Intro to Discrete Math Graded Problems

Important: Remember to read the instructions and advice in homework.html!

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§ 2.1 (10 points) Use Theorem 2.1.1 to fully simplify the logical expression below. Write a clear chain of equivalences, and justify each step by citing the law(s) used by name.

$$\sim[\sim[(p \vee q) \wedge r] \vee \sim q]$$

(Hint: The distributive law *can* be applied here, but it does not help! On the contrary, it will just make things harder for both of us. Look for another approach.)

§ 2.3 (10 points) Write the following argument in symbolic form, then use a truth table to determine whether it is valid; explain your response. If it is invalid, provide a counterexample in plain English.

> If it is chilly this Saturday, then Lisa will wear her wool dress if the hem has been repaired. The forecast for the weekend calls for cool weather, but the hem has not been repaired. Therefore Lisa won't be wearing her wool dress this Saturday.

(Hint: The hardest part of this problem is probably the translation to a symbolic representation, so I strongly recommend checking with me to confirm that you've done that correctly before proceeding!)

§ 2.4 (5 points) Design a circuit for the following I/O table, using only the three basic gate types NOT, AND, and/or OR.

The direct/standard approach, if done correctly, yields a circuit with five gates. For a bonus point, find a correct circuit that uses only *four* gates.

P	Q	R
0	0	1
0	1	0
1	0	0
1	1	1

§ 3.3 (10 points) Consider the predicate $P(x, y) =$ "$y - x = y + x^2$", where the domain for each of the variables x and y is $\mathbb{Z}$, the set of integers. For each of the following statements, determine its truth value (i.e., state whether it is true or false), then state its negation.

1. $\forall y, P(0, y)$
2. $\exists y$ s.t. $P(1, y)$
3. $\forall x, \exists y$ s.t. $P(x, y)$
4. $\exists y$ s.t. $\forall x, P(x, y)$
5. $\forall y, \exists x$ s.t. $P(x, y)$

§ 4.3 (10 points) Using the definition of rational, prove that for any rational numbers r and s, $2r + 3s$ is rational.

(Note: This is the same claim that you'll find in an exercise, but the solution is not the same, since here, you are not permitted to use other theorems or exercises. Also, unlike most other graded problems, this one is comparable to what you may find on a test! I have it here to give you practice and feedback.)

§ 4.5 (10 points) Prove that for all integers a, $a^3 - a$ is divisible by 3. (hint in footnote below[1])

[1] I've thought of two very different proofs for the given statement, both of which are perfectly acceptable. One of them is a relatively direct application of the material of this section. The other approach appears shorter and more elegant, but it hinges on a fact about divisibility that is not proved in the textbook. (Indeed, proving that fact is left as an exercise in this section.) If you wish to take this route, you must include a proof of that key fact.

In other words, make sure that everything you state in a proof is firmly established. If it's already proved or assumed by the textbook, you're good; if it is not, you must support it with a proof of your own.

§ 4.7A (12 points) In exercise #4.7.31.a, it is established that for all positive integers n, r, and s, if $rs \leq n$, then $r \leq \sqrt{n}$ or $s \leq \sqrt{n}$. (I suggest attempting to prove this yourself, then compare your answer to that given in the textbook. Do not turn this part in, since the proof is given to you; here, we will look at the remaining parts of this exercise.)

Use this fact, along with other theorems (see this footnote[2] if you'd like some hints), to prove the following:

> For each integer $n > 1$, if n is not prime then there exists a prime number p such that $p \leq \sqrt{n}$ and n is divisible by p.

After you have done this, state the contrapositive of the statement you just proved. (Note that this suggests a semi-efficient test for primality; we will use this in the next problem.)

§ 4.7B (5 points) Produce a Sieve of Eratosthenes up through 50. (If you are unfamiliar with the Sieve, consult exercise #4.7.33 and/or search online.) Use the results of this and the previous problem to determine whether the numbers 1613 and 2021 are prime or composite; explain your answers.

(If you have already done #33 and #34 from § 4.7, I will accept those results in place of this problem, if you wish, since those include this content but are more extensive.)

§ 5.2 (10 points) Use induction to prove that the standard summation formula for the first n cubes,

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2,$$

holds for all positive integers n. (Hint[3])

§ 5.3 (10 points) Use induction to prove that $10^{n+1} + 3 \cdot 10^n + 5$ is a multiple of 9 for every positive integer n.

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[2] The hint in the textbook suggests using Theorems 4.4.1, 4.4.3, and 4.4.4, along with the transitive property of order.

I would further point out that this problem is in this section because the first part of the exercise (the one you're not turning in) uses an indirect proof; for this part, though, a direct proof is much easier than an indirect approach! In other words, I do *not* recommend trying to apply contradiction or contraposition here. (It's possible, but harder.)

One more small hint: If you take the expected approach, you'll probably need to either use cases or make use of the concept of "without loss of generality" (WLOG). (The latter is recommended for being shorter, easier, and cleaner.)

[3] Recall that in § 5.2, a similar formula for the sum of the first n integers was proved both in class and in the textbook; furthermore, our textbook includes a proof of a formula for the sum of the first n squares (see the solution to exercise #10). If you are unsure how to proceed, you may find it useful to review those proofs, as these all share certain structural similarities.

§ 5.4 (10 points) One way to define the Fibonacci sequence is as follows:

$$F_0 = 1, \quad F_1 = 1, \quad F_k = F_{k-1} + F_{k-2} \text{ for every integer } k \geq 2.$$

This sequence begins 1, 1, 2, 3, 5, 8, 13, 21, ..., where each term is the sum of the two previous. (You may wish to confirm this for yourself.)

Given this background, use strong mathematical induction to prove that $F_n < 2^n$ for every integer $n \geq 1$.

§ 6.2 (10 points) Use element arguments to prove that for all sets A and B subsets of a universal set U, $A \subseteq B$ if and only if $A \cap B^c = \varnothing$.

(Hint: To prove that p iff q, it is usually easiest to first prove that if p then q, and then prove that if q then p.)