Online Appendix to "Household Income Inequality and Optimal Trend Inflation"

1 Tractable HANK (THANK) model

In this section, we present the full set of equilibrium conditions of the model.

1.1 Households

The household joint welfare maximization problem reads

$$U(B_t^S, B_t^H) = \max_{C_t^S, C_t^H, Z_{t+1}^S, Z_{t+1}^H} (1-\lambda)\frac{(C_t^S)^{1-\sigma}-1}{1-\sigma} + \lambda\frac{(C_t^H)^{1-\sigma}-1}{1-\sigma} - \chi\frac{N_t^{1+\varphi}}{1+\varphi} + \beta\mathbf{E}_t\left[U(B_{t+1}^S, B_{t+1}^H)\right]$$

subject to the flows of bonds and budget constraints:

$$\begin{aligned}
C_t^S + Z_{t+1}^S + TG_t^S &= \theta_t^S((1-\nu)W_tN_t + D_t) + \frac{R_{t-1}q_{t-1}}{\Pi_t}B_t^S + \mathcal{T}_t^S \\
C_t^H + Z_{t+1}^H + TG_t^H &= \theta_t^H((1-\nu)W_tN_t + D_t) + \frac{R_{t-1}q_{t-1}}{\Pi_t}B_t^H + \mathcal{T}_t^H \\
B_{t+1}^S &= sZ_{t+1}^S + (1-s)Z_{t+1}^H \\
B_{t+1}^H &= (1-h)Z_{t+1}^S + hZ_{t+1}^H \\
Z_{t+1}^S, Z_{t+1}^H &\geq 0
\end{aligned}$$

The first-order conditions of the above dynamic optimization problem are:

$$\begin{aligned}
\Lambda_t^S &= (1-\lambda)(C_t^S)^{-\sigma} \\
\Lambda_t^H &= \lambda(C_t^H)^{-\sigma} \\
\Lambda_t^S &= \beta s\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^S}\right] + \beta(1-h)\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^H}\right] + \Theta_t^S \\
\Lambda_t^H &= \beta(1-s)\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^S}\right] + \beta h\mathbf{E}_t\left[\frac{\partial U_{t+1}}{\partial B_{t+1}^H}\right] + \Theta_t^H \\
\Theta_t^S Z_{t+1}^S &= 0 \\
\Theta_t^H Z_{t+1}^H &= 0,
\end{aligned}$$

where Λ_t^S, Λ_t^H, Θ_t^S, and Θ_t^H are Lagrangian multipliers associated with budget constraints and borrowing constraints for each type. Additionally, the envelope conditions yields $\frac{\partial U_t}{\partial B_t^S} = \Lambda_t^S \frac{R_{t-1}q_{t-1}}{\Pi_t}$, $\frac{\partial U_t}{\partial B_t^H} = \Lambda_t^H \frac{R_{t-1}q_{t-1}}{\Pi_t}$. Combining both first-order optimality and envelope conditions with stationary type distribution assumption, we obtain:

$$\begin{aligned}
(C_t^S)^{-\sigma} &= \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left(s(C_{t+1}^S)^{-\sigma} + (1-s)(C_{t+1}^H)^{-\sigma} \right) \right] + \frac{\Theta_t^S}{1-\lambda} \\
(C_t^H)^{-\sigma} &= \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left((1-h)(C_{t+1}^S)^{-\sigma} + h(C_{t+1}^H)^{-\sigma} \right) \right] + \frac{\Theta_t^H}{\lambda} \\
\Theta_t^S Z_{t+1}^S &= 0 \\
\Theta_t^H Z_{t+1}^H &= 0.
\end{aligned}$$

Since we focus on the zero liquidity equilibrium where $\Theta_t^S = 0, \Theta_t^H > 0$, household consumption saving behavior is characterized by the savers' consumption Euler equation and hand-to-mouth spenders' budget constraint:

$$\begin{aligned}
(C_t^S)^{-\sigma} &= \beta \mathbf{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left(s(C_{t+1}^S)^{-\sigma} + (1-s)(C_{t+1}^H)^{-\sigma} \right) \right] \\
C_t^H + TG_t^H &= \theta_t^H((1-\nu)W_t N_t + D_t) + \mathcal{T}_t^H = \theta_t^H Y_t + \mathcal{T}_t^H.
\end{aligned}$$

Finally, a labor union sets wages on behalf of both household types such that the labor-supply-like wage schedule is:

$$W_t C_t^{-\sigma} = \chi N_t^{\varphi}.$$

1.2 Firms

1.2.1 Final Goods Producers

The final goods producers use intermediate goods as inputs and produce final goods according to CES technology:

$$Y_t = \left(\int Y_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right)^{\frac{\varepsilon_p}{\varepsilon_p - 1}}.$$

