PreCalculus Review

Review Questions

1. The following transformations are applied (in the given order) to the graph of $y = x^3$.

   I. Vertical Stretch by a factor of 3.

   II. Horizontal shift to the right by 2 units.

   III. Vertical shift up by 4 units.

   What is the equation of the transformed graph?

2. Solve for x: $\dfrac{x^2 + 2x - 8}{x^2 - x} < 0$.

3. Use absolute value notation to write an inequality that represents the statement:

   "x is within 3 units of 2 on the real line".

4. Solve the inequality $|3 - 4x| \geq 5$. Express your answer in interval notation.

5. Use the definition of absolute value to write $f(x) = |3(x - 1) - 2|$ as a piecewise-defined function which *does not* contain absolute values.

6. Consider the function $f(x) = 4x^2 - 8x + 1$. If possible, solve the following problems algebraically. Check your results graphically.

   (a) *Complete the square* to write this polynomial in *standard form*.

   (b) Draw a complete graph, using the standard form found in (a).

   (c) State which transformations (shifts, stretches, etc.) that must be applied to the graph of $y = g(x) = x^2$ to obtain the graph of $y = f(x)$.

7. Let $f(x) = \begin{cases} x+1 & \text{if } \ x < 0 \\ x-1 & \text{if } \ 0 \leq x \end{cases}$ and $g(x) = \begin{cases} -x & \text{if } \ x \leq 1 \\ x-1 & \text{if } \ 1 < x \end{cases}$

   Determine:

   (a) $(f - g)(x)$.

   (b) $\left(\dfrac{f}{g}\right)(x)$.

8. Let $f(x) = \sqrt{x} - 3$ and $g(x) = x^2 + x + \dfrac{1}{4}$.

   (a) Determine the domain and range of both f and g.

(b) Determine $f(g(x))$, including its domain.

(c) Determine $g(f(x))$, including its domain.

9. Let $P(x) = x^5 - 3x^4 - 9x + 27$.

   (a) Find all rational roots of $P(x)$.

   (b) Find all real linear factors of $P(x)$.

   (c) Find any irreducible quadratic factors of $P(x)$.

   (d) Solve $P(x) \geq 0$.

10. Let $r(x) = \dfrac{x^3 + 2x^2 - 2x - 3}{x^2 - 4x - 5}$.

    (a) What is the domain of this function?

    (b) Find all zeros of this function (x-intercepts).

    (c) Find all vertical asymptotes of this function.

    (d) Determine where r is positive, and where it is negative.

    (e) Determine any horizontal and slant asymptotes.

    (f) Draw a complete graph of $y = r(x)$.

    (g) Solve $r(x) < 0$.

11. Solve for x : $48 = \dfrac{2}{x-1} + \dfrac{9}{x+1} + \dfrac{253}{x+5}$.

12. Use the *unit circle* definition of the standard trigonometric functions to explain the following:

    (a) $\sin^2 t + \cos^2 t = 1$.

    (b) $\sin(\pi - t) = \sin t$.

    (c) $\tan(t + \pi) = \tan t$.

    (d) The range of $f(t) = \sin t$ is $[-1, 1]$.

    (e) The domain of $g(t) = \tan t$ is $t \neq \dfrac{(2k+1)\pi}{2}$ for integer k, i.e. all odd multiples of $\dfrac{\pi}{2}$.

    (f) $h(t) = \cos t$ decreases for $t \in (0, \pi)$. Where does it increase?

    (g) $\sec^2 t - \tan^2 t = 1$.

13. For the function $f(x) = 1 + 5\cos\left(2x - \dfrac{\pi}{2}\right)$, determine the following:

    (a) Domain(f).

    (b) Range(f).

    (c) A *complete period* for f.

    (d) The *amplitude* of f.

(e) The *phase shift* of f.

(f) The zeros of f in the interval $[-2\pi, 2\pi]$.

(g) The graph of f, showing two complete periods.

14. Solve the following equations *exactly* for $x \in [-2\pi, 2\pi]$:

    (a) $\sin x \leq -\dfrac{1}{2}$.

    (b) $3\sec^2 x = 4$.

    (c) $\tan x = -1$.

