3450:335 Ordinary Differential Equations

Review of Integration and Trigonometry

Here is a review of the integration and trigonometric skills you need in this course.

**Integration by substitution**

$$\begin{aligned} \int e^{2x}\,dx &= \frac{1}{2}e^{2x} + c \quad u = 2x, du = 2dx \\ \int 2x^3\sqrt{1+x^2}\,dx &= \int (u-1)u^{1/2}du = \int u^{3/2} - u^{1/2}\,du \quad u = 1+x^2, du = 2xdx, x^2 = u-1 \end{aligned}$$

**Integration by parts**

There are 3 classes of integrals that can be evaluated using IBP:

A) $\int$(polynomial)(sin or cos) $dx$, $\int$(polynomial)(exp) $dx$. Let ($u$ = polynomial) to kill off the polynomial one degree at a time.

B) $\int$(polynomial)(logarithm) $dx$, $\int$ arctrig $dx$. Let $u$ = logarithm or $u$ = arctrig.

C) $\int$(exp)(sin or cos) $dx$. IBP twice, using $u$ = trig both times, or $u$ = exp both times. Called FOLDING.

**Basic Trigonometric Skills**

A) Definitions of the 6 trig functions in terms of right triangles and in terms of sines and cosines.

B) Pythagorean identities, 3 flavors. $\sin^2 x + \cos^2 x = 1$, $1 + \cot^2 x = \csc^2 x$, $\tan^2 x + 1 = \sec^2 x$.

C) Half angle identities. $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$.

D) Double angle identities. $2\sin x\cos x = \sin(2x)$, $\cos^2 x - \sin^2 x = \cos(2x)$, $2\cos^2 x - 1 = \cos(2x)$, $1 - 2\sin^2 x = \cos(2x)$.

E) Sum and difference identities. $\sin(x \pm y) = \sin x\cos y \pm \sin y\cos x$, $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$.

F) Product identities. $\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$, $\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$, $\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$.

G) Elementary antiderivatives. $\int \sin x\,dx = -\cos x + c$, $\int \cos x\,dx = \sin x + c$, $\int \sec^2 x\,dx = \tan x + c$, $\int \csc^2 x\,dx = -\cot x + c$, $\int \sec x\tan x\,dx = \sec x + c$.

H) Antiderivatives using logarithmic form. $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{1}{u}\,du = -\ln|\cos x| + c$, $\displaystyle\int \sec x\,dx = \int \sec x\frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{1}{u}\,du = \ln|\sec x + \tan x| + c$.

**Trigonometric Integrals**

A) $\int \sin^m x\cos^n x\,dx$. If $n$ or $m$ is odd, pull off the odd power of sin or cos to use in $du$, let $u$ equal the other trig function, and apply Pythagorean identities if needed. If both $n$ and $m$ are odd, you can let $u$ equal $\sin x$ or $\cos x$. If $n$ and $m$ are both even, apply half angle identities and Pythagorean identities.

Examples:

$$
\begin{aligned}
\int \cos^3 x\,dx &= \int (1-\sin^2 x)(\cos x dx) = \int (1-u^2)du \\
\int \sin^3 x \cos^3 x\,dx &= \int (\sin^3 x)(1-\sin^2 x)(\cos x dx) = \int (u^3)(1-u^2)(du) \\
\int \sin^2 x\,dx &= \int \frac{1}{2}(1-\cos(2x))\,dx
\end{aligned}
$$

B) $\displaystyle\int \tan^m x \sec^n x\,dx$. If $n$ is even, pull off a $\sec^2 x$ to use in $du$, let $u = \tan x$ and apply Pythagorean identity if needed. If $m$ is odd, pull off a $\sec x \tan x$ to use in $du$, let $u = \sec x$, apply Pythagorean identity if needed. If both $n$ is even and $m$ is odd, you can do either of the above. If $n$ is odd and $m$ is even, God help you. Use Pythagorean identities to reduce to nothing but powers of $\sec x$. Integrals of $\sec x$ and $\sec^3 x$ are standard, and can be looked up in your old Calculus text (you didn't sell it back to the Bookstore, did you?). Higher odd powers of $\sec x$ require careful use of IBP. Even powers can be handled by the second sentence in this paragraph.

Examples:

$$
\begin{aligned}
\int \tan^2 x \sec^4 x\,dx &= \int \tan^2 x(\tan^2 x + 1)(\sec^2 x\,dx) = \int u^2(u^2+1)\,du \\
\int \tan x \sec^{20} x\,dx &= \int \sec^{19} x(\sec x \tan x\,dx) = \int u^{19}\,du \\
\int \tan^2 x \sec x\,dx &= \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x - \sec x\,dx
\end{aligned}
$$

**Trigonometric Substitution**

Look for expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$, or one of these to a higher power (for example, $(x^2-4)^2$ is the square root raised to the fourth power). If there is a minus under the square root, the positive term is the square of the hypotenuse and the negative term is the square of one of the legs; the square root is the other leg. If there is a plus under the square root, then the square root is the hypotenuse, and the terms inside are the squares of the legs. Use the triangle to make a substitution, which converts the integral to a trig integral (see above).