They sell the final products to households in a competitive market. The profit maximization of final goods producers yields a general relative demand function of intermediate good i and the aggregate price index.

$$Y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} Y_t$$
$$P_t = \left(\int P_t(i)^{1-\varepsilon_p} di\right)^{\frac{1}{1-\varepsilon_p}}.$$

1.2.2 Intermediate Goods Producers

Firstly, the cost-minimization problem of intermediate goods firms reads:

$$\min_{N_t(i)} \ (1-\nu)W_t N_t(i)$$
$$s.t$$
$$Y_t(i) = A_t N_t(i),$$

where ν is a labor cost subsidy that eliminates steady state distortion arising from monopolistic competition. The first-order condition yields an expression for the real marginal cost:

$$MC_t = \frac{(1-\nu)W_t}{A_t}.$$

Secondly, the profit maximization problem of the intermediate goods firms is:

$$\max_{P_t(i)} E_t \left[\sum_{k=0}^{\infty} (\theta_p \beta)^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \left(\frac{P_t(i)}{P_{t+k}} Y_{t+k}(i) - MC_{t+k} Y_{t+k}(i)\right)\right].$$

The optimality condition for reset price yields:

$$X_{1,t} = (C_t)^{-\sigma} MC_t Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi_{t+1}^{\varepsilon_p} X_{1,t+1}\right]$$
$$X_{2,t} = (C_t)^{-\sigma} Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi_{t+1}^{\varepsilon_p - 1} X_{2,t+1}\right]$$
$$\Pi_t^* = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} u_t,$$

where u_t is the price markup shock.

1.3 Government

The central bank sets the gross policy rate R_t following the Taylor rule subject to a ZLB constraint:

$$R_t^* = (R_{t-1}^*)^{\phi_r} \left(\left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y} \right)^{\phi_y} \right)^{1-\phi_r}$$

$$R_t = \begin{cases} R_t^* & \text{if } R_t^* \geq 1 \\ 1 & \text{if } R_t^* < 1, \end{cases}$$

where R_t^* is the shadow rate. ϕ_r governs the interest rate smoothing, while ϕ_π and ϕ_y measure the sensitivity of the policy rates with respect to deviations of inflation and output from their respective steady state. The fiscal authority undertakes government spending following a feedback rule that responds to deviations of output from its steady state:

$$G_t = G \left(\frac{Y_t}{Y} \right)^{\gamma_y^G}$$

$$\mathcal{T}_t^H = \mathcal{T}^H + \gamma_Y^T \left(Y_t - Y \right),$$

where γ_Y^G and γ_Y^T govern the cyclicality of government spending and transfers, respectively. The financing scheme for government spending and targeted transfers is determined by the following rules:

$$G_t = (1-\lambda) TG_t^S + \lambda TG_t^H$$

$$TG_t^S = TG^S + \frac{1-\alpha}{1-\lambda} \left(G_t - G \right)$$

$$(1-\lambda)\mathcal{T}_t^S + \lambda \mathcal{T}_t^H = 0.$$

Government spending is financed through lump-sum taxes levied on both Savers and HtMs. These taxes consist of two components: a constant steady-state component that funds steady-state government spending, and a cyclical component that adjusts with deviations of government spending from its steady-state level. The parameter α governs the allocation of the cyclical tax burden between the two household groups. When $\alpha = 0$, Savers bear the entire tax burden associated with cyclical government spending. In contrast, $\alpha = \lambda$ implies an equal tax burden across Savers and HtMs. Additionally, the targeted transfer policy requires Savers to finance the full amount of transfers to HtMs.

1.4 Full Set of Equilibrium Conditions

$$W_t C_t^{-\sigma} = \chi N_t^{\varphi} \tag{1}$$

$$(C_t^S)^{-\sigma} = \beta \mathrm{E}_t \left[\frac{R_t q_t}{\Pi_{t+1}} \left(s(C_{t+1}^S)^{-\sigma} + (1-s)(C_{t+1}^H)^{-\sigma} \right) \right] \tag{2}$$

$$C_t^H + TG_t^H = \theta_t^H Y_t + \mathcal{T}_t^H \tag{3}$$

$$\theta_t^H = \theta^H \left(\frac{Y_t}{Y} \right)^{\gamma_Y^H} \tag{4}$$

$$(1-\lambda)\theta_t^S + \lambda \theta_t^H = 1 \tag{5}$$

$$(1-\nu) W_t = A_t MC_t \tag{6}$$

$$X_{1,t} = (C_t^S)^{-\sigma} MC_t Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi_{t+1}^{\varepsilon_p} X_{1,t+1} \right] \tag{7}$$

