1.1 Definitions and Terminology

INTRODUCTION The derivative dy/dx of a function $y = \phi(x)$ is itself another function $\phi'(x)$ found by an appropriate rule. The exponential function $y = e^{0.1x^2}$ is differentiable on the interval $(-\infty, \infty)$ and by the Chain Rule its first derivative is $dy/dx = 0.2xe^{0.1x^2}$. If we replace $e^{0.1x^2}$ on the right-hand side of the last equation by the symbol y, the derivative becomes

$$\frac{dy}{dx} = 0.2xy. \tag{1}$$

Now imagine that a friend of yours simply hands you equation (1)—you have no idea how it was constructed—and asks, *What is the function represented by the symbol y?* You are now face to face with one of the basic problems in this course:

How do you solve an equation such as (1) *for the function* $y = \phi(x)$?

A DEFINITION The equation that we made up in (1) is called a **differential equation.** Before proceeding any further, let us consider a more precise definition of this concept.

DEFINITION 1.1.1 Differential Equation

An equation containing the derivatives of one or more unknown functions (or dependent variables), with respect to one or more independent variables, is said to be a **differential equation (DE).**

To talk about them, we shall classify differential equations according to **type, order,** and **linearity.**

CLASSIFICATION BY TYPE If a differential equation contains only ordinary derivatives of one or more unknown functions with respect to a *single* independent variable, it is said to be an **ordinary differential equation (ODE).** An equation involving partial derivatives of one or more unknown functions of two or more independent variables is called a **partial differential equation (PDE).** Our first example illustrates several of each type of differential equation.

EXAMPLE 1 Types of Differential Equations

(a) The equations

an ODE can contain more than one unknown function

$$\frac{dy}{dx} + 5y = e^x, \quad \frac{d^2y}{dx^2} - \frac{dy}{dx} + 6y = 0, \quad \text{and} \quad \frac{dx}{dt} + \frac{dy}{dt} = 2x + y \tag{2}$$

are examples of ordinary differential equations.

(b) The following equations are partial differential equations:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{3}$$

*Except for this introductory section, only ordinary differential equations are considered in *A First Course in Differential Equations with Modeling Applications*, Eleventh Edition. In that text the word *equation* and the abbreviation DE refer only to ODEs. Partial differential equations or PDEs are considered in the expanded volume *Differential Equations with Boundary-Value Problems*, Ninth Edition.

Notice in the third equation that there are two unknown functions and two independent variables in the PDE. This means u and v must be functions of *two or more* independent variables. ■

NOTATION Throughout this text ordinary derivatives will be written by using either the **Leibniz notation** $dy/dx, d^2y/dx^2, d^3y/dx^3, \ldots$ or the **prime notation** $y', y'', y''', \ldots$. By using the latter notation, the first two differential equations in (2) can be written a little more compactly as $y' + 5y = e^x$ and $y'' - y' + 6y = 0$. Actually, the prime notation is used to denote only the first three derivatives; the fourth derivative is written $y^{(4)}$ instead of y''''. In general, the nth derivative of y is written d^ny/dx^n or $y^{(n)}$. Although less convenient to write and to typeset, the Leibniz notation has an advantage over the prime notation in that it clearly displays both the dependent and independent variables. For example, in the equation

unknown function
or dependent variable

$$\frac{d^2x}{dt^2} + 16x = 0$$

independent variable

it is immediately seen that the symbol x now represents a dependent variable, whereas the independent variable is t. You should also be aware that in physical sciences and engineering, Newton's **dot notation** (derogatorily referred to by some as the "flyspeck" notation) is sometimes used to denote derivatives with respect to time t. Thus the differential equation $d^2s/dt^2 = -32$ becomes $\ddot{s} = -32$. Partial derivatives are often denoted by a **subscript notation** indicating the independent variables. For example, with the subscript notation the second equation in (3) becomes $u_{xx} = u_{tt} - 2u_t$.

CLASSIFICATION BY ORDER The **order of a differential equation** (either ODE or PDE) is the order of the highest derivative in the equation. For example,

$$\overset{\text{second order}}{\frac{d^2y}{dx^2}} + 5\overset{\text{first order}}{\left(\frac{dy}{dx}\right)^3} - 4y = e^x$$

is a second-order ordinary differential equation. In Example 1, the first and third equations in (2) are first-order ODEs, whereas in (3) the first two equations are second-order PDEs. A first-order ordinary differential equation is sometimes written in the **differential form**

$$M(x, y)\,dx + N(x, y)\,dy = 0.$$

EXAMPLE 2 Differential Form of a First-Order ODE

If we assume that y is the dependent variable in a first-order ODE, then recall from calculus that the differential dy is defined to be $dy = y'dx$.

(a) By dividing by the differential dx an alternative form of the equation $(y - x)\,dx + 4x\,dy = 0$ is given by

$$y - x + 4x\frac{dy}{dx} = 0 \quad \text{or equivalently} \quad 4x\frac{dy}{dx} + y = x.$$

(b) By multiplying the differential equation

$$6xy\frac{dy}{dx} + x^2 + y^2 = 0$$

by dx we see that the equation has the alternative differential form

$$(x^2 + y^2)\,dx + 6xy\,dy = 0.$$

In symbols we can express an nth-order ordinary differential equation in one dependent variable by the general form

$$F(x, y, y', \ldots, y^{(n)}) = 0, \tag{4}$$

where F is a real-valued function of $n + 2$ variables: $x, y, y', \ldots, y^{(n)}$. For both practical and theoretical reasons we shall also make the assumption hereafter that it is possible to solve an ordinary differential equation in the form (4) uniquely for the highest derivative $y^{(n)}$ in terms of the remaining $n + 1$ variables. The differential equation

$$\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}), \tag{5}$$

where f is a real-valued continuous function, is referred to as the **normal form** of (4). Thus when it suits our purposes, we shall use the normal forms

$$\frac{dy}{dx} = f(x, y) \qquad \text{and} \qquad \frac{d^2y}{dx^2} = f(x, y, y')$$

to represent general first- and second-order ordinary differential equations.

EXAMPLE 3 Normal Form of an ODE

(a) By solving for the derivative dy/dx the normal form of the first-order differential equation

$$4x\frac{dy}{dx} + y = x \quad \text{is} \quad \frac{dy}{dx} = \frac{x - y}{4x}.$$

(b) By solving for the derivative y'' the normal form of the second-order differential equation

$$y'' - y' + 6 = 0 \quad \text{is} \quad y'' = y' - 6y.$$

CLASSIFICATION BY LINEARITY An nth-order ordinary differential equation (4) is said to be **linear** if F is linear in $y, y', \ldots, y^{(n)}$. This means that an nth-order ODE is linear when (4) is $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y - g(x) = 0$ or

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{6}$$

Two important special cases of (6) are linear first-order ($n = 1$) and linear second-order ($n = 2$) DEs:

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad \text{and} \qquad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x). \tag{7}$$

In the additive combination on the left-hand side of equation (6) we see that the characteristic two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y', y'', \ldots, y^{(n)}$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \ldots, a_n$ of $y, y', \ldots, y^{(n)}$ depend at most on the independent variable x.

A **nonlinear** ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$, cannot appear in a linear equation.

EXAMPLE 4 Linear and Nonlinear ODEs

(a) The equations

$$(y - x)\,dx + 4x\,dy = 0, \quad y'' - 2y + y = 0, \quad x^3\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 5y = e^x$$

are, in turn, *linear* first-, second-, and third-order ordinary differential equations. We have just demonstrated in part (a) of Example 2 that the first equation is linear in the variable y by writing it in the alternative form $4xy' + y = x$.

(b) The equations

$$\underbrace{(1 - y)}_{\substack{\text{nonlinear term:}\\ \text{coefficient depends on } y}}y' + 2y = e^x, \qquad \frac{d^2y}{dx^2} + \underbrace{\sin y}_{\substack{\text{nonlinear term:}\\ \text{nonlinear function of } y}} = 0, \qquad \text{and} \qquad \frac{d^4y}{dx^4} + \underbrace{y^2}_{\substack{\text{nonlinear term:}\\ \text{power not 1}}} = 0$$

are examples of *nonlinear* first-, second-, and fourth-order ordinary differential equations, respectively. ■

SOLUTIONS As was stated on page 2, one of the goals in this course is to solve, or find solutions of, differential equations. In the next definition we consider the concept of a solution of an ordinary differential equation.

