{h_{1,3}}k\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,3}}\right)}{K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,3}}\right)}-\frac{\frac{1}{h_{1,2}}\sum_{\ell\in\mathcal{L}_I}k\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}{\sum_{\ell\in\mathcal{L}_I}K\left(\frac{b_1-B_{1\ell}^{max}}{h_{1,2}}\right)K\left(\frac{z-Z_\ell}{h_{Z,2}}\right)}\right).
\end{aligned} \tag{56}$$

Regarding $\partial\lambda_\ell^{1\neq 2}(b_1)/\partial b_1$, we should use (40). But since the right-hand side of (40) has many terms, writing a closed form of $\partial\lambda_\ell^{1\neq 2}(b_1)/\partial b_1$ is demanding. Thus, we may consider using the following numerical derivative for $\partial\lambda_\ell^{1\neq 2}(b_1)/\partial b_1$:

$$\frac{\partial\lambda_\ell^{1\neq 2}(b_1)}{\partial b_1} = \frac{\lambda_\ell^{1\neq 2}(b_1+\varepsilon)-\lambda_\ell^{1\neq 2}(b_1)}{\varepsilon}, \tag{57}$$

where ε is some small number. For the future use, I can also define the numerical derivative for $\partial\lambda_\ell^{1=2}(b_1)/\partial b_1$:

$$\frac{\partial\lambda_\ell^{1=2}(b_1)}{\partial b_1} = \frac{\lambda_\ell^{1=2}(b_1+\varepsilon)-\lambda_\ell^{1=2}(b_1)}{\varepsilon}, \tag{58}$$

where $\lambda_\ell^{1=2}(b_1)$ is defined in (39).

C.5 Derivations of $\hat{g}_{2|1}^{w}\left(\cdot|b_1,z,I\right)$ and $\hat{g}_{B_2^l(b_1)}\left(\cdot|B_1\leq b_1,z,I\right)$

[Back to ToC] Recall the equation (53), namely $\hat{G}_{2|1}^{w}\left(b_2|b_1,z,I\right)$ — taking the derivative of both sides with respect to b_2 yields the following estimator.

$$\hat{g}_{2|1}^{w}\left(b_2|b_1,z,I\right) = \frac{\hat{m}_2^{w}\left(b_2\mid b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2\mid b_1,z,I\right)}\hat{G}_{2|1}^{w}\left(b_2\mid b_1,z,I\right) \tag{59}$$

Also, recall (54), namely $\hat{G}_{B_2^l(b_1)}\left(b_2|B_1\leq b_1,z,I\right)$, and take the derivative of both sides with respect to b_2 — it yields the following estimator.

$$\hat{g}_{B_2^l(b_1)}\left(b_2|B_1\leq b_1,z,I\right) = \frac{1}{I-1}\frac{\hat{m}_2^{l}\left(b_2\mid b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2\mid b_1,z,I\right)}\hat{G}_{B_2^l(b_1)}(b_2|B_1\leq b_1,z,I) \tag{60}$$

Note that the right-hand sides of (59) and (60) consist of the estimators that we already know from C.1-C.3.

C.6 Derivations of $\hat{\xi}_2^w(\cdot, b_1; z, I)$ and $\hat{\xi}_2^l(\cdot, b_1; z, I)$

[Back to ToC] Given that the function ξ_2^w is defined in equation (5), its plug-in estimator $\hat{\xi}_2^w$ is as follows:

$$
\begin{aligned}
\hat{\xi}_2^w(b_2, b_1; z, I) &= b_2 + \frac{\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-1}}{\partial \hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-1}/\partial b_2} \\
&= b_2 + \frac{\hat{G}_{B_2^l(b_1)}\left(b_2|B_1 \le b_1, z, I\right)^{I-1}}{\frac{\hat{m}_2^l(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)}\hat{G}_{B_2^l(b_1)}\left(b_2|B_1 \le b_1, z, I\right)^{I-1}} \\
&= b_2 + \frac{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)}{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)} \\
&= b_2 + \frac{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \overline{K}_{2\ell}(b_2)}{\sum_{\ell \in \mathcal{L}_I^{1\neq 2}} \lambda_\ell^{1\neq 2}(b_1) \frac{1}{h_{2,3}^{1\neq 2}} K\left(\frac{b_2 - B_{2\ell}^{max}}{h_{2,3}^{1\neq 2}}\right)},
\end{aligned}
\tag{61}
$$

where the second equality of (61) holds by following derivation:

$$
\begin{aligned}
&\partial \hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-1}/\partial b_2 \\
&= (I-1)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2}\hat{g}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I) \\
&= (I-1)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2}\frac{1}{I-1}\frac{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)}\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I) \\
&= \frac{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)}\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-1},
\end{aligned}
\tag{62}
$$

where the second equality holds by (60).

Given that the function ξ_2^l is defined in equation (6), its plug-in estimator $\hat{\xi}_2^l$ is as follows:

$$
\begin{aligned}
\hat{\xi}_2^l(b_2, b_1; z, I) &= b_2 + \frac{\hat{G}_{2|1}^w(b_2|b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2}}{\partial(\hat{G}_{2|1}^w(b_2|b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2})/\partial b_2} \\
&= b_2 + \frac{1}{\frac{\hat{m}_2^w(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)}},
\end{aligned}
\tag{63}
$$

where the last equality of (63) holds by the following derivation:

$$
\begin{aligned}
&\partial(\hat{G}_{2|1}^w(b_2|b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2})/\partial b_2 \\
&= \hat{g}_{2|1}^w(b_2 \mid b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2} \\
&\quad + \hat{G}_{2|1}^w(b_2 \mid b_1, z, I)(I-2)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-3}\hat{g}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I) \\
&= \hat{G}_{2|1}^w(b_2 \mid b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2} \\
&\quad \times \left(\frac{\hat{g}_{2|1}^w(b_2 \mid b_1, z, I)}{\hat{G}_{2|1}^w(b_2 \mid b_1, z, I)} + (I-2)\frac{\hat{g}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)}{\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)}\right) \\
&= \hat{G}_{2|1}^w(b_2 \mid b_1, z, I)\hat{G}_{B_2^l(b_1)}(b_2|B_1 \le b_1, z, I)^{I-2}
\end{aligned}
$$

$$\times \left(\frac{\hat{m}_2^w\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)} + \frac{I-2}{I-1} \frac{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)} \right), \tag{64}$$

where the third equality holds by (59) and (60).

C.7 Derivation of $\widehat{\Pr}[\delta(v_1, V_2) \leq \cdot | V_1 = v_1, z, I]$

[Back to ToC] The plug-in estimator of $\Pr[\delta(v_1, V_2) \leq \cdot | V_1 = v_1, z, I]$ by noting equation (15) is as follows: B.4 shows the equivalence between the '$\Pr[\delta(v_1, V_2) \leq \cdot | V_1 = v_1, z, I]$' and '$\Pr[\tilde{\delta}(B_1, V_2) \leq \cdot | B_1 = b_1, z, I] = \Pr[\delta(\xi_1(B_1), V_2) \leq \cdot | B_1 = b_1, z, I]$.'

$$\begin{aligned} &\widehat{\Pr}[\delta(v_1, V_2) \leq d | V_1 = v_1, z, I] \\ &= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\hat{\xi}_2^w(x, b_1; z, I) \leq d\right] d\hat{G}_{2|1}^w\left(x | b_1, z, I\right), \end{aligned} \tag{65}$$

which consumes a lot of time in implementing Monte Carlo simulation. A less time-consuming approach for the estimator $\widehat{\Pr}[\delta(v_1, V_2) \leq d | V_1 = v_1, z, I]$ uses the monotonicity of ξ_2^w in b_2. Because of the monotoncity, in an ideal situation, there exists a *unique* $b^{w,*}(d; b_1) \equiv b^{w,*}(d; b_1, z, I) \in [\underline{b_2}, \overline{b_2}]$ for some d such that it satisfies the following,[54]

$$\begin{aligned} &\hat{\xi}_2^w\left(b^{w,*}(d; b_1), b_1; z, I\right) \\ &\equiv b^{w,*}(d; b_1) + \frac{\sum_{\ell \in \mathcal{L}_I} \lambda_\ell(b_1) \overline{K}_{2\ell}\left(b^{w,*}(d; b_1)\right)}{\sum_{\ell \in \mathcal{L}_I^{1\neq 2}} \lambda_\ell^{1\neq 2}(b_1) \frac{1}{h_{2,3}^{1\neq 2}} K\left(\frac{b^{w,*}(d; b_1) - B_{2\ell}^{max}}{h_{2,3}^{1\neq 2}}\right)} \\ &= d. \end{aligned} \tag{66}$$

But, since $\hat{\xi}_2^w(x, b_1; z, I)$ is the empirical analog of $\xi_2^w(x, b_1; z, I)$, the empirical analog(or estimate) may not necessarily be strictly increasing in b_2 and so $b^{w,*}(d; b_1)$ may not be unique. Thus, I define $\hat{b}^{w,*}(d; b_1)$ as *the smallest* solution of the following:

$$\hat{b}^{w,*}(d; b_1) \equiv \hat{b}^{w,*}(d; b_1, z, I) \equiv \underset{x}{\operatorname{argmin}} \left(\hat{\xi}_2^w(x, b_1; z, I) \; - \; d\right)^2.$$

Then, $\hat{b}^{w,*}(d; b_1)$ will always be a unique number for some d, so we can transform (65) into the following estimator:

$$\widehat{\Pr}[\delta(v_1, V_2) \leq d | V_1 = v_1, z, I] = \hat{G}_{2|1}^w\left(\hat{b}^{w,*}(d; b_1) | b_1, z, I\right), \tag{67}$$

whose calculation is faster than that of (65).

[54] Each $b^{w,*}(d; b_1) - \varepsilon$ and $b^{w,*}(d; b_1) + \varepsilon$ will cause "$< d$", "$> d$" respectively in (66).

C.8 Derivation of $\widehat{\Pr}[V_2 \le \cdot | V_1 = v_1, z, I]$

[Back to ToC] The plug-in estimator of $\Pr[V_2 \le v_2 | V_1 = v_1, z, I]$ by noting equation (14) is as follows:

$$\begin{aligned}
&\widehat{\Pr}[V_2 \le v_2 | V_1 = v_1, z, I] \\
&= \widehat{\Pr}[V_2 \le v_2 | B_1 = b_1, z, I] \\
&= \widehat{\Pr}[V_2 \le v_2 | B_1 \le b_1, z, I] + \frac{\hat{G}_1(b_1|z,I)}{g_1(b_1|z,I)} \left(\frac{\partial}{\partial b_1} \widehat{\Pr}[V_2 \le v_2 | B_1 \le b_1, z, I] \right) \\
&= \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\hat{\xi}_2^l(x, b_1; z, I) \le v_2 \right] d\hat{G}_{B_2^l(b_1)}(x | B_1 \le b_1, z, I) \\
&\quad + \frac{\hat{G}_1(b_1|z,I)}{g_1(b_1|z,I)} \left(\frac{\partial}{\partial b_1} \int_{\underline{b_2}}^{\overline{b_2}} \mathbb{1}\left[\hat{\xi}_2^l(x, b_1; z, I) \le v_2 \right] d\hat{G}_{B_2^l(b_1)}(x | B_1 \le b_1, z, I) \right). \qquad (68)
\end{aligned}$$

But, both integrals inside (68) cause heavy computational burden. Thus, a less-time consuming approach for the estimator $\widehat{\Pr}[V_2 \le v_2 | V_1 = v_1, z, I]$ uses the monotonicity of ξ_2^l in b_2, which is analogous to what I did in C.7. Because of the monotoncity, in an ideal situation, there exists a *unique* $b^{l,*}(v_2; b_1) \equiv b^{l,*}(v_2; b_1, z, I) \in [\underline{b_2}, \overline{b_2}]$ for some v_2 such that it satisfies the following,

$$\begin{aligned}
&\hat{\xi}_2^l(b^{l,*}(v_2; b_1), b_1; z, I) \\
&\quad \equiv b^{l,*}(v_2; b_1) + \frac{1}{\frac{\hat{m}_2^w(b^{l,*}(v_2;b_1)|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b^{l,*}(v_2;b_1)|b_1,z,I)} + \frac{I-2}{I-1} \frac{\hat{m}_2^l(b^{l,*}(v_2;b_1)|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b^{l,*}(v_2;b_1)|b_1,z,I)}} \\
&\quad = v_2. \qquad (69)
\end{aligned}$$

But, since $\hat{\xi}_2^l(x, b_1; z, I)$ is the empirical analog of $\xi_2^l(x, b_1; z, I)$, the empirical analog(or estimate) may not necessarily be strictly increasing in b_2 and so $b^{l,*}(v_2; b_1)$ may not be unique. Thus, I define $\hat{b}^{l,*}(v_2; b_1)$ as *the smallest* solution of the following:

$$\hat{b}^{l,*}(v_2; b_1) \equiv \hat{b}^{l,*}(v_2; b_1, z, I) \equiv \underset{x}{\operatorname{argmin}} \left(\hat{\xi}_2^l(x, b_1; z, I) \; - \; v_2 \right)^2.$$

Then, $\hat{b}^{l,*}(v_2; b_1)$ will always be a unique number, so we can transform both integrals inside the right-hand side of (68) as follows:

$$\begin{aligned}
&\widehat{\Pr}[V_2 \le v_2 | V_1 = v_1, z, I] \\
&= \hat{G}_{B_2^l(b_1)} \left(\hat{b}^{l,*}(v_2; b_1) \mid B_1 \le b_1, z, I \right) \\
&\quad + \frac{\hat{G}_1(b_1|z,I)}{g_1(b_1|z,I)} \left(\frac{\partial}{\partial b_1} \hat{G}_{B_2^l(b_1)} \left(\hat{b}^{l,*}(v_2; b_1) \mid B_1 \le b_1, z, I \right) \right) \\
&= \hat{G}_{B_2^l(b_1)} \left(\hat{b}^{l,*}(v_2; b_1) \mid B_1 \le b_1, z, I \right) \\
&\quad + \frac{\hat{G}_1(b_1|z,I)}{\hat{g}_1(b_1|z,I)} \Bigg(\hat{g}_{B_2^l(b_1)}(\hat{b}^{l,*}(v_2; b_1) \mid B_1 \le b_1, z, I) \frac{\partial \hat{b}^{l,*}(v_2; b_1)}{\partial b_1} + \\
&\qquad\qquad \frac{\partial}{\partial b_1} \hat{G}_{B_2^l(b_1)} \left(\hat{b}^{l,*} \mid B_1 \le b_1, z, I \right) \Bigg), \qquad (70)
\end{aligned}$$

where the second equality holds by the chain rule. All the estimators in the right-hand side of (70) have been defined in C.1-C.7 except $\frac{\partial \hat{b}^{l,*}(v_2;b_1)}{\partial b_1}$; I derive it in the following mini section.

C.8.1 Derivation of $\partial \hat{b}^{l,*}(v_2; b_1)/\partial b_1$

[Back to ToC] Recall that the value $\xi_2^l\left(b^{l,*}(v_2;b_1), b_1; z, I\right)$ and the value v_2 must be the same. Thus, I can come up with the following new notation:

$$\xi_2^l\left(b^{l,*}(v_2;b_1), b_1; z, I\right) - v_2 = 0 \iff \phi(v_2, b_1, z, I) = 0.$$

I introduced a new notation, $\phi(v_2, b_1, z, I)$, to invoke Implicit Function Theorem — take a derivative of the left-hand side and the right-hand side of '$\phi(v_2, b_1, z, I) = 0$' with respect to b_1, then we have the following:

$$\begin{aligned}
&\frac{\partial}{\partial b_1}\phi(v_2, b_1, z, I) = 0 \\
\iff &\left(\frac{\partial}{\partial b_2}\xi_2^l(b^{l,*}, b_1; z, I)\right)\left(\frac{\partial}{\partial b_1}b^{l,*}(v_2;b_1)\right) + \frac{\partial}{\partial b_1}\xi_2^l(b^{l,*}, b_1; z, I) = 0 \\
\iff &\frac{\partial}{\partial b_1}b^{l,*}(v_2;b_1) = -\frac{\partial}{\partial b_1}\xi_2^l(b^{l,*}, b_1; z, I)\Big/\frac{\partial}{\partial b_2}\xi_2^l(b^{l,*}, b_1; z, I).
\end{aligned}$$

Now we have a closed form solution of $\frac{\partial}{\partial b_1}b^{l,*}(v_2;b_1)$ which consists of $\frac{\partial}{\partial b_1}\xi_2^l(\hat{b}^{l,*}, b_1; z, I)$ and $\frac{\partial}{\partial b_2}\xi_2^l(\hat{b}^{l,*}, b_1; z, I)$. Since $\hat{\xi}_2^l(b_2, b_1; z, I) = (63)$, we have the following estimators.

$$\begin{aligned}
&\partial\hat{\xi}_2^l(b_2, b_1; z, I)/\partial b_1 \\
&= \frac{\partial}{\partial b_1}\left(b_2 + \frac{1}{\frac{\hat{m}_2^w(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)}}\right) \\
&= \frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^w(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right)^{-1} \\
&= -\left(\frac{\hat{m}_2^w(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right)^{-2} \times \\
&\left(\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^w(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right) + \frac{I-2}{I-1}\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^l(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right)\right),
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\partial\hat{\xi}_2^l(b_2, b_1; z, I)/\partial b_2 \\
&= \frac{\partial}{\partial b_2}\left(b_2 + \frac{1}{\frac{\hat{m}_2^w(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2|b_1,z,I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2|b_1,z,I)}}\right) \\
&= 1 + \frac{\partial}{\partial b_2}\left(\frac{\hat{m}_2^w(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right)^{-1} \\
&= 1 - \left(\frac{\hat{m}_2^w(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)} + \frac{I-2}{I-1}\frac{\hat{m}_2^l(b_2 \mid b_1, z, I)}{\hat{G}_{B_2^{\max}|B_1^{\max}}(b_2 \mid b_1, z, I)}\right)^{-2} \times
\end{aligned}$$

$$\left(\frac{\partial}{\partial b_2}\left(\frac{\hat{m}_2^w\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)}\right)+\frac{I-2}{I-1}\frac{\partial}{\partial b_2}\left(\frac{\hat{m}_2^l\left(b_2 \mid b_1, z, I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2 \mid b_1, z, I\right)}\right)\right). \tag{72}$$

Given (71) and (72), the plug-in estimator $\frac{\partial}{\partial b_1}\hat{b}^{l,*}(v_2;b_1)$ is as follows.

$$\begin{aligned}
&\frac{\partial}{\partial b_1}\hat{b}^{l,*}(v_2;b_1)\\
&=-\frac{\partial}{\partial b_1}\hat{\xi}_2^l(\hat{b}^{l,*},b_1;z,I)\Big/\frac{\partial}{\partial b_2}\hat{\xi}_2^l(\hat{b}^{l,*},b_1;z,I)=-(71)/(72)\\
&=\frac{① \times\left(\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^w\left(\hat{b}^{l,*}|b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}|b_1,z,I\right)}\right)+\frac{I-2}{I-1}\frac{\partial}{\partial b_1}\left(\frac{\hat{m}_2^l\left(\hat{b}^{l,*}|b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}|b_1,z,I\right)}\right)\right)}{1-① \times\left(\frac{\partial}{\partial b_2}\left(\frac{\hat{m}_2^w\left(\hat{b}^{l,*}|b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}|b_1,z,I\right)}\right)+\frac{I-2}{I-1}\frac{\partial}{\partial b_2}\left(\frac{\hat{m}_2^l\left(\hat{b}^{l,*}|b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}|b_1,z,I\right)}\right)\right)} \qquad (73)\\
&\quad\text{where } ① \equiv\left(\frac{\hat{m}_2^w\left(\hat{b}^{l,*}\mid b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}\mid b_1,z,I\right)}+\frac{I-2}{I-1}\frac{\hat{m}_2^l\left(\hat{b}^{l,*}\mid b_1,z,I\right)}{\hat{G}_{B_2^{\max}|B_1^{\max}}\left(\hat{b}^{l,*}\mid b_1,z,I\right)}\right)^{-2}.
\end{aligned}$$

To calculate (73), we need the following estimators.

$$\begin{aligned}
\partial\hat{m}_2^w(b_2|b_1,z,I)/\partial b_2 &= \sum_{\ell\in\mathcal{L}_I^{1=2}}\lambda_\ell^{1=2}(b_1)\left(\frac{1}{h_{2,3}^{1=2}}\right)^2 k\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right),\\
\partial\hat{m}_2^w(b_2|b_1,z,I)/\partial b_1 &= \sum_{\ell\in\mathcal{L}_I^{1=2}}\frac{\partial\lambda_\ell^{1=2}(b_1)}{\partial b_1}\frac{1}{h_{2,3}^{1=2}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1=2}}\right),\\
\partial\hat{m}_2^l(b_2|b_1,z,I)/\partial b_2 &= \sum_{\ell\in\mathcal{L}_I^{1\neq2}}\lambda_\ell^{1\neq2}(b_1)\left(\frac{1}{h_{2,3}^{1\neq2}}\right)^2 k\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right),\\
\partial\hat{m}_2^l(b_2|b_1,z,I)/\partial b_1 &= \sum_{\ell\in\mathcal{L}_I^{1\neq2}}\frac{\partial\lambda_\ell^{1\neq2}(b_1)}{\partial b_1}\frac{1}{h_{2,3}^{1\neq2}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}^{1\neq2}}\right),\\
\partial\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2\mid b_1,z,I\right)/\partial b_2 = (46) &= \sum_{\ell\in\mathcal{L}_I}\lambda_\ell(b_1)\frac{1}{h_{2,3}}K\left(\frac{b_2-B_{2\ell}^{max}}{h_{2,3}}\right),\\
\partial\hat{G}_{B_2^{\max}|B_1^{\max}}\left(b_2\mid b_1,z,I\right)/\partial b_1 &= \sum_{\ell\in\mathcal{L}_I}\frac{\partial\lambda_\ell(b_1)}{\partial b_1}\overline{K}_{2\ell}(b_2),
\end{aligned}$$

where (56), (57), and (58) are used in these six estimators.

