I. INTEGRATION BY PARTS: $\int u\,dv = uv - \int v\,du$

A. Classes of Integrals IBP are used on.

1. $\int (\text{Poly}) \sin ax\,dx$, $\int (\text{Poly}) \cos ax\,dx$, $\int (\text{Poly}) e^{ax}\,dx$

2. $\int (\text{Poly}) \ln ax\,dx$, $\int \text{arctrig}\,dx$

3. $\int e^{ax} \cos bx\,dx$, $\int e^{ax} \sin bx\,dx$

For class ① Kill off Poly - no. of I.B.P. Applications = deg(Poly).

For class ② Kill off $\ln x$ or arctrig

For class ③ Two applications of parts & original integral appears on R.H.S. Solve for the original integral in the eqn.

II. Basic TRIG Skills Required in This Course:

A. Rt Δ defns. & reciprocal identities.

B. Pythagorean Identities:

1. $\sin^2\theta + \cos^2\theta = 1$, 2. $1+\cot^2\theta = \csc^2\theta$, 3. $\tan^2\theta + 1 = \sec^2\theta$.

C. $\frac{1}{2}$ angle identities

1. $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$, 2. $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

D. Double angle Identities:

1. $2\sin x\cos x = \sin(2x)$, 2. $\cos^2 x - \sin^2 x = \cos(2x)$, 3. $2\cos^2 x - 1 = \cos 2x$,

4. $1 - 2\sin^2 x = \cos(2x)$.

E. Sum of angles Identities:

1. $\sin(x \pm y) = \sin x\cos y \pm \sin y\cos x$

2. $\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

F. Product Identities:

1. $\sin A\cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$; 2. $\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

3. $\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

G. Calculus You Know!

1. $\int \sin x\,dx = \cos x + C$; 2. $\int \cos x\,dx = \sin x + C$; 3. $\int \sec^2 x\,dx = \tan x + C$;
4. $\int \csc^2 x\,dx = -\cot x + C$; 5. $\int \sec x \tan x\,dx = \sec x + C$.

F. With advent of $\int \frac{1}{u}\,du = \ln|u| + C$ & u-substitution.

1. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \underset{u=\cos x}{=} -\int \frac{1}{u}\,du = -\ln|\cos x| + C.$

2. $\int \sec x\,dx = \int \sec x \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx \underset{u=\sec x+\tan x}{=} \int \frac{1}{u}\,du$

$= \ln|\sec x + \tan x| + C.$

III. Trig Integration:

A. Examples:

1. $\int \sin^2 x\,dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x\,dx$, $\frac{1}{2}$ angle identity

$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C.$

2. $\int \cos^3 x\,dx = \int \cos^2 x \cdot \cos x\,dx$, pull off cosine term.

$= \int (1 - \sin^2 x)\cos x\,dx$, Pyth. Thm.

$= \int \cos x - \sin^2 x \cdot \cos x\,dx$, distribute thru

$= \sin x - \frac{1}{3}\sin^3 x + C.$

3. $\int \sin^4 x\,dx = \int \sin^2 x \cdot \sin^2 x\,dx$

$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\,dx$

$= \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$

$= \int \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\,dx$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$

4. $\int \sin^m x \cos^n x \, dx$

a) n odd Pull off cosine, apply Pyth. Thm & let $u = \sin x$

$$\underbrace{\int \sin^m x \cdot \cos^{n-1} x}_{\text{Now in terms of sine, let } u = \sin x} \cdot \cos x \, dx$$

Pyth. Thm

b) m odd: Pull off sine, apply Pyth. Thm, let $u = \cos x$:

$$\underbrace{\int \sin^{m-1} x \cdot \cos^n x}_{\text{now a fcn of } \cos^2 x \text{. Let } u = \cos x} \cdot \sin x \, dx$$

Pyth Thm: $1 - \cos^2 x = \sin^2 x$

c) m & n even: Apply Pyth. Thm & $\frac{1}{2}$ angle identities.

$$\int \sin^2 x \cdot \cos^2 x \, dx = \int (1 - \cos^2 x) \cdot \cos^2 x \, dx$$

$$= \int \cos^2 x \, dx - \int \cos^4 x \, dx$$

$$= \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \, dx - \int (\tfrac{1}{2} + \tfrac{1}{2}\cos 2x)(\tfrac{1}{2} + \tfrac{1}{2}\cos 2x) \, dx$$

$$= \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \, dx - \int \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x \, dx$$

$$= \int \tfrac{1}{2} + \tfrac{1}{2}\cos 2x \, dx - \int \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}(\tfrac{1}{2} + \tfrac{1}{2}\cos 4x) \, dx$$

Now you can integrate each term.

