AN ALTERNATE METHOD FOR EXAMINING INFINITE LIMITS

1. Introduction

First, in Stewart (9e ET), carefully read and consider Example 7 in § 2.2 (p. 91). If that line of reasoning makes sense to you, feel free to use it whenever appropriate. However, this approach might not be transparent to you—and even if it is, it may become more difficult to correctly apply in some situations, particularly with functions that are much more complicated in form.

To illustrate a bookkeeping method that might make the analysis slightly simpler in some cases, consider the first limit in that example,

$$\lim_{x \to 3^+} \frac{2x}{x-3}$$

To represent the approach to 3 from the right, let $x = 3 + h$; we will think of h as being a very small (and shrinking) positive value, the distance between x and 3. With this substitution, we can write

$$\begin{aligned} \lim_{x \to 3^+} \frac{2x}{x-3} &= \lim_{h \to 0^+} \frac{2(3+h)}{(3+h)-3} \\ &= \lim_{h \to 0^+} \frac{6+2h}{h} \end{aligned}$$

As h approaches 0 from the right—i.e., as it takes on very, very tiny positive values—we see that the numerator is essentially 6, while the denominator is a vanishingly small positive quantity; off to the side, we might indicate this with an extremely informal notation such as

$$\frac{6}{(+)}$$

where "(+)" represents a tiny but unspecified positive value.

The number 6 divided by a very small positive number is a very large positive number, so we conclude that

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

(We must still apply the same basic type of reasoning about division, but at least we can do so with simpler expressions.)

A very similar process allows us to compute the limit from the other side, the second part of the example; here, we will instead let $x = 3 - h$.

$$\begin{aligned}
\lim_{x\to 3^-} \frac{2x}{x-3} &= \lim_{h\to 0^+} \frac{2(3-h)}{(3-h)-3} \\
&= \lim_{h\to 0^+} \frac{6-2h}{-h} \qquad \left[\frac{6}{(-)}\right] \\
&= -\infty
\end{aligned}$$

(This time, we used the notation "$(-)$" to indicate a tiny *negative* value.)

2. Elaboration

This is a very powerful technique, but it should not be applied indiscriminantly. It works in most cases of one-sided limits, but there are often much simpler techniques, particularly if there is no vertical asymptote at the point under consideration.

To see why, consider a § 2.3 example I worked out in class—or more precisely, a variation on it, in which we only consider the limit from the right. To approach the problem with this new technique, since x is approaching 4 from the right, we would substitute $x = 4 + h$ and simplify:

$$\begin{aligned}
\lim_{x\to 4^+} \frac{x^2-4x}{x^2-3x-4} &= \lim_{h\to 0^+} \frac{(4+h)^2-4(4+h)}{(4+h)^2-3(4+h)-4} \\
&= \lim_{h\to 0^+} \frac{16+8h+h^2-16-4h}{16+8h+h^2-12-3h-4} \\
&= \lim_{h\to 0^+} \frac{4h+h^2}{5h+h^2} \\
&= \lim_{h\to 0^+} \frac{4+h}{5+h} \\
&= \frac{4}{5}
\end{aligned}$$

As you can see, this works, but it's quite cumbersome compared to the solution from class; the lesson here is that when the limit exists, it should be taken directly. A substitution of the type introduced here is probably best reserved for those cases where we have reason to suspect a vertical asymptote (typically when an attempt at direct substitution yields a nonzero value divided by zero), not merely a gap (which is most common when direct substitution yields the indeterminate form 0/0).

3. Refinement

How do we know in advance whether this technique will be useful? Frankly, we usually don't. Thankfully, we don't have to make that decision ahead of time, instead just dragging it in whenever it seems appropriate.

Consider the function from the previous example, but seeking the limit at a different point. Since the original exercise requested a two-sided limit, let's start that way here:

$$\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4}$$

We attempt direct substitution, but that produces the expression $5/0$, so we suspect a vertical asymptote. If we need more details, we'll have to consider the one-sided limits; however, that looks a little difficult here. Let's try some simplification first, such as by factoring.

$$\lim_{x\to -1} \frac{x(x-4)}{(x+1)(x-4)} = \lim_{x\to -1} \frac{x}{x+1}$$

(That last step is possible because when x is near -1, $x-4 \neq 0$.)

To proceed further, we must consider a one-sided limit, which is now much simpler, fortunately.

$$\begin{aligned}
\lim_{x\to -1^+} \frac{x}{x+1} &= \lim_{h\to 0^+} \frac{-1+h}{(-1+h)+1} \\
&= \lim_{h\to 0^+} \frac{-1+h}{h} && \left[\frac{-1}{(+)}\right] \\
&= -\infty
\end{aligned}$$

Similarly, on the other side,

$$\begin{aligned}
\lim_{x\to -1^-} \frac{x}{x+1} &= \lim_{h\to 0^+} \frac{-1-h}{(-1-h)+1} \\
&= \lim_{h\to 0^+} \frac{-1-h}{-h} && \left[\frac{-1}{(-)}\right] \\
&= \infty
\end{aligned}$$

(These last two limits were evaluated by observing that a nonzero constant divided by a number very close to zero yields a number very far from zero; furthermore, a negative divided by a positive is a negative, and a negative divided by a negative is a positive, hence the limits of $-\infty$ and ∞, respectively.)

This illustrates the basic technique in some simple cases; if you have any questions, or would like to work through some examples together, feel free to speak with me personally on this topic.