Equation (73) shows us that the close-form expression of the estimator $\frac{\partial}{\partial b_1}\hat{b}^{l,*}(v_2;b_1)$ is complicated. Instead, I could come up with a numerical derivative as follows:

$$\begin{aligned}
&\frac{\partial}{\partial b_1}\hat{b}^{l,*}(v_2;b_1)\\
&=-\frac{\partial}{\partial b_1}\hat{\xi}_2^l(\hat{b}^{l,*},b_1;z,I)\Big/\frac{\partial}{\partial b_2}\hat{\xi}_2^l(\hat{b}^{l,*},b_1;z,I)=-\frac{\left(\frac{\hat{\xi}_2^l\left(\hat{b}^{l,*},b_1+\varepsilon;z,I\right)-\hat{\xi}_2^l\left(\hat{b}^{l,*},b_1;z,I\right)}{\varepsilon}\right)}{\left(\frac{\hat{\xi}_2^l\left(\hat{b}^{l,*}+\varepsilon,b_1;z,I\right)-\hat{\xi}_2^l\left(\hat{b}^{l,*},b_1;z,I\right)}{\varepsilon}\right)},
\end{aligned} \tag{74}$$

where ε is some small number.

C.9 Estimation of a function δ

[Back to ToC] B.5 shows both the original and the alternative identification strategy for the function δ. If I were to choose the original strategy, then coming up with the estimator $\hat{\delta}$ is as follows; I borrow notations from B.5.

✓ Make a grid of α. Then for each α in the grid, compute the quantiles $\hat{v}_{2|1}(\alpha|b_1, z, I)$ by using (68) or (70) — this will be computationally taxing. Also, for each α compute the quantiles $\hat{\tilde{d}}_{2|1}(\alpha|b_1, z, I)$ defined as follows.

$$\hat{\tilde{d}}_{2|1}(\alpha|b_1, z, I) = \hat{\xi}_2^w\left(\hat{b}_{2|1}^w(\alpha|b_1, z, I), b_1; z, I\right), \tag{75}$$

where (75) is the plug-in estimator of $\tilde{d}_{2|1}(\alpha|b_1) = \xi_2^w(b_{2|1}^w(\alpha|b_1), b_1)$, which I also used in (33). $\hat{\xi}_2^w(\cdot, \cdot; \cdot, \cdot)$ is already defined in (61), and $\hat{b}_{2|1}^w(\alpha|b_1, z, I)$ stands for the α-quantile of $\hat{G}_{2|1}^w(\cdot|b_1, z, I)$ from (53), which is as follows.

$$\hat{b}_{2|1}^w(\alpha|b_1, z, I) \equiv \underset{x}{\operatorname{argmin}} \left(\hat{G}_{2|B_1^{max}}^w(x|b_1, z, I) \; - \; \alpha\right)^2.$$

Given $\hat{v}_{2|1}(\alpha|b_1, z, I)$ and $\hat{\tilde{d}}_{2|1}(\alpha|b_1, z, I)$ for each value of αs, I can compare them and nonparametrically estimate a function $\hat{\tilde{\delta}}(b_1, v_2, z, I)$. Since the estimator is dependent on the value of I, I can come up with the following estimator.

$$\hat{\tilde{\delta}}(b_1, v_2, z) = \frac{1}{\sum_{\tilde{I}=2}^{N} L_{\tilde{I}}} \times \sum_{I=2}^{N} \left(L_I \times \hat{\tilde{\delta}}(b_1, v_2, z, I)\right), \tag{76}$$

where L_I is the number of auction pairs with I bidders. What (76) does is doing the weighted average of $\hat{\tilde{\delta}}(b_1, v_2, z, I)$ to get $\hat{\tilde{\delta}}(b_1, v_2, z)$.

The same estimator $\hat{\delta}$ exploiting alternative strategy is as follows.

✓ Make a grid of b_2. Then for each b_2 in the grid, apply $\hat{\xi}_2^l(b_2, b_1; z, I) = (63)$ so that we now have the grid of $\hat{\xi}_2^l(b_2, b_1; z, I)$. For this new grid, evaluate each grid point with $\hat{\tilde{F}}_{2|1}(\cdot|b_1, z, I) = (70)$ so that now you have grid of α. Given this grid of α, do (75) so that you get the grid of $\hat{\tilde{d}}_{2|1}(\alpha|b_1, z, I)$. Now, compare 'the grid of $\hat{\xi}_2^l(b_2, b_1; z, I)$' with 'the grid of $\hat{\tilde{d}}_{2|1}(\alpha|b_1, z, I)$' so that you nonparametrically estimate $\hat{\tilde{\delta}}(b_1, v_2, z, I)$, and exploiting (76) finishes the estimation.

C.10 Derivations of $\hat{\xi}_1(\cdot; z, I)$ and $\hat{F}_1(\cdot|z, I)$

[Back to ToC] C.1-C.8 introduced all the estimators that constitute the plug-in estimator of a function ξ_1; namely, replacing all the bid distribution in (9) with its estimators produces $\hat{\xi}_1$.

I construct the estimator $\hat{F}_1(\cdot|z, I)$ based on $\hat{\xi}_1$. Identification section asserts $V_1^{\max} = \xi_1(B_1^{\max})$, and because a function ξ_1 is monotone, it must be that the α-quantile of $V_1^{\max}$ equals $\xi_1(\alpha$-quantile of $B_1^{\max})$.

Thus, I first come up with the estimator $\widehat{\Pr}[B_1^{\max} \leq \cdot|z, I]$, which I already introduced in (16). Then for any $\alpha \in [0, 1]$, I calculate the quantile $\hat{b}_1^{\max}(\alpha|z, I)$ that comes from "$\operatorname{argmin}_x(\widehat{\Pr}[B_1^{\max} \leq$

$\alpha|z, I] - x)^2$", and transform it to $\hat{\xi}_1(\hat{b}_1^{\max}(\alpha|z, I); z, I)$, which is the α-quantile of $\widehat{\Pr}[V_1^{\max} \leq \cdot|z, I] = \widehat{\Pr}[V_1 \leq \cdot|z, I]^I$. Varying α will pin down all the quantiles of $\widehat{\Pr}[V_1 \leq \cdot|z, I]^I$, then I take the I-th root of each α to get $\widehat{\Pr}[V_1 \leq \cdot|z, I] \equiv \hat{F}_1(\cdot|z, I)$.

C.11 Monte Carlo Setting

[Back to ToC] The bid distributions that I started with have the following form:

$$\begin{aligned} G_{2|1}^w(b_2|b_1) &= b_2^{b_1^q + w}, \quad & b_2, b_1 \in [0,1]^2 \\ G_{B_2^l(b_1)}(b_2|B_1 \leq b_1) &= b_2^{b_1^q + l}, \quad & b_2, b_1 \in [0,1]^2 \\ G_1(b_1) &= b_1^p, \quad & b_1 \in [0,1] \end{aligned} \tag{77}$$

There are four parameters (p, q, w, l) inside (77), and I assume $I = 2$. Because bids are restricted within the unit interval, the supports of both random variables $B_1^{\max}$ and $B_2^{\max}$ are also unit interval; moreover, the support of $B_2^{\max}$ given some $B_1^{\max} = b_1$ is also $[0, 1]$ irrespective of b_1.

Condition (i) of Theorem 1 is met because it asserts that the bid distributions must be absolutely continuous, which holds true under (77). Remaining conditions to be met are (ii) and (iii), which are the reason why I had to come up with parameter values ($q = \frac{1}{70}$, $p = 0.5$, $l = 0.2$, $w = 0.1$); descriptive statistics coming from (77) given the parameters are shown in C.11.1.

Why should the parameter q be a small number? — It has to do with condition (ii), which asserts that $G_{B_2^l(b)|b}(\cdot|b)$ derived from equation (10) must be a valid distribution. C.11.2 shows that $G_{B_2^l(b)|b}(\cdot|b)$ is the same as $F_{2|1}(\cdot|v_1)$, implying that guaranteeing the validity of $G_{B_2^l(b)|b}(\cdot|b)$ is equivalent to guaranteeing the validity of $F_{2|1}(\cdot|v_1)$.

Given (77), equation (10) outputs the following function.

$$G_{B_2^l(b_1)|B_1}(b_2|b_1) = \frac{b_1}{p} \left(b_1^{q-1} b_2^{b_1^q + l} q \log(b_2) - \frac{-b_1^{q-1} q b_2 b_2^{b_1^q + l - 1} (b_1^q + l)}{b_1^{2q} + 2b_1^q w + b_1^q + w^2 + w} \right) + b_2^{b_1^q + l}, \tag{78}$$

whose terms have one-to-one relationship with those in (10): red corresponds to $G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)$; blue corresponds to $G_1(b_1)/g_1(b_1)$; teal corresponds to $\partial G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)/\partial b_1$; purple corresponds to $\frac{\partial \xi_2^l(b_2; b_1)/\partial b_1}{\partial \xi_2^l(b_2; b_1)/\partial b_2}$; orange corresponds to $\partial G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)/\partial b_2 = g_{B_2^l(b_1)}(b_2|B_1 \leq b_1)$.

I can rearrange (78) as follows.

$$G_{B_2^l(b_1)|B_1}(b_2|b_1) = \frac{{b_1}^q}{p} \left({b_2}^{{b_1}^q + l} q \log(b_2) + q \frac{{b_2}^{{b_1}^q + l} ({b_1}^q + l)}{({b_1}^q + w)({b_1}^q + w + 1)} \right) + {b_2}^{{b_1}^q + l}, \tag{79}$$

where I use red and blue colors for b_2 and b_1 in (79); some description about follows.

1. If $b_1 = 0$, then (79) becomes b_2^l, so (77) outputs a valid distribution, unless $l = 0$.

2. If $b_1 = 1$, then (79) becomes as follows.

$$G_{B_2^l(1)|B_1}(b_2|1) = \frac{1}{p}\left(b_2^{1+l} q \log(b_2) + q\frac{b_2^{1+l}(1+l)}{(1+w)(1+w+1)}\right) + b_2^{1+l}.$$

As $b_2 \downarrow 0$, we have 0. As $b_2 \uparrow 1$, we get the following result.

$$\underbrace{\frac{1}{p}\left(q\frac{(1+l)}{(1+w)(1+w+1)}\right)} +1,$$

which is larger than 1, as long as $[p, q, w, l]$ have values larger than or equal to 0; if q gets small or w gets large, then the underbrace gets closer to zero but never exactly zero.

3. If $b_1 \in (0, 1)$, then (79) still maintains its form. As $b_2 \downarrow 0$, the whole equation becomes 0, and as $b_2 \uparrow 1$, (79) becomes as follows.

$$\underbrace{\frac{b_1^q}{p}\left(q\frac{(b_1^q+l)}{(b_1^q+w)(b_1^q+w+1)}\right)} +1,$$

where the underbrace is positive, so the whole is larger than 1. To make the underbrace zero, we can consider the following measures:

- *Let p be a very large number*: This is undesirable because, say $p = 100$, then we will have $G_1(b_1) = b_1^{100}$, which outputs the first auction bids that are very close to 1; it worsens the Monte Carlo simulation.
- *Let w be a very large number*: This is also undesirable because, say $w = 3$, then we will have $G^w_{2|B_1^{max}}(b_2|b_1) = b_2^{b_1^q+3}$, which outputs B_2^ws that are very close to 1; it also worsens the Monte Carlo simulation.
- *Let q be a very small number*: This is desirable, so I choose this route.

Thus, what I want to conclude from (79) is that, both underbraces must be zero to make (79) a valid distribution. Making those exactly zero is possible when $q = 0$, but it nullifies the effect of the first auction bid as can be seen from (77).
This is why I choose a small value for q, such as 1/70, even though it does not *perfectly* satisfy condition (ii) of Theorem 1.

Remaining parameters, p, l, and w — However, with a suitable choice for the values of remaining parameters $[p, l, w]$, the distribution coming from (79) *appear* to be a valid distribution shown in figure below.

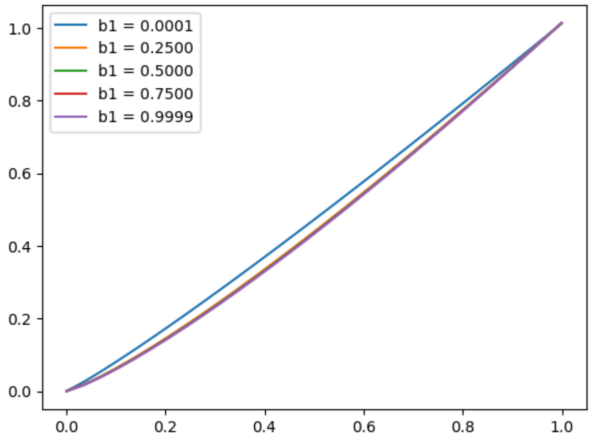


Figure — $G_{B_2^l(b_1)|B_1}(\cdot|b_1)$ for five values of b_1

Values for the remaining parameters are set at $p = 0.5$, $l = 0.2$, and $w = 0.1$, which I describe below.

- $p = 0.5$: Now $G_1(b_1)$ inside (77) becomes $b_1^{0.5}$, so the density of $B_1^{\max}$ is not skewed toward the upper bound of b_1, namely 1 — this enhances Monte Carlo simulation.
- $(l = 0.2, w = 0.1)$: Both l and w shouldn't be too large, say 100 and 200. It is because (77) given the large values output b_2^l and b_2^w that are very close to 1; it worsens the Monte Carlo simulation. This is why I chose small values for l and w.

  Moreover, under (77) I get the following functions for ξ_2^w and ξ_2^l.

$$\xi_2^l(b_2, b_1) = b_2 + \frac{b_2}{b_1^q + w}, \tag{80}$$

$$\xi_2^w(b_2, b_1) = b_2 + \frac{b_2}{b_1^q + l}. \tag{81}$$

  We notice that as long as $l > w$, ξ_2^l is greater than ξ_2^w for any given (b_1, b_2). It is equivalent to saying that under this setup, v_2 is greater than $\tilde{\delta}(b_1, v_2)$ for any given b_1; this setup is important since it guarantees $\xi_1(b_1) = v_1 > 0$ for every b_1 — otherwise, if $\tilde{\delta}(b_1, v_2)$ is much greater than v_2, I face $\xi_1(b_1) = v_1 < 0$ for some b_1[55]. As a result, I decided that l should be greater than w.

  Lastly, if l and w differs much, the Monte Carlo simulation didn't work well. Thus, I decided that l and w should be close to each other — this is why I chose $l = 0.2$ and $w = 0.1$.

As a result, the values I set for each parameter is ($q = \frac{1}{70}$, $p = 0.5$, $l = 0.2$, $w = 0.1$), which transforms (77) to the following distributions.

$$G_{2|1}^w(b_2|b_1) = b_2^{b_1^{1/70}+0.1}, \quad b_2, b_1 \in [0,1]^2$$

$$G_{B_2^l(b_1)}(b_2|B_1 \le b_1) = b_2^{b_1^{1/70}+0.2}, \quad b_2, b_1 \in [0,1]^2$$

[55] The reason I think we face negative v_1 when $w \gg l$ is that if synergy is very high, a bidder is willing to bid a positive amount even if his v_1 below 0.

$$G_1(b_1) = b_1^{0.5}. \qquad b_1 \in [0, 1] \tag{82}$$

Condition (iii) is satisfied because any positive values for (q, w, l) make both functions ξ_2^w and ξ_2^l from (81) and (80) monotone in b_2 for every b_1; monotonicity of ξ_1 is shown as a solid line in Figure 3.

I used Python and its packages, such as Numba and Multiprocessing, to boost up its running time.

Lastly, even with $I = 3$, finding the adequate bid distributions that satisfy conditions (i)-(iii) of the model become nearly impossible, which is why I settled down with $I = 2$. With $I = 2$, one could think of substituting Beta or Gamma distributions for the right-hand sides of (77). But, this substitution outputs extremely noisy Monte Carlo estimates.