$$X_{2,t} = (C_t^S)^{-\sigma} Y_t + \theta_p \beta \mathrm{E}_t \left[\Pi_{t+1}^{\varepsilon_p - 1} X_{2,t+1} \right] \tag{8}$$

$$\Pi_t^* = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}}{X_{2,t}} u_t \tag{9}$$

$$1 = (1-\theta_p)(\Pi_t^*)^{1-\varepsilon_p} + \theta_p \Pi_t^{\varepsilon_p - 1} \tag{10}$$

$$v_t^p = (1-\theta_p)(\Pi_t^*)^{-\varepsilon_p} + \theta_p \Pi_t^{\varepsilon_p} v_{t-1}^p \tag{11}$$

$$Y_t = \frac{A_t N_t}{v_t^p} \tag{12}$$

$$(1-\lambda) C_t^S + \lambda C_t^H + G_t = Y_t \tag{13}$$

$$R_t^* = (R_{t-1}^*)^{\phi_r^1} \left(R_{ss}^* \left(\frac{\Pi_t}{\Pi} \right)^{\phi_\pi} \left(\frac{Y_t}{Y} \right)^{\phi_y} \right)^{1-\phi_r^1} \tag{14}$$

$$R_t = max\left\{1, R_t^*\right\} \tag{15}$$

$$G_t = G \left(\frac{Y_t}{Y} \right)^{\gamma_y^G} \tag{16}$$

$$TG_t^S = TG^S + \frac{1-\alpha}{1-\lambda} \left(G_t - G \right) \tag{17}$$

$$G_t = (1-\lambda) TG_t^S + \lambda TG_t^H \tag{18}$$

$$\mathcal{T}_t^H = \mathcal{T}^H + \gamma_Y^T (Y_t - Y) \tag{19}$$

$$(1-\lambda)\mathcal{T}_t^S + \lambda \mathcal{T}_t^H = 0 \tag{20}$$

$$\log(A_t) = \rho_a \log(A_{t-1}) + \sigma_a \varepsilon_{a,t} \tag{21}$$

$$\log(u_t) = \rho_u \log(u_{t-1}) + \sigma_u \varepsilon_{u,t} \tag{22}$$

$$\log(q_t) = \rho_q \log(q_{t-1}) + \sigma_q \varepsilon_{q,t}, \tag{23}$$

where $v_t^p = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon_p} di$ is the measure of price dispersion. Equations (1) to (5) characterize household consumption-saving and labor-supply behavior. Equations (6) to (9) constitute the optimality conditions of intermediate goods firms. Equations (10) and (12) come from the definition of the aggregate price index and the aggregation of intermediate goods, respectively. Equation (11) and (13) describes the dynamics of price dispersion and the goods market clearing condition. Equations (14) to (20) define the monetary and fiscal policy rules. Finally, equations (21) to (23) specify exogenous TFP, price markup, and risk premium shock processes.

2 Second-Order Approximation of Household Welfare

In this section, we describe how to approximate the per-period welfare of households, assuming the log-utility function for consumption. The hatted variables in this appendix denote the log deviation from steady state (i.e. $\hat{x}_t = \log(X_t) - \log(X)$). Instantaneous household welfare for type $j \in \{S, H\}$ is

$$U_t^j \equiv \frac{(C_t^j)^{1-\sigma} - 1}{1-\sigma} - \chi \frac{N_t^{1+\varphi}}{1+\varphi}.$$

Firstly, we will explain the method of approximating the price dispersion. Starting from the relationship between reset price inflation and inflation, we get the second-order approximation of inflation dynamics.

$$\begin{aligned} 1 &= (1-\theta_p)(\Pi_t^*)^{1-\varepsilon_p} + \theta_p(\Pi_t)^{\varepsilon_p - 1} \\ \rightarrow \hat{\pi}_t^* &= \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\left(\hat{\pi}_t + \frac{1}{2}(\varepsilon_p - 1)\hat{\pi}_t^2\right) + \frac{1}{2}(\varepsilon_p - 1)\hat{\pi}_t^{*2}, \end{aligned} \tag{24}$$

where $\overline{\theta}_p = \theta_p \Pi^{\varepsilon_p - 1}$. Up to first order, (24) becomes $\hat{\pi}^*_t = \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t$. Substituting out $\hat{\pi}^{*2}_t$ in (24), we obtain the second-order expression for reset price inflation:

$$\hat{\pi}^*_t = \frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t + \frac{1}{2}(\varepsilon_p - 1)\frac{\overline{\theta}_p}{(1-\overline{\theta}_p)^2}\hat{\pi}^2_t.$$

