3450:223 Calculus III, Final Sample Problems

These problems provide a sample of typical problems you are expected to be able to solve.

1. Vector Operations. Let $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find
   (a) $\mathbf{u} \cdot \mathbf{v}$
   (b) $\mathbf{u} \times \mathbf{v}$
   (c) $3\mathbf{u} \cdot \mathbf{v} - (2\mathbf{u} + 3\mathbf{v}) \times \mathbf{v}$
   (d) the angle between $\mathbf{u}$ and $\mathbf{v}$
   (e) $\text{proj}_{\mathbf{v}}\mathbf{u}$
   (f) $\text{proj}_{\mathbf{u}}\mathbf{v}$

2. Lines, Planes and Quadric Surfaces
   (a) Find the equation of the line going through the points (1,2,0) and (4,5,6).
   (b) Find the equation of the plane going through the points $(2, 0, -1)$, $(0, 4, 1)$ and $(-1, -1, 0)$.
   (c) Find the point of intersection of the line $\mathbf{r}(t) = \langle 2t + 1, 3t - 2, t + 1\rangle$ and the plane $x - 3y + 2z = 7$.
   (d) Find the angle between the line $\mathbf{r}(t) = \langle 2t+1, 3t-2, t+1\rangle$ and the plane $x-3y+2z = 7$.
   (e) Find the distance between the point $\text{P} = (3, -2, 6)$ and the plane $-2x + y - 3z = 4$.
   (f) Use intercepts to sketch the plane $2x + 4y + 8z = 16$.
   (g) Sketch the surface $z = 5 - (x^2 + y^2)$. What type of quadric surface is this?
   (h) Sketch the surface $z = -3 + \sqrt{x^2 + y^2}$. What type of quadric surface is this?
   (i) Sketch the surface $x^2 + 4y^2 + 9z^2 = 36$. What type of quadric surface is this?
   (j) What is the difference between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 \leq 1$?

3. Motion along a 1D curve in R^3. Let an object follow the path $\mathbf{r}(t) = \langle 2\cos 4t, 6t - 2, 2\sin 4t\rangle$. Find
   (a) the velocity, speed and acceleration
   (b) the unit tangent and normal vectors
   (c) the curvature
   (d) the tangential and normal components of acceleration

4. Partial derivatives
   (a) For the functions below, evaluate all first and second derivatives, and simplify as much as possible.
      i. $f(x, y) = \ln\left(\dfrac{x}{y^2} + 1\right)$
      ii. $f(x, y, z) = e^{xy^2/z}$
      iii. $f(x, y, z) = \dfrac{(z + y\sin x)^{4/9}}{(x^2 + y^3 + z^4)^{1/3}}$

(b) Let $f(x,y) = x^2y$. Use the chain rule to find the polar derivatives $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial^2 f}{\partial r \partial \theta}$.

(c) Show that $u(x,t) = e^{-D\pi^2 t}\sin(\pi x)$ satisfies the diffusion equation $u_t = Du_{xx}$, where D is a constant.

(d) Find the tangent plane to $f(x,y) = \ln(x^2/y)$ at the point $(3,4)$.

(e) Use a linear approximation to estimate $Q = \dfrac{2.02^4}{1.97^3 + 4}$.

(f) Find the rate of change of the function $f(x,y,z) = x^2y + z^3$ at the point $(1,2,3)$ in the direction given by $\mathbf{v} = \langle -1, 4, 5 \rangle$.

(g) Find the upward pointing normal vector to the curve $y = e^x$ in R^2.

(h) Find the downward pointing normal vector to the surface $z = x^2 + y^2$.

(i) Find and classify the critical points of $f(x,y) = x^3 - 3x^2y + y^2$.

(j) Find the extreme values of $f(x,y) = x^2 + 2xy^2$ on the domain $x^2 + y^2 \leq 2$.

(k) Find the maximum and minimum values of $f(x,y) = ye^x$ along the constraint curve $x^2 + 2y^2 = 4$.

(l) Find the maximum value of $f(x,y) = x^2 - 6y^2 + 2$ along the constraint curve $2x + 3y = 5$.

5. Integrals

(a) Evaluate $\displaystyle\iint_D (y(x-3)^3 + 1)\, dA$, where $D = \{(x,y) | 1 \leq x \leq 4, x - 3 \leq y \leq 1\}$

(b) Consider $\displaystyle\int_0^2 \int_{\sqrt[3]{y}}^8 e^{x^4}\, dx\, dy$. Change the order of integration then evaluate the integral.