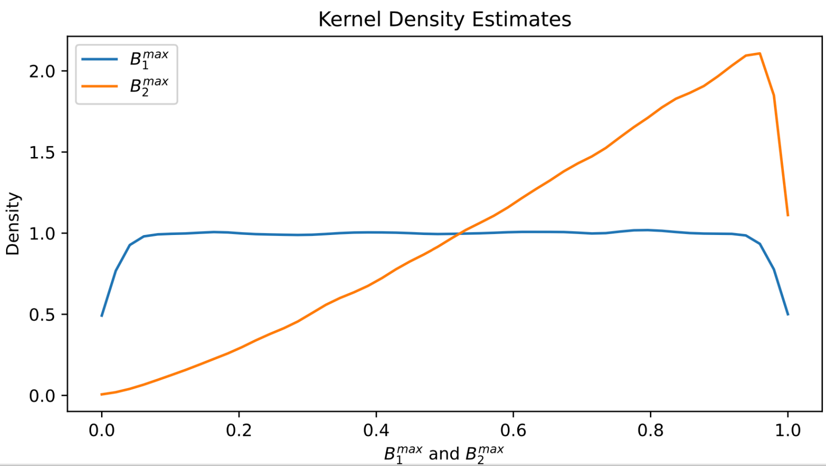
C.11.1 Descriptive Statistics

[Back to ToC] Recall what I wrote in 4.1; given (82) in hand, I set the number of samples as 200 and let each sample have 1,000 auction pairs (i.e., $L = 1,000$). Thus, I have $200 \times 1,000 = 200,000$ observations of $B_1^{\max}$ and $B_2^{\max}$ — its statistics follows:

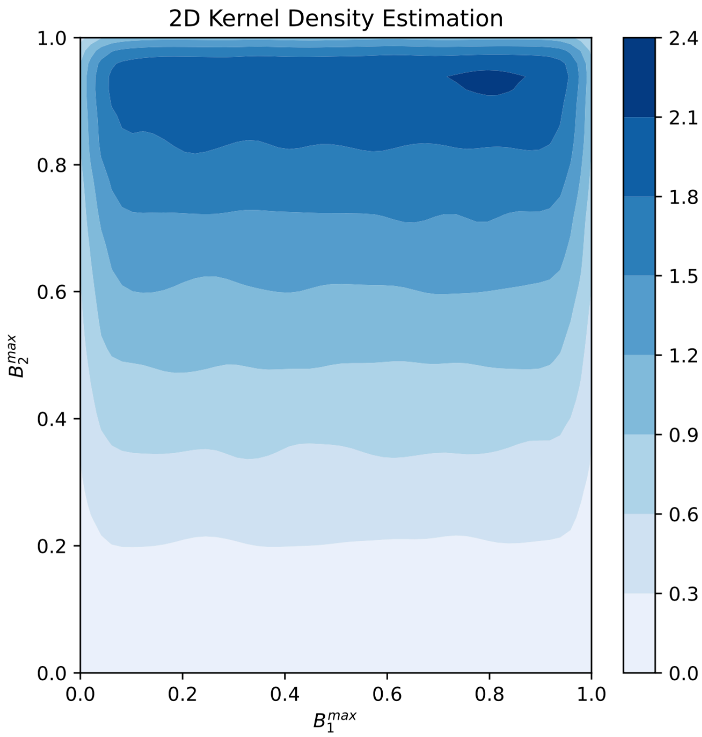
- Mean and Standard deviation of $B_1^{\max}$: 0.501 and 0.288
- Mean and Standard deviation of $B_2^{\max}$: 0.695 and 0.223
- Correlation of $B_1^{\max}$ and $B_2^{\max}$: 0.01
- A probability that the first auction winner wins the second auction (i.e., $L_{1=2}/200,000$): 47.9%

When describing Figures 5 and 6, I noted that V_1 and V_2 are nearly independent, which is expressed here as a low correlation of 0.01. Also, the repeated winning rate less than fifty percent, namely 47.9%, indicates a negative synergy between V_1 and V_2, which is expressed in Figure 6.

The following two figures show the kernel density estimates of $B_1^{\max}$ and $B_2^{\max}$: the supports of both random variables are [0,1], which I once pointed out in (77).



Kernel Density Estimates 2



Kernel Density Estimates 1

C.11.2 Equivalence of $G_{B_2^l(b_1)|B_1}$ and $F_{2|1}$

[Back to ToC] For reference, I copy and paste (10).

$$\begin{aligned}
&G_{B_2^l(b_1)|B_1}(b_2|b_1) = G_{B_2^l(b_1)}(b_2|B_1 \leq b_1) \\
&\quad + \frac{G_1(b_1)}{g_1(b_1)} \left\{ \frac{\partial G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)}{\partial b_1} - \frac{\partial \xi_2^l(b_2, b_1)/\partial b_1}{\partial \xi_2^l(b_2, b_1)/\partial b_2} g_{B_2^l(b_1)}(b_2|B_1 \leq b_1) \right\},
\end{aligned} \tag{83}$$

where $g_{B_2^l(b_1)}(b_2|B_1 \leq b_1) = \partial G_{B_2^l(b_1)}(b_2|B_1 \leq b_1)/\partial b_2$. I also copy and paste (14) for reference.

$$\Pr[V_2 \leq \cdot|V_1 = v_1] = \Pr[V_2 \leq \cdot|B_1 < b_1] + \frac{G_1(b_1)}{g_1(b_1)} \left(\frac{\partial}{\partial b_1} \Pr[V_2 \leq \cdot|B_1 < b_1] \right). \tag{84}$$

I show the right-hand sides of both equations are equivalent. First, note the following equality, which I used in C.8 to derive (70).

$$\Pr[V_2 \leq v_2 \mid B_1 < b_1] = G_{B_2^l(b_1)}\big(b^{l,*}(v_2; b_1) \mid B_1 \leq b_1\big), \tag{85}$$

where $b^{l,*}(v_2; b_1)$ satisfies $\xi_2^l\left(b^{l,*}(v_2; b_1), b_1\right) = v_2$. Given (85), I can change the right-hand side of (84) as follows — for notational simplicity, I use the equality $G_{B_2^l(b_1)}(\cdot|B_1 \leq b_1) = G^l_{2|B_1^{\max}}(\cdot|b_1)$ that I established in Lemma 1:

$$\begin{aligned}
&G^l_{2|B_1^{max}}\left(b^{l,*}(v_2; b_1) \mid b_1\right) + \frac{G_1(b_1)}{g_1(b_1)} \left\{ \frac{\partial}{\partial b_1} G^l_{2|B_1^{max}}\left(b^{l,*}(v_2; b_1) \mid b_1\right) \right\} \\
&= G^l_{2|B_1^{max}}\left(b^{l,*}(v_2; b_1) \mid b_1\right) + \frac{G_1(b_1)}{g_1(b_1)}
\end{aligned}$$

$$
\times \left\{ g^l_{2|B_1^{max}} \left(b^{l,*}(v_2; b_1) \mid b_1 \right) \frac{\partial b^{l,*}(v_2; b_1)}{\partial b_1} + \frac{\partial}{\partial b_1} G^l_{2|B_1^{max}} \left(b^{l,*} \mid b_1 \right) \right\}
$$

$$
= G^l_{2|B_1^{max}} \left(b^{l,*}(v_2; b_1) \mid b_1 \right) + \frac{G_1(b_1)}{g_1(b_1)}
$$

$$
\times \left\{ -g^l_{2|B_1^{max}} \left(b^{l,*}(v_2; b_1) \mid b_1 \right) \frac{\partial \xi^l_2(b^{l,*}, b_1)/\partial b_1}{\partial \xi^l_2(b^{l,*}, b_1)/\partial b_2} + \frac{\partial}{\partial b_1} G^l_{2|B_1^{max}} \left(b^{l,*} \mid b_1 \right) \right\}, \tag{86}
$$

where the first equality holds by the chain rule, and the second equality holds by $\frac{\partial}{\partial b_1} b^{l,*}(v_2; b_1) = -\frac{\partial}{\partial b_1} \xi^l_2 \left(b^{l,*}, b_1 \right) \Big/ \frac{\partial}{\partial b_2} \xi^l_2 \left(b^{l,*}, b_1 \right)$ which I introduced in C.8.1. Given the equality, I can replace the right-hand side of (84) with (86), which yields the following.

$$
\Pr[V_2 \leq v_2 | V_1 = v_1] = G^l_{2|B_1^{max}} \left(b^{l,*}(v_2; b_1) \mid b_1 \right)
$$

$$
+ \frac{G_1(b_1)}{g_1(b_1)} \left\{ \frac{\partial G^l_{2|B_1^{max}} \left(b^{l,*} \mid b_1 \right)}{\partial b_1} - \frac{\partial \xi^l_2(b^{l,*}, b_1)/\partial b_1}{\partial \xi^l_2(b^{l,*}, b_1)/\partial b_2} g^l_{2|B_1^{max}} \left(b^{l,*}(v_2; b_1) \mid b_1 \right) \right\},
$$

whose right-hand side is the same as the right-hand side of (83).

D Section 5, Application

D.1 About Korean Fruit Auction

[Back to ToC] Even though I used the term 'Korean Fruit Auction' for the ease of exposition, the precise term is 'Agricultural Produce Auction in Garak Market.' As the name indicates, it not only sells fruits but also vegetables, and also is one of the largest agricultural produce markets, in the following sense:

- As of 2022, *thirty-three public wholesale markets* account for 99.4%[56] of volume and 99.2%[57] of value in vegetables and fruits being traded in South Korea. *Garak market*, one of thirty-three public wholesale markets, is the leading market in the sense that its trade volume and value were 2,235,696 tons and \$3.848 billion[58], which account for 34.5% and 36.1% of the trade volume and value of thirty-three public wholesale markets.

Garak market is governed by, not only limited to, the following regulations.

- Act on Distribution and Price Stabilization of Agricultural and Fishery Products (농수산물 유통 및 가격안정에 관한 법률) ≡ Ⓐ
- Enforcement Decree of the Act on Distribution and Price Stabilization of Agricultural and Fishery Products (농수산물 유통 및 가격안정에 관한 법률 시행령) ≡ Ⓑ
- Enforcement Rule of the Act on Distribution and Price Stabilization of Agricultural and Fishery Products (농수산물 유통 및 가격안정에 관한 법률 시행규칙) ≡ Ⓒ
- Seoul Metropolitan Government Ordinance on Agricultural and Fishery Products Wholesale Markets (서울특별시 농수산물도매시장 조례) ≡ Ⓓ

[56] 6,473,286tons/6,514,595tons
[57] ₩13,863billion/₩13,969billion
[58] ₩5,002.4billion

- Seoul Metropolitan Government Enforcement Rule of the Ordinance on Agricultural and Fishery Products Wholesale Markets (서울특별시 농수산물도매시장 조례 시행규칙) ≡ Ⓔ

One of the objectives of these regulations is to maintain stable price for agricultural products, mandated by Article 123 (4) of Constitution of the Republic of Korea.

I discuss only the information that seems relevant to this paper — more details can be found in various Korean reports, which I mention in D.1.5

D.1.1 Market and Auction characteristics

[Back to ToC] 79.6% of 2,235,696 tons, or equivalently 76.7% of $3.848 billion, were sold via auction. Other means of trading such as bargaining(정가수의매매) or special transaction(상장예외) exist, but they are not the main channel as they account for 12.3% and 8.1% of transaction volumes, and 11.2% and 12.1% of transaction values.

Auctions were held 306 days in 2022 (i.e., on Sundays and certain chosen holidays, the auction is not held). Ⓔ stipulates which goods at what time should be sold via auction: I attach the corresponding part of Ⓔ for reference.

<서울특별시 가락동농수산물도매시장>

구분	품 목	경매개시 시각	경매 장소
과일류	포도, 복숭아, 감귤, 자두, 딸기, 메론, 참외, 토마토, 박스수박	02:00	과실 경매장
	사과, 배, 유자, 단감, 떫은 감, 기타 수박, 수입과실(바나나, 오렌지)	08:30	과실 경매장
채소류	당근	제주산 08:00 기 타 15:30	채소 경매장
	상추, 쑥갓, 시금치, 아욱, 근대, 열무, 청경채, 치커리, 얼갈이 중 박스포장품, 대파	19:00	채소 경매장
	시금치, 아욱, 근대, 열무, 얼갈이, 옥수수, 봄동 중 비규격 출하품(일명 짝짐)	20:00	채소 경매장
	감자, 깻잎	21:30	채소 경매장
	버섯류, 부추, 미나리, 양배추, 고추	22:00	채소 경매장
	무, 배추, 포장쪽파	22:00	청과배송주차장
	피망, 가지, 호박, 오이, 양파, 시금치, 봄동(남부지방 비규격 출하품)	23:00	채소 경매장

I do not translate it into English, but one can see that certain goods should be auctioned starting from 2:00 am, and other goods from 8:30 am, and etc.

Six auction houses exist under Garak Market: five of them were mentioned in Table 2 and another one is Daeah. Five houses deal both vegetables and fruits while Daeah only focuses on vegetables, which is why I exclude it from the analysis. Farmers decide one of six auction houses and request to sell their product: I do not discuss the details of farmers' decision.

Given 2,235,696 tons of trade volume in 2022, the following statistics show the *percentage of the trade volume* each auction house accounts for, and its *daily average trading volume*.

- Seoul: 14.7%, 1,071 tons
- Joongang: 14.5%, 1,060 tons
- Donghwa: 17.0%, 1,241 tons

- Hankook: 17.0%, 1,239 tons
- Daeah: 20.1%, 1,468 tons
- Nonghyup: 8.7%, 634 tons
- Special Transaction: 8.1%, 589 tons

As can be seen from the daily average trading volume, sizable amount of goods has to be sold on a given day which is why each auction lasts typically three to ten seconds.

Items to be auctioned are delivered to the auction site before the auction begins. For example, in the case of fruits, since the fruit auction starts at 2:00 am, most items are delivered by or around 12:00 am. Most fruit bidders begin checking the quality of the items starting from 12:00 am. At least for fruit bidders, the type and quantity of fruit they need to win are decided before the auction starts, as things become too hectic during the auction for them to take orders from their customers.

Since bidders observe the produce to be auctioned before the auction begins, my model—which assumes that a bidder does not have perfect knowledge of his v_2 (i.e., the value in the last auction) during the first period (i.e., the second-to-last auction)—does not capture the full reality. Therefore, I needed to verify whether the necessary conditions of my model are satisfied, as demonstrated in the Application section.

D.1.2 Bidder

[Back to ToC] Ⓔ limits the total number of bidders within the Garak Market to be 1,187. Six auction houses split this number and fill their bidders; the process of splitting remains uncertain, but is irrelevant to know at least for this paper. Bidders are wholesaler and each has his own refrigerator in Garak Market; it is known that each bidder has at least five to six customers, who are mostly retailers or department stores.

When a bidder delivers the items he won in the auction to his customers, the price he receives from the customers is determined in one of three ways:

1. The bidder sets the price on his own and informs the customer.
2. The bidder discloses the price at which he won the item and negotiates with the customer on the margin he will receive.
3. The bidder and the customer have a forward contract (typically ranging from one week to one year), which sets a fixed price that the bidder receives for delivering the item to the customer.

The first method is used when the transaction volume between the bidder and the customer is small, while the second and third methods are applied in large volume transactions. It is evident that there is no common, unanimous price governing how much bidders receive for delivering the item to the customer, thereby justifying the use of the private value paradigm.

Bidders enter into contracts with one of six auction houses to participate in the auction. According to a bidder I contacted, they can only contract with one auction house at a time, although I could not verify this from sources Ⓐ-Ⓔ. While bidders are allowed to switch auction houses, this rarely occurs, and most of them remain in the same auction house and renew a contract; source Ⓐ specifies that once a bidder signs a contract with an auction house, the contract duration ranges from three to ten years. Additionally, source Ⓔ sets a minimum monthly transaction value that bidders must meet —$62,000 (₩80 million) for fruit —which is considered non-restrictive, as most bidders are known to easily satisfy this threshold. Due to this leniency, no bidders were reportedly expelled from any auction houses in 2022.

In general, ninety percent of the fruits or vegetables won by a bidder on a given day is delivered to their customers, while only ten percent is kept for the bidder's own use. As said, each bidder has their own refrigerator within the market, but the size typically ranges from three to five cubic meters, making it difficult for a bidder to store large quantities of produce won in a single day. In addition to their individual refrigerators, bidders have access to a large shared refrigerator to store the items they have won; however, they must pay rental costs to use it.

According to source Ⓔ, with some exceptions, a bidder can only purchase fruits or vegetables from other bidders if they have not yet exceeded 20% of the previous year's transaction volume. This regulation is intended to encourage bidders to purchase through auctions, bargaining, or special transactions, rather than relying on a secondary market between bidders.

Indeed, bidders are asymmetric. Among the 87,349 apple auctions I observe, there are 264 unique winning bidders, and the top 10% (26 bidders) win 33.7% of the auctions. Although my two-period model assumes that all bidders are symmetric at the start of the first auction, in reality, the bidders are asymmetric.

Focusing again on the five auction houses that sell fruits, two auction houses, 'Joongang' and 'Donghwa,' disclose[59] that Joongang has 120 veggie bidders and 79 fruit bidders, while Donghwa has 256 veggie bidders and 98 fruit bidders.

D.1.3 Auctioneer

[Back to ToC] Paragraph 1 of Article 33 in Ⓐ specifies that the basic principle for sequencing the order of objects in an auction follows the order of consignment. However, Paragraph 2 states that, if necessary for efficient distribution, the wholesale market (i.e., Garak market in our case) can deviate from the principle outlined in Paragraph 1. It is through this deviation that the Auction Houses (and the auctioneer who works in one of the Auction Houses) can adjust the order of items to be auctioned. Article 46 of Ⓔ specifies that the auctioneer may prioritize selling products that are in large quantities or of high quality (further details can be found in Article 46).

An auctioneer works for one of six auction houses, each typically specializing in a narrow range of varieties (e.g., selling only fruits but not vegetables). Although I couldn't get all the answers to my questions from the auctioneer I contacted, I can safely assume that the auctioneer usually begins the auction by selling high-quality products, as indicated by D.1.5 and various sources. If high-quality products generally come from Place A, this ordering method could be seen as discriminating against farmers from other regions. To address this issue, some

[59] As of Aug 26, 2024

auctioneers group locations into a few categories and rotate the order daily, so that on some days produce from Place A is auctioned off first, and on other days, produce from Place B is auctioned off first.

Typically, if a farmer from Place A requests to sell his produce (e.g., Fuji apples), there are three to four objects, such as '32 boxes of large apples,' '45 boxes of medium apples,' and '56 boxes of small apples.' Almost always, the auctioneer first auctions off the large apples, followed by the medium apples, and then the small apples, before moving on to the next farmer from Place A.

What I gathered from a thread written by an auctioneer is that not all farmers prefer their produce to be sold earlier in the auction. Typically, when there is a large supply of items to be sold at auction and farmers anticipate lower winning bids, they generally prefer their items to be sold earlier. However, in the opposite case, when the supply of items is expected to be small, farmers tend to prefer that their items be sold in the middle or later part of the auction.

A bidder I contacted told me that a farmer cannot request which auctioneer sells his item (i.e., the farmer can choose the Auction House, but not a specific auctioneer within that House). Once the item is sold, the auctioneer notifies the farmer of the winning bid.

As of 2022 (the period covered by my dataset), paper invoices were the predominant means of communication between bidders and the Auction House. As noted in D.1.5 , this hampers the Auction House's ability to make accurate predictions about the type and quantity of produce expected to arrive the next day or the day after. To address this issue, the government has been encouraging farmers and Auction Houses to adopt electronic invoices, enabling the Auction House to make more precise forecasts.

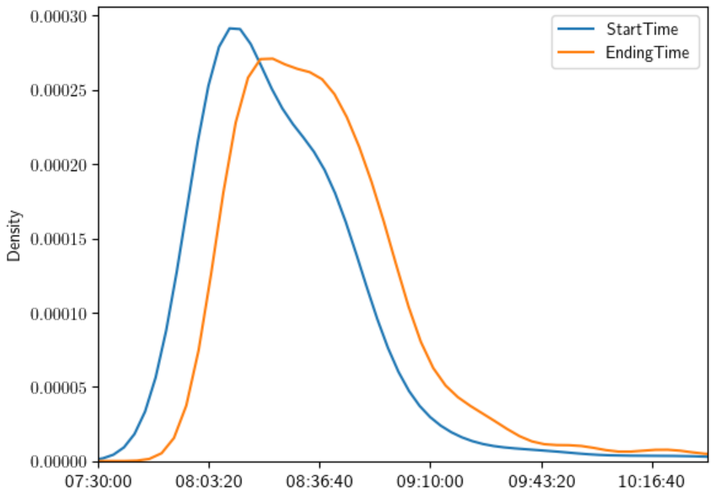
D.1.4 Other Descriptive Statistics or Features of Apple Auction

[Back to ToC] Further information available upon inquiry: I intend to describe how I merged the two separate datasets.