Next, we use the price dispersion dynamics:

$$v^p_t = (1-\theta_p)(\Pi^*_t)^{-\varepsilon_p} + \theta_p \Pi^{\varepsilon_p}_t v^p_{t-1}.$$

The second-order approximation of the price dispersion dynamics is:

$$\hat{v}^p_t + \frac{1}{2}(\hat{v}^p_t)^2 = (1-\overline{\theta}_p\Pi)\left(-\varepsilon_p\hat{\pi}^*_t + \frac{1}{2}\varepsilon^2_p(\hat{\pi}^*_t)^2\right) + \overline{\theta}_p\Pi\left(\varepsilon_p\hat{\pi}_t + \hat{v}^p_{t-1} + \frac{1}{2}\varepsilon^2_p\hat{\pi}^2_t + \frac{1}{2}(\hat{v}^p_{t-1})^2 + \varepsilon_p\hat{\pi}_t\hat{v}^p_{t-1}\right). \tag{25}$$

Up to first order, (25) is:

$$\hat{v}^p_t = \frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}^p_{t-1}.$$

By replacing the reset price inflation and squared price dispersion term with their first-order expression, (25) can be rewritten as:

$$\begin{aligned}
\hat{v}^p_t = &-\frac{1}{2}\left(\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}^p_{t-1}\right)^2 \\
&+ (1-\overline{\theta}_p\Pi)\left(-\varepsilon_p\left(\frac{\overline{\theta}_p}{1-\overline{\theta}_p}\hat{\pi}_t + \frac{1}{2}(\varepsilon_p-1)\frac{\overline{\theta}_p}{(1-\overline{\theta}_p)^2}\hat{\pi}_t{}^2\right) + \frac{1}{2}\varepsilon^2_p\left(\frac{\overline{\theta}_p}{1-\overline{\theta}_p}\right)^2\hat{\pi}^2_t\right) \\
&+ \overline{\theta}_p\Pi\left(\varepsilon_p\hat{\pi}_t + \hat{v}^p_{t-1} + \frac{1}{2}\varepsilon^2_p\hat{\pi}^2_t + \frac{1}{2}(\hat{v}^p_{t-1})^2 + \varepsilon_p\hat{\pi}_t\hat{v}^p_{t-1}\right).
\end{aligned}$$

With some mechanical algebra and collecting like terms, we obtain:

$$\begin{aligned}
\hat{v}^p_t = &\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{1-\overline{\theta}_p}\hat{\pi}_t + \overline{\theta}_p\Pi\hat{v}^p_{t-1} + \frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p\Pi - \varepsilon_p + 1)(1-\overline{\theta}_p\Pi)}{(1-\overline{\theta}_p)^2}\hat{\pi}^2_t \\
&+ \frac{\varepsilon_p(1-\overline{\theta}_p\Pi)}{1-\overline{\theta}_p}\hat{\pi}_t\hat{v}^p_{t-1} + \frac{1}{2}\overline{\theta}_p\Pi(1-\overline{\theta}_p\Pi)(\hat{v}^p_{t-1})^2.
\end{aligned}$$

As in *Alves* (2014), we assume that the price dispersion $\hat{v}^p_t$ is a second-order term, and, hence, will regard any terms taking the form of $\hat{v}^p_t\hat{x}_t$ as higher than the second order.

Under this assumption, the second-order expression of price dispersion becomes:

$$\hat{v}_t^p = \overline{\theta}_p \Pi \hat{v}_{t-1}^p + \frac{\varepsilon_p \overline{\theta}_p (\Pi - 1)}{1 - \overline{\theta}_p} \hat{\pi}_t + \frac{1}{2} \frac{\varepsilon_p \overline{\theta}_p (\varepsilon_p \Pi - \varepsilon_p + 1)(1 - \overline{\theta}_p \Pi)}{(1 - \overline{\theta}_p)^2} \hat{\pi}_t^2.$$

Taking the unconditional expectation, we have:

$$\mathrm{E}(\hat{v}_t^p) = \frac{\varepsilon_p \overline{\theta}_p (\Pi - 1)}{(1 - \overline{\theta}_p)(1 - \overline{\theta}_p \Pi)} \mathrm{E}(\hat{\pi}_t) + \frac{1}{2} \frac{\varepsilon_p \overline{\theta}_p (\varepsilon_p \Pi - \varepsilon_p + 1)}{(1 - \overline{\theta}_p)^2} \mathrm{E}(\hat{\pi}_t^2). \tag{26}$$

