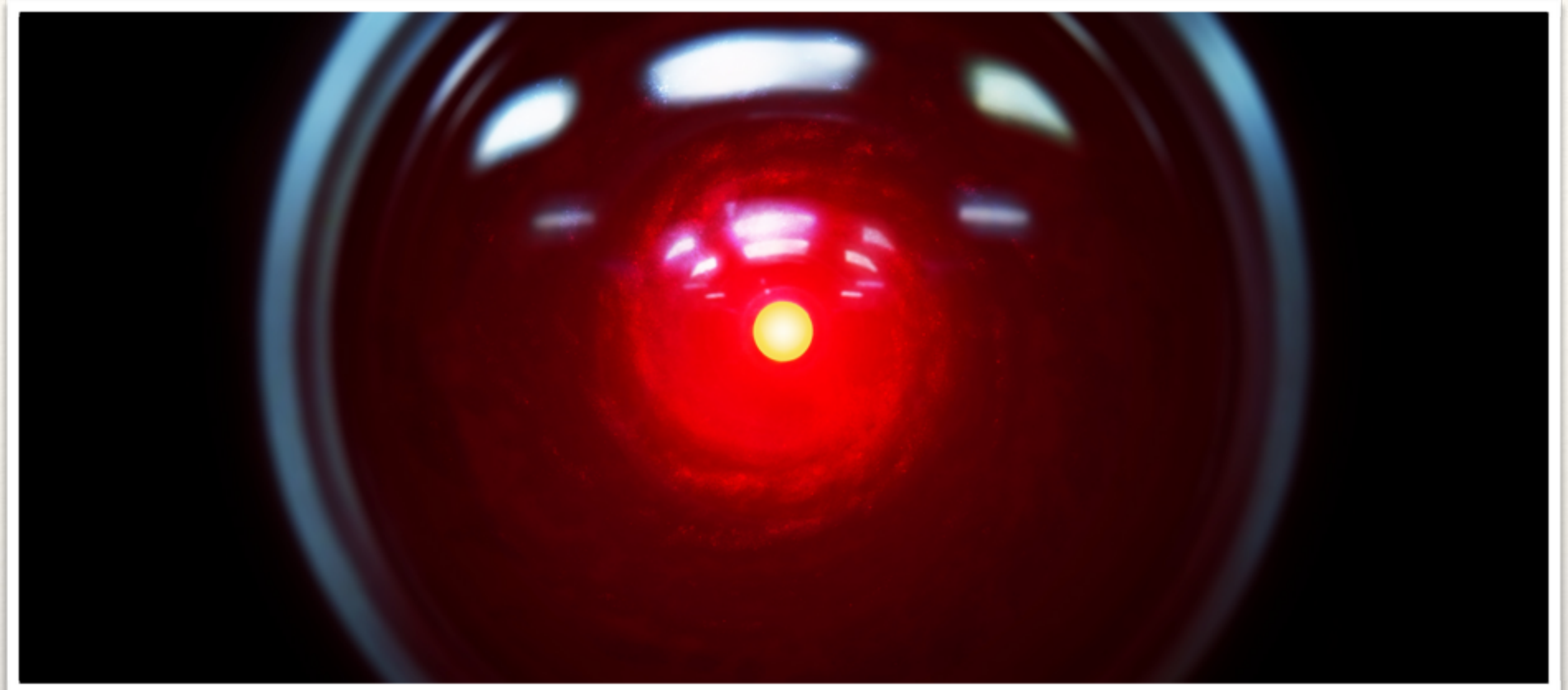


*COMP3005: Computer Vision*

# Covariance and Principal Components

Jonathon Hare
jsh2@ecs.soton.ac.uk

# Variance and Covariance

# Random Variables and Expected Values

- Mathematicians talk variance (and covariance) in terms of *random variables* and *expected values*
- A random variable is a variable that takes on different values due to chance.
  - The set of sample values from a single dimension of a featurespace can be considered to be a random variable
- The expected value (denoted E[X]) is the most likely value a random variable will take.
  - For this course we'll **assume** that the values an element of a feature can take are all equally likely
    - The expected value is thus just the **mean value**

# Variance

$$\sigma^2(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

Variance ($\sigma^2$) is the mean squared difference from the mean ($\mu$).

It's a measure of how spread-out the data is.