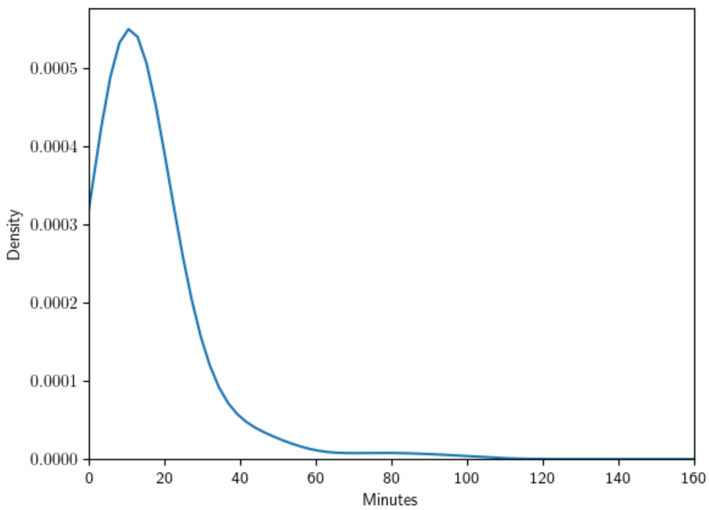
- ✓ Reserve Price: One of Ⓐ-Ⓔ states that a farmer can set a reserve price when requesting his produce to be sold. However, no farmers are known to use this reserve price. If the produce he requests is not sold at the Garak Market, the farmer either moves it to another small-sized auction market at his own expense or uses an external refrigerator to keep it for a day or two before requesting that it be sold again at the Garak Market, also at his own expense.

  Thus, when a farmer requests his produce to be sold at a certain auction house, the auctioneer typically informs him whether it is a good or bad time to sell. However, the problem is that even if it is a bad time, a farmer cannot keep his produce for an extended period because farmers are typically small-sized operations. One of the issues is that the supply of produce in the Garak Market is volatile, leading to instances where too much produce enters the market, causing prices to dip (or vice-versa too). Items listed in D.1.5 commonly highlight that "the ability of farmers (or producers) to delay or expedite their shipment (산지 출하물량 조절 역량)" is weak.

- ✓ Apple Auctions Start/Ending Time: For each five auction and for each day, I can observe the first and the last apple auctions (835 auctions each; I excluded Saturday), leading me to create the following two figures (in plotting these figures, I excluded Saturdays, as there are typically fewer apples auctioned on that day).



Density Plot of Starting and Ending Times: Apples



Density Plot of Time Spent on Apple Auctions

The top panel shows the density plots of starting and ending times. The descriptive statistics are as follows:

— Starting Time: mean(8:04:59), standard deviation(1:22:17), median(8:17:14)

— Ending Time: mean(8:33:11), standard deviation(57:55), median(8:34:23)

The bottom panel shows the density plot of the time spent on apple auctions: 835 observations that I used represent the time difference between the starting and ending times.

— mean(28:13), standard deviation(1:05:11), median(12:04)

D.1.5 Korean Reports

[Back to ToC] List of press releases follow:

- 가락시장 수산부류 '응찰자 가리기' 경매 전면 시행 (서울시농수산식품공사,24.6) // 세계 최초 농산물 온라인도매시장 출범 (농림축산식품부,23.11) // 농산물 산지유통센터 (APC) 스마트화·광역화 추진 계획 (농림축산식품부,23.7) // 2022 년 2 월 가락시장 청과부류 거래실적 분석 (서울시농수산식품공사,22.3) // 2022 년 가락시장 청과부류 거래실적 (서울시농수산식품공사,23.2) // 가락시장 농산물 경매 공정성 강화한다 (서울시농수산식품공사,20.11) // 가락시장 농산물 경매 진행 방식 개선 (서울시농수산식품공사,20.8) // 농식품부, 농산물 도매시장 유통환경을 바꾸겠습니다!(농림축산식품부,23.1) // 먹거리 물가안정과 함께 과수산업 경쟁력 제고 및 유통구조 개선 노력 강화 (기획재정부,24.4) // 농산물 도매시장 거래제도의 쟁점과 과제 (국회입법조사처,15.9) // 농수산물 유통경로 다양화와 경쟁 촉진을 통해 유통비용 10% 이상 절감 (관계부처합동,24.5)

List of papers or reports follows:

- 명절 과일 수요 및 가격 분석 (한국농촌경제연구원,16.8) // 농수산물도매시장 주요 쟁점과 정책적 함의 (한국농촌경제연구원,21.6) // 농수산물 유통구조 개선방안 (KDI,24.5) // 2022 년도 농수산물 도매시장 통계연보 (23.1, 농림축산식품부) // 채소 수급 및 가격안정화 방안 연구 (한국농촌경제연구원,11.11) // 전자식 경매 도입이 가락시장의 가격효율성에 미치는 영향분석 (농업경영 정책연구 제 38 권 제 2 호,11.6) // 제 7 차 농어업분과위원회 결과 보고 (대통령직속 농어업 농어촌 특별위원회,21.2) // 세계 도매시장별 가격변동성 비교 연구 (농식품신유통연구원,21.5) // 가락시장 청과부류 정가·수의매매 거래실태 분석 및 개선방안 도출 연구 (농식품신유통연구원,19.11) // 2022 농수산물 도매시장 통계연보 (농림축산식품부,24.3)

List of symposiums or seminars that I watched follows:

- [NBS 초대석] 공영도매시장의 올바른 개선방향은? 순천대 농경제학과 이춘수 교수, 농산물 도매시장의 공익적 역할 재정립을 위한 심포지엄 (종합본)

Miscellaneous reports, news articles, and miscellaneous videos that I read or watched are excluded.

Documents and seminars here recommend policies that encourage bidders and farmers to use bargaining instead of auctions as a main transaction channel, although this approach has not achieved significant success so far. Introducing a mandatory reserve price as a circuit breaker has also been considered but was discarded due to the risk of bidders forming a bidding ring to drive winning bids down to the reserve price level.

D.1.6 Oscillating and Decreasing Winning Bids

[Back to ToC] Figure C below corresponds to Figure 15. Figures A, B, and D to F illustrate the results when the number of order bins is adjusted or when the number of boxes in each auction is used as the weight.

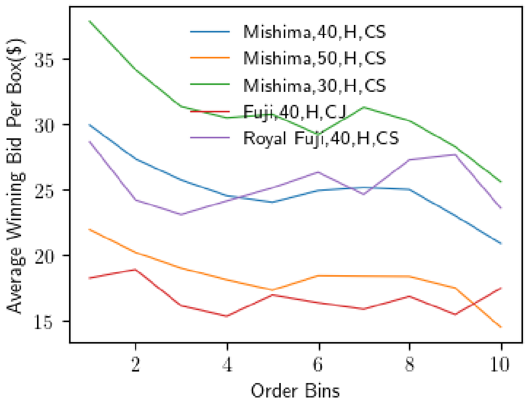


Fig. A – 10 order bins; Simple average winning bid.

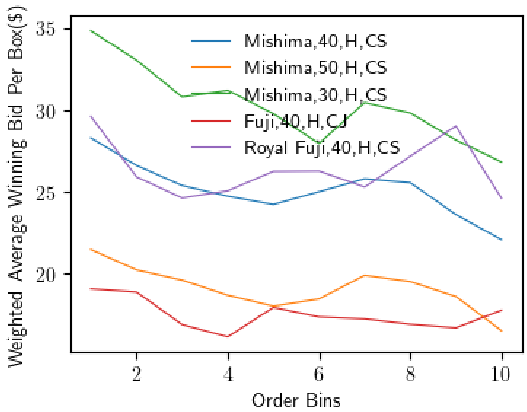


Fig. B – 10 order bins; Weighted average winning bid, with the number of boxes in each auction as the weight.

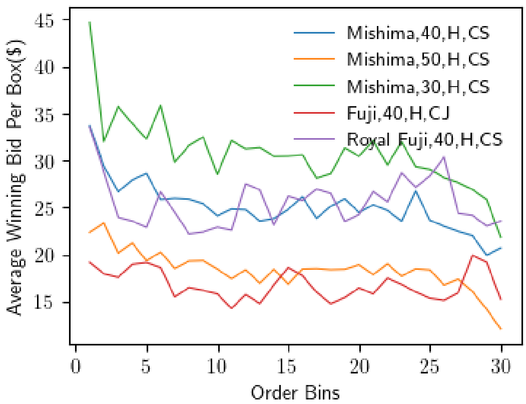


Fig. C – 30 order bins; Simple average winning bid.

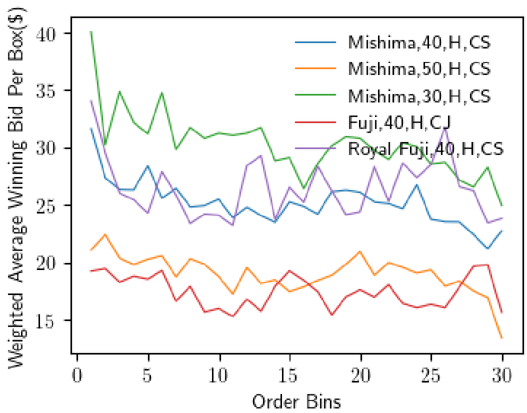


Fig. D – 30 order bins; Weighted average winning bid, with the number of boxes in each auction as the weight.

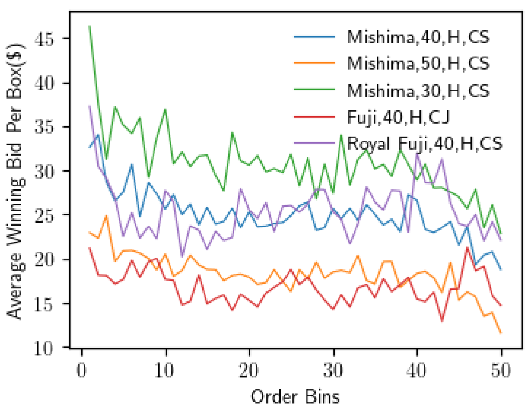


Fig. E – 50 order bins; Simple average winning bid.

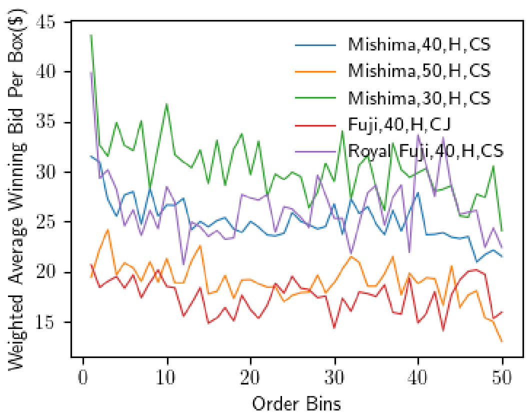


Fig. F – 50 order bins; Weighted average winning bid, with the number of boxes in each auction as the weight.

We observe that the average winning bid oscillates throughout the day and tends to decline; the regression results in D.5.1 also show a negative coefficient on the order variable. This decline in winning bids is inconsistent with another auction theory[60], which predicts that winning bids should follow a martingale process—oscillating without a downward trend. According to this theory, the winning bid remains stable because the forces driving it up (i.e., the decreasing number of remaining objects) and those driving it down (i.e., the decreasing number of bidders) are perfectly balanced.

The observed decline in winning bids can be attributed to the lower value of the remaining objects, making bidders less inclined to place higher bids and thereby disrupting the balance. This decrease in value is related to the sequence in which each object is auctioned. As mentioned in 5.1, the auctioneer has a certain degree of discretion in setting the auction order and is known to prioritize high-quality products or those from reputable farmers at the beginning to anchor bidders' perceptions. Starting with high-quality products suggests that the quality of items in the middle and latter parts of the auction is lower than at the beginning. Additionally, objects auctioned later may experience some deterioration due to inadequate air conditioning at the auction site, which is why D.1.5 reports similar patterns of decreasing and oscillatory winning bids for other types of produce

D.2 Model of Asymmetric Bidders

[Back to ToC] Recall that in the original model, where I bidders are symmetric at the onset of the auction, the main restrictions are that three functions ξ_2^w, ξ_2^l and ξ_1 be monotone.

This appendix serves to show that the number of restrictions exponentially increases as the number of asymmetric bidders also increase. I assume three bidders, $\{i, j, k\}$; If I were to expand it to five or six bidders, still the same logic applies.

The parameters of interest are as follows, which are in contrast with the parameters of the original model, $[F_1, F_{2|1}, \delta]$.

- $F_{1i}(\cdot)$, $F_{1j}(\cdot)$, $F_{1k}(\cdot)$, $F_{2|1}(\cdot|\cdot)$, $\delta(\cdot,\cdot)$.

One could replace a set $\{F_{2|1}(\cdot|\cdot), \delta(\cdot,\cdot)\}$ above with $\{F_{2n|1n}, \delta_n,\ \ n \in \{i, j, k\}\}$. I assume that I am a bidder i, and derive equilibrium strategies; if I were to assume either bidders j or k, still the same logic applies too.

Before specifying equilibrium strategies, I assume that the parameters above are common knowledge among bidders. This implies that bidder i knows that j and k are subject to F_{1j}, F_{1k}, $F_{2|1}$, and δ.

D.2.1 When i is the first auction winner

[Back to ToC] When bidder i wins the first auction with a winning bid of $\tilde{b}_{1i} = \tilde{s}_1(v_{1i})$ and enters the second auction, he has to choose the optimal amount of b_{2i}^w — the reason I use tilde for $\tilde{s}_1$ is that at this stage, i's first bid need not be the equilibrium strategy.

$$[\delta(v_{1i}, v_{2i}) - b_{2i}^w]\ \Pr[B_{2j}^{li} \leq b_{2i}^w, B_{2k}^{li} \leq b_{2i}^w \mid B_{1j} \leq \tilde{b}_{1i}, B_{1k} \leq \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}],$$

[60] Refer to chapter 15.1.3 of Krishna (2010b) and the first page of Van Den Berg et al. (2001). Indeed, chapter 15.1.3 of Krishna (2010b) assumes a bidder with unit demand, which differs from my setting, but the notion that the two countervailing forces exactly offsetting each other is what I am concentrating on. The result of this offsetting is that the path of the winning bid is martingale. Thus, the decline in winning bid is regarded to be anomalous, which is discussed in Ashenfelter (1989) and Mcafee and Vincent (1993).

in which B^{li}_{2j} denotes that it is bidder j's second auction bid who lost the first auction to bidder i. The probability term above is equivalent to the following.

$$\begin{aligned}&\Pr[s^l_{2j}(V_{1j}, V_{2j}, \tilde{b}_{1i}; \{i\}) \leq b^w_{2i}, s^l_{2k}(V_{1k}, V_{2k}, \tilde{b}_{1i}; \{i\}) \leq b^w_{2i} \\ &\quad \mid s_{1j}(V_{1j}) \leq \tilde{b}_{1i}, s_{1k}(V_{1k}) \leq \tilde{b}_{1i}, V_{1i} = v_{1i}, V_{2i} = v_{2i}]\end{aligned} \tag{87}$$

For example, $s^l_{2j}(\cdot, \cdot; \{i\})$ indicates that this equilibrium strategy is for bidder j who faces first auction winner i. (87) is equivalent to the following:

$$\begin{aligned}&\Pr[s^l_{2j}(V_{1j}, V_{2j}, \tilde{b}_{1i}; \{i\}) \leq b^w_{2i} \mid s_{1j}(V_{1j}) \leq \tilde{b}_{1i}]\times \\ &\quad \Pr[s^l_{2k}(V_{1k}, V_{2k}, \tilde{b}_{1i}; \{i\}) \leq b^w_{2i} \mid s_{1k}(V_{1k}) \leq \tilde{b}_{1i}]\end{aligned} \tag{88}$$

(88) holds because,

1. The condition $\{V_{1i} = v_{1i}, V_{2i} = v_{2i}\}$ inside (87) is independent with the random variables $\{V_{1j}, V_{2j}, V_{1k}, V_{2k}\}$, so it can be omitted.
2. After the omission of the condition $\{V_{1i} = v_{1i}, V_{2i} = v_{2i}\}$, the only random variables left are $V_{1j}, V_{2j}, V_{1k}, V_{2k}$. Note that V_{1j} being less than some number provides no information about what V_{1k}, V_{2k} will be, and also note that V_{1k} being less than some number provides no information about what V_{1j}, V_{2j} will be. Therefore, the conditional independence holds so that (87) changes to (88).

I can rewrite (88) as follows.

$$\Pr[B^{li}_{2j} \leq b^w_{2i} | B_{1j} \leq \tilde{b}_{1i}] \times \Pr[B^{li}_{2k} \leq b^w_{2i} | B_{1k} \leq \tilde{b}_{1i}].$$

So, what bidder i has to solve is the following problem.

$$[\delta(v_{1i}, v_{2i}) - b^w_{2i}] \;\; G_{B^{li}_{2j}(\tilde{b}_{1i})}(b^w_{2i} | B_{1j} \leq \tilde{b}_{1i}) G_{B^{li}_{2k}(\tilde{b}_{1i})}(b^w_{2i} | B_{1k} \leq \tilde{b}_{1i}), \tag{89}$$

in which if I take a derivative with respect to b^w_{2i}, I get the following first order condition.

$$\delta(v_{1i}, v_{2i}) = b^w_{2i} + \frac{G_{B^{li}_{2j}(\tilde{b}_{1i})}(b^w_{2i} | B_{1j} \leq \tilde{b}_{1i}) G_{B^{li}_{2k}(\tilde{b}_{1i})}(b^w_{2i} | B_{1k} \leq \tilde{b}_{1i})}{\frac{\partial}{\partial b^w_{2i}} G_{B^{li}_{2j}(\tilde{b}_{1i})}(b^w_{2i} | B_{1j} \leq \tilde{b}_{1i}) G_{B^{li}_{2k}(\tilde{b}_{1i})}(b^w_{2i} | B_{1k} \leq \tilde{b}_{1i})} \equiv \xi^w_{2i}(b^w_{2i}; \tilde{b}_{1i}) \tag{90}$$

For future use, I define the following:

$$H^w_{2i}(b^w_{2i}; \tilde{b}_{1i}) = G_{B^{li}_{2j}(\tilde{b}_{1i})}(b^w_{2i} | B_{1j} \leq \tilde{b}_{1i}) G_{B^{li}_{2k}(\tilde{b}_{1i})}(b^w_{2i} | B_{1k} \leq \tilde{b}_{1i}) \tag{91}$$

I am aware that b^w_{2i} in (90) need not be the equilibrium bid, but I leave the notation as it is.