Next, the budget constraints of each household type are as follows:

$$C_t^S + TG_t^S = \theta_t^S Y_t + \mathcal{T}_t^S \tag{27}$$

$$C_t^H + TG_t^H = \theta_t^H Y_t + \mathcal{T}_t^H \tag{28}$$

Substituting out the lump-sum tax TG_t^S, transfers $\mathcal{T}_t^S$, and income share θ_t^S, the Savers' budget constraint can be rewritten as:

$$C_t^S + TG^S + \frac{1 - \alpha}{1 - \lambda} \left(G \left(\frac{Y_t}{Y} \right)^{\gamma_Y^G} - G \right) = \frac{1 - \lambda \theta^H \left(\frac{Y_t}{Y} \right)^{\gamma_Y^H}}{1 - \lambda} Y_t - \frac{\lambda \left(\mathcal{T}^H + \gamma_Y^T \left(Y_t - Y \right) \right)}{1 - \lambda}. \tag{29}$$

Taking a second-order approximation, we have:

$$\begin{aligned}
&\frac{C^S}{Y} \left(\hat{c}_t^S + \frac{1}{2} (\hat{c}_t^S)^2 \right) + \frac{1 - \alpha}{1 - \lambda} s_g \left(\gamma_Y^G \hat{y}_t + \frac{1}{2} (\gamma_Y^G)^2 \hat{y}_t^2 \right) \\
&= \frac{1}{1 - \lambda} \left(\hat{y}_t + \frac{1}{2} \hat{y}_t^2 \right) - \frac{\lambda \theta^H}{1 - \lambda} \left((1 + \gamma_Y^H) \hat{y}_t + \frac{1}{2} (1 + \gamma_Y^H)^2 \hat{y}_t^2 \right) - \frac{\lambda \gamma_Y^T}{1 - \lambda} \left(\hat{y}_t + \frac{1}{2} \hat{y}_t^2 \right)
\end{aligned}$$

$$\begin{aligned}
\rightarrow \frac{C^S}{Y} \left(\hat{c}_t^S + \frac{1}{2} (\hat{c}_t^S)^2 \right) &= \left(\frac{1 - \lambda \theta^H (1 + \gamma_Y^H)}{1 - \lambda} - \frac{\lambda \gamma_Y^T}{1 - \lambda} - \frac{1 - \alpha}{1 - \lambda} s_g \gamma_Y^G \right) \hat{y}_t \\
&\quad + \frac{1}{2} \left(\frac{1 - \lambda \theta^H (1 + \gamma_Y^H)^2}{1 - \lambda} - \frac{\lambda \gamma_Y^T}{1 - \lambda} - \frac{1 - \alpha}{1 - \lambda} s_g (\gamma_Y^G)^2 \right) \hat{y}_t^2.
\end{aligned} \tag{30}$$

Up to first order, Saver's budget constraint can be written as:

$$\hat{c}_t^S = \left(\frac{C^S}{Y} \right)^{-1} \left(\frac{1 - \lambda \theta^H (1 + \gamma_Y^H)}{1 - \lambda} - \frac{\lambda \gamma_Y^T}{1 - \lambda} - \frac{1 - \alpha}{1 - \lambda} s_g \gamma_Y^G \right) \hat{y}_t. \tag{31}$$

Substituting out $(c_t^S)^2$ in (30) using (31) yields:

$$
\begin{aligned}
\hat{c}_t^S &= \left(\frac{C^S}{Y}\right)^{-1}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_Y^G\right)\hat{y}_t \\
&+ \frac{1}{2}\left(\frac{C^S}{Y}\right)^{-1}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)^2}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g(\gamma_Y^G)^2\right. \\
&- \left.\left(\frac{C^S}{Y}\right)^{-1}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_Y^G\right)^2\right)\hat{y}_t^2 \\
&= \theta_{cSy}\hat{y}_t + \theta_{cSy2}\hat{y}_t^2.
\end{aligned} \tag{32}
$$

Substituting out the lump-sum tax TG_t^H, transfers $\mathcal{T}_t^H$, and income share θ_t^H, the HtMs' budget constraint can be rewritten as:

$$
C_t^H + \frac{\alpha}{\lambda}G\left(\frac{Y_t}{Y}\right)^{\gamma_Y^G} - \frac{1-\lambda}{\lambda}TG^S + \frac{1-\alpha}{\lambda}G = \theta^H\left(\frac{Y_t}{Y}\right)^{\gamma_Y^H}Y_t + \mathcal{T}^H + \gamma_Y^T(Y_t - Y). \tag{33}
$$