DEFINITION 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

In other words, a solution of an nth-order ordinary differential equation (4) is a function ϕ that possesses at least n derivatives and for which

$$F(x, \phi(x), \phi'(x), \ldots, \phi^{(n)}(x)) = 0 \qquad \text{for all } x \text{ in } I.$$

We say that ϕ *satisfies* the differential equation on I. For our purposes we shall also assume that a solution ϕ is a real-valued function. In our introductory discussion we saw that $y = e^{0.1x^2}$ is a solution of $dy/dx = 0.2xy$ on the interval $(-\infty, \infty)$.

Occasionally, it will be convenient to denote a solution by the alternative symbol $y(x)$.

INTERVAL OF DEFINITION You cannot think *solution* of an ordinary differential equation without simultaneously thinking *interval*. The interval I in Definition 1.1.2 is variously called the **interval of definition,** the **interval of existence,** the **interval of validity,** or the **domain of the solution** and can be an open interval (a, b), a closed interval $[a, b]$, an infinite interval (a, ∞), and so on.

EXAMPLE 5 Verification of a Solution

Verify that the indicated function is a solution of the given differential equation on the interval $(-\infty, \infty)$.

(a) $\dfrac{dy}{dx} = xy^{1/2}$; $\quad y = \frac{1}{16}x^4$ $\qquad$ **(b)** $y'' - 2y' + y = 0$; $\quad y = xe^x$

SOLUTION One way of verifying that the given function is a solution is to see, after substituting, whether each side of the equation is the same for every x in the interval.

(a) From

$$\textit{left-hand side:} \quad \frac{dy}{dx} = \frac{1}{16}(4 \cdot x^3) = \frac{1}{4}x^3,$$

$$\textit{right-hand side:} \quad xy^{1/2} = x \cdot \left(\frac{1}{16}x^4\right)^{1/2} = x \cdot \left(\frac{1}{4}x^2\right) = \frac{1}{4}x^3,$$

we see that each side of the equation is the same for every real number x. Note that $y^{1/2} = \frac{1}{4}x^2$ is, by definition, the nonnegative square root of $\frac{1}{16}x^4$.

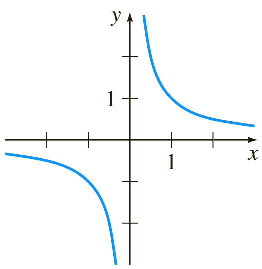
(b) From the derivatives $y' = xe^x + e^x$ and $y'' = xe^x + 2e^x$ we have, for every real number x,

$$\textit{left-hand side:} \quad y'' - 2y' + y = (xe^x + 2e^x) - 2(xe^x + e^x) + xe^x = 0,$$

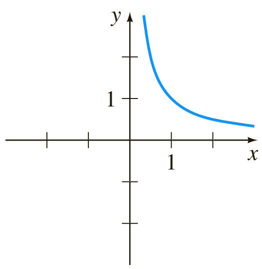
$$\textit{right-hand side:} \quad 0. \qquad \blacksquare$$

Note, too, that each differential equation in Example 5 possesses the constant solution $y = 0$, $-\infty < x < \infty$. A solution of a differential equation that is identically zero on an interval I is said to be a **trivial solution.**

SOLUTION CURVE The graph of a solution ϕ of an ODE is called a **solution curve.** Since ϕ is a differentiable function, it is continuous on its interval I of definition. Thus there may be a difference between the graph of the *function* ϕ and the graph of the *solution* ϕ. Put another way, the domain of the function ϕ need not be the same as the interval I of definition (or domain) of the solution ϕ. Example 6 illustrates the difference.



(a) function $y = 1/x$, $x \neq 0$



(b) solution $y = 1/x$, $(0, \infty)$

FIGURE 1.1.1 In Example 6 the function $y = 1/x$ is not the same as the solution $y = 1/x$

EXAMPLE 6 Function versus Solution

(a) The domain of $y = 1/x$, considered simply as a *function*, is the set of all real numbers x except 0. When we graph $y = 1/x$, we plot points in the xy-plane corresponding to a judicious sampling of numbers taken from its domain. The rational function $y = 1/x$ is discontinuous at 0, and its graph, in a neighborhood of the origin, is given in Figure 1.1.1(a). The function $y = 1/x$ is not differentiable at $x = 0$, since the y-axis (whose equation is $x = 0$) is a vertical asymptote of the graph.

(b) Now $y = 1/x$ is also a solution of the linear first-order differential equation $xy' + y = 0$. (Verify.) But when we say that $y = 1/x$ is a *solution* of this DE, we mean that it is a function defined on an interval I on which it is differentiable and satisfies the equation. In other words, $y = 1/x$ is a solution of the DE on *any* interval that does not contain 0, such as $(-3, -1)$, $\left(\frac{1}{2}, 10\right)$, $(-\infty, 0)$, or $(0, \infty)$. Because the solution curves defined by $y = 1/x$ for $-3 < x < -1$ and $\frac{1}{2} < x < 10$ are simply segments, or pieces, of the solution curves defined by $y = 1/x$ for $-\infty < x < 0$ and

$0 < x < \infty$, respectively, it makes sense to take the interval I to be as large as possible. Thus we take I to be either $(-\infty, 0)$ or $(0, \infty)$. The solution curve on $(0, \infty)$ is shown in Figure 1.1.1(b). ■

EXPLICIT AND IMPLICIT SOLUTIONS You should be familiar with the terms *explicit functions* and *implicit functions* from your study of calculus. A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an **explicit solution.** For our purposes, let us think of an explicit solution as an explicit formula $y = \phi(x)$ that we can manipulate, evaluate, and differentiate using the standard rules. We have just seen in the last two examples that $y = \frac{1}{16}x^4$, $y = xe^x$, and $y = 1/x$ are, in turn, explicit solutions of $dy/dx = xy^{1/2}$, $y'' - 2y' + y = 0$, and $xy' + y = 0$. Moreover, the trivial solution $y = 0$ is an explicit solution of all three equations. When we get down to the business of actually solving some ordinary differential equations, you will see that methods of solution do not always lead directly to an explicit solution $y = \phi(x)$. This is particularly true when we attempt to solve nonlinear first-order differential equations. Often we have to be content with a relation or expression $G(x, y) = 0$ that defines a solution ϕ implicitly.

DEFINITION 1.1.3 Implicit Solution of an ODE

A relation $G(x, y) = 0$ is said to be an **implicit solution** of an ordinary differential equation (4) on an interval I, provided that there exists at least one function ϕ that satisfies the relation as well as the differential equation on I.

It is beyond the scope of this course to investigate the conditions under which a relation $G(x, y) = 0$ defines a differentiable function ϕ. So we shall assume that if the formal implementation of a method of solution leads to a relation $G(x, y) = 0$, then there exists at least one function ϕ that satisfies both the relation (that is, $G(x, \phi(x)) = 0$) and the differential equation on an interval I. If the implicit solution $G(x, y) = 0$ is fairly simple, we may be able to solve for y in terms of x and obtain one or more explicit solutions. See (*iv*) in the *Remarks.*

EXAMPLE 7 Verification of an Implicit Solution

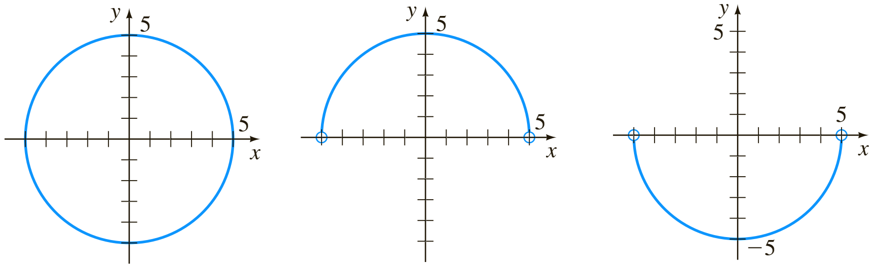
The relation $x^2 + y^2 = 25$ is an implicit solution of the differential equation

$$\frac{dy}{dx} = -\frac{x}{y} \tag{8}$$

on the open interval $(-5, 5)$. By implicit differentiation we obtain

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}25 \qquad \text{or} \qquad 2x + 2y\frac{dy}{dx} = 0. \tag{9}$$

Solving the last equation in (9) for the symbol dy/dx gives (8). Moreover, solving $x^2 + y^2 = 25$ for y in terms of x yields $y = \pm\sqrt{25 - x^2}$. The two functions $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ satisfy the relation (that is, $x^2 + \phi_1^2 = 25$ and $x^2 + \phi_2^2 = 25$) and are explicit solutions defined on the interval $(-5, 5)$. The solution curves given in Figures 1.1.2(b) and 1.1.2(c) are segments of the graph of the implicit solution in Figure 1.1.2(a).