D.2.2 When i loses the first auction and the winner is j

[Back to ToC] Bidder i has to solve the following problem. As the title of this subsection says, the first auction winner here is assumed to be bidder j — according to the disclosure policy, j's winning bid b_{1j} is publicly known.

$$[v_{2i} - b_{2i}^{lj}]\ \Pr[B_{2j}^{w} \leq b_{2i}^{lj}, B_{2k}^{lj} \leq b_{2i}^{lj} \mid B_{1j} = b_{1j}, B_{1k} \leq B_{1j}, \tilde{b}_{1i} \leq B_{1j}, V_{1i} = v_{1i}, V_{2i} = v_{2i}],$$

in which the probability term is equivalent to the following.

$$\begin{aligned}&\Pr[s_{2j}^{w}(V_{1j}, V_{2j}) \leq b_{2i}^{lj}, s_{2k}^{l}(V_{2k}, b_{1j}, \{j\}) \leq b_{2i}^{lj} \\ &\qquad \mid s_{1j}(V_{1j}) = b_{1j}, s_{1k}(V_{1k}) \leq b_{1j}, \tilde{b}_{1i} \leq b_{1j}, V_{1i} = v_{1i}, V_{2i} = v_{2i}].\end{aligned} \tag{92}$$

Following the same logic(i.e., independence of the random variables), (92) is the same as the following — in (92), what is known from i's perspective is $\{V_{1i} = v_{1i}, V_{2i} = v_{2i}, V_{1j} = s_1^{-1}(b_{1j})\}$

$$H_{2i}^{l}(b_{2i}^{lj}; b_{1j}, \{j\}) \equiv G_{B_{2j}^{w}|B_{1j}}(b_{2i}^{lj}|b_{1j}) \times G_{B_{2k}^{lj}(b_{1j})}(b_{2i}^{lj}|B_{1k} \leq b_{1j}) \tag{93}$$

for $\tilde{b}_{1i} \leq b_{1j}$. Thus the first order condition for bidder i with respect to his second bid b_{2i}^{lj} is as follows.

$$v_{2i} = b_{2i}^{lj} + \frac{G_{B_{2j}^{w}|B_{1j}}(b_{2i}^{lj}|b_{1j}) G_{B_{2k}^{lj}(b_{1j})}(b_{2i}^{lj}|B_{1k} \leq b_{1j})}{\frac{\partial}{\partial b_{2i}^{lj}} G_{B_{2j}^{w}|B_{1j}}(b_{2i}^{lj}|b_{1j}) G_{B_{2k}^{lj}(b_{1j})}(b_{2i}^{lj}|B_{1k} \leq b_{1j})} \equiv \xi_{2i}^{l}(b_{2i}^{lj}; b_{1j}, \{j\}) \tag{94}$$

for $\tilde{b}_{1i} \leq b_{1j}$.
I am aware that b_{2i}^{lj} in (94) need not be the equilibrium bid, but I leave the notation as it is.

D.2.3 When i loses the first auction and the winner was k

[Back to ToC] Unlike the previous subsection, the first auction winner here is bidder k, not bidder j. If I use the same logic, I get the following First order condition.

$$v_{2i} = b_{2i}^{lk} + \frac{G_{B_{2k}^{w}|B_{1k}}(b_{2i}^{lk}|b_{1k}) G_{B_{2j}^{lk}(b_{1k})}(b_{2i}^{lk}|B_{1j} \leq b_{1k})}{\frac{\partial}{\partial b_{2i}^{lk}} G_{B_{2k}^{w}|B_{1k}}(b_{2i}^{lk}|b_{1k}) G_{B_{2j}^{lk}(b_{1k})}(b_{2i}^{lk}|B_{1j} \leq b_{1k})} \equiv \xi_{2i}^{l}(b_{2i}^{lk}; b_{1k}, \{k\}) \tag{95}$$

for $\tilde{b}_{1i} \leq b_{1k}$. For future use, I define the following:

$$H_{2i}^{l}(b_{2i}^{lk}; b_{1k}, \{k\}) \equiv G_{B_{2k}^{w}|B_{1k}}(b_{2i}^{lk}|b_{1k}) \times G_{B_{2j}^{lk}(b_{1k})}(b_{2i}^{lk}|B_{1j} \leq b_{1k}) \tag{96}$$

I am aware that b_{2i}^{lk} in (95) need not be the equilibrium bid, but I leave the notation as it is.

D.2.4 Continuation Values

[Back to ToC] In the next subsection, I will use three continuation values, $\mathcal{V}_i^{w}(v_{1i}, \tilde{b}_{1i})$, $\mathcal{V}_i^{l}(v_{1i}, \tilde{b}_{1i}, \{j\})$, and $\mathcal{V}_i^{l}(v_{1i}, \tilde{b}_{1i}, \{k\})$, so I need to define those — the last two continuation values denote the case in which either bidder j or bidder k being the first auction winner.

First, if bidder i is the first auction winner, and given (90), then his (optimal) expected profit in the second auction is:

$$\frac{G_{B_{2j}^{li}(\tilde{b}_{1i})}(b_{2i}^{w}|B_{1j}\leq \tilde{b}_{1i})^2 G_{B_{2k}^{li}(\tilde{b}_{1i})}(b_{2i}^{w}|B_{1k}\leq \tilde{b}_{1i})^2}{\frac{\partial}{\partial b_{2i}^{w}} G_{B_{2j}^{li}(\tilde{b}_{1i})}(b_{2i}^{w}|B_{1j}\leq \tilde{b}_{1i}) G_{B_{2k}^{li}(\tilde{b}_{1i})}(b_{2i}^{w}|B_{1k}\leq \tilde{b}_{1i})}$$
$$\equiv \frac{H_{2i}^{w}(b_{2i}^{w};\tilde{b}_{1i})^2}{h_{2i}^{w}(b_{2i}^{w};\tilde{b}_{1i})},$$

in which b_{2i}^{w} is $\tilde{s}_{2i}^{w}(v_{1i}, v_{2i}, \tilde{b}_{1i})$.

Right after the first auction and before the start of the second auction, what bidder i knows is that he has v_{1i} but is uncertain of his V_{2i}. Therefore, his $\mathcal{V}_i^{w}(v_{1i}, \tilde{b}_{1i})$ is as follows.

$$\mathbb{E}_{V_2|V_1}\left[\left.\frac{H_{2i}^{w}(\tilde{s}_{2i}^{w}(v_{1i}, V_2, \tilde{b}_{1i});\tilde{b}_{1i})^2}{h_{2i}^{w}(\tilde{s}_{2i}^{w}(v_{1i}, V_2, \tilde{b}_{1i});\tilde{b}_{1i})}\right| v_{1i}\right]. \tag{97}$$

Note here that the expectation is taken with respect to $\Pr[V_2 \leq \cdot|V_1 = v_{1i}]$ not $\Pr[V_{2i} \leq \cdot|V_{1i} = v_{1i}]$ because the given parameter is $F_{2|1}$.

Second, if bidder i is the first auction loser and if the winner was j, then given (94), bidder i's (optimal) expected profit in the second auction is:

$$\frac{G_{B_{2j}^{w}|B_{1j}}(b_{2i}^{lj}|b_{1j})^2 G_{B_{2k}^{lj}(b_{1j})}(b_{2i}^{lj}|B_{1k}\leq b_{1j})^2}{\frac{\partial}{\partial b_{2i}^{lj}} G_{B_{2j}^{w}|B_{1j}}(b_{2i}^{lj}|b_{1j}) G_{B_{2k}^{lj}(b_{1j})}(b_{2i}^{lj}|B_{1k}\leq b_{1j})}$$
$$\equiv \frac{H_{2i}^{l}(b_{2i}^{lj}; b_{1j}, \{j\})^2}{h_{2i}^{l}(b_{2i}^{lj}; b_{1j}, \{j\})},$$

in which b_{2i}^{lj} is $\tilde{s}_{2i}^{l}(v_{2i}, b_{1j}; \{j\})$. Right after the first auction and before the start of the second auction, what bidder i knows is that he has v_{1i} but is uncertain of his V_{2i} and what the winning bid will be. Thus, before calculating $\mathcal{V}_i^{l}(v_{1i}, \tilde{b}_{1i}, \{j\})$, we need to know the following conditional distribution.

$$\begin{aligned}
&\Pr[V_{2i}\leq \cdot, B_{1j}\leq \cdot|V_{1i}=v_{1i}, B_{1j}>\tilde{b}_{1i}, B_{1j}>B_{1k}]\\
&=\frac{\Pr[V_{2i}\leq \cdot, B_{1j}\leq \cdot, V_{1i}=v_{1i}, B_{1j}>\tilde{b}_{1i}, B_{1j}>B_{1k}]}{\Pr[V_{1i}=v_{1i}, B_{1j}>\tilde{b}_{1i}, B_{1j}>B_{1k}]}\\
&=F_{2|1}(\cdot|v_{1i})\frac{\Pr[B_{1j}\leq \cdot, B_{1j}>\tilde{b}_{1i}, B_{1j}>B_{1k}]}{\Pr[B_{1j}>\tilde{b}_{1i}, B_{1j}>B_{1k}]}\\
&=F_{2|1}(\cdot|v_{1i})\frac{\Pr[B_{1j}\leq \cdot, B_{1j}>B_{1k}|B_{1j}>\tilde{b}_{1i}]}{\Pr[B_{1j}>B_{1k}|B_{1j}>\tilde{b}_{1i}]}\\
&=F_{2|1}(\cdot|v_{1i})\Pr[B_{1j}\leq \cdot, B_{1j}>B_{1k}|B_{1j}>\tilde{b}_{1i}]\frac{1-G_{B_{1j}}(\tilde{b}_{1i})}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b_{1j})g_{B_{1j}}(b_{1j})db_{1j}}\\
&=F_{2|1}(\cdot|v_{1i})\frac{\int_{\tilde{b}_{1i}}^{\cdot} G_{B_{1k}}(b_{1j})g_{B_{1j}}(b_{1j})db_{1j}}{1-G_{B_{1j}}(\tilde{b}_{1i})}\frac{1-G_{B_{1j}}(\tilde{b}_{1i})}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b_{1j})g_{B_{1j}}(b_{1j})db_{1j}}\\
&=F_{2|1}(\cdot|v_{1i})\frac{\int_{\tilde{b}_{1i}}^{\cdot} G_{B_{1k}}(b_{1j})g_{B_{1j}}(b_{1j})db_{1j}}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b_{1j})g_{B_{1j}}(b_{1j})db_{1j}},
\end{aligned}$$

in which the second equality holds by the independence assumption, and that $\Pr[V_{2i} \leq \cdot | V_{1i} = \cdot]$ is the same as $F_{2|1}$.

Given that we know the conditional distribution needed for $\mathcal{V}_i^l$, I calculate $\mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{j\})$ as follows.

$$\frac{1}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b_{1j}) g_{B_{1j}}(b_{1j}) db_{1j}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{j\}); b_1, \{j\})^2}{h_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{j\}); b_1, \{j\})} dF_{2|1}(v_2|v_{1i}) d \int_{\tilde{b}_{1i}}^{b_1} G_{B_{1k}}(b_{1j}) g_{B_{1j}}(b_{1j}) db_{1j}$$
$$= \frac{1}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b_{1j}) dG_{B_{1j}}(b_{1j})} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{j\}); b_1, \{j\})^2}{h_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{j\}); b_1, \{j\})} dF_{2|1}(v_2|v_{1i}) G_{B_{1k}}(b_1) g_{B_{1j}}(b_1) db_1 \tag{98}$$

Then using the same logic, $\mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{k\})$ in which now bidder k is the first winner is as follows.

$$\frac{1}{\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1j}}(b_{1k}) dG_{B_{1k}}(b_{1k})} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{k\}); b_1, \{k\})^2}{h_{2i}^l(\tilde{s}_{2i}^l(v_2, b_1; \{k\}); b_1, \{k\})} dF_{2|1}(v_2|v_{1i}) G_{B_{1j}}(b_1) g_{B_{1k}}(b_1) db_1 \tag{99}$$

D.2.5 i in the first auction

[Back to ToC] Bidder i has to solve the following problem — he has to choose the optimal $\tilde{b}_{1i}$.

$$\begin{aligned}
&[v_{1i} - \tilde{b}_{1i} + \mathcal{V}_i^w(v_{1i}, \tilde{b}_{1i})] \Pr[B_{1j} \leq \tilde{b}_{1i}, B_{1k} \leq \tilde{b}_{1i} | V_{1i} = v_{1i}] \\
&+ \mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{j\}) \Big[\Pr[B_{1j} > \tilde{b}_{1i}, B_{1k} \leq \tilde{b}_{1i} | V_{1i} = v_{1i}] + \Pr[B_{1j} > \tilde{b}_{1i}, B_{1k} > \tilde{b}_{1i}, B_{1j} > B_{1k} | V_{1i} = v_{1i}]\Big] \\
&+ \mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{k\}) \Big[\Pr[B_{1j} \leq \tilde{b}_{1i}, B_{1k} > \tilde{b}_{1i} | V_{1i} = v_{1i}] + \Pr[B_{1j} > \tilde{b}_{1i}, B_{1k} > \tilde{b}_{1i}, B_{1j} \leq B_{1k} | V_{1i} = v_{1i}]\Big],
\end{aligned} \tag{100}$$

in which five probability terms appear. The *first* probability equals the following.

$$\Pr[s_{1j}(V_{1j}) \leq \tilde{b}_{1i}, s_{1k}(V_{1k}) \leq \tilde{b}_{1i} | V_{1i} = v_{1i}].$$

Given the structure of the random variable, I can express above as follows.

$$G_{B_{1j}}(\tilde{b}_{1i}) G_{B_{1k}}(\tilde{b}_{1i}).$$

The *third* probability equals the following.

$$\Pr[s_{1j}(V_{1j}) > \tilde{b}_{1i}, s_{1k}(V_{1k}) > \tilde{b}_{1i}, s_{1j}(V_{1j}) > s_{1k}(V_{1k}) | V_{1i} = v_{1i}].$$

I can rewrite this *third* probability as follows. Note that I omit the condition $\{V_{1i} = v_{1i}\}$ because the omission has no effect on the probability (Let $U_j \equiv s_{1j}(V_{1j})$ and $U_k \equiv s_{1k}(V_{1k})$).

$$\begin{aligned}
&\Pr[U_{1j} > \tilde{b}_{1i}, U_{1k} > \tilde{b}_{1i}, U_{1j} > U_{1k}] \\
&= \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{y}^{\overline{b_1}} \Pr[U_{1j} = x] \Pr[U_{1k} = y] \, dx dy
\end{aligned}$$

$$= \int_{\tilde{b}_{1i}}^{\overline{b_1}} \Pr[U_{1j} > y] \Pr[U_{1k} = y] dy$$

Thus, I can rewrite the *third* probability as follows.

$$\int_{\tilde{b}_{1i}}^{\overline{b_1}} (1 - G_{B_{1j}}(b)) g_{B_{1k}}(b) db.$$

I can also rewrite the *fifth* probability, $\Pr[B_{1j} > \tilde{b}_{1i}, B_{1k} > \tilde{b}_{1i}, B_{1j} \leq B_{1k} | V_{1i} = v_{1i}]$, as follows.

$$\int_{\tilde{b}_{1i}}^{\overline{b_1}} (1 - G_{B_{1k}}(b)) g_{B_{1j}}(b) db$$

Given the derivations, I rewrite (100) as follows.

$$\begin{aligned} &[v_{1i} - \tilde{b}_{1i} + \mathcal{V}_i^w(v_{1i}, \tilde{b}_{1i})] G_{B_{1j}}(\tilde{b}_{1i}) G_{B_{1k}}(\tilde{b}_{1i}) \\ &+ \mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{j\}) \left[\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b) dG_{B_{1j}}(b) \right] \\ &+ \mathcal{V}_i^l(v_{1i}, \tilde{b}_{1i}, \{k\}) \left[\int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1j}}(b) dG_{B_{1k}}(b) \right], \end{aligned} \tag{101}$$

by noting the following equivalences:

$$\begin{aligned} (1 - G_{B_{1j}}(\tilde{b}_{1i})) G_{B_{1k}}(\tilde{b}_{1i}) + \int_{\tilde{b}_{1i}}^{\overline{b_1}} (1 - G_{B_{1j}}(b)) g_{B_{1k}}(b) db &= \int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1k}}(b) dG_{B_{1j}}(b) \\ (1 - G_{B_{1k}}(\tilde{b}_{1i})) G_{B_{1j}}(\tilde{b}_{1i}) + \int_{\tilde{b}_{1i}}^{\overline{b_1}} (1 - G_{B_{1k}}(b)) g_{B_{1j}}(b) db &= \int_{\tilde{b}_{1i}}^{\overline{b_1}} G_{B_{1j}}(b) dG_{B_{1k}}(b) \end{aligned}$$

(101) is what bidder i has to solve by choosing optimal $\tilde{b}_{1i}$. The continuation values inside (101) can be replaced with (97), (98), and (99). By this replacement, (101) becomes as follows.

$$\begin{aligned} &\left[v_{1i} - \tilde{b}_{1i} + \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^w(\tilde{b}_{2i}^w; \tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w; \tilde{b}_{1i})} dF_{2|1}(v_2|v_{1i}) \right] G_{B_{1j}}(\tilde{b}_{1i}) G_{B_{1k}}(\tilde{b}_{1i}) \\ &+ \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lj}; b_1, \{j\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lj}; b_1, \{j\})} dF_{2|1}(v_2|v_{1i}) G_{B_{1k}}(b_1) g_{B_{1j}}(b_1) db_1 \\ &+ \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lk}; b_1, \{k\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lk}; b_1, \{k\})} dF_{2|1}(v_2|v_{1i}) G_{B_{1j}}(b_1) g_{B_{1k}}(b_1) db_1, \end{aligned} \tag{102}$$

in which $\tilde{b}_{2i}^w \equiv \tilde{s}_{2i}^w(v_{1i}, v_2, \tilde{b}_{1i})$, $\tilde{b}_{2i}^{lj} \equiv \tilde{s}_{2i}^l(v_2, b_1; \{j\})$, and $\tilde{b}_{2i}^{lk} \equiv \tilde{s}_{2i}^l(v_2, b_1; \{k\})$. Note that these three tildes are the optimizer(or argmax) of a profit function, which enables me to think of Envelope theorem.

If I take a derivative of (102) with respect to $\tilde{b}_{1i}$ and let it equal 0, I get the following equation.

$$
\begin{aligned}
&\left[-1+\int_{\underline{v_2}}^{\overline{v_2}} \frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} \right) dF_{2|1}(v_2|v_{1i})\right] G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i}) \\
&+\left[v_{1i}-\tilde{b}_{1i}+\int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} dF_{2|1}(v_2|v_{1i})\right] \frac{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial \tilde{b}_{1i}} \\
&+\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})} dF_{2|1}(v_2|v_{1i})G_{B_{1k}}(b_1)g_{B_{1j}}(b_1)db_1 \\
&+\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})} dF_{2|1}(v_2|v_{1i})G_{B_{1j}}(b_1)g_{B_{1k}}(b_1)db_1 = 0.
\end{aligned}
$$

Rearranging the equation yields the following:

$$
\begin{aligned}
&\left[v_{1i}-\tilde{b}_{1i}+\int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} dF_{2|1}(v_2|v_{1i})\right] \frac{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial \tilde{b}_{1i}} \\
&=\left[1-\int_{\underline{v_2}}^{\overline{v_2}} \frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} \right) dF_{2|1}(v_2|v_{1i})\right] G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i}) \\
&-\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})} dF_{2|1}(v_2|v_{1i})G_{B_{1k}}(b_1)g_{B_{1j}}(b_1)db_1 \\
&-\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})} dF_{2|1}(v_2|v_{1i})G_{B_{1j}}(b_1)g_{B_{1k}}(b_1)db_1
\end{aligned}
$$

Rearranging the equation *again* yields the following:

$$
\begin{aligned}
v_{1i} &= \tilde{b}_{1i} + \frac{G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \\
&- \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} dF_{2|1}(v_2|v_{1i}) - \frac{G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} \right) dF_{2|1}(v_2|v_{1i}) \\
&- \frac{1}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \times \\
&\quad \left[\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})} dF_{2|1}(v_2|v_{1i})G_{B_{1k}}(b_1)g_{B_{1j}}(b_1)db_1 + \right. \\
&\quad \left. \frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})} dF_{2|1}(v_2|v_{1i})G_{B_{1j}}(b_1)g_{B_{1k}}(b_1)db_1 \right] \equiv \xi_1(\tilde{b}_{1i}),
\end{aligned} \tag{103}
$$

in which $H_{2i}^w(\cdot)$, $H_{2i}^l(\cdot;\{j\})$, and $H_{2i}^l(\cdot;\{k\})$ come from (91), (93), and (96).

D.2.6 Equilibrium Strategies

[Back to ToC] Up to now, I haven't described in detail what the equilibrium strategies are. By referring to the first page of D.2, I define *monotone* strategies as follows.