5. $\int \tan^m x \sec^n x \, dx$

a) n even: Split off $\sec^2 x$, apply pyth. identity ($\sec^2 x = \tan^2 x - 1$), let $u = \tan x$.

b) m odd: Split off factor of $\sec x \tan x$, apply $\tan^2 x = \sec^2 x - 1$
let $u = \sec x$.

c) m even, n odd: use identities, including $\tan^2 x = \sec^2 x - 1$ to reduce to powers of $\sec x$ alone

TRIG Substitution:

IV

Purpose: To Remove Square roots in integrands

COST: Evaluate a trig. integral

Pyth. Identity

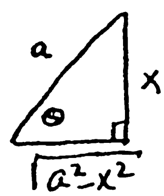
$1-\sin^2\theta = \cos^2\theta$, If $x = a\sin\theta$, $\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = a\cos\theta$ & $dx = a\cos\theta\, d\theta$.

$1+\tan^2\theta = \sec^2\theta$, If $x = a\tan\theta$, $\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = a\sec\theta$

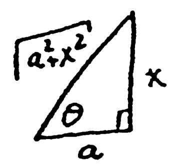
$\sec^2\theta - 1 = \tan^2\theta$, If $x = a\sec\theta$, $\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\tan^2\theta} = a\tan\theta$.

Recall:

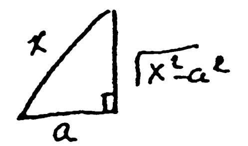
If $x = a\sin\theta$:



If $x = a\tan\theta$:



If $x = a\sec\theta$:

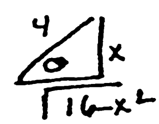


Ex:

① $\displaystyle\int \frac{1}{x^2\sqrt{16-x^2}}\,dx \underset{\substack{x=4\sin\theta \\ dx = 4\cos\theta\, d\theta}}{=} \int \frac{1}{16\sin^2\theta\sqrt{16-16\sin^2\theta}} \cdot 4\cos\theta\, d\theta$

$= \frac{1}{16}\int \frac{1}{\sin^2\theta}\,d\theta = \frac{1}{16}\int \csc^2\theta\, d\theta = -\frac{1}{16}\cot\theta + C$

$= -\frac{1}{16}\frac{\sqrt{16-x^2}}{x} + C$



② $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx \underset{\substack{x=3\sin\theta \\ dx=3\cos\theta}}{=} \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\, d\theta$

$= 9\int \sin^2\theta\, d\theta = 9\int \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta$

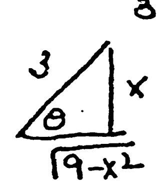
$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + C$

$x = 3\sin\theta$

$\sin\theta = \frac{x}{3}$ or $\theta = \sin^{-1}(\frac{x}{3})$.

$= \frac{9}{2}\theta - \frac{9}{4}\cdot 2\cdot\sin\theta\cdot\cos\theta + C$

$= \frac{9}{2}\sin^{-1}(\frac{x}{3}) - \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C$.



Purpose: To Integrate RATIONAL FCNS i.e. $\int \frac{\text{Poly}}{\text{Poly}}\,dx$.

USED: IN LAPLACE TRANSFORM Techniques (ODE'S) & Control Theory

OBSERVATION: $\frac{1}{x^2-4} = \frac{\frac{1}{4}}{x-2} - \frac{\frac{1}{4}}{x+2}$ So $\int \frac{1}{x^2-4}dx = \int \frac{\frac{1}{4}}{x-2}dx - \int \frac{\frac{1}{4}}{x+2}dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2|$

FACTS:

① Any Poly can be decomposed into its linear & irreducible quadratic factors: Ex: $Q(x) = x^3 - 3x^2 + x - 3 = (x-3)(x^2+1)$

② Fundamental Theorem of Algebra: Every Rational FCN $\frac{P(x)}{Q(x)}$, where $\deg P(x) < \deg Q(x)$ can be expressed as

$$\frac{P(x)}{Q(x)} = R_1(x) + R_2(x) + \cdots + R_n(x),$$ where each $R_i(x)$ is of the form $\frac{A}{(ax+b)^k}$ or $\frac{Ax+B}{(ax^2+bx+c)^k}$ (NOTE: ax^2+bx+c is an irred. Quad).