    (d) $2\sin^2(2x) - 1 = 0$.

15. *Prove* that

$$\frac{\sin(x+h) - \sin x}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right), \quad \text{for all } h \neq 0.$$

16. Find δ such that $\sqrt{2}\sin x - \cos x = \sqrt{3}\sin(x+\delta)$ for all x.

17. Find A such that $\sqrt{2}\sin x + \cos x = A\sin\left(x + \cos^{-1}\frac{\sqrt{2}}{\sqrt{3}}\right)$, for all x.

18. Find A and δ such that $\sin x + \sqrt{2}\cos x = A\sin(x+\delta)$ for all x.

19. Solve the following equations *exactly*:

    (a) $\sin^2 t + \tan t + \cos^2 t = 2$, $0 \leq t \leq 2\pi$.

    (b) $\left(\sec\theta - \dfrac{1}{2}\right)(2\cos\theta - \sqrt{3}) = 0$, $0 \leq \theta \leq 2\pi$.

    (c) $\cos(2x) + \cos x + 1 = 0$.

    (d) $4\sin x\cos x = \sqrt{3}$, $-2\pi < x < 2\pi$.

    (e) $\sin(3x) - \cos(3x) = 0$.

    (f) $\sqrt{2}\sec t = -2$, $0 \leq t \leq 2\pi$.

    (g) $\sin t - \cos t = \frac{1}{3}$. (**Hint:** Write this in the form $A\sin(t+\alpha) = \dfrac{1}{3}$).

    (h) $\tan^2 t - 3\tan t = 1$, $-2\pi \leq t \leq 2\pi$.

Solutions

1. $y = 3(x-2)^3 + 4$.

2. $(-4, 0) \cup (1, 2)$.

3. $|x-2| \le 3$.

4. $\left(-\infty, -\frac{1}{2}\right] \cup [2, \infty)$.

5. $f(x) = \begin{cases} -3x+5 & \text{if} \quad x < \frac{5}{3} \\ 3x-5 & \text{if} \quad \frac{5}{3} \le x \end{cases}$

6. (a) $f(x) = 4(x-1)^2 - 3$.

   (b) Your graph should be a parabola with vertex at $(1, -3)$, which opens up. The graph crosses the x-axis at $x = 1 \pm \frac{\sqrt{3}}{2}$.

   (c) Vertical stretch by a factor of 4; vertical shift 3 units down; horizontal shift 1 unit to the right.

7. (a) $(f-g)(x) = \begin{cases} 2x+1 & \text{if} \quad x < 0 \\ 2x-1 & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{if} \quad 1 < x \end{cases}$

   (b) $\left(\frac{f}{g}\right)(x) = \begin{cases} -\frac{x+1}{x} & \text{if} \quad x < 0 \\ \frac{1-x}{x} & \text{if} \quad 0 \le x \le 1 \\ 1 & \text{if} \quad 1 < x \end{cases}$

8. (a) Note: $g(x) = \left(x + \frac{1}{2}\right)^2$, so $\text{Domain}(f) = [0, \infty)$, $\text{Range}(f) = [-3, \infty)$, $\text{Domain}(g) = (-\infty, \infty)$, $\text{Range}(g) = [0, \infty)$.

   (b) $f(g(x)) = \sqrt{g(x)} - 3 = \left|x + \frac{1}{2}\right| - 3$, $-\infty < x < \infty$.

   (c) $g(f(x)) = \left(\sqrt{x} - \frac{5}{2}\right)^2 = x - 5\sqrt{x} + \frac{25}{4}$, $x \ge 0$.

9. (a) $x = 3$.

   (b) $(x-3)$, $(x-\sqrt{3})$, $(x+\sqrt{3})$.

   (c) (x^2+3).

   (d) $[-\sqrt{3}, \sqrt{3}] \cup [3, \infty)$.

10. (a) $x \ne -1, 5$.

    (b) $x = \frac{-1 \pm \sqrt{13}}{2}$.