(a) implicit solution
$x^2 + y^2 = 25$

(b) explicit solution
$y_1 = \sqrt{25 - x^2},\ -5 < x < 5$

(c) explicit solution
$y_2 = -\sqrt{25 - x^2},\ -5 < x < 5$

FIGURE 1.1.2 An implicit solution and two explicit solutions of (8) in Example 7

Because the distinction between an explicit solution and an implicit solution should be intuitively clear, we will not belabor the issue by always saying, "Here is an explicit (implicit) solution."

FAMILIES OF SOLUTIONS The study of differential equations is similar to that of integral calculus. When evaluating an antiderivative or indefinite integral in calculus, we use a single constant c of integration. Analogously, we shall see in Chapter 2 that when solving a first-order differential equation $F(x, y, y') = 0$ we *usually* obtain a solution containing a single constant or parameter c. A solution of $F(x, y, y') = 0$ containing a constant c is a set of solutions $G(x, y, c) = 0$ called a **one-parameter family of solutions.** When solving an nth-order differential equation $F(x, y, y', \ldots, y^{(n)}) = 0$ we seek an ***n*-parameter family of solutions** $G(x, y, c_1, c_2, \ldots, c_n) = 0$. This means that a single differential equation can possess an infinite number of solutions corresponding to an unlimited number of choices for the parameter(s). A solution of a differential equation that is free of parameters is called a **particular solution.**

The parameters in a family of solutions such as $G(x, y, c_1, c_2, \ldots, c_n) = 0$ are *arbitrary* up to a point. For example, proceeding as in (9) a relation $x^2 + y^2 = c$ formally satisfies (8) for any constant c. However, it is understood that the relation should always make sense in the real number system; thus, if $c = -25$ we cannot say that $x^2 + y^2 = -25$ is an implicit solution of the differential equation.

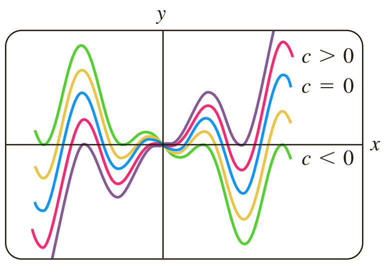


FIGURE 1.1.3 Some solutions of DE in part (a) of Example 8

EXAMPLE 8 Particular Solutions

(a) For all real values of c, the one-parameter family $y = cx - x\cos x$ is an explicit solution of the linear first-order equation

$$xy' - y = x^2 \sin x$$

on the interval $(-\infty, \infty)$. (Verify.) Figure 1.1.3 shows the graphs of some particular solutions in this family for various choices of c. The solution $y = -x\cos x$, the blue graph in the figure, is a particular solution corresponding to $c = 0$.

(b) The two-parameter family $y = c_1e^x + c_2xe^x$ is an explicit solution of the linear second-order equation

$$y'' - 2y' + y = 0$$

in part (b) of Example 5. (Verify.) In Figure 1.1.4 we have shown seven of the "double infinity" of solutions in the family. The solution curves in red, green, and blue are the graphs of the particular solutions $y = 5xe^x$ ($c_1 = 0$, $c_2 = 5$), $y = 3e^x$ ($c_1 = 3$, $c_2 = 0$), and $y = 5e^x - 2xe^x$ ($c_1 = 5$, $c_2 = 2$), respectively.

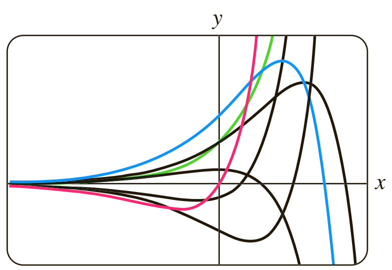


FIGURE 1.1.4 Some solutions of DE in part (b) of Example 8

Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation—that is, a solution that cannot be obtained by specializing *any* of the parameters in the family of solutions. Such an extra solution is called a **singular solution.** For example, we have seen that $y = \frac{1}{16}x^4$ and $y = 0$ are solutions of the differential equation $dy/dx = xy^{1/2}$ on $(-\infty, \infty)$. In Section 2.2 we shall demonstrate, by actually solving it, that the differential equation $dy/dx = xy^{1/2}$ possesses the one-parameter family of solutions $y = \left(\frac{1}{4}x^2 + c\right)^2$, $c \geq 0$. When $c = 0$, the resulting particular solution is $y = \frac{1}{16}x^4$. But notice that the trivial solution $y = 0$ is a singular solution since it is not a member of the family $y = \left(\frac{1}{4}x^2 + c\right)^2$; there is no way of assigning a value to the constant c to obtain $y = 0$.

In all the preceding examples we used x and y to denote the independent and dependent variables, respectively. But you should become accustomed to seeing and working with other symbols to denote these variables. For example, we could denote the independent variable by t and the dependent variable by x.

EXAMPLE 9 Using Different Symbols

The functions $x = c_1 \cos 4t$ and $x = c_2 \sin 4t$, where c_1 and c_2 are arbitrary constants or parameters, are both solutions of the linear differential equation

$$x'' + 16x = 0.$$

For $x = c_1 \cos 4t$ the first two derivatives with respect to t are $x' = -4c_1 \sin 4t$ and $x'' = -16c_1 \cos 4t$. Substituting x'' and x then gives

$$x'' + 16x = -16c_1 \cos 4t + 16(c_1 \cos 4t) = 0.$$

In like manner, for $x = c_2 \sin 4t$ we have $x'' = -16c_2 \sin 4t$, and so

$$x'' + 16x = -16c_2 \sin 4t + 16(c_2 \sin 4t) = 0.$$

Finally, it is straightforward to verify that the linear combination of solutions, or the two-parameter family $x = c_1 \cos 4t + c_2 \sin 4t$, is also a solution of the differential equation. ■

The next example shows that a solution of a differential equation can be a piecewise-defined function.

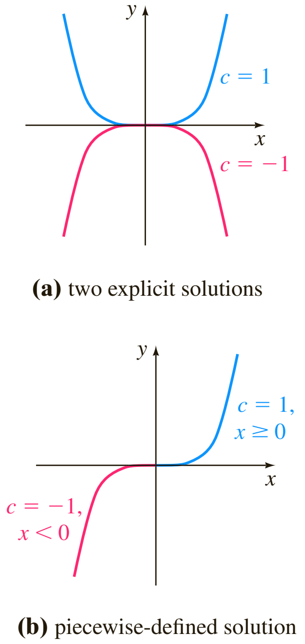


FIGURE 1.1.5 Some solutions of DE in Example 10

EXAMPLE 10 Piecewise-Defined Solution

The one-parameter family of quartic monomial functions $y = cx^4$ is an explicit solution of the linear first-order equation

$$xy' - 4y = 0$$

on the interval $(-\infty, \infty)$. (Verify.) The blue and red solution curves shown in Figure 1.1.5(a) are the graphs of $y = x^4$ and $y = -x^4$ and correspond to the choices $c = 1$ and $c = -1$, respectively.

The piecewise-defined differentiable function

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

is also a solution of the differential equation but cannot be obtained from the family $y = cx^4$ by a single choice of c. As seen in Figure 1.1.5(b) the solution is constructed from the family by choosing $c = -1$ for $x < 0$ and $c = 1$ for $x \geq 0$. ■

SYSTEMS OF DIFFERENTIAL EQUATIONS Up to this point we have been discussing single differential equations containing one unknown function. But often in theory, as well as in many applications, we must deal with systems of differential equations. A **system of ordinary differential equations** is two or more equations involving the derivatives of two or more unknown functions of a single independent variable. For example, if x and y denote dependent variables and t denotes the independent variable, then a system of two first-order differential equations is given by

$$\begin{aligned}\frac{dx}{dt} &= f(t, x, y)\\ \frac{dy}{dt} &= g(t, x, y).\end{aligned} \tag{10}$$

A **solution** of a system such as (10) is a pair of differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, defined on a common interval I, that satisfy each equation of the system on this interval.

REMARKS

(i) It might not be apparent whether a first-order ODE written in differential form $M(x, y)\,dx + N(x, y)\,dy = 0$ is linear or nonlinear because there is nothing in this form that tells us which symbol denotes the dependent variable. See Problems 9 and 10 in Exercises 1.1.

