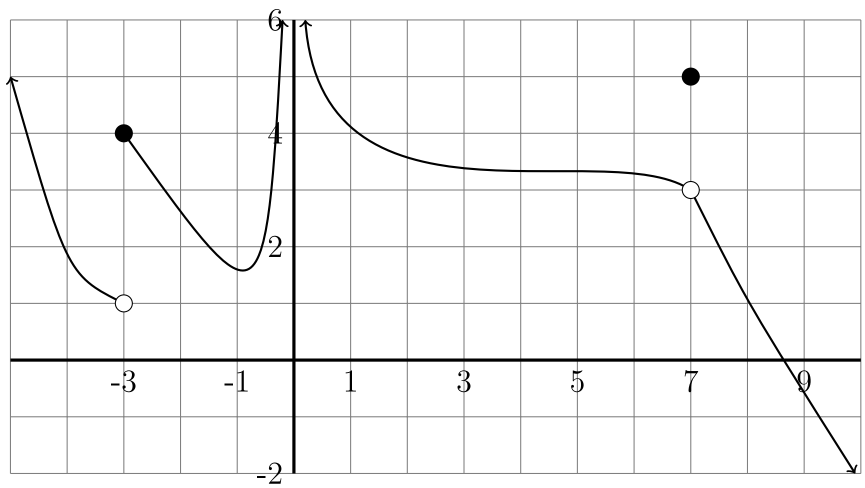
3450:221 Calculus I, Final Sample Problems

These problems provide a sample of typical problems you are expected to be able to solve.

A. Limits

1. Graphical

   a. Use the graph of $f(x)$ to evaluate the limits
   $L_1 = \lim_{x\to -3^-} f(x)$, $L_2 = \lim_{x\to -3^+} f(x)$, $L_3 = \lim_{x\to -3} f(x)$
   $L_4 = \lim_{x\to 0^-} f(x)$, $L_5 = \lim_{x\to 0^+} f(x)$, $L_6 = \lim_{x\to 0} f(x)$
   $L_7 = \lim_{x\to 7^-} f(x)$, $L_8 = \lim_{x\to 7^+} f(x)$, $L_9 = \lim_{x\to 7} f(x)$



   b. Sketch the graph of a function $f(x)$ with the following properties: $f(0) = 3$, $\lim_{x\to 1^-} f(x) = 4$, $\lim_{x\to 1^+} f(x) = 2$, $\lim_{x\to 4^-} f(x) = -\infty$, $\lim_{x\to 4^+} f(x) = \infty$, $\lim_{x\to -\infty} f(x) = -\infty$, $\lim_{x\to \infty} f(x) = 3$.

2. Computational

   a. $\lim_{x\to 0} \dfrac{|x|}{x}$

   b. $\lim_{x\to 5} \dfrac{x^2 - 6x + 5}{x - 5}$

   c. $\lim_{x\to 3} \dfrac{\frac{1}{x} - \frac{1}{3}}{x - 3}$

   d. $\lim_{x\to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$

   e. $\lim_{x\to 0} \ln(1 + e^{-x})$

   f. $\lim_{x\to 0} \ln(1 - e^{-x})$

   g. $\lim_{x\to \infty} \ln(1 + e^{-x})$

   h. $\lim_{x\to -\infty} \ln(1 + e^{-x})$

   i. $\lim_{x\to -\infty} \dfrac{\sqrt{1 + 4x^6}}{2 - x^3}$

j. $\lim_{x \to \infty} \sqrt{9x^2 + x} - 3x$

k. $\lim_{x \to 0} \dfrac{\sin(5x)}{4x}$

l. $\lim_{x \to 0} (1 + x)^{1/x}$

m. $\lim_{x \to 0} (1 + ax)^{1/x}$

n. $\lim_{t \to 0} \dfrac{e^{2t} - 1}{\sin t}$

o. $\lim_{x \to 0} \dfrac{\sqrt{1 + 2x} - \sqrt{1 - 4x}}{x}$

3. Continuity

a. Find the value of k that makes the function $f(x)$ continuous at $x = 2$:

$$f(x) = \begin{cases} 3x + k, & \text{if } x \leq 2 \\ x^2 - x, & \text{if } x > 2 \end{cases}$$

b. Is $f(x)$ below continuous at $x = 1$? Why or why not?