*technically it's $E[(X - E[X])^2]$*

# Covariance

$$\sigma(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x - \mu_x)(y - \mu_y)$$

Covariance ($\sigma(x,y)$) measures how two variables change together

*technically it's $E[(x - E[x])(y-E[y])]$*

# Covariance

$$\sigma(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x - \mu_x)(y - \mu_y)$$

The variance is the covariance when the two variables are the same ($\sigma(x,x)=\sigma^2(x)$)

# Covariance

$$\sigma(x, y) = \frac{1}{n}\sum_{i=1}^{n}(x - \mu_x)(y - \mu_y)$$

A covariance of 0 means the variables are uncorrelated

*(Covariance is related to Correlation… see notes)*

# Covariance Matrix

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma(X_1, X_1) & \sigma(X_1, X_2) & \dots & \sigma(X_1, X_n) \\ \sigma(X_2, X_1) & \sigma(X_2, X_2) & \dots & \sigma(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(X_n, X_1) & \sigma(X_n, X_2) & \dots & \sigma(X_n, X_n) \end{bmatrix}$$

A covariance matrix encodes how all possible pairs of dimensions in an *n*-dimensional dataset vary together

# Covariance Matrix

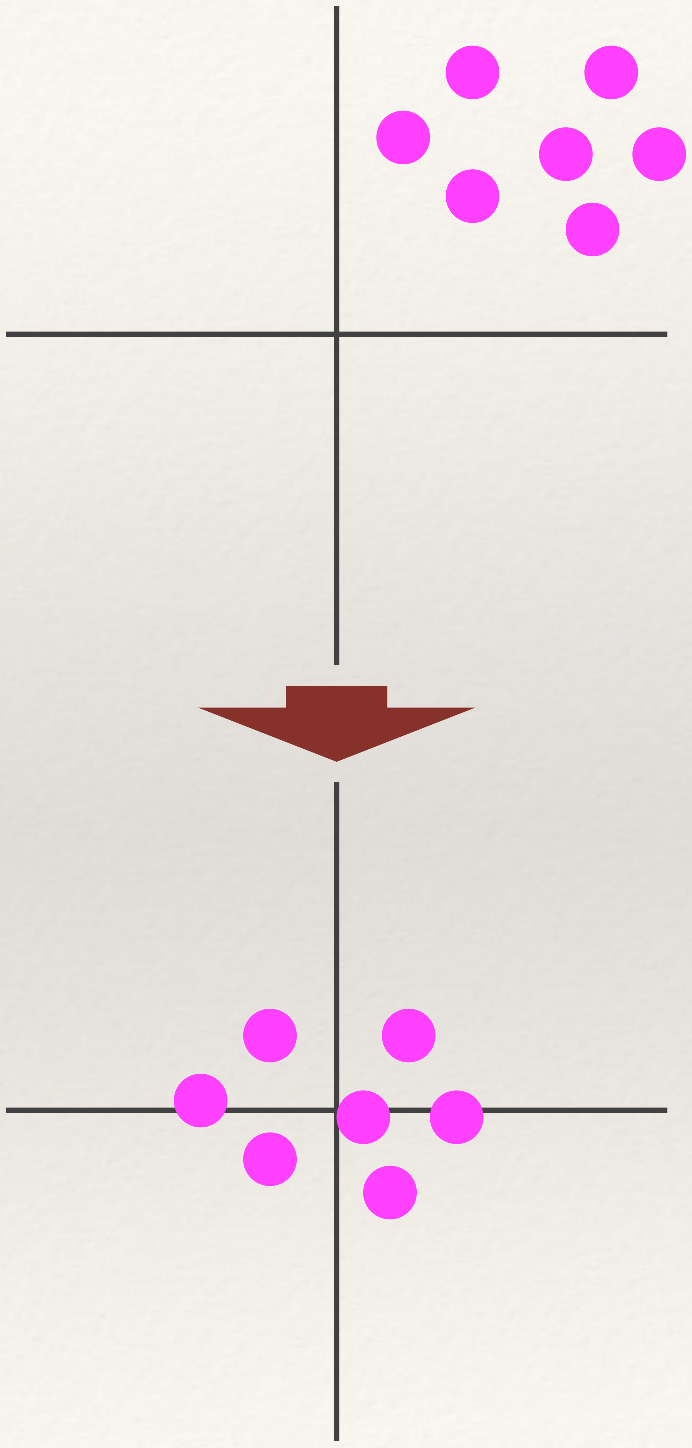
$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma(X_1, X_1) & \sigma(X_1, X_2) & \cdots & \sigma(X_1, X_n) \\ \sigma(X_2, X_1) & \sigma(X_2, X_2) & \cdots & \sigma(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(X_n, X_1) & \sigma(X_n, X_2) & \cdots & \sigma(X_n, X_n) \end{bmatrix}$$