Information sets of any bidder *at the beginning of the first and the second auctions* are as follows.

$$\mathcal{F}_1 = \{id, V_{1id}, F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I\}$$
$$\mathcal{F}_2 = \{id, V_{1id}, F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I, V_{2id}, B_{1id}, B_1^{\max}, W_1\},$$

where id denotes a bidder himself, and I denotes a set $\{i, j, k\}$. If we were to extend it to more than three bidders, then both $\mathcal{F}_1$ and $\mathcal{F}_2$ have to be updated accordingly.

A bidder's strategy is a pair of strategies $[s_1, s_2]$, one for each auction that depends on the information available to him *at the beginning of each auction.* So for any bidders, $\{i, j, k\}$, the first auction equilibrium bid, B_1, is as follows.

$$\begin{aligned} B_1 &= s_1(id, V_{1id}, F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I) \\ &= \begin{cases} s_1(i, V_{1i}) \equiv s_{1i}(V_{1i}) & \text{if } id = i \\ s_1(j, V_{1j}) \equiv s_{1i}(V_{1j}) & \text{if } id = j \\ s_1(j, V_{1k}) \equiv s_{1i}(V_{1k}) & \text{if } id = k, \end{cases} \end{aligned} \tag{104}$$

in which a strategy s_1 is strictly monotone in V_{1id}, and I omitted a set $\{F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I\}$ as the set is a common knowledge and invariant.

Before discussing equilibrium second auction bid, B_2, note the following:

> $\mathcal{F}_2$ includes B_{1id}, and B_{1id} comes from s_1 defined above. Since s_1 includes arguments $\{id$, V_{1id}, F_{1i}, F_{1j}, F_{1k}, $F_{2|1}$, δ, $I\}$, and given that these arguments are already inside $\mathcal{F}_2$, this implies that I can omit B_{1id} from $\mathcal{F}_2$.

Thus, equilibrium bid, B_2, is as follows.

$$\begin{aligned} B_2 &= s_2(id, V_{1id}, F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I, V_{2id}, B_1^{\max}, W_1) \\ &= \begin{cases} s_2(i, V_{1i}, V_{2i}, B_1^{\max}, i) \equiv s_{2i}^w(V_{1i}, V_{2i}) & \text{if } id = i, W_1 = i \\ s_2(i, V_{1i}, V_{2i}, B_1^{\max}, j) \equiv s_{2i}^l(V_{2i}, B_1^{\max}; \{j\}) & \text{if } id = i, W_1 = j \\ s_2(i, V_{1i}, V_{2i}, B_1^{\max}, k) \equiv s_{2i}^l(V_{2i}, B_1^{\max}; \{k\}) & \text{if } id = i, W_1 = k \\ s_2(j, V_{1j}, V_{2j}, B_1^{\max}, j) \equiv s_{2j}^w(V_{1j}, V_{2j}) & \text{if } id = j, W_1 = j \\ s_2(j, V_{1j}, V_{2j}, B_1^{\max}, i) \equiv s_{2j}^l(V_{2j}, B_1^{\max}; \{i\}) & \text{if } id = j, W_1 = i \\ s_2(j, V_{1j}, V_{2j}, B_1^{\max}, k) \equiv s_{2j}^l(V_{2j}, B_1^{\max}; \{k\}) & \text{if } id = j, W_1 = k \\ s_2(k, V_{1k}, V_{2k}, B_1^{\max}, k) \equiv s_{2k}^w(V_{1k}, V_{2k}) & \text{if } id = k, W_1 = k \\ s_2(k, V_{1k}, V_{2k}, B_1^{\max}, i) \equiv s_{2k}^l(V_{2k}, B_1^{\max}; \{i\}) & \text{if } id = k, W_1 = i \\ s_2(k, V_{1k}, V_{2k}, B_1^{\max}, j) \equiv s_{2k}^l(V_{2k}, B_1^{\max}; \{j\}) & \text{if } id = k, W_1 = j \end{cases} \end{aligned} \tag{105}$$

The strategy s_2 is monotone in V_{2id}, and I omitted a set $\{F_{1i}, F_{1j}, F_{1k}, F_{2|1}, \delta, I\}$ as the set is a common knowledge and invariant. I want to comment on the following points for (105):

- When id and W_1 are the same, I omitted $B_1^{\max}$ inside $s_{2id}^w(V_{1id}, V_{2id})$. The reason is that in this case, $B_1^{\max} = B_{1id}$ holds, and I already described above the justification of removing B_{1id} from $\mathcal{F}_2$.

- When id and W_1 are *not* the same, I omitted V_{1id} from $s^l_{2id}(V_{2id}, B_1^{\max}; \{W_1\})$. The reason is that V_{1id} provides no information because bidders are independent, and the second objects' value is solely decided by V_{2id}.

Given $[s_1, s_2]$ from (104) and (105), I describe how first order conditions change in equilibrium for bidder i. In equilibrium, bidder i's following bids, which were used in the previous subsections,

$$b^w_{2i} = \tilde{s}^w_{2i}(v_{1i}, v_{2i}, \tilde{b}_{1i}) \text{ from } (90)$$

$$b^{lj}_{2i} = \tilde{s}^l_{2i}(v_{2i}, b_{1j}; \{j\}) \text{ from } (94)$$

$$b^{lk}_{2i} = \tilde{s}^l_{2i}(v_{2i}, b_{1k}; \{k\}) \text{ from } (95)$$

$$\tilde{b}_{1i} = \tilde{s}_1(v_{1i}) \text{ from } (103)$$

must equal to his competitors strategies $[s_1, s_2]$. Thus, in equilibrium, the following holds.

$$b^w_{2i} = \tilde{s}^w_{2i}(v_{1i}, v_{2i}, \tilde{b}_{1i}) = s^w_{2i}(v_{1i}, v_{2i})$$

$$b^{lj}_{2i} = \tilde{s}^l_{2i}(v_{2i}, b_{1j}; \{j\}) = s^l_{2i}(v_{2i}, b_{1j}; \{j\})$$

$$b^{lk}_{2i} = \tilde{s}^l_{2i}(v_{2i}, b_{1k}; \{k\}) = s^l_{2i}(v_{2i}, b_{1k}; \{k\})$$

$$\tilde{b}_{1i} = \tilde{s}_1(v_{1i}) = s_1(v_{1i})$$

Given this equilibrium restriction put on bidder i, then bidder i's equilibrium bids $[s_1, s_2]$ satisfies the following equalities — they come from (90), (94), and (95).

$$\delta(v_{1i}, v_{2i}) = b^w_{2i} + \frac{G_{B^{li}_{2j}(b_{1i})}(b^w_{2i}|B_{1j} \le b_{1i}) G_{B^{li}_{2k}(b_{1i})}(b^w_{2i}|B_{1k} \le b_{1i})}{\frac{\partial}{\partial b^w_{2i}} G_{B^{li}_{2j}(b_{1i})}(b^w_{2i}|B_{1j} \le b_{1i}) G_{B^{li}_{2k}(b_{1i})}(b^w_{2i}|B_{1k} \le b_{1i})} \equiv \xi^w_{2i}(b^w_{2i}; b_{1i}) \tag{106}$$

$$v_{2i} = b^{lj}_{2i} + \frac{G_{B^w_{2j}|B_{1j}}(b^{lj}_{2i}|b_{1j}) G_{B^{lj}_{2k}(b_{1j})}(b^{lj}_{2i}|B_{1k} \le b_{1j})}{\frac{\partial}{\partial b^{lj}_{2i}} G_{B^w_{2j}|B_{1j}}(b^{lj}_{2i}|b_{1j}) G_{B^{lj}_{2k}(b_{1j})}(b^{lj}_{2i}|B_{1k} \le b_{1j})} \equiv \xi^l_{2i}(b^{lj}_{2i}; b_{1j}, \{j\}) \tag{107}$$

$$v_{2i} = b^{lk}_{2i} + \frac{G_{B^w_{2k}|B_{1k}}(b^{lk}_{2i}|b_{1k}) G_{B^{lk}_{2j}(b_{1k})}(b^{lk}_{2i}|B_{1j} \le b_{1k})}{\frac{\partial}{\partial b^{lk}_{2i}} G_{B^w_{2k}|B_{1k}}(b^{lk}_{2i}|b_{1k}) G_{B^{lk}_{2j}(b_{1k})}(b^{lk}_{2i}|B_{1j} \le b_{1k})} \equiv \xi^l_{2i}(b^{lk}_{2i}; b_{1k}, \{k\}), \tag{108}$$

in which b_{1j} and b_{1k} in (107) and (108) are higher than b_{1i}. To guarantee the equilibrium, the right-hand side of (106) must be increasing in b^w_{2i} for every b_{1i}, the right-hand sides of (107) and (108) must be increasing in b^{lj}_{2i} and b^{lk}_{2i} for every b_{1j} and b_{1k}. These monotonicities ensure that $[s^w_{2i}, s^l_{2i}(\cdot; \{j\}), s^l_{2i}(\cdot; \{k\})]$ inside (105) are indeed monotone strategies as desired.
(Note: (107) and (108) already imply testable restrictions. If b^{lj}_{2i} and b^{lk}_{2i} are the same numbers, and if b_{1j} and b_{1k} are the same numbers, and if the resulting v_{2i}s are different, then indeed it means that bidder i reacts differently according to the identity of the first auction winner.)

Because $b_{1i} = s_{1i}(v_{1i})$ holds, and because s_1 in (104) is a monotone strategy, I can do the following things on (103) in the *following order*:

1. Replace $\tilde{b}_{1i}$, $\tilde{b}^w_{2i}$, $\tilde{b}^{lj}_{2i}$, $\tilde{b}^{lk}_{2i}$ inside (103) with b_{1i}, b^w_{2i}, b^{lj}_{2i}, b^{lk}_{2i} — recall that b^w_{2i}, b^{lj}_{2i}, b^{lk}_{2i} are the same as $s^w_{2i}(v_{1i}, v_2)$, $s^l_{2i}(v_2, b_1; \{j\})$, $s^l_{2i}(v_2, b_1; \{k\})$ because bidder i plays equilibrium strategies.

2. Note the term $dF_{2|1}(v_2|v_{1i})$ inside (103). I attach the screenshot below for exposition.

$$
\begin{aligned}
v_{1i} = \tilde{b}_{1i} &+ \frac{G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \\
&- \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} \underbrace{dF_{2|1}(v_2|v_{1i})}_{1} - \frac{G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{\partial}{\partial \tilde{b}_{1i}} \left(\frac{H_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})^2}{h_{2i}^w(\tilde{b}_{2i}^w;\tilde{b}_{1i})} \right) \underbrace{dF_{2|1}(v_2|v_{1i})}_{1} \\
&- \frac{1}{\partial G_{B_{1j}}(\tilde{b}_{1i})G_{B_{1k}}(\tilde{b}_{1i})/\partial \tilde{b}_{1i}} \times \\
&\quad \left[\frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lj};b_1,\{j\})} \underbrace{dF_{2|1}(v_2|v_{1i})}_{2} G_{B_{1k}}(b_1) g_{B_{1j}}(b_1) db_1 + \right. \\
&\quad \left. \frac{\partial}{\partial \tilde{b}_{1i}} \int_{\tilde{b}_{1i}}^{\overline{b_1}} \int_{\underline{v_2}}^{\overline{v_2}} \frac{H_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})^2}{h_{2i}^l(\tilde{b}_{2i}^{lk};b_1,\{k\})} \underbrace{dF_{2|1}(v_2|v_{1i})}_{3} G_{B_{1j}}(b_1) g_{B_{1k}}(b_1) db_1 \right] \equiv \xi_1(\tilde{b}_{1i}), \qquad (17)
\end{aligned}
$$

in which $H_{2i}^w(\cdot)$, $H_{2i}^l(\cdot;\{j\})$, and $H_{2i}^l(\cdot;\{k\})$ come from (6), (7), and (10).

There are three types of $dF_{2|1}$, namely ①, ②, and ③.

3. I can replace ① with $dG_{B_{2i}^w|B_{1i}}(b_{2i}^w|b_{1i})$ by noting the following facts.

   - Integrating $b_{2i}^w = s_{2i}^w(v_{1i}, V_2)$ over $[\underline{v_2}, \overline{v_2}]$ with $\Pr[V_2 \leq \cdot|V_1 = v_{1i}]$ is the same as integrating b_{2i}^w over $[s_{2i}^w(v_{1i}, \underline{v_2}), s_{2i}^w(v_{1i}, \overline{v_2})]$ with $\Pr[s_{2i}^w(V_1, V_2) \leq \cdot|V_1 = v_{1i}]$.
   - Given $\Pr[s_{2i}^w(V_1, V_2) \leq \cdot|V_1 = v_{1i}]$, I can change it to $\Pr[s_{2i}^w(V_1, V_2) \leq \cdot|s_{1i}(V_1) = s_{1i}(v_{1i})]$ which doesn't change any result.

4. I can replace ② with $G_{B_{2i}^{lj}(b_{1j})|B_{1i}}(b_{2i}^{lj}|b_{1i})$ by noting the following:

   - Recall that b_1 inside $b_{2i}^{lj} = s_{2i}^l(v_2, b_1; \{j\})$ is j's first auction bid, not i's bid.
   - Integrating $b_{2i}^{lj} = s_{2i}^l(v_2, b_1; \{j\})$ over $[\underline{v_2}, \overline{v_2}]$ with $\Pr[V_2 \leq \cdot|V_1 = v_{1i}]$ while leaving b_1 intact is the same as integrating b_{2i}^{lj} over $[s_{2i}^l(\underline{v_2}, b_1; \{j\}),\ s_{2i}^l(\overline{v_2}, b_1; \{j\})]$ with $\Pr[s_{2i}^l(V_2, b_1; \{j\}) \leq \cdot|V_1 = v_{1i}]$.
   - Given $\Pr[s_{2i}^l(V_2, b_1; \{j\}) \leq \cdot|V_1 = v_{1i}]$, I can change it to $\Pr[s_{2i}^l(V_2, b_1; \{j\}) \leq \cdot|s_{1i}(V_1) = s_{1i}(v_{1i})]$, which doesn't change any result.

5. Similarly, I can replace ③ with $G_{B_{2i}^{lk}(b_{1k})|B_{1i}}(b_{2i}^{lk}|b_{1i})$.

After all these changes, i's equilibrium bids must satisfy the following equality, which comes from (103).

$$
\begin{aligned}
v_{1i} = b_{1i} &+ \frac{G_{B_{1j}}(b_{1i})G_{B_{1k}}(b_{1i})}{\partial G_{B_{1j}}(b_{1i})G_{B_{1k}}(b_{1i})/\partial b_{1i}} \\
&- \int_{\underline{b_2}}^{\overline{b_2}} \frac{H_{2i}^w(b_{2i}^w;b_{1i})^2}{h_{2i}^w(b_{2i}^w;b_{1i})} dG_{B_{2i}^w|B_{1i}}(b_{2i}^w|b_{1i}) \\
&- \frac{G_{B_{1j}}(b_{1i})G_{B_{1k}}(b_{1i})}{\partial G_{B_{1j}}(b_{1i})G_{B_{1k}}(b_{1i})/\partial b_{1i}} \int_{\underline{b_2}}^{\overline{b_2}} \frac{\partial}{\partial b_{1i}} \left(\frac{H_{2i}^w(b_{2i}^w;b_{1i})^2}{h_{2i}^w(b_{2i}^w;b_{1i})} \right) dG_{B_{2i}^w|B_{1i}}(b_{2i}^w|b_{1i}) \\
&- \frac{1}{\partial G_{B_{1j}}(b_{1i})G_{B_{1k}}(b_{1i})/\partial b_{1i}} \times \\
&\quad \left[\frac{\partial}{\partial b_{1i}} \int_{b_{1i}}^{\overline{b_1}} \int_{\underline{b_2}}^{\overline{b_2}} \frac{H_{2i}^l(b_{2i}^{lj};b_1,\{j\})^2}{h_{2i}^l(b_{2i}^{lj};b_1,\{j\})} dG_{B_{2i}^{lj}(b_{1j})|B_{1i}}(b_{2i}^{lj}|b_{1i}) G_{B_{1k}}(b_1) g_{B_{1j}}(b_1) db_1 + \right.
\end{aligned}
$$

$$\frac{\partial}{\partial b_{1i}}\int_{b_{1i}}^{\overline{b_1}}\int_{\underline{b_2}}^{\overline{b_2}}\frac{H^l_{2i}(b^{lk}_{2i};b_1,\{k\})^2}{h^l_{2i}(b^{lk}_{2i};b_1,\{k\})}dG_{B^{lk}_{2i}(b_{1k})|B_{1i}}(b^{lk}_{2i}|b_{1i})G_{B_{1j}}(b_1)g_{B_{1k}}(b_1)db_1\Bigg] \equiv \xi_1(b_{1i}), \tag{109}$$

in which H^w_{2i}, $H^l_{2i}(\cdot;\{i\})$, and $H^l_{2i}(\cdot;\{k\})$ inside (109) come from (91), (93), and (96). The right-hand side of (109) must be increasing in b_{1i}, and this restriction ensures s_{1i} from (104) is monotone.

In conclusion, unlike only three restrictions of monotonicity in the original model, we have four restrictions of monotonicity for bidder i applied to (106), (107), (108), and (109). As there are three bidders, the total number of restrictions becomes twelve, and one can easily see that the number will exponentially increase as the number of bidders goes up.

One could categorize bidders into two groups, such as 'regular or fringe' as in Jofre-Bonet and Pesendorfer (2003) or 'loggers or mills' as in Athey et al. (2011), and progress asymmetric bidder model, which I leave for future research.

D.3 Product-Mix Auction

[Back to ToC] I illustrate the implementation of the Product-Mix Auction using a simple example.

✓ (Item 1) The Bank of England (hereafter, BOE) continues to employ the Product-Mix Auction (hereafter, PMA) in its open market operations: Link 1(2024) link 2(2023), and link 3(2015)

✓ (Item 2) PMA is a variant of the 'simultaneous multi-unit uniform price auction,' and the *Handbook of Market Design* (p. 315) notes that the ascending-clock auction is superior to the one-shot sealed-bid auction.

The reason I am considering the PMA is that 1) it concludes faster than the ascending-clock auction, which is critical in agricultural produce auctions; 2) since bidders bid truthfully in the PMA, estimated value from the bids can directly be used in counterfactual analysis; and 3) as noted in the third paragraph on page 2 of Klemperer (2018), the PMA identifies the competitive equilibrium.

✓ (Item 3) Assume the auctioneer is selling two varieties, apples and pears, which are substitutes; as noted in footnote 36 of Klemperer (2018), if the two varieties were complements, a competitive equilibrium might not exist.

A bidder can express his preference as follows — I will assume '3.0 kg + high-quality produce' for both apples and pears.

– If he *only* wants to purchase 25 boxes of apples at \$10 per box and is *not* interested in buying any boxes of pears, he would bid as follows:

$$\begin{aligned}&(25, \$10, \$0)\\ &\equiv \text{(Number of boxes, bid for apple per box, bid for pear per box)}\end{aligned}$$

If the uniform price for the apple becomes \$7, he pays \$7 per box and wins 25 boxes; if the uniform price for the apple is exactly \$10, then he pays \$10 per box but may only receive a fraction of the 25 boxes; if the uniform price for the apple exceeds \$10, he wins nothing.

– If he *only* wants to buy 30 boxes of pears at \$10 per box and is *not* interested in buying any boxes of apples, then he bids as follows:

$$(30, \$0, \$10)$$

If the uniform price for the pear becomes \$7, then he pays \$7 per box and wins 30 boxes; if the uniform price for the pear is exactly \$10, then he pays \$10 per box but may get a fraction of 30 boxes; if the uniform price for the pear exceeds \$10, he wins nothing.