    (c) Vertical asymptote $x = 5$. Note that $x = -1$ is a *removeable discontinuity.*

(d) Positive on $\left(\frac{-1-\sqrt{13}}{2},-1\right)\cup\left(-1,\frac{-1+\sqrt{13}}{2}\right)\cup(5,\infty)$; Negative on $\left(-\infty,\frac{-1-\sqrt{13}}{2}\right)\cup\left(\frac{-1+\sqrt{13}}{2},5\right)$.

(e) Slant asymptote $y = x+6$ (or $y = x$).

(f) Be sure to include at least $-10 < x < 15$, noting the vertical asymptote, and the "hole" in the graph at $x = -1$.

(g) $\left(-\infty,\frac{-1-\sqrt{13}}{2}\right)\cup\left(\frac{-1+\sqrt{13}}{2},5\right)$.

11. $x = \frac{1}{2}, \pm\sqrt{2}$.

12. (a) The point $(\cos t, \sin t)$ is on the unit circle $x^2+y^2 = 1$.

(b) $\pi - t$ and t are *supplementary angles*, that is, they have the same *reference angle*.

(c) The straight line with angle t cuts two places on the unit circle, giving the same value for $\tan(t+\pi)$ and $\tan t$.

(d) $\sin t$ is just the y-coordinate on the unit circle.

(e) At odd multiples of $\frac{\pi}{2}$, the line is vertical, and so the slope $(\tan t)$ is undefined.

(f) $\cos t$ is the x-coordinate, which decreases from 1 to -1 as t goes from 0 to π. As t goes from π to 2π, this coordinate increases from -1 back to 1.

(g) Divide the *Pythagorean identity* in part (a) by $\cos t$, and use the definitions of $\tan t$ and $\sec t$.

13. (a) $\text{Domain}(f) = (-\infty,\infty)$.

(b) $\text{Range}(f) = [-4,6]$.

(c) $[0,\pi]$.

(d) 5.

(e) $\frac{\pi}{4}$.

(f) Two solutions are $\alpha = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}\left(-\frac{1}{5}\right) = \frac{3\pi}{4}-\frac{1}{2}\tan^{-1}(2)$, and $\beta = \frac{5\pi}{4}-\frac{1}{2}\cos^{-1}\left(-\frac{1}{5}\right) = \frac{3\pi}{4}+\frac{1}{2}\tan^{-1}(2)$, which satisfy $\frac{\pi}{2} < \alpha,\beta < \pi$, so the full set is $\alpha-2\pi, \beta-2\pi, \alpha-\pi, \beta-\pi, \alpha, \beta, \alpha+\pi, \beta+\pi$.

14. (a) $\left[-\frac{5\pi}{6},-\frac{\pi}{6}\right]\cup\left[\frac{7\pi}{6},\frac{11\pi}{6}\right]$.

(b) $\left\{-\frac{11\pi}{6},-\frac{7\pi}{6},-\frac{5\pi}{6},-\frac{\pi}{6},\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}\right\}$.

(c) $\left\{-\frac{5\pi}{4},-\frac{\pi}{4},\frac{3\pi}{4},\frac{7\pi}{4}\right\}$.

(d) $\left\{\pm\frac{\pi}{8}, \pm\frac{3\pi}{8}, \pm\frac{5\pi}{8}, \pm\frac{7\pi}{8}, \pm\frac{9\pi}{8}, \pm\frac{11\pi}{8}, \pm\frac{13\pi}{8}, \pm\frac{15\pi}{8}\right\}$.

15. Use the formula : $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

16. $\delta = 2\pi - \cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) \neq -\frac{\pi}{4}$

17. $A = \sqrt{3}$.

18. $A = \sqrt{3}$, $\delta = \sin^{-1}\left(\sqrt{\frac{2}{3}}\right) = \tan^{-1}(\sqrt{2})$.

19. (a) $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

(b) $\left\{\frac{\pi}{6}, \frac{11\pi}{6}\right\}$.

(c) $\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n$, n an integer.

(d) $\left\{-\frac{11\pi}{6}, -\frac{5\pi}{3}, -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}\right\}$.

(e) $x = \frac{\pi}{12} + \frac{\pi n}{3}$, n an integer.

(f) $\frac{3\pi}{4}, \frac{5\pi}{4}$.