(ii) We will see in the chapters that follow that a solution of a differential equation may involve an **integral-defined function.** One way of defining a function F of a single variable x by means of a definite integral is:

$$F(x) = \int_a^x g(t)\,dt. \tag{11}$$

If the integrand g in (11) is continuous on an interval $[a, b]$ and $a \le x \le b$, then the derivative form of the Fundamental Theorem of Calculus states that F is differentiable on (a, b) and

$$F'(x) = \frac{d}{dx}\int_a^x g(t)\,dt = g(x) \tag{12}$$

The integral in (11) is often **nonelementary,** that is, an integral of a function g that does not have an elementary-function antiderivative. Elementary functions include the familiar functions studied in a typical precalculus course:

constant, *polynomial*, *rational*, *exponential*, *logarithmic*, *trigonometric*, *and inverse trigonometric functions*,

as well as rational powers of these functions; finite combinations of these functions using addition, subtraction, multiplication, division; and function compositions. For example, even though e^{-t^2}, $\sqrt{1 + t^3}$, and $\cos t^2$ are elementary functions, the integrals $\int e^{-t^2}\,dt$, $\int \sqrt{1 + t^3}\,dt$, and $\int \cos t^2\,dt$ are nonelementary. See Problems 25–28 in Exercises 1.1. Also see Appendix A.

(iii) Although the concept of a solution of a differential equation has been emphasized in this section, you should be aware that a DE does not necessarily have to possess a solution. See Problem 43 in Exercises 1.1. The question of whether a solution exists will be touched on in the next section.

(*continued*)

(iv) A few last words about implicit solutions of differential equations are in order. In Example 7 we were able to solve the relation $x^2 + y^2 = 25$ for y in terms of x to get two explicit solutions, $\phi_1(x) = \sqrt{25 - x^2}$ and $\phi_2(x) = -\sqrt{25 - x^2}$, of the differential equation (8). But don't read too much into this one example. Unless it is easy or important or you are instructed to, there is usually no need to try to solve an implicit solution $G(x, y) = 0$ for y explicitly in terms of x. Also do not misinterpret the second sentence following Definition 1.1.3. An implicit solution $G(x, y) = 0$ can define a perfectly good differentiable function ϕ that is a solution of a DE, yet we might not be able to solve $G(x, y) = 0$ using analytical methods such as algebra. The solution curve of ϕ may be a segment or piece of the graph of $G(x, y) = 0$. See Problems 49 and 50 in Exercises 1.1. Also, read the discussion following Example 4 in Section 2.2.

(v) It might not seem like a big deal to assume that $F(x, y, y', \ldots, y^{(n)}) = 0$ can be solved for $y^{(n)}$, but one should be a little bit careful here. There are exceptions, and there certainly are some problems connected with this assumption. See Problems 56 and 57 in Exercises 1.1.

(vi) If *every* solution of an nth-order ODE $F(x, y, y', \ldots, y^{(n)}) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \ldots, c_n) = 0$ by appropriate choices of the parameters c_i, $i = 1, 2, \ldots, n$, we then say that the family is the **general solution** of the DE. In solving linear ODEs, we shall impose relatively simple restrictions on the coefficients of the equation; with these restrictions one can be assured that not only does a solution exist on an interval but also that a family of solutions yields all possible solutions. Nonlinear ODEs, with the exception of some first-order equations, are usually difficult or impossible to solve in terms of elementary functions. Furthermore, if we happen to obtain a family of solutions for a nonlinear equation, it is not obvious whether this family contains all solutions. On a practical level, then, the designation "general solution" is applied only to linear ODEs. Don't be concerned about this concept at this point, but store the words "general solution" in the back of your mind—we will come back to this notion in Section 2.3 and again in Chapter 4.

EXERCISES 1.1

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1–8 state the order of the given ordinary differential equation. Determine whether the equation is linear or nonlinear by matching it with (6).

1. $(1 - x)y'' - 4xy' + 5y = \cos x$

2. $x\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^4 + y = 0$

3. $t^5y^{(4)} - t^3y'' + 6y = 0$

4. $\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + u = \cos(r + u)$

5. $\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

6. $\dfrac{d^2R}{dt^2} = -\dfrac{k}{R^2}$

7. $(\sin\theta)y''' - (\cos\theta)y' = 2$

8. $\ddot{x} - \left(1 - \dfrac{\dot{x}^2}{3}\right)\dot{x} + x = 0$

In Problems 9 and 10 determine whether the given first-order differential equation is linear in the indicated dependent variable by matching it with the first differential equation given in (7).

9. $(y^2 - 1)\,dx + x\,dy = 0$; in y; in x

10. $u\,dv + (v + uv - ue^u)\,du = 0$; in v; in u

In Problems 11–14 verify that the indicated function is an explicit solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

11. $2y' + y = 0$; $\quad y = e^{-x/2}$

12. $\dfrac{dy}{dt} + 20y = 24; \quad y = \dfrac{6}{5} - \dfrac{6}{5}e^{-20t}$

13. $y'' - 6y' + 13y = 0; \quad y = e^{3x}\cos 2x$

14. $y'' + y = \tan x; \quad y = -(\cos x)\ln(\sec x + \tan x)$

In Problems 15–18 verify that the indicated function $y = \phi(x)$ is an explicit solution of the given first-order differential equation. Proceed as in Example 6, by considering ϕ simply as a *function* and give its domain. Then by considering ϕ as a *solution* of the differential equation, give at least one interval I of definition.

15. $(y - x)y' = y - x + 8; \quad y = x + 4\sqrt{x + 2}$

16. $y' = 25 + y^2; \quad y = 5\tan 5x$

17. $y' = 2xy^2; \quad y = 1/(4 - x^2)$

18. $2y' = y^3\cos x; \quad y = (1 - \sin x)^{-1/2}$

In Problems 19 and 20 verify that the indicated expression is an implicit solution of the given first-order differential equation. Find at least one explicit solution $y = \phi(x)$ in each case. Use a graphing utility to obtain the graph of an explicit solution. Give an interval I of definition of each solution ϕ.

19. $\dfrac{dX}{dt} = (X - 1)(1 - 2X); \quad \ln\left(\dfrac{2X - 1}{X - 1}\right) = t$

20. $2xy\,dx + (x^2 - y)\,dy = 0; \quad -2x^2y + y^2 = 1$

In Problems 21–24 verify that the indicated family of functions is a solution of the given differential equation. Assume an appropriate interval I of definition for each solution.

21. $\dfrac{dP}{dt} = P(1 - P); \quad P = \dfrac{c_1e^t}{1 + c_1e^t}$

22. $\dfrac{dy}{dx} + 4xy = 8x^3; \quad y = 2x^2 - 1 + c_1e^{-2x^2}$

23. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0; \quad y = c_1e^{2x} + c_2xe^{2x}$

24. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 12x^2;$

$y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$

In Problems 25–28 use (12) to verify that the indicated function is a solution of the given differential equation. Assume an appropriate interval I of definition of each solution.

25. $x\dfrac{dy}{dx} - 3xy = 1; \quad y = e^{3x}\displaystyle\int_1^x \frac{e^{-3t}}{t}\,dt$

26. $2x\dfrac{dy}{dx} - y = 2x\cos x; \quad y = \sqrt{x}\displaystyle\int_4^x \frac{\cos t}{\sqrt{t}}\,dt$

27. $x^2\dfrac{dy}{dx} + xy = 10\sin x; \quad y = \dfrac{5}{x} + \dfrac{10}{x}\displaystyle\int_1^x \frac{\sin t}{t}\,dt$

28. $\dfrac{dy}{dx} + 2xy = 1; \quad y = e^{-x^2} + e^{-x^2}\displaystyle\int_0^x e^{t^2}\,dt$

29. Verify that the piecewise-defined function

$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$$

is a solution of the differential equation $xy' - 2y = 0$ on $(-\infty, \infty)$.

30. In Example 7 we saw that $y = \phi_1(x) = \sqrt{25 - x^2}$ and $y = \phi_2(x) = -\sqrt{25 - x^2}$ are solutions of $dy/dx = -x/y$ on the interval $(-5, 5)$. Explain why the piecewise-defined function

$$y = \begin{cases} \sqrt{25 - x^2} & -5 < x < 0 \\ -\sqrt{25 - x^2}, & 0 \le x < 5 \end{cases}$$

is *not* a solution of the differential equation on the interval $(-5, 5)$.

In Problems 31–34 find values of m so that the function $y = e^{mx}$ is a solution of the given differential equation.

31. $y' + 2y = 0$

32. $5y' = 2y$

33. $y'' - 5y' + 6y = 0$

34. $2y'' + 7y' - 4y = 0$

In Problems 35 and 36 find values of m so that the function $y = x^m$ is a solution of the given differential equation.

35. $xy'' + 2y' = 0$

36. $x^2y'' - 7xy' + 15y = 0$

In Problems 37–40 use the concept that $y = c$, $-\infty < x < \infty$, is a constant function if and only if $y' = 0$ to determine whether the given differential equation possesses constant solutions.