– If he wants to buy 30 boxes of apples *or* pears, but wants to buy apples at \$12 per box while pears at \$15 per box, then he bids as follows:

$$(30, \$12, \$15)$$

If the uniform prices for apple and pear become (\$10, \$10), then we have $(\$12, \$15) - (\$10, \$10) = (\$2, \$5)$. This '(\$2,\$5)' means that he gets a surplus of $\$2 \times 30$ if he gets apples while a surplus of $\$5 \times 30$ if he gets pears instead. Since the PMA tries to maximize a bidder's surplus, it allocates 30 boxes of *pears* to him if the uniform price is (\$10, \$10).

If the uniform prices for apple and pear are (\$10,\$15) instead, then the PMA gives 30 boxes of *apple* to him at a price of \$10 per box. This is because giving apples to him leads to a surplus of '$\$(12-10) \times 30$' while giving pears to him leads to a surplus '$\$(15 - 15) \times 30$.'

If the uniform prices for apple and pear are exactly (\$12,\$15), then the auctioneer rations — I will describe the detail in Item 7.

As noted in footnote 11 of Grace (2024), this *or* bid is rarely used in practice, which is why I do not consider the *or* bid in coming up with Figures 16 and 17 in Application section.

– A bidder can submit multiple bids too; there is no restriction on how many bids a bidder can submit to the auctioneer.

For example:

$$(20, \$5, \$0), (30, \$0, \$10).$$

This means that a bidder has submitted two bids, each of which is 'wants to buy 20 boxes of apple for \$5 per box' and 'wants to buy 30 boxes of pears for \$ 10 per box.' In this case, if the uniform prices become (\$6,\$8), then this bidder will only get 30 boxes of pears paying $\$8 \times 30$; instead, if the uniform prices are (\$2, \$2), then this bidder gets both 20 boxes of apple and 30 boxes of pear.

- Of course, he can submit the following bids too.

  For example:

$$(20, \$5, \$0), (30, \$0, \$10), (10, \$6, \$7).$$

  This expresses that this bidder has added another bid, which is 'wants to buy 10 boxes of apple at \$6 per box *or* 10 boxes of pear at \$7 per box.'

✓ (Item 4) Now, suppose a bidder i submitted three bids, A, B, and C, as follows:

- A: (1, \$0, \$7)
- B: (7, \$5, \$3)
- C: (5, \$2, \$0)

Then we can draw the following plot.

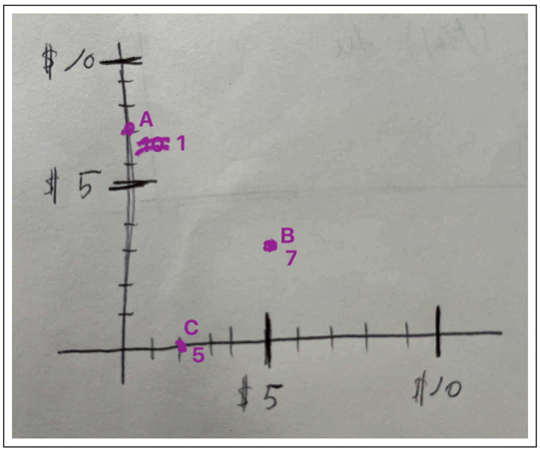


Figure 1

Each X- and Y-axis denotes 'bid per box for apple' and 'bid per box for pear.' If we look at B, we observe that its coordinate is (\$5, \$3). '7' written beside B expresses that it wants 7 boxes of apple *or* pear.

✓ (Item 5) Figure 1 represents i's bids. Suppose that there were other bidders $\{j, k, m\}$ and that they also submitted bids (ultimately, who bid and how much they bid does not matter.). Then, Figure 1 transforms into the following Figure 2.

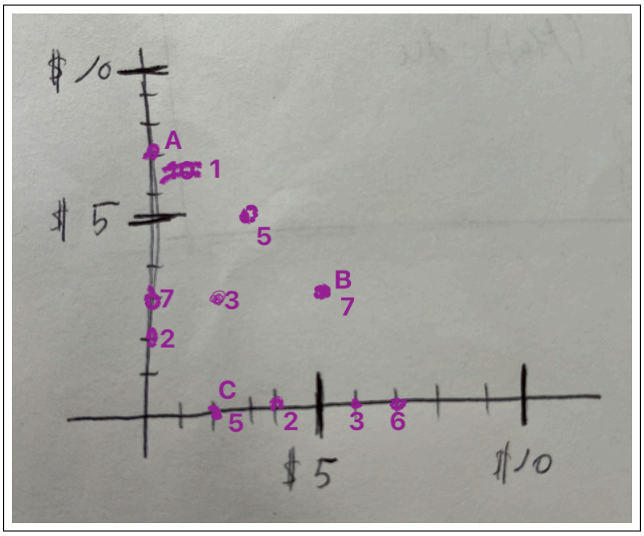


Figure 2

Figure 2 has the following property.

- Total boxes that all the bidders want are 41 because 1+5+7+3+7+2+5+2+3+6 = 41. Red-colored numbers come from A, B, and C.

✓ (Item 6) Given the submitted bids represented in Figure 2, say the auctioneer wants to sell a total of 23 boxes.

(for now, let's forget about how many boxes of apple and pear the auctioneer has.)

Thus, the auctioneer has to choose some uniform price such that the uniform price rejects '18=41-23' boxes; one example is as follows.

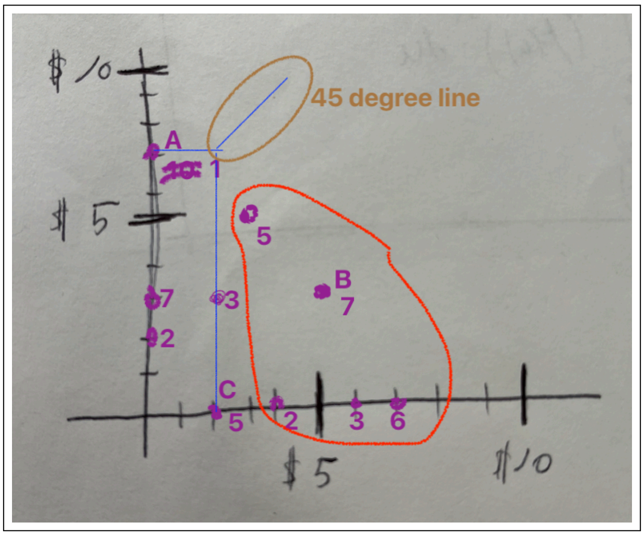


Figure 3

Some descriptions about Figure 3 follows.

- Uniform price here is **($2, $7)**, which is the cross section of blue lines; ($2, $7) means that the bidder pays $2 per box for apple and $7 per box for pear.

– Bids inside the red-circle *must* be accepted.

– Inside the red-circle, we observe the bids that lie on X-axis, namely (2,3,6). These bids express that they *only* want to buy apples. So, these bids pay a uniform price of \$2 per box and get (2 boxes of apple, 3 boxes of apple, 6 boxes of apple).

– What about 5 and 7 inside the red-circle which do not lie on X-axis? These two bids express that they want 'either apple or pear' as I wrote in Item 3. But, note that the clearing price for the pear is '\$7 per box' and that these two bids are only willing to pay \$5 and \$3. Thus, the auctioneer allocates apple to these two bids.

– Up to now, 23 boxes(5+7+2+3+6) have been sold to the bidders. Thus, the bids that *exactly* lie on the blue lines, namely (1,3,5), are all rejected. Moreover, the bids that lie inside the rectangles inside the blue lines, namely (7,2), are also rejected.

✓ (Item 7) Note that Item 6 shows only one example of a uniform price that rejects 18=41-23 boxes. Another possible uniform price is as follows (Item 9 will tell the exact allocation mechanism).

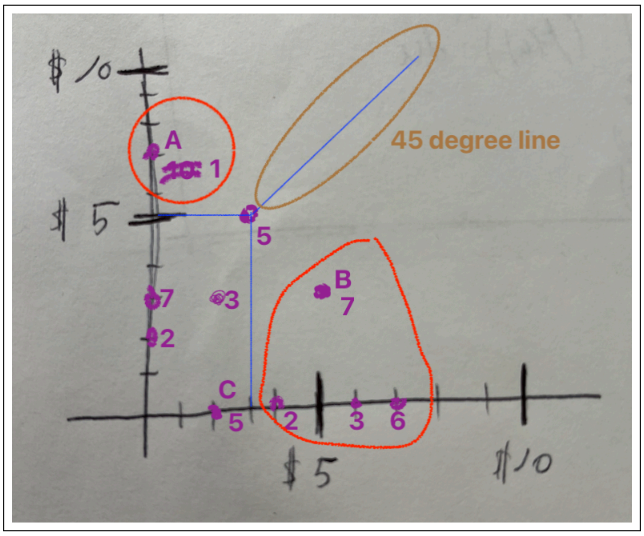


Figure 4

The uniform price in this case is **(\$3,\$5)** not (\$2,\$7). In this case, all the bids inside the red circles are accepted; it amounts to 19 boxes.

– A bid that lies to the west of the 45 degree line gets 1 box of pear and pays \$5 per box.

– Bids that lie below the 45 degree line gets 18 boxes of apple and pay \$3 per box.

What about a bid that is located at the cross section of the blue lines (namely 5)? Since 19 boxes are already sold and the auctioneer wants to sell 4 boxes more, the auctioneer can choose one of the following options:

(boxes of apple, boxes of pear) = {(4,0),(3,1),(2,2),(1,3),(0,4)}

which means the auctioneer rations the quantity.

✓ (Item 8) Now, I write down the possible uniform prices that reject 18 boxes, so that the auctioneer sells 23=41-18 boxes.

- **($3,$7)** Refer to the figure below.

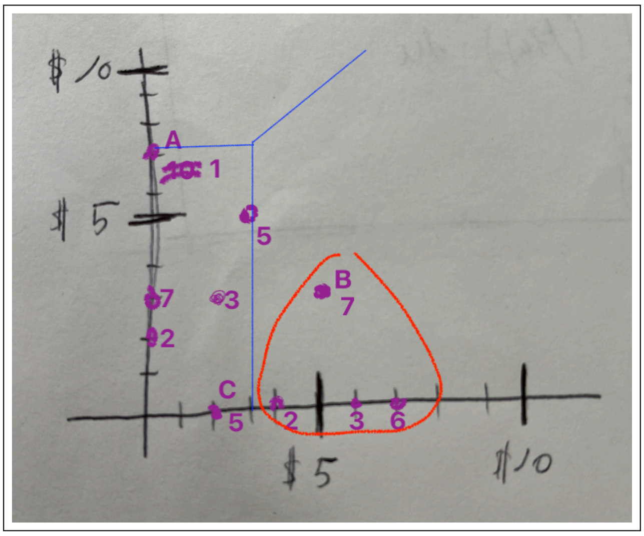


Figure 5

All the bids that are inside the red circle are accepted; they get 18 boxes of apples. The bids that lie on the blue line, namely 1 and 5, are rationed to match 23 boxes. It means that only 5=23-18 boxes should be chosen, so the possible options for the auctioneer are as follows.

{(5 apple boxes, 0 pear box), (4 apple boxes, 1 pear box)}

- **($4,$3)** Refer to the figure below.

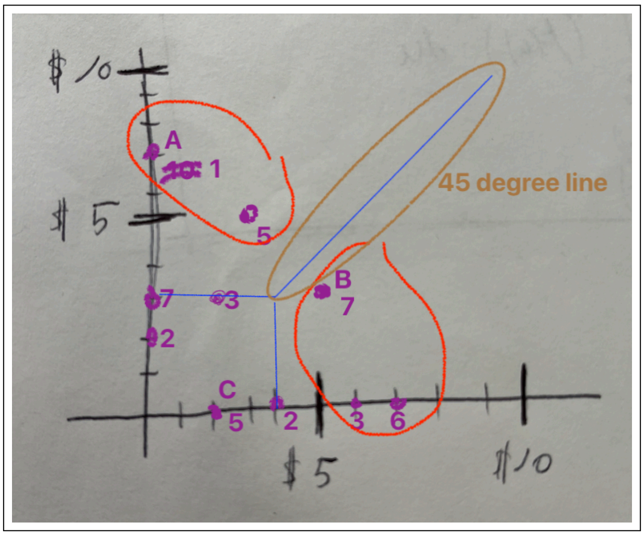


Figure 6

The bids that are left to the 45 degree line and above the uniform price(i.e., 1,5) get 6 boxes of pear; the bids that are right to the 45 degree line and above the uniform price(i.e., 7,3,6) get 16 boxes of apple.

Note that the 7 inside the red-circle expresses (7,$5,$3), which means that it wants to buy either '7 boxes of apple at $5 per box' or '7 boxes of pear at $3 per box'. Also, recall that the uniform price here is ($4,$3). So, this bid is entitled to get either apple or pear, but the auctioneer allocates apple to this bid because this maximizes a bidder's surplus.

Now, one might have noticed some regular patterns innate in the PMA. I will describe the regularity using the following figure.

✓ (Item 9) **($4,$5)**: Refer to the figure below:

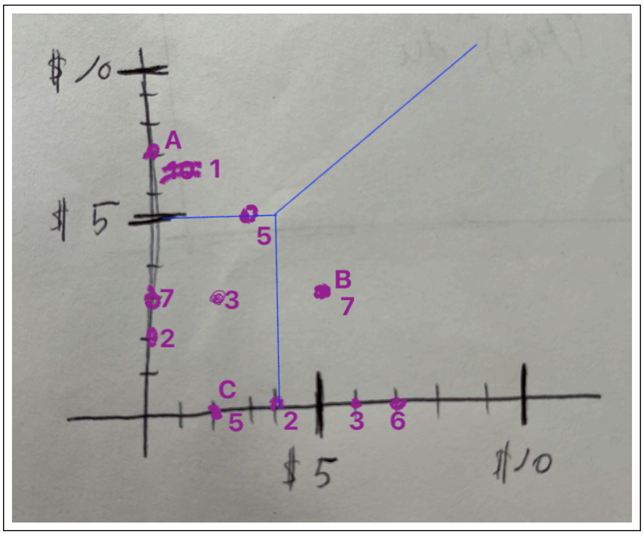


Figure 7

Allocation proceeds as follows:

1. Auctioneer fixes how many boxes to sell (still, let's forget about how many boxes of apple and pear the auctioneer has). In our case, bidders want a total of 41 boxes, and the auctioneer wants to sell 23 boxes.
2. Set a uniform price to reject 18 boxes. In the Figure 7, uniform price is set at ($4 for apple, $5 for pear). The uniform price is the cross section of the blue lines.
3. Given the cross section of the blue lines(i.e., ($4,$5)), one can imagine a rectangle where vertices are as follows:

   ($0,$0), ($4,$0), ($4,$5), ($0,$5)
4. Any bids that are *strictly* outside of this rectangle are accepted: in our case, the accepted bids are 1, 7, 3, and 6.
   - Now, 45-degree line plays a role. If a bid lies below the line then the bid is given apple; if a bid lies left to the line then the bid is given pear.

   This ensures maximizing bidders' surplus.
5. Any bids that *lie on the blue lines of the rectangle* are subject to rationing — in our case, the bids that lie on the blue lines are 5 and 2.

- First, recall that we want to sell 23 boxes of apples and already 17=1+7+3+6 boxes have been committed.
- Thus, the remaining 6 boxes should be sold. Since the bids indicate 7=5+2 boxes, we must do rationing.
- Note that 2 represents (2,\$4,\$0) and 5 represents (5,\$3,\$5). And since the uniform price is (\$4,\$3), 2 is only entitled to apple and 5 is only entitled to pear. Thus, possible rationing options are as follows:

(1 apples, 5 pears) or (2 apples, 4 pears)

6. Lastly, any bids that *lie on the black lines of the rectangle* or *lie strictly inside the rectangle* are rejected.

✓ (Item 10) In a nutshell, Items 6-8 show 'there are multiple uniform prices that reject a fixed number of boxes' and Item 9 shows 'how the allocation rule works and that rationing happens.'

Refer to the following figure:

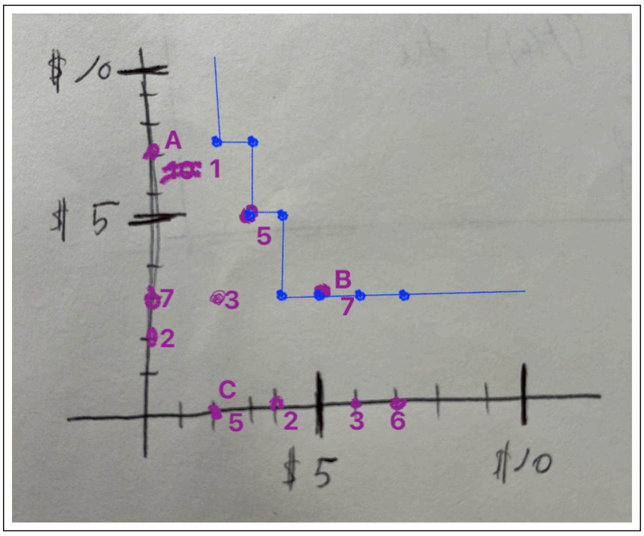


Figure 8

Figure 8 comes from repeating the allocation process described in Item 9; that is, the blue lines here denote the set of uniform prices that reject 18 boxes. In Items 6-8, I mentioned uniform prices such as '(\$2,\$7), (\$3,\$7), (\$3,\$5), (\$4,\$5), (\$4,\$3),' and these uniform prices are all expressed as blue dots.

✓ (Item 11) I will name each blue dot in Figure 8 as follows:

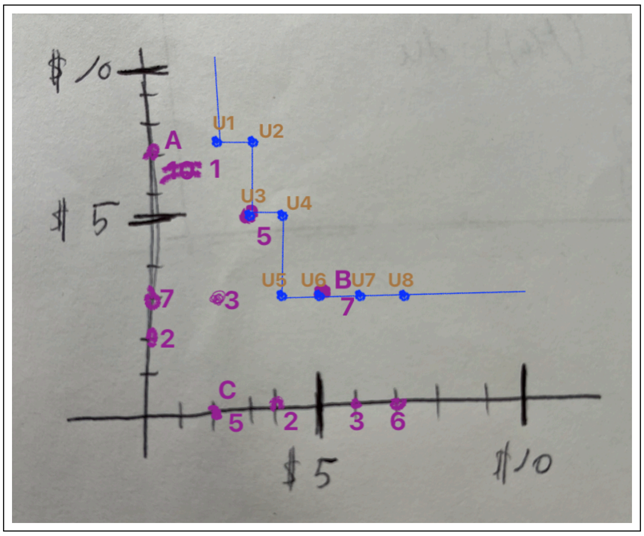


Figure 9

I will describe the properties of each U1-U8:

- Name of the dot: (Uniform Price), Price ratio, Related Figure, {Possible ratios of apple to pear}.
- U1: ($2 per apple, $7 per pear), $2/$7, Figure 3, {23apple/0pear}.
- U2: ($3 per apple, $7 per pear), $3/$7, Figure 5, {23apple/0pear, 22apple/1pear}.
- U3: ($3 per apple, $5 per pear), $3/$5, Figure 4,
  {22apple/1pear, 21apple/2pear, 20apple/3pear, 19apple/4pear, 18apple/5pear}.
- U4: ($4 per apple, $5 per pear), $4/$5, Figure 7, {18apple/5pear, 17apple/6pear}
- U5: ($4 per apple, $3 per pear), $4/$3, Figure 6, {17apple/6pear, 16apple/7pear}
- U6: ($5 per apple, $3 per pear), $5/$3, None,
  {16apple/7pear, 15apple/8pear, 14apple/9pear,..., 9apple/14pear}
- U7: ($6 per apple, $3 per pear), $6/$3, None,
  {9apple/14pear, 8apple/15pear, 7apple/16pear, 6apple/17pear}
- U8: ($7 per apple, $3 per pear), $7/$3, None,
  {6apple/17pear, 5apple/18pear,..., 0apple/23pear}

Even though lengthy, some regularity can be found in U1-U8; as the price ratio increases from $2/$7 to $7/$3, the ratio of apple to pear decreases from 23apple/0pear to 0apple/23pear.

