Integral Curves of a Vector Field with a Fractal Discontinuity

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Nonsmooth systems studied most often have smooth or piecewise-smooth boundaries between smooth vector fields, and especially linear or planar boundaries. What happens when there is a boundary that is not as simple? What if the boundary is a fractal? Can a solution to such a system slide or "chatter" along this boundary?

As an initial back-of-the-envelope problem, we will take a look at a very simple 2-dimensional system with a discontinuity boundary which is one side of the Koch snowflake spanning the points (0,0) and (1,0). Above the boundary, the vector field points to the left, and below it points upwards, each with a magnitude of 1. We consider the initial value problem with $x(0) = 1, y(0) = 0$.

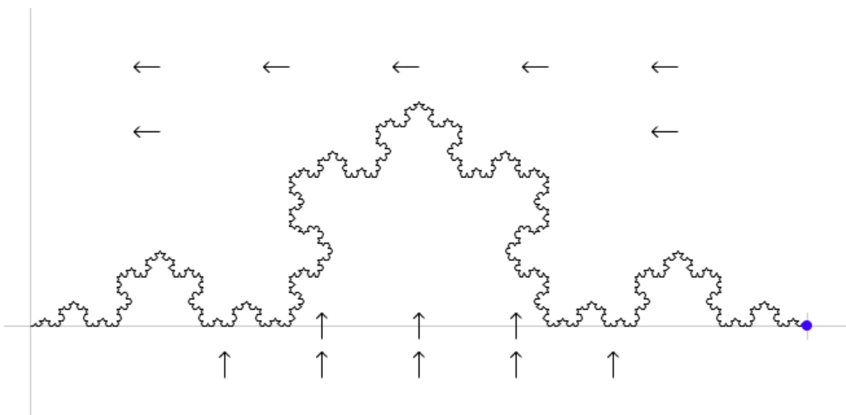

Figure 1: The two vector fields separated by the Koch snowflake fractal boundary. The initial condition $x(0) = 1, y(0) = 0$ is shown as the dot on the right.

The boundary of the snowflake has infinite length, and indeed between any two points on the curve the length is infinite, so a solution which slides along the boundary seems impossible in our situation. A question for the future is whether a system exists in which a solution can slide along a fractal boundary. We can find the solution to our system, albeit in a recursive fashion.

The solution can only move up and to the left (sliding having been ruled out). So the solution must reach the largest peak of the snowflake, since if it reached the line $x = 1/2$ or $y = \sqrt{3}/3$, it would move directly to that peak. As it must move in orthogonal directions only, it must go a total length of $1/2 + \sqrt{3}/3$. A similar argument can be made to show the solution must reach the smaller peak shown in Figure 2 below. From that peak, the solution will move left until it reaches the fractal boundary again, and then upward until it again hits the fractal boundary shown below. From there the fractal is a copy of the previous step, so we can recursively iterate this path, scaling by 1/3 each time, until it reaches the largest peak.

Finally, we can see the path through the first third is a replica of the path we just drew, since the fractal contains a scaled copy of itself. Recursively filling in this path we can generate the full integral curve, which from the start to the peak has a total length of $1/2 + \sqrt{3}/3$ and only moves to the left or upward. The integral curve is itself a fractal, but one of finite length.

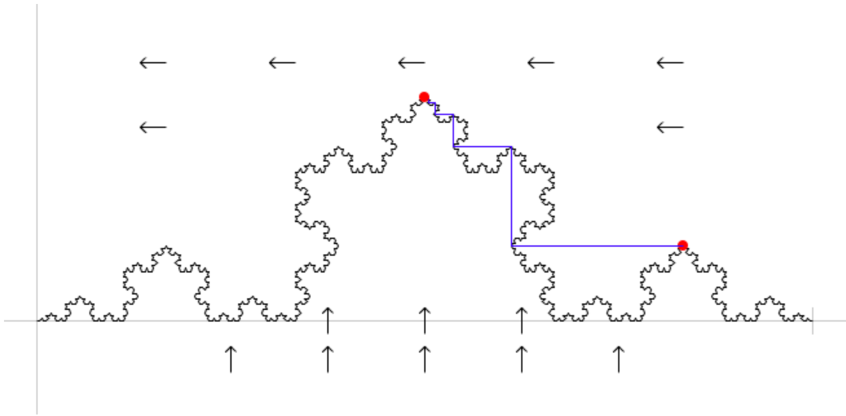

Figure 2: The solution between the largest peak and second largest peak (dotted), iteratively moving left and upward.

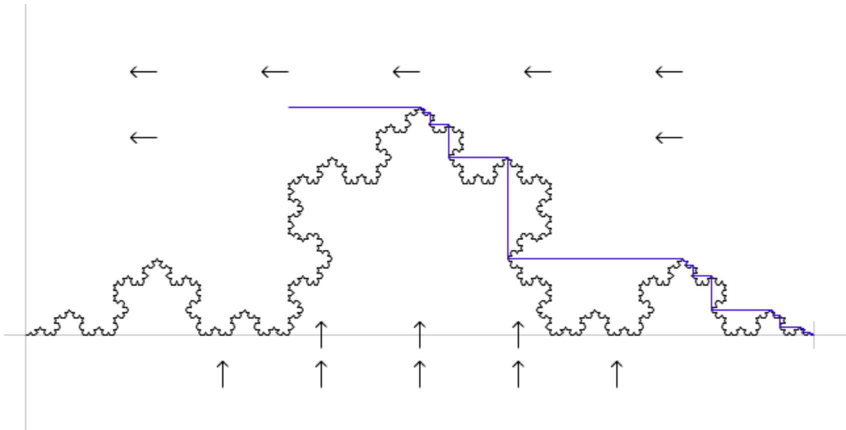

Figure 3: The full solution with initial condition $x(0) = 1, y(0) = 0$.

This example an entertaining problem, and certainly there are deeper and more challenging questions to be answered about fractal discontinuities in nonsmooth vector fields. Even with this example, finding the integral curves for other initial conditions may be more difficult, and changes in the orientation of the vector field may also produce different results.