By taking a similar approach to derive (32), the second-order approximation of (33) is:

$$
\begin{aligned}
\hat{c}_t^H &= \left(\frac{C^H}{Y}\right)^{-1}\left(\theta^H(1+\gamma_Y^H) + \gamma_Y^T - \frac{\alpha}{\lambda}s_g\gamma_Y^G\right)\hat{y}_t \\
&+ \frac{1}{2}\left(\frac{C^H}{Y}\right)^{-1}\left(\theta^H(1+\gamma_Y^H)^2 + \gamma_Y^T - \frac{\alpha}{\lambda}s_g(\gamma_Y^G)^2\right. \\
&- \left.\left(\frac{C^H}{Y}\right)^{-1}\left(\theta^H(1+\gamma_Y^H) + \gamma_Y^T - \frac{\alpha}{\lambda}s_g\gamma_Y^G\right)^2\right)\hat{y}_t^2 \\
&= \theta_{cHy}\hat{y}_t + \theta_{cHy2}\hat{y}_t^2.
\end{aligned} \tag{34}
$$

Finally, per-period welfare of household type $j \in \{S, H\}$ can be approximated as:

$$
\begin{aligned}
U_t^j &\equiv \frac{(C_t^j)^{1-\sigma} - 1}{1-\sigma} - \chi\frac{N_t^{1+\varphi}}{1+\varphi} \\
&\approx \frac{(C^j)^{1-\sigma} - 1}{1-\sigma} - \chi\frac{N^{1+\varphi}}{1+\varphi} \\
&+ (C^j)^{1-\sigma}\left(\hat{c}_t^j + \frac{1}{2}(1-\sigma)(\hat{c}_t^j)^2\right) \\
&- \chi(Yv^p)^{1+\varphi}\left(\hat{y}_t + \hat{v}_t^p - \hat{a}_t + \frac{1}{2}(1+\varphi)\hat{y}_t^2 + \frac{1}{2}(1+\varphi)\hat{a}_t^2 - (1+\varphi)\hat{y}_t\hat{a}_t\right).
\end{aligned}
$$

Substituting out $\hat{c}_t^j$ using (32) and (34),

$$
\begin{aligned}
U_t^j \approx\ & \frac{(C_t^j)^{1-\sigma}-1}{1-\sigma} - \chi \frac{N^{1+\phi}}{1+\phi} \\
& + \left((C^j)^{1-\sigma}\theta_{cjy} - \chi (Yv^p)^{1+\varphi}\right)\hat{y}_t \\
& + \chi (Yv^p)^{1+\varphi}\hat{a}_t \\
& + \left((C^j)^{1-\sigma}\left(\theta_{cjy2} + \frac{1}{2}(1-\sigma)\theta_{cjy}^2\right) - \frac{1}{2}(1+\varphi)\chi (Yv^p)^{1+\varphi}\right)\hat{y}_t^2 \\
& + \chi (Yv^p)^{1+\varphi}(1+\varphi)\hat{y}_t\hat{a}_t \\
& - \chi (Yv^p)^{1+\varphi}\frac{1}{2}(1+\varphi)\hat{a}_t^2 \\
& - \chi (Yv^p)^{1+\varphi}\hat{v}_t^p.
\end{aligned}
$$

Taking the unconditional expectations and substituting out the price dispersion using (26), we write the expected utility of type j:

$$
\begin{aligned}
EU^j \approx\ & \frac{(C^j)^{1-\sigma}-1}{1-\sigma} - \chi \frac{N^{1+\phi}}{1+\phi} - \chi (Yv^p)^{1+\varphi}\frac{1}{2}(1+\varphi)\frac{\sigma_a^2}{1-\rho_a^2} \\
& - \chi (Yv^p)^{1+\varphi}\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{(1-\overline{\theta}_p)(1-\overline{\theta}_p\Pi)}\mathbf{E}(\hat{\pi}_t) \\
& + \left((C^j)^{1-\sigma}\theta_{cjy} - \chi (Yv^p)^{1+\varphi}\right)\mathbf{E}(\hat{y}_t) \\
& - \chi (Yv^p)^{1+\varphi}\frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p(\Pi-1)+1)}{(1-\overline{\theta}_p)^2}\mathbf{E}(\hat{\pi}_t^2) \\
& + \left((C^j)^{1-\sigma}\left(\theta_{cjy2} + \frac{1}{2}(1-\sigma)\theta_{cjy}^2\right) - \frac{1}{2}(1+\varphi)\chi (Yv^p)^{1+\varphi}\right)\mathbf{E}(\hat{y}_t^2) \\
& - \chi (Yv^p)^{1+\varphi}(1+\varphi)\mathbf{E}(\hat{y}_t\hat{a}_t).
\end{aligned}
$$