(g) $\frac{\pi}{4} + \sin^{-1}\left(\frac{1}{3\sqrt{2}}\right) + 2\pi n = \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{\sqrt{17}}\right) + 2\pi n$, and

$\frac{5\pi}{4} - \sin^{-1}\left(\frac{1}{3\sqrt{2}}\right) + 2\pi n = \frac{5\pi}{4} - \tan^{-1}\left(\frac{1}{\sqrt{17}}\right) + 2\pi n$, n an integer.

(h) Let $\alpha = \tan^{-1}\left(\frac{3+\sqrt{13}}{2}\right), \beta = \tan^{-1}\left(\frac{3-\sqrt{13}}{2}\right)$.

The solutions are $\alpha - 2\pi, \alpha - \pi, \alpha, \alpha + \pi, \beta - \pi, \beta, \beta + \pi, \beta + 2\pi$.

Completing the Square

A very useful tool in your algebra arsenal is *completing the square*, that is, to re-write a quadratic expression as follows:

$$ax^2 + bx + c = a(x-h)^2 + k$$

We start by dividing by $a \neq 0$:

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c,$$

then use the identity $(\alpha+\beta)^2 - \beta^2 = \alpha^2 + 2\alpha\beta$ to write

$$\begin{aligned} a\left[x^2 + \tfrac{b}{a}x\right] + c &= \\ &= a\left[\left(x + \tfrac{b}{2a}\right)^2 - \left(\tfrac{b}{2a}\right)^2\right] + c \\ &= a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a} + c \\ &= a\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2-4ac}{4a} \end{aligned}$$

example: $2x^2 - 3x + 5$
$= 2\left(x^2 - \dfrac{3}{2}x\right) + 5$
$= 2\left[\left(x - \dfrac{3}{4}\right)^2 - \left(\dfrac{3}{4}\right)^2\right] + 5$
$= 2\left(x - \dfrac{3}{4}\right)^2 - \dfrac{9}{8} + 5$
$= 2\left(x - \dfrac{3}{4}\right)^2 + \dfrac{31}{8}.$

example: $\dfrac{1}{2}x^2 + 4x + 3$
$= \dfrac{1}{2}\left(x^2 + 8x\right) + 3$
$= \dfrac{1}{2}\left[(x+4)^2 - 4^2\right] + 3$
$= \dfrac{1}{2}(x+4)^2 - 8 + 3 = \dfrac{1}{2}(x+4)^2 - 5.$

example: $7 - 5x - 2x^2 = -2x^2 - 5x + 7$
$= -2\left[x^2 + \dfrac{5}{2}x\right] + 7$
$= -2\left[\left(x + \dfrac{5}{4}\right)^2 - \left(\dfrac{5}{4}\right)^2\right] + 7$
$= -2\left(x + \dfrac{5}{4}\right)^2 + \dfrac{25}{8} + 7$

$$= -2\left(x+\frac{5}{4}\right)^2 + \frac{81}{8}.$$

example: Find the centre and radius of the circle with equation $x^2 - 6x + y^2 + 4y = 12$.
$x^2 - 6x + y^2 + 4y = (x-3)^2 - 3^2 + (y+2)^2 - 2^2 = 12$, from which
$(x-3)^2 + (y+2)^2 = 25$, giving a centre of (3,-2), radius 5.

Quadratic Equations

As you proceed through your mathematics coursework, from 221 Analytical Geometry - Calculus I, through 222 Calculus II, 223 Calculus III and 335 Introduction to Ordinary Differential Equations, you will be surprised to find out how often we need to solve *quadratic equations*, of the form

$$ax^2 + bx + c = 0, \;\; a \neq 0 \; .$$

To derive the *quadratic equation*, we *complete the square* as previously, obtaining

$$0 = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a},$$

from which

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \; .$$

Taking square roots,

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; .$$

Sinusoidals

Another useful tool is to re-write sums of sines and cosines as single values, as in

$$a \sin \omega t + b \cos \omega t = A \sin(\omega t + \delta),$$

(using the identity $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$), where $A = \sqrt{a^2 + b^2}$ and $\cos\delta = \dfrac{a}{A}$, $\sin\delta = \dfrac{b}{A}$.