(c) Evaluate $\displaystyle\iint_D (1 + x^2 + y^2)\, dA$, where D is the circle of radius 2 centered at the origin.

(d) Find the surface area of the portion of the surface $z = 10 - 2x^2 - 2y^2$ that lies above the xy-plane.

(e) Evaluate $\displaystyle\iiint_D xy\, dV$, where D is the region in the first octant bounded by the coordinate planes and $x + 2y + 3z = 6$.

(f) Evaluate $\displaystyle\iiint_D (x^2 + y^2)^2\, dV$, where D is the region bounded above by $z = 10 - x^2 - y^2$ and below by $z = 8$.

(g) Modify the standard cylindrical coordinates to evaluate $\displaystyle\iiint_D (x^2 + z^2)\, dV$, where D is the cylinder of radius 1 with axis centered along the y-axis, with $1 \leq y \leq 3$.

(h) Use spherical coordinates to evaluate $\displaystyle\iiint_D (x^2 + y^2 + z^2)\, dV$, where D is the region bounded above by $x^2 + y^2 + z^2 = 25$ and below by $z = \sqrt{x^2 + y^2}$.

(i) Use a change of variables to evaluate $\iint_D xy\, dA$, where D is the diamond-shaped domain bounded by $y = x + 1$, $y = 1 - x$, $y = x - 1$ and $y = -1 - x$.

6. Vector Calculus

(a) Evaluate $I = \int_C (x + 2y + 3z)\, ds$, where $C = C_1 + C_2$. The first segment of the path, C_1, is the arc of the circle $x^2 + y^2 = 1$ lying in the xy-plane, beginning at the point $(1, 0, 0)$ and ending at the point $(0, 1, 0)$. The second segment of the path, C_2, is the arc of the circle $y^2 + z^2 = 1$ lying in the yz-plane, beginning at the point $(0, 1, 0)$ and ending at the point $(0, 0, 1)$.

(b) Find the work done (the vector line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$) by a particle moving through the vector field $\mathbf{F} = y\mathbf{i} + z^2\mathbf{j} - x\mathbf{k}$ along the path $C = C_1 + C_2$, where C_1 is the straight line segment from $(1, 1, 1)$ to $(2, 3, 4)$ and C_2 is the straight line segment from $(2, 3, 4)$ to $(0, -1, 5)$.

(c) Show that the vector field
$\mathbf{F} = \langle e^y + \sec x \tan x \tan y + y \cos(xy) + 2x, xe^y + \sec x \sec^2 y + x \cos(xy) - 3y^2 \rangle$ is conservative, and find a potential for it.

(d) The vector field $\mathbf{F} = \left\langle y^2 z^4 + \frac{1}{x} + \frac{z}{y} e^{xz}, 2xyz^4 + \frac{1}{y} - \frac{1}{y^2} e^{xz}, 4xy^2 z^3 - \frac{1}{z} + \frac{x}{y} e^{xz} \right\rangle$ is conservative (you don't need to verify it). Find a potential for $\mathbf{F}$.

(e) Use Green's Theorem to evaluate the vector line integral of $\mathbf{F} = \left\langle \frac{y^2}{2} + \tan(e^{x^2}), \ln(y^4) - \frac{x^2}{2} \right\rangle$ over the closed curve C, which is the circle of radius 2 centered at the origin oriented in the counterclockwise direction.

(f) Find the divergence and curl of $\mathbf{F} = \sin(x^2 + y)\mathbf{i} - e^{xz}\mathbf{j} + \ln(x + 2y + 3z)\mathbf{k}$.

(g) Find the flux, $\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, d\mathbf{S}$, of the vector field $\mathbf{F} = \mathbf{i} - z\mathbf{j} + x\mathbf{k}$ across the surface S, where S is the portion of the plane $x + 2y + z = 2$ that lies in the first octant, with upward pointing normal.

(h) Use Stokes' Theorem to compute $\iint_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = \langle xy, xz, yz \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies inside the cylinder $x^2 + y^2 = 1$ and is above the xy-plane.

(i) Use the Divergence Theorem to find the net flux of $\mathbf{F} = \langle x, 2y, 4z \rangle$ out of the sphere of radius 3 centered at the origin.