37. $3xy' + 5y = 10$

38. $y' = y^2 + 2y - 3$

39. $(y - 1)y' = 1$

40. $y'' + 4y' + 6y = 10$

In Problems 41 and 42 verify that the indicated pair of functions is a solution of the given system of differential equations on the interval $(-\infty, \infty)$.

41. $\dfrac{dx}{dt} = x + 3y$

$\dfrac{dy}{dt} = 5x + 3y;$

$x = e^{-2t} + 3e^{6t},$

$y = -e^{-2t} + 5e^{6t}$

42. $\dfrac{d^2x}{dt^2} = 4y + e^t$

$\dfrac{d^2y}{dt^2} = 4x - e^t;$

$x = \cos 2t + \sin 2t + \frac{1}{5}e^t,$

$y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$

Discussion Problems

43. Make up a differential equation that does not possess any real solutions.

44. Make up a differential equation that you feel confident possesses only the trivial solution $y = 0$. Explain your reasoning.

45. What function do you know from calculus is such that its first derivative is itself? Its first derivative is a constant multiple k of itself? Write each answer in the form of a first-order differential equation with a solution.

46. What function (or functions) do you know from calculus is such that its second derivative is itself? Its second derivative is the negative of itself? Write each answer in the form of a second-order differential equation with a solution.

47. The function $y = \sin x$ is an explicit solution of the first-order differential equation $\dfrac{dy}{dx} = \sqrt{1 - y^2}$. Find an interval I of definition. [*Hint*: I is *not* the interval $(-\infty, \infty)$.]

48. Discuss why it makes intuitive sense to presume that the linear differential equation $y'' + 2y' + 4y = 5 \sin t$ has a solution of the form $y = A \sin t + B \cos t$, where A and B are constants. Then find specific constants A and B so that $y = A \sin t + B \cos t$ is a particular solution of the DE.

In Problems 49 and 50 the given figure represents the graph of an implicit solution $G(x, y) = 0$ of a differential equation $dy/dx = f(x, y)$. In each case the relation $G(x, y) = 0$ implicitly defines several solutions of the DE. Carefully reproduce each figure on a piece of paper. Use different colored pencils to mark off segments, or pieces, on each graph that correspond to graphs of solutions. Keep in mind that a solution ϕ must be a function and differentiable. Use the solution curve to estimate an interval I of definition of each solution ϕ.

49.

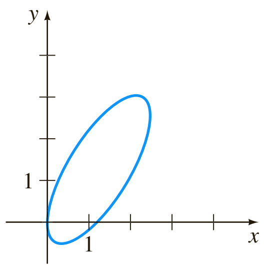


FIGURE 1.1.6 Graph for Problem 49

50.

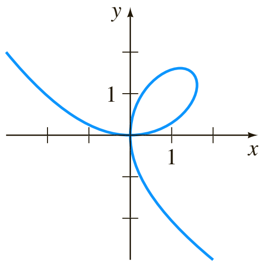


FIGURE 1.1.7 Graph for Problem 50

51. The graphs of members of the one-parameter family $x^3 + y^3 = 3cxy$ are called **folia of Descartes.** Verify that this family is an implicit solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}.$$

52. The graph in Figure 1.1.7 is the member of the family of folia in Problem 51 corresponding to $c = 1$. Discuss: How can the DE in Problem 51 help in finding points on the graph of $x^3 + y^3 = 3xy$ where the tangent line is vertical? How does knowing where a tangent line is vertical help in determining an interval I of definition of a solution ϕ of the DE? Carry out your ideas and compare with your estimates of the intervals in Problem 50.

53. In Example 7 the largest interval I over which the explicit solutions $y = \phi_1(x)$ and $y = \phi_2(x)$ are defined is the open interval $(-5, 5)$. Why can't the interval I of definition be the closed interval $[-5, 5]$?

54. In Problem 21 a one-parameter family of solutions of the DE $P' = P(1 - P)$ is given. Does any solution curve pass through the point $(0, 3)$? Through the point $(0, 1)$?

55. Discuss, and illustrate with examples, how to solve differential equations of the forms $dy/dx = f(x)$ and $d^2y/dx^2 = f(x)$.

56. The differential equation $x(y')^2 - 4y' - 12x^3 = 0$ has the form given in (4). Determine whether the equation can be put into the normal form $dy/dx = f(x, y)$.

57. The normal form (5) of an nth-order differential equation is equivalent to (4) whenever both forms have exactly the same solutions. Make up a first-order differential equation for which $F(x, y, y') = 0$ is not equivalent to the normal form $dy/dx = f(x, y)$.

58. Find a linear second-order differential equation $F(x, y, y', y'') = 0$ for which $y = c_1x + c_2x^2$ is a two-parameter family of solutions. Make sure that your equation is free of the arbitrary parameters c_1 and c_2.

Qualitative information about a solution $y = \phi(x)$ of a differential equation can often be obtained from the equation itself. Before working Problems 59–62, recall the geometric significance of the derivatives dy/dx and d^2y/dx^2.

59. Consider the differential equation $dy/dx = e^{-x^2}$.

(a) Explain why a solution of the DE must be an increasing function on any interval of the x-axis.

(b) What are $\lim_{x\to-\infty} dy/dx$ and $\lim_{x\to\infty} dy/dx$? What does this suggest about a solution curve as $x \to \pm\infty$?

(c) Determine an interval over which a solution curve is concave down and an interval over which the curve is concave up.

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

60. Consider the differential equation $dy/dx = 5 - y$.

(a) Either by inspection or by the method suggested in Problems 37–40, find a constant solution of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. Find intervals on the y-axis on which $y = \phi(x)$ is decreasing.

61. Consider the differential equation $dy/dx = y(a - by)$, where a and b are positive constants.

(a) Either by inspection or by the method suggested in Problems 37–40, find two constant solutions of the DE.

(b) Using only the differential equation, find intervals on the y-axis on which a nonconstant solution $y = \phi(x)$ is increasing. Find intervals on which $y = \phi(x)$ is decreasing.

(c) Using only the differential equation, explain why $y = a/2b$ is the y-coordinate of a point of inflection of the graph of a nonconstant solution $y = \phi(x)$.

(d) On the same coordinate axes, sketch the graphs of the two constant solutions found in part (a). These constant solutions partition the xy-plane into three regions. In each region, sketch the graph of a nonconstant solution $y = \phi(x)$ whose shape is suggested by the results in parts (b) and (c).

62. Consider the differential equation $y' = y^2 + 4$.

(a) Explain why there exist no constant solutions of the DE.

(b) Describe the graph of a solution $y = \phi(x)$. For example, can a solution curve have any relative extrema?

(c) Explain why $y = 0$ is the y-coordinate of a point of inflection of a solution curve.

(d) Sketch the graph of a solution $y = \phi(x)$ of the differential equation whose shape is suggested by parts (a)–(c).

Computer Lab Assignments

In Problems 63 and 64 use a CAS to compute all derivatives and to carry out the simplifications needed to verify that the indicated function is a particular solution of the given differential equation.

63. $y^{(4)} - 20y''' + 158y'' - 580y' + 841y = 0;$

$y = xe^{5x}\cos 2x$

64. $x^3y''' + 2x^2y'' + 20xy' - 78y = 0;$

$$y = 20\frac{\cos(5\ln x)}{x} - 3\frac{\sin(5\ln x)}{x}$$

1.2 Initial-Value Problems

INTRODUCTION We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ also satisfies certain prescribed side conditions, that is, conditions that are imposed on the unknown function $y(x)$ and its derivatives at a number x_0. On some interval I containing x_0 the problem of solving an nth-order differential equation subject to n side conditions specified at x_0:

$$\begin{aligned} &\textit{Solve:} && \frac{d^ny}{dx^n} = f(x, y, y', \ldots, y^{(n-1)}) \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \tag{1}$$

where $y_0, y_1, \ldots, y_{n-1}$ are arbitrary constants, is called an ***n*th-order initial-value problem (IVP).** The values of $y(x)$ and its first $n-1$ derivatives at x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$ are called **initial conditions (IC).**

Solving an nth-order initial-value problem such as (1) frequently entails first finding an n-parameter family of solutions of the differential equation and then using the initial conditions at x_0 to determine the n constants in this family. The resulting particular solution is defined on some interval I containing the number x_0.