Under the assumption of $TG^j = s_g Y^j$ and $\mathcal{T}^H = 0$, we obtain:

$$
\begin{aligned}
EU^S \approx & \frac{(C^S)^{1-\sigma}-1}{1-\sigma} - \chi\frac{N^{1+\phi}}{1+\phi} - \chi(Yv^p)^{1+\varphi}\frac{1}{2}(1+\varphi)\frac{\sigma_a^2}{1-\rho_a^2} \\
& - \chi(Yv^p)^{1+\varphi}\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{(1-\overline{\theta}_p)(1-\overline{\theta}_p\Pi)}\mathbf{E}(\hat{\pi}_t) \\
& + \left(Y^{1-\sigma}\left(\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\right)^{-\sigma}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_Y^G\right) - \chi(Yv^p)^{1+\phi}\right)\mathbf{E}(\hat{y}_t) \\
& - \chi(Yv^p)^{1+\varphi}\frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p(\Pi-1)+1)}{(1-\overline{\theta}_p)^2}\mathbf{E}(\hat{\pi}_t^2) \\
& + \left(\frac{1}{2}Y^{1-\sigma}\left(\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\right)^{-\sigma}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)^2}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g(\gamma_Y^G)^2\right)\right. \\
& - \frac{\sigma}{2}Y^{1-\sigma}\left(\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\right)^{-(1+\sigma)}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_Y^G\right)^2 \\
& \left. - \chi(Yv^p)^{1+\phi}\frac{1}{2}(1+\varphi)\right)\mathbf{E}(\hat{y}_t^2) \\
& - \chi(Yv^p)^{1+\varphi}(1+\varphi)\mathbf{E}(\hat{y}_t\hat{a}_t)
\end{aligned}
$$

$$
\begin{aligned}
EU^H \approx & \frac{(C^H)^{1-\sigma}-1}{1-\sigma} - \chi\frac{N^{1+\phi}}{1+\phi} - \chi(Yv^p)^{1+\varphi}\frac{1}{2}(1+\varphi)\frac{\sigma_a^2}{1-\rho_a^2} \\
& - \chi(Yv^p)^{1+\varphi}\frac{\varepsilon_p\overline{\theta}_p(\Pi-1)}{(1-\overline{\theta}_p)(1-\overline{\theta}_p\Pi)}\mathbf{E}(\hat{\pi}_t) \\
& + \left(Y^{1-\sigma}\left(\frac{(1-s_g)\mu_y}{(1-\lambda)\mu_y+\lambda}\right)^{-\sigma}\left(\frac{1-\lambda\theta^H(1+\gamma_Y^H)}{1-\lambda} - \frac{\lambda\gamma_Y^T}{1-\lambda} - \frac{1-\alpha}{1-\lambda}s_g\gamma_Y^G\right) - \chi(Yv^p)^{1+\phi}\right)\mathbf{E}(\hat{y}_t) \\
& - \chi(Yv^p)^{1+\varphi}\frac{1}{2}\frac{\varepsilon_p\overline{\theta}_p(\varepsilon_p(\Pi-1)+1)}{(1-\overline{\theta}_p)^2}\mathbf{E}(\hat{\pi}_t^2) \\
& + \left(\frac{1}{2}Y^{1-\sigma}\left(\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\right)^{-\sigma}\left(\theta^H(1+\gamma_Y^H)^2 + \gamma_Y^T - \frac{\alpha}{\lambda}s_g(\gamma_Y^G)^2\right)\right. \\
& - \frac{\sigma}{2}Y^{1-\sigma}\left(\frac{(1-s_g)}{(1-\lambda)\mu_y+\lambda}\right)^{-(1+\sigma)}\left(\theta^H(1+\gamma_Y^H) + \gamma_Y^T - \frac{\alpha}{\lambda}s_g\gamma_Y^G\right)^2 \\
& \left. - \chi(Yv^p)^{1+\phi}\frac{1}{2}(1+\varphi)\right)\mathbf{E}(\hat{y}_t^2) \\
& - \chi(Yv^p)^{1+\varphi}(1+\varphi)\mathbf{E}(\hat{y}_t\hat{a}_t)
\end{aligned}
$$

The unconditional value under steady-state inflation $\overline{\Pi}$ for each type is given as follows.