Examples:

1) $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$. The hypotenuse is 4 and the sides are $x$ and $\sqrt{16-x^2}$. Let $x = 4\sin\theta$, so $dx = 4\cos\theta$. The integral becomes $\displaystyle\int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}} 4\cos\theta\,d\theta = \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta = -\frac{1}{16}\cot\theta + c = -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + c.$

2) $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}\,dx$. The hypotenuse is $\sqrt{9+x^2}$ and the legs are $x$ and 3. Let $x = 3\tan\theta$, so $dx = 3\sec^2\theta d\theta$. The integral becomes $\displaystyle\int \frac{27\tan^3\theta}{3\sec\theta} 3\sec^2\theta\,d\theta = 27\int \frac{\sin^3\theta}{\cos^4\theta}\,d\theta$. Write the numerator as

$\sin^2\theta(\sin\theta\,d\theta) = (1-u^2)du$

with $u = \cos\theta$. The integral becomes

$\displaystyle 27\int (1-u^2)u^{-4}(-du) = 27\left(-\frac{1}{u} + \frac{1}{3u^3}\right) = \left(-9\sqrt{9+x^2} + \frac{1}{3}(9+x^2)^{3/2}\right).$

Note that $u = \cos\theta = 3/\sqrt{9+x^2}$. Oh, and don't forget the constant of integration.

**Partial Fraction Decomposition**

Uses: In Laplace Transform Techniques (ODE's) and Control Theory.

Observation: $\dfrac{1}{x^2-4} = \dfrac{\frac{1}{4}}{x-2} - \dfrac{\frac{1}{4}}{x+2}$.

So, $\displaystyle\int \frac{1}{x^2-4}\,dx = \int \frac{\frac{1}{4}}{x-2}\,dx - \int \frac{\frac{1}{4}}{x+2}\,dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C.$

Important Facts:

1. Any polynomial can be decomposed into its linear and irreducible quadratic factors.

   Example: $Q(x) = x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$.

2. Fundamental Theorem of Algebra: Every rational function $\frac{P(x)}{Q(x)}$, where $\deg[P(x)] < \deg[Q(x)]$ can be expressed as

$$\frac{P(x)}{Q(x)} = R_1(x) + R_2(x) + \cdots + R_n(x),$$

   where each $R_i(x)$ is of the form $\dfrac{A}{(ax+b)^k}$ or $\dfrac{Ax+B}{(ax^2+bx+c)^k}$. [Note: $(ax^2+bx+c)$ is an irreducible quadratic.] The $R_i$'s are called the partial fraction decomposition.

3. If $\deg[P(x)] \geq \deg[Q(x)]$, after long division $\dfrac{P(x)}{Q(x)} =$ polynomial $+ \dfrac{p(x)}{q(x)}$, where $\deg[p(x)] < \deg[q(x)]$.

   Partial fraction decomposition can be applied to $\dfrac{p(x)}{q(x)}$.

Example:

$\dfrac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \left(\dfrac{1}{x^2+x-2}\right)$. Here the polynomial is $3x^2+1$ and

$$\frac{p(x)}{q(x)} = \frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{-\frac{1}{3}}{x+2} + \frac{\frac{1}{3}}{x-1}.$$

- To find the partial fraction decomposition in the case where $\deg[p(x)] < \deg[q(x)]$ for $\dfrac{p(x)}{q(x)}$:

1. Factor $q(x)$ into its linear and irreducible quadratics.
2. Linear Factor Rule: For each factor $(ax+b)^m$, the partial fraction decomposition contains

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n},$$

   where $A_1, A_2, \ldots, A_n$ are constants to be determined.

3. Irreducible Quadratic Factor Rule: For each $(ax^2+bx+c)^m$, the partial fraction decomposition has the form
$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n},$$
where $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are constants to be determined.

Examples:

1. $\displaystyle\int \frac{4}{2x+3}\,dx$. Let $u=2x+3 \implies du = 2\,dx$

   So, $\displaystyle\int \frac{4}{2x+3}\,dx = \int \frac{4}{u}\cdot\frac{1}{2}\,du = \int \frac{2}{u}\,du = 2\ln|2x+3|+C.$

2. $\displaystyle\int \frac{4}{(2x+3)^5}\,dx$. Let $u=2x+3 \implies du = 2\,dx$

   So, $\displaystyle\int \frac{4}{(2x+3)^5}\,dx = \int \frac{4}{u^5}\cdot\frac{1}{2}\,du = \int 2u^{-5}\,du = -\frac{2}{4}u^{-4}+C = -\frac{1}{2(2x+3)^4}+C.$

3. $\displaystyle\int \frac{2}{2x^2+4x+3}\,dx = \int \frac{2}{2(x+1)^2+1}\,dx$. Let $u=\sqrt{2}(x+1) \implies du=\sqrt{2}\,dx.$

   So, $\displaystyle\int \frac{2}{2x^2+4x+3}\,dx = \sqrt{2}\int \frac{1}{u^2+1}\,du = \sqrt{2}\tan^{-1}(\sqrt{2}(x+1))+C.$

4. $\displaystyle\int \frac{2x+1}{2x^2+4x+3}\,dx$

   Note: If you let $u=2x^2+4x+2 \implies du=(4x+4)\,dx$. The trouble is the numerator is $2x+1$, not $4x+4$. Some manipulation is needed to deal with this situation.

$$\begin{aligned}
&= \frac{1}{2}\int \frac{2(2x+1)}{2x^2+4x+3}\,dx \\
&= \frac{1}{2}\int \frac{4x+2}{2x^2+4x+3}\,dx \\
&= \frac{1}{2}\int \frac{4x+2+(2-2)}{2x^2+4x+3}\,dx \\
&= \frac{1}{2}\int \frac{(4x+4)-2}{2x^2+4x+3}\,dx \\
&= \frac{1}{2}\left[\int \frac{4x+4}{2x^2+4x+3} - \frac{2}{2x^2+4x+3}\right]dx
\end{aligned}$$

   The first integral is a natural log and the second integral is an inverse tangent (example #3).

   So, $\displaystyle\int \frac{2x+1}{2x^2+4x+3}\,dx = \frac{1}{2}\ln|2x^2+4x+3| - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}(x+1))+C.$