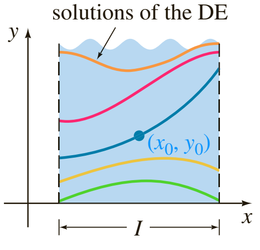


FIGURE 1.2.1 Solution curve of first-order IVP

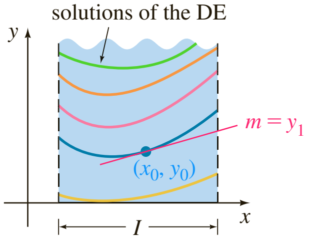


FIGURE 1.2.2 Solution curve of second-order IVP

GEOMETRIC INTERPRETATION The cases $n = 1$ and $n = 2$ in (1),

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\textit{Solve:} && \frac{d^2y}{dx^2} = f(x, y, y') \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1 \end{aligned} \tag{3}$$

are examples of **first-** and **second-order** initial-value problems, respectively. These two problems are easy to interpret in geometric terms. For (2) we are seeking a solution $y(x)$ of the differential equation $y' = f(x, y)$ on an interval I containing x_0 so that its graph passes through the specified point (x_0, y_0). A solution curve is shown in blue in Figure 1.2.1. For (3) we want to find a solution $y(x)$ of the differential equation $y'' = f(x, y, y')$ on an interval I containing x_0 so that its graph not only passes through (x_0, y_0) but the slope of the curve at this point is the number y_1. A solution curve is shown in blue in Figure 1.2.2. The words *initial conditions* derive from physical systems where the independent variable is time t and where $y(t_0) = y_0$

and $y'(t_0) = y_1$ represent the position and velocity, respectively, of an object at some beginning, or initial, time t_0.

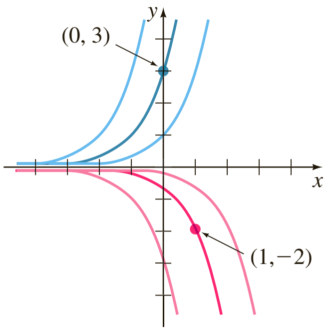


FIGURE 1.2.3 Solution curves of two IVPs in Example 1

EXAMPLE 1 Two First-Order IVPs

(a) In Problem 45 in Exercises 1.1 you were asked to deduce that $y = ce^x$ is a one-parameter family of solutions of the simple first-order equation $y' = y$. All the solutions in this family are defined on the interval $(-\infty, \infty)$. If we impose an initial condition, say, $y(0) = 3$, then substituting $x = 0$, $y = 3$ in the family determines the constant $3 = ce^0 = c$. Thus $y = 3e^x$ is a solution of the IVP

$$y' = y, \quad y(0) = 3.$$

(b) Now if we demand that a solution curve pass through the point $(1, -2)$ rather than $(0, 3)$, then $y(1) = -2$ will yield $-2 = ce$ or $c = -2e^{-1}$. In this case $y = -2e^{x-1}$ is a solution of the IVP

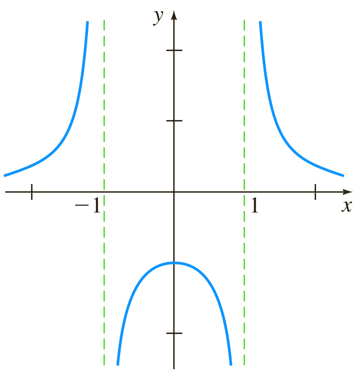
$$y' = y, \quad y(1) = -2.$$

The two solution curves are shown in dark blue and dark red in Figure 1.2.3. ■

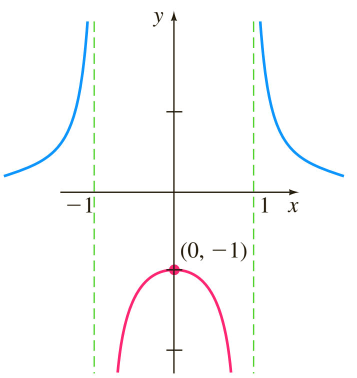
The next example illustrates another first-order initial-value problem. In this example notice how the interval I of definition of the solution $y(x)$ depends on the initial condition $y(x_0) = y_0$.

EXAMPLE 2 Interval *I* of Definition of a Solution

In Problem 6 of Exercises 2.2 you will be asked to show that a one-parameter family of solutions of the first-order differential equation $y' + 2xy^2 = 0$ is $y = 1/(x^2 + c)$. If we impose the initial condition $y(0) = -1$, then substituting $x = 0$ and $y = -1$ into the family of solutions gives $-1 = 1/c$ or $c = -1$. Thus $y = 1/(x^2 - 1)$. We now emphasize the following three distinctions:



(a) function defined for all x except $x = \pm 1$



(b) solution defined on interval containing $x = 0$

FIGURE 1.2.4 Graphs of function and solution of IVP in Example 2

- Considered as a *function*, the domain of $y = 1/(x^2 - 1)$ is the set of real numbers x for which $y(x)$ is defined; this is the set of all real numbers except $x = -1$ and $x = 1$. See Figure 1.2.4(a).
- Considered as a *solution of the differential equation* $y' + 2xy^2 = 0$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined and differentiable. As can be seen in Figure 1.2.4(a), the largest intervals on which $y = 1/(x^2 - 1)$ is a solution are $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.
- Considered as a *solution of the initial-value problem* $y' + 2xy^2 = 0$, $y(0) = -1$, the interval I of definition of $y = 1/(x^2 - 1)$ could be taken to be any interval over which $y(x)$ is defined, differentiable, *and* contains the initial point $x = 0$; the largest interval for which this is true is $(-1, 1)$. See the red curve in Figure 1.2.4(b). ■

See Problems 3–6 in Exercises 1.2 for a continuation of Example 2.

EXAMPLE 3 Second-Order IVP

In Example 9 of Section 1.1 we saw that $x = c_1 \cos 4t + c_2 \sin 4t$ is a two-parameter family of solutions of $x'' + 16x = 0$. Find a solution of the initial-value problem

$$x'' + 16x = 0, \quad x\left(\frac{\pi}{2}\right) = -2, \quad x'\left(\frac{\pi}{2}\right) = 1. \tag{4}$$

SOLUTION We first apply $x(\pi/2) = -2$ to the given family of solutions: $c_1 \cos 2\pi + c_2 \sin 2\pi = -2$. Since $\cos 2\pi = 1$ and $\sin 2\pi = 0$, we find that $c_1 = -2$. We next apply $x'(\pi/2) = 1$ to the one-parameter family $x(t) = -2 \cos 4t + c_2 \sin 4t$. Differentiating and then setting $t = \pi/2$ and $x' = 1$ gives $8 \sin 2\pi + 4c_2 \cos 2\pi = 1$, from which we see that $c_2 = \frac{1}{4}$. Hence $x = -2 \cos 4t + \frac{1}{4} \sin 4t$ is a solution of (4). ■

EXISTENCE AND UNIQUENESS Two fundamental questions arise in considering an initial-value problem:

Does a solution of the problem exist? If a solution exists, is it unique?

For the first-order initial-value problem (2) we ask:

Existence { *Does the differential equation $dy/dx = f(x, y)$ possess solutions?*
Do any of the solution curves pass through the point (x_0, y_0)?

Uniqueness { *When can we be certain that there is precisely one solution curve passing through the point (x_0, y_0)?*

Note that in Examples 1 and 3 the phrase "*a* solution" is used rather than "*the* solution" of the problem. The indefinite article "a" is used deliberately to suggest the possibility that other solutions may exist. At this point it has not been demonstrated that there is a single solution of each problem. The next example illustrates an initial-value problem with two solutions.

EXAMPLE 4 An IVP Can Have Several Solutions

Each of the functions $y = 0$ and $y = \frac{1}{16}x^4$ satisfies the differential equation $dy/dx = xy^{1/2}$ and the initial condition $y(0) = 0$, so the initial-value problem

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

has at least two solutions. As illustrated in Figure 1.2.5, the graphs of both functions, shown in red and blue pass through the same point (0, 0). ■

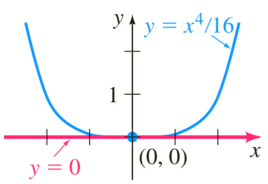


FIGURE 1.2.5 Two solution curves of the same IVP in Example 4

Within the safe confines of a formal course in differential equations one can be fairly confident that *most* differential equations will have solutions and that solutions of initial-value problems will *probably* be unique. Real life, however, is not so idyllic. Therefore it is desirable to know in advance of trying to solve an initial-value problem whether a solution exists and, when it does, whether it is the only solution of the problem. Since we are going to consider first-order differential equations in the next two chapters, we state here without proof a straightforward theorem that gives conditions that are sufficient to guarantee the existence and uniqueness of a solution of a first-order initial-value problem of the form given in (2). We shall wait until Chapter 4 to address the question of existence and uniqueness of a second-order initial-value problem.

THEOREM 1.2.1 Existence of a Unique Solution

Let R be a rectangular region in the xy-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\partial f/\partial y$ are continuous on R, then there exists some interval I_0: $(x_0 - h, x_0 + h)$, $h > 0$, contained in $[a, b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem (2).

