AN ALGEBRAIC METHOD FOR COMPUTING CROSS PRODUCTS

This method of computing cross products relies only on the algebraic and geometric properties of the cross product itself, without appeal to the calculation of determinants of 3×3 matrices.

To illustrate, let us use Example 1 from § 12.4 of Stewart's *Calculus*, 7/8/9e, in which we are asked to compute the cross product of $\mathbf{a} = \langle 1, 3, 4 \rangle$ and $\mathbf{b} = \langle 2, 7, -5 \rangle$. We begin by writing these in terms of the standard basis vectors, then expanding with the algebraic properties of the cross product (chiefly distribution) outlined in Theorem 11:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \langle 1, 3, 4 \rangle \times \langle 2, 7, -5 \rangle \\
&= (1\,\hat{\mathbf{\imath}} + 3\,\hat{\mathbf{\jmath}} + 4\,\hat{\mathbf{k}}) \times (2\,\hat{\mathbf{\imath}} + 7\,\hat{\mathbf{\jmath}} - 5\,\hat{\mathbf{k}}) \\
&= 2(\hat{\mathbf{\imath}} \times \hat{\mathbf{\imath}}) + 7(\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}}) - 5(\hat{\mathbf{\imath}} \times \hat{\mathbf{k}}) + 6(\hat{\mathbf{\jmath}} \times \hat{\mathbf{\imath}}) + 21(\hat{\mathbf{\jmath}} \times \hat{\mathbf{\jmath}}) \\
&\qquad - 15(\hat{\mathbf{\jmath}} \times \hat{\mathbf{k}}) + 8(\hat{\mathbf{k}} \times \hat{\mathbf{\imath}}) + 28(\hat{\mathbf{k}} \times \hat{\mathbf{\jmath}}) - 20(\hat{\mathbf{k}} \times \hat{\mathbf{k}})
\end{aligned}$$

Now we need to simplify all those little cross products of unit vectors. A few facts are useful here:

First, the cross product of parallel vectors is $\mathbf{0}$. In particular, the special case relevant here, the cross product of any vector with itself is $\mathbf{0}$.

Second, using the right-hand rule, we see that $\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}} = \hat{\mathbf{k}}$, $\hat{\mathbf{\jmath}} \times \hat{\mathbf{k}} = \hat{\mathbf{\imath}}$, and $\hat{\mathbf{k}} \times \hat{\mathbf{\imath}} = \hat{\mathbf{\jmath}}$. If thinking about the spatial geometry is slow and difficult, you might find it useful to observe that these follow a cycle in alphabetical order:

$$\cdots \to \hat{\mathbf{\imath}} \to \hat{\mathbf{\jmath}} \to \hat{\mathbf{k}} \to \hat{\mathbf{\imath}} \to \hat{\mathbf{\jmath}} \to \hat{\mathbf{k}} \to \cdots$$

Third, again using the right-hand rule, we see that $\hat{\mathbf{\jmath}} \times \hat{\mathbf{\imath}} = -\hat{\mathbf{k}}$, $\hat{\mathbf{k}} \times \hat{\mathbf{\jmath}} = -\hat{\mathbf{\imath}}$, and $\hat{\mathbf{\imath}} \times \hat{\mathbf{k}} = -\hat{\mathbf{\jmath}}$. These can be obtained from the previous by the anticommutative property of cross products. In other words, you can follow the cycle listed above in either direction, but if you're going *backwards* (i.e., against the arrows, to the left), you introduce a negative sign.

Now we're ready to finish up. I'll repeat the last line of our partial computation from above, then apply the observations we've just made, collect like terms, and simplify.