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 1 + x, & \text{if } x > 1 \end{cases}$$

c. Find the value of k that makes $f(x)$ below be continuous at $x = 3$.

$$f(x) = \begin{cases} 1 + x^2, & \text{if } x \leq 3 \\ 2 + kx, & \text{if } x > 3 \end{cases}$$

B. Derivatives

1. Graphical

a. For $f(x) = x^3 - 3x^2 - 9x + 4$, find the intervals where $f(x)$ is increasing/decreasing and the intervals where $f(x)$ is concave up and down. Find the coordinates of any local maximum points, local minimum points, and inflection points.

b. For $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$, find the intervals where $f(x)$ is increasing/decreasing and the intervals where $f(x)$ is concave up and down. Find the coordinates of any local maximum points, local minimum points, and inflection points.

c. Sketch the graph of a function $f(x)$ with the following properties: $f(0) = 1$, $f'(x) = 0$ at $x = 0, 2, 4$, $f'(x) > 0$ for $x < 0$ and $2 < x < 4$, $f'(x) < 0$ for $0 < x < 2$ and $4 < x$, $f''(x) > 0$ for $1 < x < 3$ and $f''(x) < 0$ for $x < 1$ and $x > 3$.

2. Definition

a. Use the definition of the derivative to find $f'(x)$ for $f(x) = \dfrac{1}{\sqrt{1 + x}}$.

3. Computational. Simplify all derivatives as much as possible.

a. Find $f'(x)$ and $f''(x)$ for $f(x) = \dfrac{5}{8}x^{8/3} - \dfrac{5}{8}x^{-3/5} + \pi^2$.

b. Find $f'(x)$ for $f(x) = e^x + x^e$.

c. Find $f'(y)$ for $f(y) = y^{1/3}(y - 2)^{2/3}$.

d. Find $f'(x)$ for $f(x) = \dfrac{1 + e^{x^2}}{1 - e^{-x^2}}$.

e. Find $f'(r)$ for $f(r) = \dfrac{r^2(r+1)^{1/3}}{(r+2)^{2/3}}$.

f. Find $g'(\theta)$ for $g(\theta) = \dfrac{\sin 3\theta}{\sin 2\theta}$.

g. Find $h'(x)$ for $h(x) = \ln(1 + e^{-x})$.

h. Find $f'(x)$ for $f(x) = 3^{x^2}$.

i. Find $r'(p)$ for $r(p) = \sec(\ln p + 1)$.

j. Find $f'(x)$ for $f(x) = \ln(\tan(e^{x^2} + 2x))$.

k. Find $c'(x)$ for $c(x) = \displaystyle\int_1^{e^x} t \ln t^2 \, dt$.

l. Find $f'(t)$ for $f(t) = e^{3t} \cos 5t$.

m. Find $f'(x)$ for $f(x) = (x^2+1)^4(x + \sin(\ln x))^{1/3}$.

n. Find $\alpha'(\theta)$ for $\alpha(\theta) = \sin^2(\theta^2)$.

o. Find $f'(x)$ for $f(x) = \arctan(e^x)$.

p. Find $f'(x)$ for $f(x) = \cot^{-1}(e^x)$.

q. Find $f'(x)$ for $f(x) = \arctan(3x)$.

r. Find $f'(x)$ for $f(x) = \arctan\left(\dfrac{x}{2}\right)$.

s. Find $g'(x)$ for $g(x) = \dfrac{\sinh x}{\cosh x + 1}$.