The R_i's are called the partial fraction decomposition.

③ If $\deg P(x) \geq \deg Q(x)$, after long division $\frac{P(x)}{Q(x)} = \text{Poly} + \frac{p(x)}{g(x)}$ where $\deg p(x) < \deg g(x)$. Partial fraction decomposition can be applied to $\frac{p(x)}{g(x)}$.

Ex: $$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = (3x^2+1) + \left(\frac{1}{x^2+x-2}\right)$$

$\text{Poly} = 3x^2+1$; $\frac{p(x)}{g(x)} = \frac{1}{x^2+x-2} = \frac{1}{(x+2)(x-1)} = \frac{-1/3}{x+2} + \frac{1/3}{x-1}$

How Does one find THE P.F.D. In The Case $\deg p(x) < \deg g(x)$ for $\frac{p(x)}{g(x)}$:

① First factor $g(x)$ into its linear & irreducible quadratics:

② Linear FACTOR Rule: For each factor $(ax+b)^m$, the partial fraction decomp. contains $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$, where $A_1, \ldots, A_m$ are to be determined.

③ Irreducible Quadratic Factor Rule: For each $(ax^2+bx+c)^m$, The P.F.D. has the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_mx+B_m}{(ax^2+bx+c)^m}$$

where $A_1, \ldots, A_m$ & $B_1, \ldots, B_m$ are constants to be determined.

AFTER P.F.D. You'll need to be able to integrate.

1. $\int \frac{A_k}{(ax+b)^k}\,dx$

2. $\int \frac{a_k+b_k}{(ax^2+bx+c)^k}\,dx$, You DO KNOW HOW TO INTEGRATE THESE!

Examples:

1) $$\int \frac{4}{2x+3}\,dx \underset{\substack{u=2x+3\\ du=2dx}}{=} \int \frac{4}{u}\cdot\frac{1}{2}\,du = \int \frac{2}{u}\,du = 2\ln|2x+3| + C.$$

2) $$\int \frac{4}{(2x+3)^5}\,dx \underset{\substack{u=2x+3\\ du=2dx}}{=} \int \frac{4}{u^5}\cdot\frac{1}{2}\,du = \int 2u^{-5}\,du = -\frac{2}{4}u^{-4}+C = -\frac{1}{2(2x+3)^4}+C$$

3) $$\int \frac{2}{2x^2+4x+3}\,dx = \int \frac{2}{2(x+1)^2+1}\,dx \underset{\substack{u=\sqrt{2}(x+1)\\ du=\sqrt{2}dx}}{=} \int \frac{2}{u^2+1}\cdot\frac{1}{\sqrt{2}}\,du$$
$$= \sqrt{2}\int \frac{1}{u^2+1}\,du = \sqrt{2}\tan^{-1}(\sqrt{2}(x+1)) + C$$

4) $$\int \frac{2x+1}{2x^2+4x+3}\,dx =$$

NOTE: If you let $u=2x^2+4x+3$, $du=(4x+4)dx$. The trouble is the numerator is $2x+1$, not $4x+4$. We make it so & deal w/ the consequences.

$$= \frac{1}{2}\int \frac{2(2x+1)}{2x^2+4x+3}\,dx = \frac{1}{2}\int \frac{4x+2}{2x^2+4x+3}\,dx$$
$$= \frac{1}{2}\int \frac{4x+2+(2-2)}{2x^2+4x+3}\,dx$$
$$= \frac{1}{2}\int \frac{(4x+4)-2}{2x^2+4x+3}\,dx = \frac{1}{2}\int \frac{4x+4}{2x^2+4x+3}\,dx - \int \frac{1}{2x^2+4x+3}\,dx.$$

The 1st integral is a log & the 2nd integral is an inverse tangent.

$$= \frac{1}{2}\ln|2x^2+4x+3| - \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}(x+1)) + C.$$

5) Question: How would you handle?

i) $$\int \frac{2}{(2x^2+4x+3)^2}\,dx = \int \frac{2}{(2(x+1)^2+1)^2}\,dx$$

ii) $$\int \frac{2x+1}{(2x^2+4x+3)^2}\,dx$$