O.D.E.'S 335 INTEGRATION REVIEW Name:____________________

1. $\frac{dy}{dx} = e^{-x}\cos x$ Ans. $y = \frac{1}{2}e^{-x}(\sin x - \cos x) + c$

2. $\frac{dy}{dx} = \frac{x^2 - 3x + 3}{x^2 - 3x + 2}$ Ans. $y = x + \ln\left|\frac{x-2}{x-1}\right| + c$

3. $\frac{dy}{dx} = \frac{1}{2x^2 + 3x + 5}$ Ans. $y = \frac{2}{\sqrt{31}}\arctan\left(\frac{4x+3}{\sqrt{31}}\right) + c$

4. $\frac{dy}{dx} = \frac{x^2}{(x+2)^2}$ Ans. $y = x - \frac{4}{x+2} - 4\ln|x+2| + c$

5. $\frac{dy}{dx} = \frac{x^2}{a^2 - x^3}$ Ans. $y = -\frac{1}{3}\ln\left|a^2 - x^3\right| + c$

6. $\frac{dy}{dx} = \frac{2}{x(x^2+1)^2}$ Ans. $y = \frac{1}{x^2+1} + \ln\left|\frac{x^2}{x^2+1}\right| + c$

7. $\frac{dy}{dx} = \frac{x}{1+\sqrt{x}}$ Ans. $y = \frac{2}{3}x\sqrt{x} - x + 2\sqrt{x} - 2\ln|\sqrt{x}+1| + c$

8. $\frac{dy}{dx} = \frac{1}{x\sqrt{x+1}}$ Ans. $y = 2\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x}}\right| + c$

9. $\frac{dy}{dx} = \frac{a}{\sqrt{a^2 - x^2}}$ Ans. $y = a\arctan\frac{x}{\sqrt{a^2-x^2}} + c$

10. $\frac{dy}{dx} = \frac{5\cos x}{6 + \sin x - \sin^2 x}$ Ans. $y = \ln\left|\frac{2+\sin x}{3-\sin x}\right| + c$

INTEGRATION REVIEW Solutions

1. $y = \int e^{-x}\cos x\,dx + C$

$\int e^{-x}\cos x\,dx$

$u = e^{-x} \quad dv = \cos x\,dx$

$du = -e^{-x} \quad v = \sin x$

$= e^{-x}\sin x + \int e^{-x}\sin x\,dx + C$

$u = e^{-x} \quad dv = \sin x\,dx$

$du = -e^{-x} \quad v = -\cos x\,dx$

$= e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx + C$

$2\int e^{-x}\cos x\,dx = e^{-x}(\sin x - \cos x) + C$

$\int e^{-x}\cos x\,dx = \frac{1}{2}e^{-x}(\sin x - \cos x) + \underbrace{C/2}_{C}$

2. $\frac{dy}{dx} = \frac{x^2-3x+3}{x^2-3x+2} = 1 + \frac{1}{(x-2)(x-1)}$

$= 1 + \frac{A}{x-2} + \frac{B}{x-1}$

$= 1 + \frac{1}{x-2} - \frac{1}{x-1} =$

$y = \int \frac{x^2-3x+3}{x^2-3x+2}\,dx = \int 1 + \frac{1}{x-2} - \frac{1}{x-1}\,dx = x + \ln|x-2| - \ln|x+1| + C = x + \ln\left|\frac{x-2}{x+1}\right| + C$

3. $y = \int \frac{1}{2x^2+3x+5}\,dx$

$= \int \frac{1}{2(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16})+5}\,dx$

$= \int \frac{1}{2(x+\frac{3}{4})^2 - \frac{9}{8}+5}\,dx$

$= \int \frac{1}{2(x+\frac{3}{4})^2 + \frac{31}{8}} = \frac{8}{31}\int \frac{1}{\frac{16}{31}(x+\frac{3}{4})^2 + 1}$

$= \frac{8}{31}\cdot\frac{\sqrt{31}}{4}\int \frac{1}{\left(\frac{4x+3}{\sqrt{31}}\right)^2+1}\,dx \qquad u = \frac{4}{\sqrt{31}}x + \frac{3}{\sqrt{31}}$

$du = \frac{4}{\sqrt{31}}\,dx$

$= \frac{2}{\sqrt{31}}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{31}}\tan^{-1}u + C = \frac{2}{\sqrt{31}}\tan^{-1}\left(\frac{4}{\sqrt{31}} + \frac{3}{\sqrt{31}}\right) + C$

$$\frac{-4x-4}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$$

$A=-4$

$2A+B=-4 \rightarrow B=4$

4. $$y = \int \frac{x^2}{x^2+4x+4}\,dx = \int 1 + \frac{-4x-4}{(x+2)^2}\,dx = \int 1 + \frac{A}{(x+2)} + \frac{B}{(x+2)^2}\,dx$$

$$x^2+4x+4 \overline{)\,x^2} \quad 1$$

$$-(x^2+4x+4)$$

$$-4x-4$$

$$= \int 1 + \frac{-4(x+2-2)+4}{(x+2)^2}\,dx$$

$$= \int 1 - \frac{4}{x+2} + \frac{4}{(x+2)^2}\,dx = x - 4\ln|x+2| - \frac{4}{x+2} + C$$