When there is no type switching ($s = 1$),

$$EV^S(\Pi = \overline{\Pi}) = \frac{EU^S(\Pi = \overline{\Pi})}{1-\beta}$$
$$EV^H(\Pi = \overline{\Pi}) = \frac{EU^H(\Pi = \overline{\Pi})}{1-\beta}.$$

When type switching occurs, the unconditional value for each household is determined by solving:

$$EV^S(\Pi = \overline{\Pi}) = EU^S(\Pi = \overline{\Pi}) + s\beta EV^S(\Pi = \overline{\Pi}) + (1-s)\beta EV^H(\Pi = \overline{\Pi})$$
$$EV^H(\Pi = \overline{\Pi}) = EU^H(\Pi = \overline{\Pi}) + (1-h)\beta EV^S(\Pi = \overline{\Pi}) + h\beta EV^H(\Pi = \overline{\Pi}),$$

where $\lambda = \frac{1-s}{2-s-h}$.

3 Welfare Cost of Trend Inflation

This section outlines the procedure for computing the consumption-equivalent variation that equates the unconditional value between two scenarios: (1) THANK in which the inflation target is set at the THANK optimal level $\Pi^{THANK,opt}$ and (2) THANK in which the inflation target is set at the RANK optimal level $\Pi^{RANK,opt}$. The unconditional value under $\Pi^{THANK,opt}$ is computed as follows.

$$\begin{bmatrix} 1-\beta s & -\beta(1-s) \\ -\beta(1-h) & 1-\beta h \end{bmatrix} \begin{bmatrix} EV^S(\Pi^{THANK,opt} \mid \gamma_Y^H, \mu_Y, s) \\ EV^H(\Pi^{THANK,opt} \mid \gamma_Y^H, \mu_Y, s) \end{bmatrix} = \begin{bmatrix} EU^S(\Pi^{THANK,opt} \mid \gamma_Y^H, \mu_Y, s) \\ EU^H(\Pi^{THANK,opt} \mid \gamma_Y^H, \mu_Y, s) \end{bmatrix}.$$

The unconditional value under $\Pi^{RANK,opt}$ with compensating consumption variation (λ^S or λ^H) is computed as follows.

$$\begin{bmatrix} 1-\beta s & -\beta(1-s) \\ -\beta(1-h) & 1-\beta h \end{bmatrix} \begin{bmatrix} EV^S(\Pi^{RANK,opt} \mid \gamma_Y^H, \mu_Y, s) \\ EV^H(\Pi^{RANK,opt} \mid \gamma_Y^H, \mu_Y, s) \end{bmatrix}$$
$$= \begin{bmatrix} (1+\lambda^S)^{1-\sigma} \\ 1 \end{bmatrix} \odot \begin{bmatrix} E\left(\frac{(C_t^S)^{1-\sigma}}{1-\sigma} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \\ E\left(\frac{(C_t^H)^{1-\sigma}}{1-\sigma} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \end{bmatrix} - \begin{bmatrix} \frac{1}{1-\sigma} + E\left(\frac{N_t^{1+\phi}}{1+\phi} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \\ \frac{1}{1-\sigma} + E\left(\frac{N_t^{1+\phi}}{1+\phi} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \end{bmatrix}$$

$$\begin{bmatrix} 1-\beta s & -\beta(1-s) \\ -\beta(1-h) & 1-\beta h \end{bmatrix} \begin{bmatrix} EV^S(\Pi = \Pi^{RANK,opt} \mid \gamma_Y^H, \mu_Y, s) \\ EV^H(\Pi = \Pi^{RANK,opt} \mid \gamma_Y^H, \mu_Y, s) \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ (1+\lambda^H)^{1-\sigma} \end{bmatrix} \odot \begin{bmatrix} E\left(\frac{(C_t^S)^{1-\sigma}}{1-\sigma} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \\ E\left(\frac{(C_t^H)^{1-\sigma}}{1-\sigma} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \end{bmatrix} - \begin{bmatrix} \frac{1}{1-\sigma} + E\left(\frac{N_t^{1+\phi}}{1+\phi} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \\ \frac{1}{1-\sigma} + E\left(\frac{N_t^{1+\phi}}{1+\phi} \mid \Pi^{RANK,opt}, \gamma_Y^H, \mu_Y, s\right) \end{bmatrix}.$$

We compute λ^j such that $EV^j(\Pi^{THANK,opt} \mid \gamma_Y^H, \mu_Y, s) = EV^j(\Pi^{RANK,opt} \mid \gamma_Y^H, \mu_Y, s)$. λ^j denotes the proportional change in consumption required for households of type j to attain the same level of welfare in a THANK economy with the RANK-optimal trend inflation rate as in one with the THANK-optimal trend inflation rate.

References

Alves, Sergio Afonso Lago, "Lack of divine coincidence in New Keynesian models," *Journal of Monetary Economics*, 2014, *67*, 33–46.