Intro to Discrete Math Graded Problems

Important: Remember to read the instructions and advice in homework.html!

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§ 2.1 (10 points) Use Theorem 2.1.1 to fully simplify the logical expression below. Write a clear chain of equivalences, and justify each step by citing the law(s) used by name.

$$\sim[\sim[(p \vee q) \wedge r] \vee \sim q]$$

(Hint: The distribute law *can* be applied here, but it does not help! On the contrary, it will just make things harder for both of us. Look for another approach.)

§ 2.3 (10 points) Write the following argument in symbolic form, then use a truth table to determine whether it is valid; explain your response. If it is invalid, provide a counterexample in plain English.

> If it is cool this Friday, then Craig will wear his leather jacket if the pockets are mended. The forecast for Friday calls for cool weather, but the pockets have not been mended. Therefore Craig won't be wearing his leather jacket this Friday.

(Hint: The hardest part of this problem is probably the translation to a symbolic representation, so I strongly recommend checking with me to confirm that you've done that correctly before proceeding!)

§ 2.4 (10 points) Design a circuit for the following I/O table, using only the three basic gate types NOT, AND, and/or OR.

The direct/standard approach, if done correctly, yields a circuit with five gates; this is worth 9 points. For full credit, find a circuit that uses no more than four gates.

P	Q	R
0	0	1
0	1	0
1	0	0
1	1	1

§ 3.3 (10 points) Consider the predicate

$$P(x, y) = \text{“} y - x = y + x^2 \text{”}$$

where the universe (domain) for each of the variables x and y is $\mathbb{Z}$, the set of integers. For each of the following statements, determine its truth value (i.e., state whether it is true or false), then state its negation.

1. $\forall y, P(0, y)$
2. $\exists y$ s.t. $P(1, y)$
3. $\forall x, \exists y$ s.t. $P(x, y)$
4. $\exists y$ s.t. $\forall x, P(x, y)$
5. $\forall y, \exists x$ s.t. $P(x, y)$

§ 4.5 (8 points) Prove that for all integers a, $a^3 - a$ is a multiple of 3.

§ 4.7A (9 points) In exercise #4.7.31.a, it is established that for all positive integers n, r, and s, if $rs \leq n$, then $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$. (I suggest attempting to prove this yourself, then compare your answer to that given in the textbook. Do not turn this part in, since the proof is given to you; here, we will look at the remaining parts of this exercise.)

Use this fact, along with other theorems (see this footnote[1] if you'd like a hint), to prove the following:

> For each integer $n > 1$, if n is not prime then there exists a prime number p such that $p \leq \sqrt{n}$ and n is divisible by p.

After you have done this, state the contrapositive of the statement you just proved. (Note that this suggests a semi-efficient test for primality; we will use this in the next problem.)

§ 4.7B (5 points) Produce a Sieve of Eratosthenes up through 50. (If you are unfamiliar with the Sieve, consult exercise #4.7.33 and/or search online.) Use the results of this and the previous problem to determine whether the numbers 1613 and 2021 are prime or composite; explain your answers.[2]

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[1] The hint in the textbook for problem #4.7.31.b suggests using Theorems 4.4.1 (if $a \mid b$, then $a \leq b$), 4.4.3 (transitivity of divisibility), and 4.4.4 (any integer $n > 1$ is divisible by a prime), along with the transitive property of order.

[2] If you have already done #33 and #34 from § 4.7, I will accept those results in place of this problem, if you wish, since those include this content but are more extensive.

§ 5.2 (8 points) Use induction to prove that the standard summation formula for the first n cubes,

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2,$$

holds for all positive integers n. (Hint[3])

§ 5.3 (8 points) Use induction to prove that $10^{n+1} + 3 \cdot 10^n + 5$ is a multiple of 9 for every positive integer n.

[3]Recall that in § 5.2, a similar formula for the sum of the first n integers was proved both in class and in the textbook; furthermore, our textbook includes a proof of a formula for the sum of the first n squares (see the solution to exercise #10). If you are unsure how to proceed, you may find it useful to review those proofs, as these all share certain structural similarities.