Keeping this phenomenon in mind, refer to the figure below.

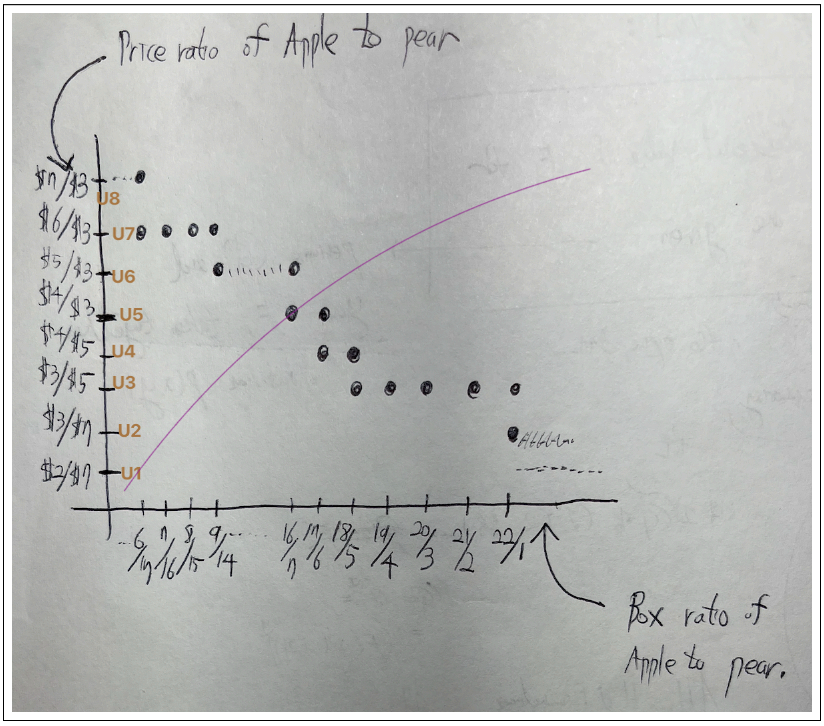


Figure 10

X-axis is the 'box ratio of apple to pear,' and the Y-axis is the 'price ratio of apple to pear.' If we look at the Y-axis, we notice U1-U8. For example, I wrote above that U4 has \$4/\$5 and {18apple/5pear, 17apple/6pear}, and these are expressed as dots in Figure 10.

Connecting these dots leads to the stepped line, which is conceptually the same as the stepped line in Figure 2 (page 10) of Klemperer (2018).

One can think of these dots as the demand curve. To be precise, these dots express the 'relative demand of apple to pear.' Thus, if the price ratio of apple to pear goes down, the box ratio of apple to pear goes up.

Note that the auctioneer sees this demand curve before setting the uniform price. Of course, if he changes his mind and wants to supply 30 boxes instead of 23 boxes, then the whole dots will shift to the right. Lastly, the purple line is the auctioneer's supply curve, which corresponds to 'Supply' in Figure 2 (page 10) of Klemperer (2018).

The benefits of using the PMA are described on page 6 and in the first paragraph of page 7 of Klemperer (2018).

✓ (Item 12) Now, I want to return to the issue of 'how many boxes of fruits the auctioneer has.' Up to now, I have ignored this issue.

As said, one will notice that Figure 2 (page 10) of Klemperer (2018) and Figure 10 are comparable. To further describe the analogous features,

- 41 boxes in total corresponds to £5.5 billion.
- 23 boxes to be sold correspond to £2.5 billion.
- Apple and pear correspond to Weak and Strong collateral.

- We try to sell 'apple' and 'pear', while Klemperer tries to buy 'weak' and 'strong' collateral.
- Y-axis of Figure 10 corresponds to Y-axis of Figure 2 of Klemperer. The only difference is that I use price ratio and Klemperer uses price difference.
- X-axis has the same story; I use the box ratio of apple to pear and Klemperer uses the ratio allocated to weak relative to strong.

So, it is clear that in the case of Klemperer's figure, no restrictions are put on the supply curve since the BOE itself prints the sterling. But, in the case of Figure 10, it might be the case that:

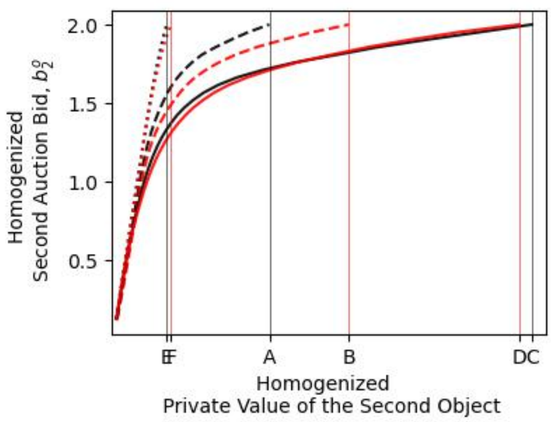
- The auctioneer wants to sell 23 boxes, and has 23 boxes in its warehouse. But, it may be the case that 16 boxes are apple and 7 boxes are pear. Then, only the coordinate (16/7, \$4/\$3) in Figure 10 is possible for the auctioneer; he can't choose other dots in the figure.
- But, say, the auctioneer wants to sell 13 boxes instead of 23 boxes. Given that the auctioneer has 16 boxes of apple and 7 boxes of pear, then he can at least choose the following 'box ratio of apple to pear':

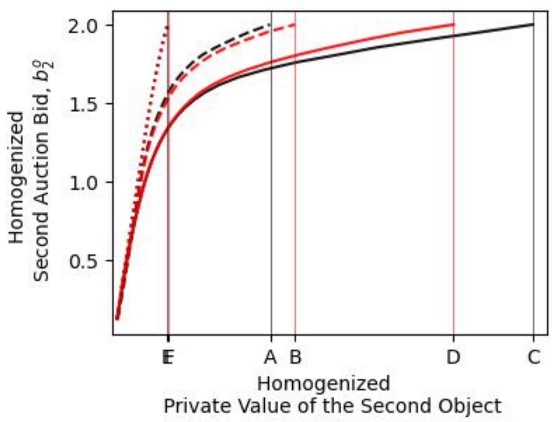
  13/0, 12/1, 11/2, 10/3, 9/4, 8/5, 7/6, 6/7.

In conclusion, unlike the BOE, the supply curve in the Korean Fruit Auction is constrained by the available number of boxes of apples and pears. Therefore, at a minimum, the auctioneer in the Korean Fruit Auction must have access to a large warehouse to store and draw significant quantities of produce —one of the objects the government aims to achieve by 2031.

D.4 Robustness Check

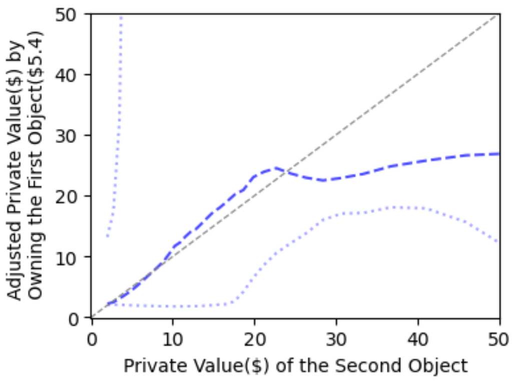
[Back to ToC] As noted in D.5, I multiply '$\exp(Z'\hat{\beta})$' by the homogenized bids and values to recover the unhomogenized bids and values, meaning that it suffices verify whether the necessary conditions hold for the homogenized bids and values.

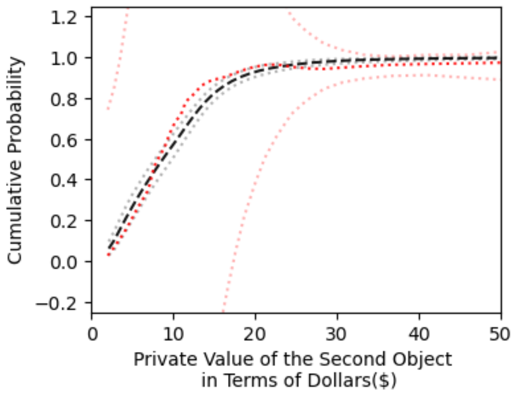




The left panels shows $\hat{\xi}_2^w$ (black) and $\hat{\xi}_2^l$ (red) when $I = 3$. Since the confidence intervals are very narrow, I have omitted them for the sake of visual clarity, only leaving the median estimates: The dotted estimates are conditioned on the ninety-fifth percentile of b_1^o, the solid estimates on the fiftieth percentile of b_1^o, and the dashed estimates on the fifth percentile of b_1^o, which is the Figure 10 presented in the body of the paper. One can observe that the functions are strictly increasing.

The right panel is identical to the left panel, except that I have changed I from 3 to 5. One can observe that as I changes from 3 to 5, only the red lines (i.e., the bidding strategy of the first auction loser in the second auction) change, while the black lines (i.e., the bidding strategy of the first auction winner in the second auction) remain unchanged. The reason for this phenomenon are that (i) I assumed $\mathcal{L}_I$ is the same for both $I = 3$ or $I = 5$, and (ii) the output of (61) is invariant to I, as seen from its third equality, whereas the output of (63) depends on I, as seen from its last equality.





Both the left and right panels are identical to Figures 12 and 13, except that I is 5 here. Unlike in Figures 12 and 13, we observe a weak negative complementarity under $I = 5$ (and, even positive complementarity at some values on the x-axis). The intuitive explanation for this phenomenon is as follows.

- 0.332, which is the probability of the first auction winner winning the second auction (I mentioned this statistic in Application section), is a fact. I denote this fact as Ⓐ.

- Another fact is Figure 14. Namely, the bid distribution of the second auction of the first auction loser stochastically dominates that of the first auction winner. I denote this fact as Ⓑ.

- Assume that Ⓑ accounts for complementarity of degree -10. (if number is negative, then it is negative complementarity).

- If I set $I = 5$, then the fair probability is 0.20 (i.e., 1/5), and the true fact is 0.332. Therefore, one could interpret this phenomenon as "the first auction winner feeling positive complementarity of degree, say 8, so that he gets aggressive in the second auction." So, Ⓐ accounts for complementarity of degree $+8$.

- As a result, $+8 - 10 = -2$ comes out, which is a weak negative complementarity shown in the left panel.

- If I set $I = 3$, then the fair probability is 0.33 (i.e., 1/3), and the true fact is still 0.332. Therefore, one could interpret this phenomenon as "the first auction winner feels almost no positive complementarity of degree, say 1, so that he is somewhat so-so in the second auction." So, Ⓐ accounts for complementarity of degree +1.
- As a result, $-8 + 1 = -7$ comes out, which is a strong negative complementarity found in Figure 12.

Lastly, I explain why I assert that a valid distribution is formed from the right-hand side of equation (10). Note that C.11.2 demonstrates the equivalence between equation (10) and $F_{2|1} = \tilde{F}_{2|B_1}$. I graphically presented $\tilde{F}_{2|B_1}$ in Figure 13, represented by the red line. Since Figure 13 assumes a value slightly below the fifth percentile of b_1^o, what remains to validate my assertions is to display the red lines for both the fifth and ninety-fifth percentiles of b_1^o. With numerous tests still ongoing, I anticipate that certain bandwidths of the kernel density estimators for each value of b_1^o will allow me to derive a valid formation of $\tilde{F}_{2|B_1}$.

D.5 Details regarding Estimation

[Back to ToC]

✓ (Bid Homogenization) Notable papers that discuss bid homogenization include Haile et al. (2003) and Perrigne and Vuong (2023). Homogenization assumes that a bidder's value for the object is a combination of (i) a value derived from the object's observable characteristics, and (ii) a value specific to the bidder, independent of the object's observable characteristics.

Variants of homogenization exist. Haile et al. (2003) assumes an additively separable structure, while more recent papers such as Asker (2010), Sant'Anna (2018), and Compiani et al. (2020) assume a multiplicatively separable structure. Specifically, they assume that a bidder's utility takes the following form:

$$u_{ik} = \exp^{x_k \gamma} \times v_{ik} \epsilon_k,$$

where u_{ik} is the valuation of a bidder i in k-th auction; x_k is a vector of covariates of the object at k-th auction; v_{ik} is the private value of a bidder i that is independent of x_k; and ϵ_k represents the unobserved heterogeneity. In Sant'Anna (2018), ϵ_k is set to 1, implying that unobserved heterogeneity is not accounted for, a simplification that I also adopt. As footnote 31 of Asker (2010) notes, this valuation structure offers more flexibility than that of Haile et al. (2003), as it is multiplicatively separable: both the mean and variance of the value distribution are influenced by the observed auction covariates.

I followed exactly the approach described in Sant'Anna (2018). In my context, it progresses as follows. The superscript o refers to homogenized bid or homogenized value.

Step 1: My dataset contains 87,349 apple auctions, and I am using 1,906 auctions, resulting in $L = 953$ auction pairs. I denote the auction covariates, which have an

$m \times 1$ dimension, as the vector Z. This vector includes not only fruit characteristics but also the precise time at which each auction concluded, which I normalize between 0 and 1 and refer to it as the 'order variable' — 0 represents the first apple auction and 1 denotes the last apple auction at a specific auction house on a given day.

STEP 2: I apply the log transformation to the winning bids from all 87,349 auctions. I then regress $\log b^{\max}$ on Z, meaning I use all 87,349 auctions in this regression. The reason for using the entire dataset, rather than just the 1,906 auctions, is that the coefficient for the 'order variable' is negative with $p < 0.05$ in the full regression. This is an important point because if I were to use only the last two auctions, namely limiting the dataset to 1,906 auctions, I would not fully capture the effect of the 'order variable.' The result of this regression is shown in D.5.1

STEP 3: Thus, we have 87,349 tuples of ($\log(b^{\max})$, $Z'\hat{\beta}$, $\log(b^{\max,o})$), where $\log(b^{\max,o})$ represents the residual from the regression. This regression is based on the equation that appears on page 18 of Sant'Anna (2018), namely:

$$\log b_{it} = x_t'\alpha + \log b_{it}^o \tag{110}$$

This equation is based on Proposition 2 of Sant'Anna (2018).

STEP 4: Next, I select the last two auctions of each day for each auction house, resulting in $(\log(b_{1\ell}^{\max,o}),\ \log(b_{2\ell}^{\max,o}), W_{1\ell}, W_{2\ell})$ where $\ell \in \{1, \ldots, 953\}$. Following Sant'Anna (2018) and Asker (2010), I then de-log these values, yielding $(b_{1\ell}^{\max,o},\ b_{2\ell}^{\max,o}, W_{1\ell}, W_{2\ell})$ where $\ell \in \{1, \ldots, 953\}$.

STEP 5: I use these homogenized bids for the estimation. It implies that the kernel density estimator that I use is at most bi-variate.

STEP 6: Suppose I want to plot the unhomogenized $\hat{\xi}_2^w$. In this case, I use $\hat{\xi}_2^{w,o}(b_2^o; b_1^o)$. Let the lowest and highest values of b_2^o range from 10 to 40, and I fix b_1^o at 3. By varying b_2^o between 10 and 40, I obtain $\hat{\hat{\delta}}^o(b_1^o, \cdot)$, and suppose the median of its estimates ranges between 30 and 50.

Thinking of figures 10 and 11, the y-axis in this case will be $b_2^o \in [10, 40]$ and the x-axis will have a range $\hat{\hat{\delta}}^o(b_1^o, \cdot) \in [30, 50]$. Then I can convert b_2^o to unhomogenized b_2 by following (110), namely:

$$b_2 = \exp^{Z_2'\hat{\beta}} \times b_2^o$$

Also, I can convert $\hat{\hat{\delta}}^o(b_1^o, \cdot)$ to $\hat{\hat{\delta}}(b_1^o, \cdot)$ by doing: (this approach is valid by Sant'Anna (2018))

$$\hat{\hat{\delta}}(b_1^o, \cdot) = \exp^{Z_2'\hat{\beta}} \times \hat{\hat{\delta}}^o(b_1^o, \cdot) \tag{111}$$

And, I can convert b_1^o to b_1 by doing:

$$b_1 = \exp^{Z_1'\hat{\beta}} \times b_1^o.$$

STEP 7: Given that I have gotten $\hat{\hat{D}}^o_{2|1}$ and $\hat{\hat{F}}^o_{2|1}$, I can convert the x-axis of $\hat{\hat{D}}^o_{2|1}$ by doing (111). Also I can convert the x-axis of $\hat{\hat{F}}^o_{2|1}$ by doing:

$$\hat{v}_2 = \exp^{Z_2'\hat{\beta}} \times \hat{v}_2^o$$

And, if I were to do $\hat{v}_1^o = \hat{\xi}_1^o(b_1^o)$, I can recover its dollar value by doing:

$$\hat{v}_1 = \exp^{Z_1'\hat{\beta}} \times \hat{v}_1^o$$
$$b_1 = \exp^{Z_1'\hat{\beta}} \times b_1^o$$

Another approach to homogenization is to use percentages, as in Kong (2021), although this method is not suitable for the setting of sequential first-price auctions. More recently, Gimenes and Guerre (2020) introduced a quantile regression approach as an alternative to bid homogenization.

✓ Further information available upon inquiry: I plan to discuss the bandwidth and the kernel that I chose in detail.

D.5.1 Regression Result

[Back to ToC]

	Log-transformed winning bid per box (Korean Won, ₩)
Constant	8.552261 (0.146632)
Number of Boxes	0.003733 (0.000086)
Number of Apple Auctions Held on a Given Day at an Auction House	-0.000112 (0.000016)
Order Variable	-0.152303 (0.004810)
List of Fixed Effects	Month, Day, Auction House Type of Apple, Grade, Size, Place of origin, Group
Observations	87,349
Degree of Freedom Residuals	87,180
R^2, Adjusted R^2	0.472, 0.471

Note that the purpose of this regression is to absorb as much heterogeneity across auction items as possible, so that the resulting homogenized bid and value are free from the influence of these covariates; the coefficients themselves are not the primary focus in this analysis.

Considering the constant term of 8.552261, applying the transformation exp(8.552261)/1300 = 3.98 yields a value in U.S. dollars, assuming an exchange rate of \$1 = ₩1300.

The numbers in parentheses are heteroskedasticity-robust standard errors. The variable not included in this regression is the weight of a single apple box, which is almost always 10.0 kilograms.

"Number of Boxes" corresponds to the quantity of apple boxes offered. "Number of Apple Auctions Held on a Given Day at an Auction House" refers to the number of apple auctions that occurred at a specific auction house (e.g., 45 apple auctions occurred at the Seoul Auction House). "Order Variable" is normalized between 0 and 1, where 0 represents the first apple auction and 1 denotes the last apple auction at a specific auction house on a given day. We observe that the coefficient for the "Order Variable" is negative.

Regarding the Fixed Effects, "Day" refers to what day it is (e.g., Monday), "Auction House" indicates one of the five auction houses (e.g., Joongang Auction House), "Type of Apple" refers to varieties (e.g., Fuji, Gala), "Size" pertains to the size of each individual apple inside the box (e.g., Very Large, Large), and "Group" denotes whether the farmer who requested this object is an individual or part of a partnership.

"Grade" denotes the quality of apples inside the box, but this variable is considered unreliable, as it is assigned by the producer (or farmer) and is typically labeled as 'best.' Consequently, bidders do not take the farmer-assigned grade seriously, which is why they inspect the fruits or vegetables before each auction.