5. $$y = \int \frac{x^2}{a^2-x^3}\,dx \qquad u = a^2-x^3 \qquad du = -3x^2\,dx$$

$$= -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|a^2-x^3| + C$$

6. $$y = \int \frac{2}{x(x^2+1)^2}\,dx$$

$$\frac{2}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$x=0 \Rightarrow A=2$

$$= A(x^4+2x^2+1) + (Bx+C)(x^3+x) + Dx^2+Ex$$

$$= A(x^4+2x^2+1) + Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex$$

$$= \underline{(A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x}$$

$B=-2 \qquad C=0 \qquad 4-2,\ -2 \qquad E=0$

$D=-2$

$$= \frac{2}{x} - 2\frac{x}{x^2+1} - \frac{2x}{(x^2+1)^2}$$

$$y = \int \frac{2}{(x+2)^2}\,dx = \int \frac{2}{x} - \frac{2x}{x^2+1} - \frac{2x}{(x^2+1)^2}\,dx$$

$u = x^2+1 \quad du = 2x\,dx$

$$= 2\ln|x| - \ln(x^2+1) + \frac{1}{x^2+1} + C$$

7. $y = \int \frac{x}{1+\sqrt{x}}\,dx$ $\qquad u = 1+\sqrt{x}$ $\qquad (u-1)^2 = x$

$2(u-1) = dx$

$$= \int \frac{(u-1)^2 \cdot 2(u-1)}{u}\,du$$

$$= 2\int \frac{u^3 - 3u^2 + 3u - 1}{u}\,du$$

$$= 2\int u^2 - 3u + 3 - \frac{1}{u}\,du$$

$$= \frac{2}{3}u^3 - \frac{2\cdot 3}{2}u^2 + 3u - \ln(u) = \frac{2}{3}(1+\sqrt{x})^3 - 3(1+\sqrt{x})^2 + 3(1+\sqrt{x})$$

$$+ \ln(1+\sqrt{x}) + C$$

v.s. $\qquad u = \sqrt{x} \qquad x = u^2 \qquad dx = 2u\,du$

$$\int \frac{u^2 \cdot 2u\,du}{1+u}$$

$$\begin{array}{r l}
 & 2u^2 - 2u + 2 \\
u+1 & \overline{)\,2u^3} \\
 & -\ (2u^3 + 2u^2) \\
 & -2u^2 \\
 & -2u^2 - 2u \\
 & 2u \\
 & 2u + 2 \\
 & -2
\end{array}$$

$$y = \int 2u^2 - 2u + 2 - \frac{2}{u}\,du$$

$$y = \frac{2}{3}u^3 - u^2 + 2u - 2\ln u$$

$$= \frac{2}{3}x^{3/2} - x + 2\sqrt{x} - 2\ln\sqrt{x} + C$$

8. $y' = \frac{1}{x\sqrt{x+1}}$ $\quad u = \sqrt{x+1}$, $u^2 - 1 = x$ $\quad 2u\,du = x\,dx$

$$y = \int \frac{2u}{(u^2-1)u}\,du = \int \frac{2}{u^2-1}\,du = \int \frac{2}{(u-1)(u+1)}\,du$$

$$= \int \frac{1}{u-1} - \frac{1}{u+1}\,du = \ln|u-1| - \ln|u+1| + C$$

$$= \ln|\sqrt{x+1} - 1| - \ln|\sqrt{x+1} + 1| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

9. $y' = \frac{a}{\sqrt{a^2-x^2}}$ $\quad y = \int \frac{a}{\sqrt{a^2-x^2}}\,dx = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\,dx$

$u = \frac{x}{a}$ $\quad dx = a\,du$ $\quad = \int \frac{a}{\sqrt{1-u^2}}\,du = a\sin^{-1}u + C$

$$= a\sin^{-1}\left(\frac{x}{a}\right) + C$$

10. $y' = \frac{5\cos x}{6 + \sin x - \sin^2 x}$ $\quad u = \sin x$ $\quad du = \cos x\,dx$

$$y = \int \frac{5}{6+u-u^2}\,du = \int \frac{-5}{u^2-u-6}\,du = \int \frac{-5}{(u-3)(u+2)}\,du$$

$$= \int \frac{-1}{u-3} + \frac{1}{u+2}\,du = -\ln|u-3| + \ln|u+2| + C$$

$$= \ln\left|\frac{u+2}{u-3}\right| + C$$

$$= \ln\left|\frac{\sin x + 2}{\sin x - 3}\right| + C$$

$$= \ln\left|\frac{\sin x + 2}{3 - \sin x}\right| + C$$