Chapter 16: Vector Calculus

§ 16.1*: Vector Fields

Definition: Let D be a set in $\mathbb{R}^n$. A **vector field** on $\mathbb{R}^n$ is a function $\mathbf{F}$ that assigns to each point $(x_1, \ldots, x_n)$ in D an n-dimensional vector $\mathbf{F}(x_1, \ldots, x_n)$.

#2: Sketch the vector field $\mathbf{F}(x, y) = \frac{1}{2}x\,\hat{\mathbf{\imath}} + y\,\hat{\mathbf{\jmath}}$.

With **component functions** P and Q:

$$\begin{aligned}\mathbf{F}(x, y) &= P(x, y)\,\hat{\mathbf{\imath}} + Q(x, y)\,\hat{\mathbf{\jmath}} \\ &= \langle P(x, y), Q(x, y)\rangle\end{aligned}$$

or, more compactly when the domain is clear, $\mathbf{F} = P\,\hat{\mathbf{\imath}} + Q\,\hat{\mathbf{\jmath}} = \langle P, Q\rangle$. Functions like P and Q are sometimes called **scalar fields**.

Example A: Sketch the vector field $\mathbf{F}(x, y, z) = \hat{\mathbf{\jmath}} - \hat{\mathbf{\imath}}$.

Example B: Find and sketch the gradient vector field ∇f of $f(x, y) = \sqrt{x^2 + y^2}$.

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar field, i.e., if $\mathbf{F} = \nabla f$ for some f. In this case, f is called a **potential function** for $\mathbf{F}$.

From § 16.3, pp. 1090–2:

Suppose $\mathbf{F} = P\,\hat{\mathbf{\imath}} + Q\,\hat{\mathbf{\jmath}}$ is conservative.

Theorem 5: If $\mathbf{F}(x, y) = P(x, y)\,\hat{\mathbf{\imath}} + Q(x, y)\,\hat{\mathbf{\jmath}}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Theorem 6: Let $\mathbf{F} = P\,\hat{\mathbf{\imath}} + Q\,\hat{\mathbf{\jmath}}$ be a vector field on an open simply-connected region D. If P and Q have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D, then $\mathbf{F}$ is conservative.

Examples: Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

C: $\mathbf{F}(x, y) = (3x^2 - 2y^2)\,\hat{\mathbf{\imath}} + (4xy + 3)\,\hat{\mathbf{\jmath}}$

D: $\mathbf{F}(x, y) = e^x \sin y\,\hat{\mathbf{\imath}} + e^x \cos y\,\hat{\mathbf{\jmath}}$