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= 2(\hat{\mathbf{\imath}} \times \hat{\mathbf{\imath}}) + 7(\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}}) - 5(\hat{\mathbf{\imath}} \times \hat{\mathbf{k}}) + 6(\hat{\mathbf{\jmath}} \times \hat{\mathbf{\imath}}) + 21(\hat{\mathbf{\jmath}} \times \hat{\mathbf{\jmath}}) \\
&\qquad - 15(\hat{\mathbf{\jmath}} \times \hat{\mathbf{k}}) + 8(\hat{\mathbf{k}} \times \hat{\mathbf{\imath}}) + 28(\hat{\mathbf{k}} \times \hat{\mathbf{\jmath}}) - 20(\hat{\mathbf{k}} \times \hat{\mathbf{k}}) \\
&= 2\,\mathbf{0} + 7\,\hat{\mathbf{k}} - 5(-\hat{\mathbf{\jmath}}) + 6(-\hat{\mathbf{k}}) + 21\,\mathbf{0} - 15\,\hat{\mathbf{\imath}} + 8\,\hat{\mathbf{\jmath}} + 28(-\hat{\mathbf{\imath}}) - 20\,\mathbf{0} \\
&= (-15 - 28)\hat{\mathbf{\imath}} + (5 + 8)\hat{\mathbf{\jmath}} + (7 - 6)\hat{\mathbf{k}} \\
&= -43\,\hat{\mathbf{\imath}} + 13\,\hat{\mathbf{\jmath}} + 1\,\hat{\mathbf{k}} \\
&= \langle -43, 13, 1 \rangle
\end{aligned}
$$

(Compare this result with Example 1 in the textbook.)

With some practice, these unit cross products can be evaluated pretty quickly—you just have to be careful with the bookkeeping, which many students find cumbersome. If you'd like to see a second example, let's try Example 3, taking a few shortcuts (like skipping over the terms containing a vector crossed with itself, since those will reduce to $\mathbf{0}$):

$$
\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= (-3\,\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}} - 7\,\hat{\mathbf{k}}) \times (-5\,\hat{\mathbf{\jmath}} - 5\,\hat{\mathbf{k}}) \\
&= 15(\hat{\mathbf{\imath}} \times \hat{\mathbf{\jmath}}) + 15(\hat{\mathbf{\imath}} \times \hat{\mathbf{k}}) - 5(\hat{\mathbf{\jmath}} \times \hat{\mathbf{k}}) + 35(\hat{\mathbf{k}} \times \hat{\mathbf{\jmath}}) \\
&= 15\,\hat{\mathbf{k}} + 15(-\hat{\mathbf{\jmath}}) - 5\,\hat{\mathbf{\imath}} + 35(-\hat{\mathbf{\imath}}) \\
&= -40\,\hat{\mathbf{\imath}} - 15\,\hat{\mathbf{\jmath}} + 15\,\hat{\mathbf{k}}
\end{aligned}
$$

Of course, with a bit more practice, you can combine the work of those middle bits into one step, writing something like this:

$$
\begin{aligned}
\overrightarrow{PQ} \times \overrightarrow{PR} &= (-3\,\hat{\mathbf{\imath}} + \hat{\mathbf{\jmath}} - 7\,\hat{\mathbf{k}}) \times (-5\,\hat{\mathbf{\jmath}} - 5\,\hat{\mathbf{k}}) \\
&= 15\,\hat{\mathbf{k}} + 15(-\hat{\mathbf{\jmath}}) - 5\,\hat{\mathbf{\imath}} + 35(-\hat{\mathbf{\imath}}) \\
&= -40\,\hat{\mathbf{\imath}} - 15\,\hat{\mathbf{\jmath}} + 15\,\hat{\mathbf{k}}
\end{aligned}
$$

With these improvements in efficiency, even the somewhat lengthier Example 1 can be rewritten considerably more compactly:

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (1\,\hat{\mathbf{\imath}} + 3\,\hat{\mathbf{\jmath}} + 4\,\hat{\mathbf{k}}) \times (2\,\hat{\mathbf{\imath}} + 7\,\hat{\mathbf{\jmath}} - 5\,\hat{\mathbf{k}}) \\
&= 7\,\hat{\mathbf{k}} - 5(-\hat{\mathbf{\jmath}}) + 6(-\hat{\mathbf{k}}) - 15\,\hat{\mathbf{\imath}} + 8\,\hat{\mathbf{\jmath}} + 28(-\hat{\mathbf{\imath}}) \\
&= -43\,\hat{\mathbf{\imath}} + 13\,\hat{\mathbf{\jmath}} + 1\,\hat{\mathbf{k}}
\end{aligned}
$$

Once you've reached that level of proficiency, this method becomes competitive with the two more popular methods in terms of speed, making your choice of method more a matter of personal preference than objective practicality. Use whichever you like best.