4. Implicit Differentiation.

   a. Find $\dfrac{dy}{dx}$ if $x^2 + x^3 y = y^2 + 1$.

   b. Find $\dfrac{dy}{dx}$ if $\tan^{-1}(x^2 y) = x + e^y$.

   c. Find $\dfrac{dy}{dx}$ if $\ln y + e^x = \ln x + e^y$.

5. Logarithmic Differentiation.

   a. Find y' for $y = \dfrac{e^{-3x}\sqrt{x^2+4}}{(x+2)^2(x+3)^3}$.

   b. Find y' for $y = x^{\sin x}$.

6. Related Rates and Linear Approximations

   a. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 2 ft higher than the bow of the boat. The rope is being pulled in at the rate of 1.5 ft/sec. How fast is the boat approaching the dock when it is 10 ft from the dock.

   b. A composite fiber mat in the shape of a rectangle with width w and height h changes dimensions as its temperature changes. Suppose the width is increasing at the rate of 0.4 cm/sec and the height is decreasing at the rate of 0.3 cm/sec. At what rate is the area of the mat changing at the instant when $w = 6$ cm and $h = 10$ cm?

   c. Use a linear approximation for $f(x) = (8+x)^{1/3}$ to estimate $8.07^{1/3}$.

7. Extreme Values and Critical Numbers

   a. Find the Critical Numbers of $f(x) = x^{2/3}(x+1)^3$.

b. Find the Critical Numbers of $f(t) = t^{6/7} - 3t^{3/7}$.

c. Find the Critical Numbers of $g(x) = x^3 + 6x^2 - 15x + 4$.

d. Find the absolute maximum and minimum values of $f(x) = \frac{1}{3}x^3 - 2x^2 + 4x + 2$ on the interval $[1, 4]$.

e. Find the absolute maximum and minimum values of $f(x) = \frac{x^3}{3} + \frac{5x^2}{2} - 6x + 3$ on the interval $[-5, 2]$.

f. Verify that the function $f(x) = 2x^2 - 3x + 1$ satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find the values of c that satisfy the conclusion of the theorem.

8. Optimization

a. Find two numbers x and y whose difference is 100 and whose product is a minimum.

b. You have 100 feet of fence to build a rectanglular pen with a divider going through the middle. Find the dimensions of the pen that has maximal area.

c. You need to build a box that has a square base, no top, and volume $V = 32000$ cm^3. Find the dimensions of the box that use the minimal amount of material.

C. Integrals

1. Find $f(x)$ if $f''(x) = \frac{15}{16}x^{1/4} - \frac{6}{125x^{11/15}}$.

2. Find $f(x)$ if $f'(x) = \frac{1}{3}x^{9/4} - \frac{1}{5}x^{-1/5}$ and $f(1) = 3$.

3. Evaluate these integrals.

a. $I = \int_1^2 \frac{1}{3}x^{3/2} + \frac{1}{2}x^{1/2}\,dx$

b. $I = \int_1^4 \sqrt{2x+1}\,dx$

c. $I = \int_0^1 (2r+1)^{17}\,dr$

d. $I = \int_0^4 |x-1|\,dx$

e. $I = \int_0^{18} \sqrt{\frac{3}{z}}\,dz$

f. $I = \int \frac{1}{4+x^2}\,dx$

g. $I = \int \frac{1}{1+9x^2}\,dx$

h. $I = \int \frac{e^x}{1+e^x}\,dx$

i. $I = \int \frac{e^x}{1+e^{2x}}\,dx$

j. $I = \int \frac{x}{4+x}\,dx$

k. $I = \int \frac{x}{7+x^2}\,dx$

l. $I = \int \tan 3x\,dx$

m. $I = \int \frac{1}{x}(\ln x + 1)\,dx$

n. $I = \int_8^{10} \frac{e^{1/w}}{w^2}\,dw$

o. $I = \int_0^2 \frac{1}{(4-2x)^{5/2}}\,dx$

p. $I = \int \frac{1}{\sqrt{1-x^2}\arcsin x}\,dx$

q. $I = \int x^5\sqrt{x^3+2}\,dx$