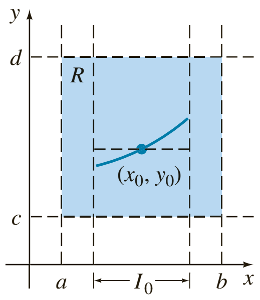


FIGURE 1.2.6 Rectangular region R

The foregoing result is one of the most popular existence and uniqueness theorems for first-order differential equations because the criteria of continuity of $f(x, y)$ and $\partial f/\partial y$ are relatively easy to check. The geometry of Theorem 1.2.1 is illustrated in Figure 1.2.6.

EXAMPLE 5 Example 4 Revisited

We saw in Example 4 that the differential equation $dy/dx = xy^{1/2}$ possesses at least two solutions whose graphs pass through (0, 0). Inspection of the functions

$$f(x, y) = xy^{1/2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{x}{2y^{1/2}}$$

shows that they are continuous in the upper half-plane defined by $y > 0$. Hence Theorem 1.2.1 enables us to conclude that through any point (x_0, y_0), $y_0 > 0$ in the upper half-plane there is some interval centered at x_0 on which the given differential equation has a unique solution. Thus, for example, even without solving it, we know that there exists some interval centered at 2 on which the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ has a unique solution. ■

In Example 1, Theorem 1.2.1 guarantees that there are no other solutions of the initial-value problems $y' = y$, $y(0) = 3$ and $y' = y$, $y(1) = -2$ other than $y = 3e^x$ and $y = -2e^{x-1}$, respectively. This follows from the fact that $f(x, y) = y$ and $\partial f/\partial y = 1$ are continuous throughout the entire xy-plane. It can be further shown that the interval I on which each solution is defined is $(-\infty, \infty)$.

INTERVAL OF EXISTENCE/UNIQUENESS Suppose $y(x)$ represents a solution of the initial-value problem (2). The following three sets on the real x-axis may not be the same: the domain of the function $y(x)$, the interval I over which the solution $y(x)$ is defined or exists, and the interval I_0 of existence *and* uniqueness. Example 6 of Section 1.1 illustrated the difference between the domain of a function and the interval I of definition. Now suppose (x_0, y_0) is a point in the interior of the rectangular region R in Theorem 1.2.1. It turns out that the continuity of the function $f(x, y)$ on R by itself is sufficient to guarantee the existence of at least one solution of $dy/dx = f(x, y)$, $y(x_0) = y_0$, defined on some interval I. The interval I of definition for this initial-value problem is usually taken to be the largest interval containing x_0 over which the solution $y(x)$ is defined and differentiable. The interval I depends on both $f(x, y)$ and the initial condition $y(x_0) = y_0$. See Problems 31–34 in Exercises 1.2. The extra condition of continuity of the first partial derivative $\partial f/\partial y$ on R enables us to say that not only does a solution exist on some interval I_0 containing x_0, but it is the *only* solution satisfying $y(x_0) = y_0$. However, Theorem 1.2.1 does not give any indication of the sizes of intervals I and I_0; *the interval I of definition need not be as wide as the region R, and the interval I_0 of existence and uniqueness may not be as large as I.* The number $h > 0$ that defines the interval I_0: $(x_0 - h, x_0 + h)$ could be very small, so it is best to think that the solution $y(x)$ is *unique in a local sense*—that is, a solution defined near the point (x_0, y_0). See Problem 51 in Exercises 1.2.

REMARKS

(i) The conditions in Theorem 1.2.1 are sufficient but not necessary. This means that when $f(x, y)$ and $\partial f/\partial y$ are continuous on a rectangular region R, it must always follow that a solution of (2) exists and is unique whenever (x_0, y_0) is a point interior to R. However, if the conditions stated in the hypothesis of

Theorem 1.2.1 do not hold, then anything could happen: Problem (2) *may* still have a solution and this solution *may* be unique, or (2) may have several solutions, or it may have no solution at all. A rereading of Example 5 reveals that the hypotheses of Theorem 1.2.1 do not hold on the line $y = 0$ for the differential equation $dy/dx = xy^{1/2}$, so it is not surprising, as we saw in Example 4 of this section, that there are two solutions defined on a common interval $(-h, h)$ satisfying $y(0) = 0$. On the other hand, the hypotheses of Theorem 1.2.1 do not hold on the line $y = 1$ for the differential equation $dy/dx = |y - 1|$. Nevertheless it can be proved that the solution of the initial-value problem $dy/dx = |y - 1|$, $y(0) = 1$, is unique. Can you guess this solution?

(ii) You are encouraged to read, think about, work, and then keep in mind Problem 50 in Exercises 1.2.

(iii) Initial conditions are prescribed at a *single* point x_0. But we are also interested in solving differential equations that are subject to conditions specified on $y(x)$ or its derivative at *two* different points x_0 and x_1. Conditions such as

$$y(1) = 0, \quad y(5) = 0 \qquad \text{or} \qquad y(\pi/2) = 0, \quad y'(\pi) = 1$$

are called **boundary conditions (BC).** A differential equation together with boundary conditions is called a **boundary-value problem (BVP).** For example,

$$y'' + y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$$

is a boundary-value problem. See Problems 39–44 in Exercises 1.2.

When we start to solve differential equations in Chapter 2 we will solve only first-order equations and first-order initial-value problems. The mathematical description of many problems in science and engineering involve second-order IVPs or two-point BVPs. We will examine some of these problems in Chapters 4 and 5.

EXERCISES 1.2

Answers to selected odd-numbered problems begin on page ANS-1.

In Problems 1 and 2, $y = 1/(1 + c_1e^{-x})$ is a one-parameter family of solutions of the first-order DE $y' = y - y^2$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition.

1. $y(0) = -\frac{1}{3}$

2. $y(-1) = 2$

In Problems 3–6, $y = 1/(x^2 + c)$ is a one-parameter family of solutions of the first-order DE $y' + 2xy^2 = 0$. Find a solution of the first-order IVP consisting of this differential equation and the given initial condition. Give the largest interval I over which the solution is defined.

3. $y(2) = \frac{1}{3}$

4. $y(-2) = \frac{1}{2}$

5. $y(0) = 1$

6. $y\left(\frac{1}{2}\right) = -4$

In Problems 7–10, $x = c_1 \cos t + c_2 \sin t$ is a two-parameter family of solutions of the second-order DE $x'' + x = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

7. $x(0) = -1, \quad x'(0) = 8$

8. $x(\pi/2) = 0, \quad x'(\pi/2) = 1$

9. $x(\pi/6) = \frac{1}{2}, \quad x'(\pi/6) = 0$

10. $x(\pi/4) = \sqrt{2}, \quad x'(\pi/4) = 2\sqrt{2}$

In Problems 11–14, $y = c_1e^x + c_2e^{-x}$ is a two-parameter family of solutions of the second-order DE $y'' - y = 0$. Find a solution of the second-order IVP consisting of this differential equation and the given initial conditions.

11. $y(0) = 1, \quad y'(0) = 2$

12. $y(1) = 0, \quad y'(1) = e$

13. $y(-1) = 5, \quad y'(-1) = -5$

14. $y(0) = 0, \quad y'(0) = 0$

In Problems 15 and 16 determine by inspection at least two solutions of the given first-order IVP.

15. $y' = 3y^{2/3}, \quad y(0) = 0$

16. $xy' = 2y, \quad y(0) = 0$

In Problems 17–24 determine a region of the xy-plane for which the given differential equation would have a unique solution whose graph passes through a point (x_0, y_0) in the region.

17. $\dfrac{dy}{dx} = y^{2/3}$

18. $\dfrac{dy}{dx} = \sqrt{xy}$

19. $x\dfrac{dy}{dx} = y$

20. $\dfrac{dy}{dx} - y = x$

21. $(4 - y^2)y' = x^2$

22. $(1 + y^3)y' = x^2$

23. $(x^2 + y^2)y' = y^2$

24. $(y - x)y' = y + x$

In Problems 25–28 determine whether Theorem 1.2.1 guarantees that the differential equation $y' = \sqrt{y^2 - 9}$ possesses a unique solution through the given point.

25. $(1, 4)$

26. $(5, 3)$

27. $(2, -3)$

28. $(-1, 1)$

29. (a) By inspection find a one-parameter family of solutions of the differential equation $xy' = y$. Verify that each member of the family is a solution of the initial-value problem $xy' = y$, $y(0) = 0$.

(b) Explain part (a) by determining a region R in the xy-plane for which the differential equation $xy' = y$ would have a unique solution through a point (x_0, y_0) in R.

(c) Verify that the piecewise-defined function

$$y = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$$

satisfies the condition $y(0) = 0$. Determine whether this function is also a solution of the initial-value problem in part (a).

30. (a) Verify that $y = \tan(x + c)$ is a one-parameter family of solutions of the differential equation $y' = 1 + y^2$.

(b) Since $f(x, y) = 1 + y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Use the family of solutions in part (a) to find an explicit solution of the first-order initial-value problem $y' = 1 + y^2$, $y(0) = 0$. Even though $x_0 = 0$ is in the interval $(-2, 2)$, explain why the solution is not defined on this interval.

(c) Determine the largest interval I of definition for the solution of the initial-value problem in part (b).

31. (a) Verify that $y = -1/(x + c)$ is a one-parameter family of solutions of the differential equation $y' = y^2$.

(b) Since $f(x, y) = y^2$ and $\partial f/\partial y = 2y$ are continuous everywhere, the region R in Theorem 1.2.1 can be taken to be the entire xy-plane. Find a solution from the family in part (a) that satisfies $y(0) = 1$. Then find a solution from the family in part (a) that satisfies $y(0) = -1$. Determine the largest interval I of definition for the solution of each initial-value problem.

(c) Determine the largest interval I of definition for the solution of the first-order initial-value problem $y' = y^2$, $y(0) = 0$. [*Hint*: The solution is not a member of the family of solutions in part (a).]

32. (a) Show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(1) = 1$, is $y = 1/(2 - x)$.

(b) Then show that a solution from the family in part (a) of Problem 31 that satisfies $y' = y^2$, $y(3) = -1$, is $y = 1/(2 - x)$.

(c) Are the solutions in parts (a) and (b) the same?

33. (a) Verify that $3x^2 - y^2 = c$ is a one-parameter family of solutions of the differential equation $y\,dy/dx = 3x$.

(b) By hand, sketch the graph of the implicit solution $3x^2 - y^2 = 3$. Find all explicit solutions $y = \phi(x)$ of the DE in part (a) defined by this relation. Give the interval I of definition of each explicit solution.

(c) The point $(-2, 3)$ is on the graph of $3x^2 - y^2 = 3$, but which of the explicit solutions in part (b) satisfies $y(-2) = 3$?

34. (a) Use the family of solutions in part (a) of Problem 33 to find an implicit solution of the initial-value problem $y\,dy/dx = 3x$, $y(2) = -4$. Then, by hand, sketch the graph of the explicit solution of this problem and give its interval I of definition.

(b) Are there any explicit solutions of $y\,dy/dx = 3x$ that pass through the origin?

In Problems 35–38 the graph of a member of a family of solutions of a second-order differential equation $d^2y/dx^2 = f(x, y, y')$ is given. Match the solution curve with at least one pair of the following initial conditions.

(a) $y(1) = 1, \quad y'(1) = -2$

(b) $y(-1) = 0, \quad y'(-1) = -4$

(c) $y(1) = 1, \quad y'(1) = 2$

(d) $y(0) = -1, \quad y'(0) = 2$

(e) $y(0) = -1, \quad y'(0) = 0$

(f) $y(0) = -4, \quad y'(0) = -2$

35.

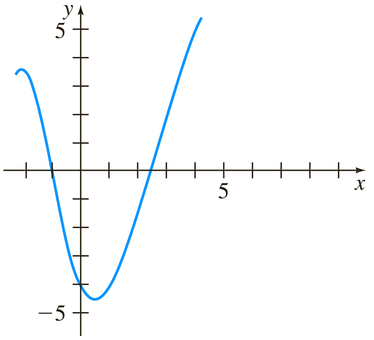


FIGURE 1.2.7 Graph for Problem 35

36.

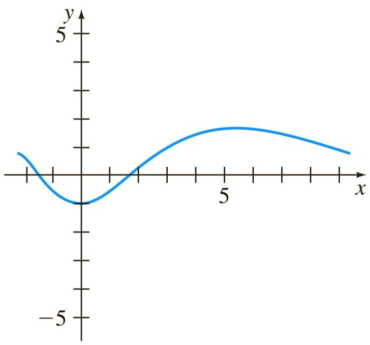


FIGURE 1.2.8 Graph for Problem 36

37.

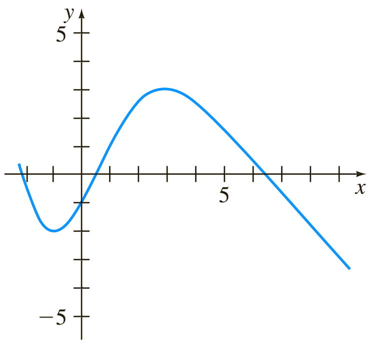


FIGURE 1.2.9 Graph for Problem 37

38.

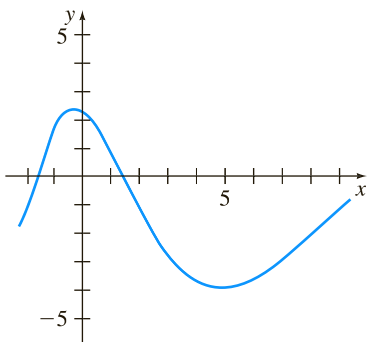


FIGURE 1.2.10 Graph for Problem 38

In Problems 39–44, $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions of the second-order DE $y'' + 4y = 0$. If possible, find a solution of the differential equation that satisfies the given side conditions. The conditions specified at two different points are called boundary conditions.

39. $y(0) = 0,\ y(\pi/4) = 3$

40. $y(0) = 0,\ y(\pi) = 0$

41. $y'(0) = 0,\ y'(\pi/6) = 0$

42. $y(0) = 1,\ y'(\pi) = 5$

43. $y(0) = 0,\ y(\pi) = 2$

44. $y'(\pi/2) = 1,\ y'(\pi) = 0$

Discussion Problems

In Problems 45 and 46 use Problem 55 in Exercises 1.1 and (2) and (3) of this section.

45. Find a function whose graph at each point (x, y) has the slope given by $8e^{2x} + 6x$ and has the y-intercept $(0, 9)$.

46. Find a function whose second derivative is $y'' = 12x - 2$ at each point (x, y) on its graph and $y = -x + 5$ is tangent to the graph at the point corresponding to $x = 1$.

47. Consider the initial-value problem $y' = x - 2y$, $y(0) = \frac{1}{2}$. Determine which of the two curves shown in Figure 1.2.11 is the only plausible solution curve. Explain your reasoning.

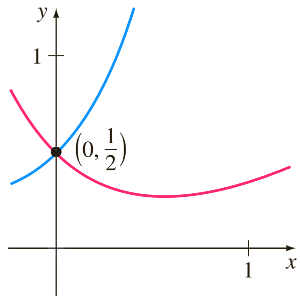


FIGURE 1.2.11 Graphs for Problem 47

48. Show that

$$x = \int_0^y \frac{1}{\sqrt{t^3 + 1}}\, dt$$

is an implicit solution of the initial-value problem

$$2\frac{d^2y}{dx^2} - 3y^2 = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Assume that $y \ge 0$. [*Hint*: The integral is nonelementary. See (*ii*) in the *Remarks* at the end of Section 1.1.]

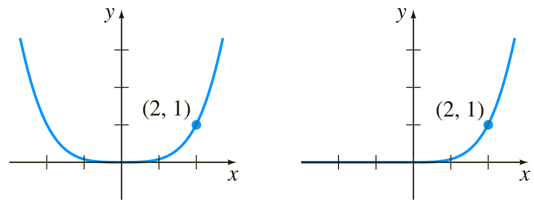
49. Determine a plausible value of x_0 for which the graph of the solution of the initial-value problem $y' + 2y = 3x - 6$, $y(x_0) = 0$ is tangent to the x-axis at $(x_0, 0)$. Explain your reasoning.

50. Suppose that the first-order differential equation $dy/dx = f(x, y)$ possesses a one-parameter family of solutions and that $f(x, y)$ satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R of the xy-plane. Explain why two different solution curves cannot intersect or be tangent to each other at a point (x_0, y_0) in R.

51. The functions $y(x) = \frac{1}{16}x^4$, $-\infty < x < \infty$ and

$$y(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \ge 0 \end{cases}$$

have the same domain but are clearly different. See Figures 1.2.12(a) and 1.2.12(b), respectively. Show that both functions are solutions of the initial-value problem $dy/dx = xy^{1/2}$, $y(2) = 1$ on the interval $(-\infty, \infty)$. Resolve the apparent contradiction between this fact and the last sentence in Example 5.



(a) **(b)**

FIGURE 1.2.12 Two solutions of the IVP in Problem 51