The covariance matrix is a **square symmetric matrix**

# *Demo: 2D covariance*

# Mean Centring

- Mean Centring is the process of computing the mean (across each dimension independently) of a set of vectors, and then subtracting the mean vector from every vector in the set.
    - All the vectors will be translated so their average position is the origin



# Covariance matrix again

$$\mathbf{Z} = \begin{bmatrix} \boxed{\mathrm{v} \quad \mathrm{v} \quad \mathrm{v} \quad \cdots} \\ \mathrm{v} \quad \mathrm{v} \quad \mathrm{v} \quad \cdots \\ \mathrm{v} \quad \mathrm{v} \quad \mathrm{v} \quad \cdots \\ \mathrm{v} \quad \mathrm{v} \quad \mathrm{v} \quad \cdots \end{bmatrix}$$

Each row is a **mean centred** featurevector

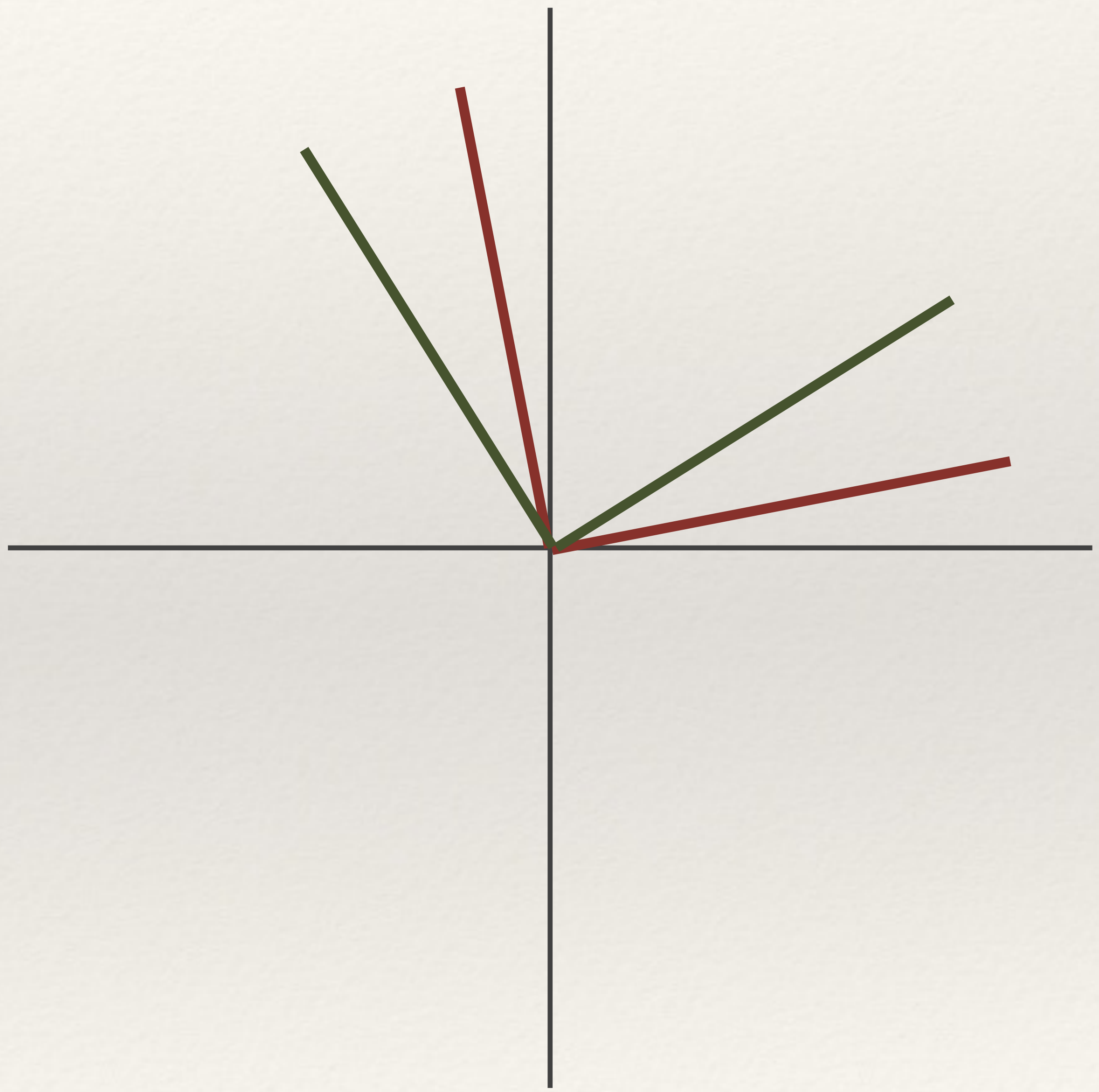
*Then*

$$\boldsymbol{\Sigma} \propto \mathbf{Z}^{\mathrm{T}}\mathbf{Z}$$

# Principal axes of variation

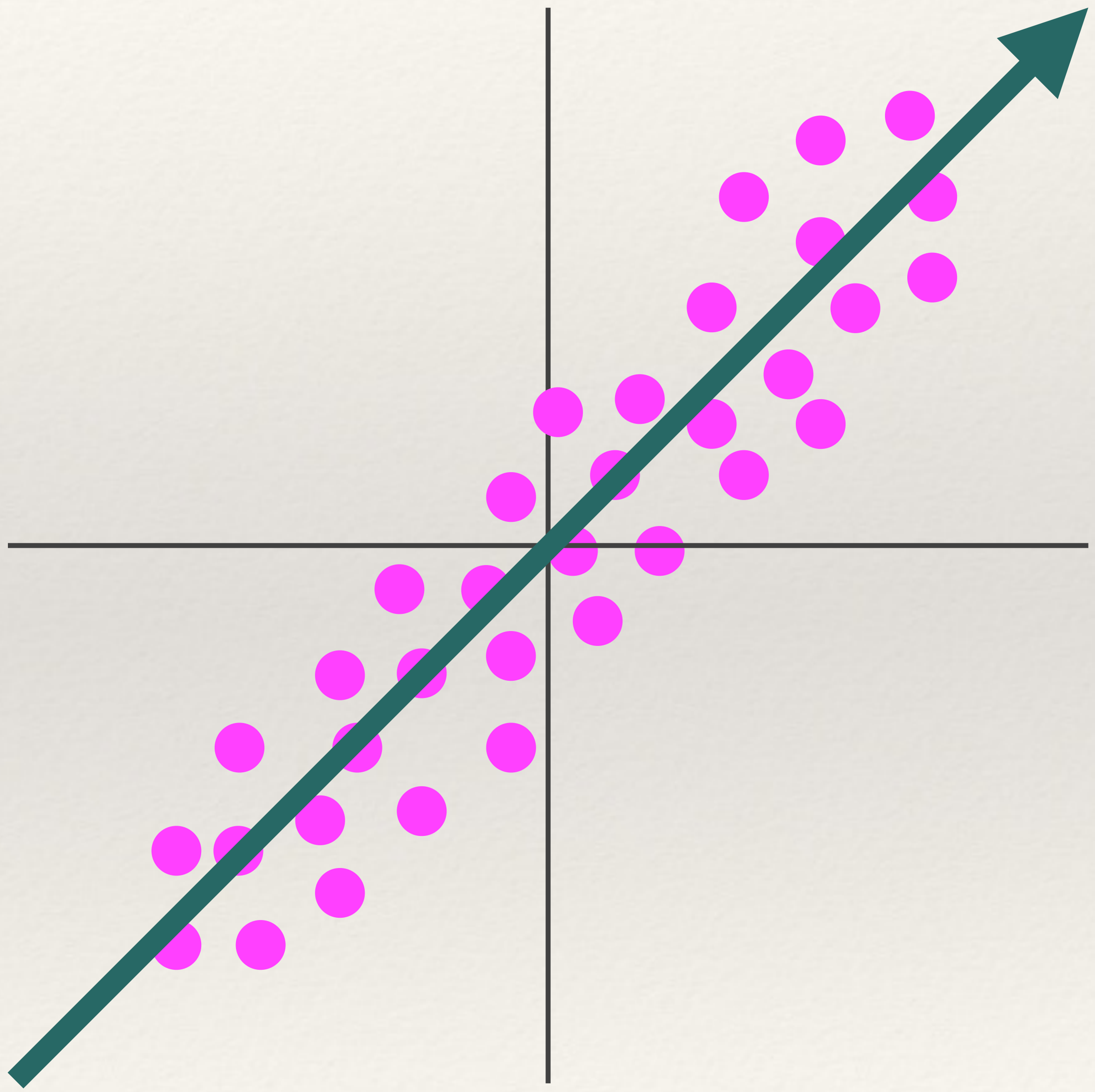
# Basis

- A basis is a set of *n* **linearly independent** vectors in an *n* dimensional space
  - The vectors are **orthogonal**
  - They form a "coordinate system"
  - There are an infinite number of possible basis



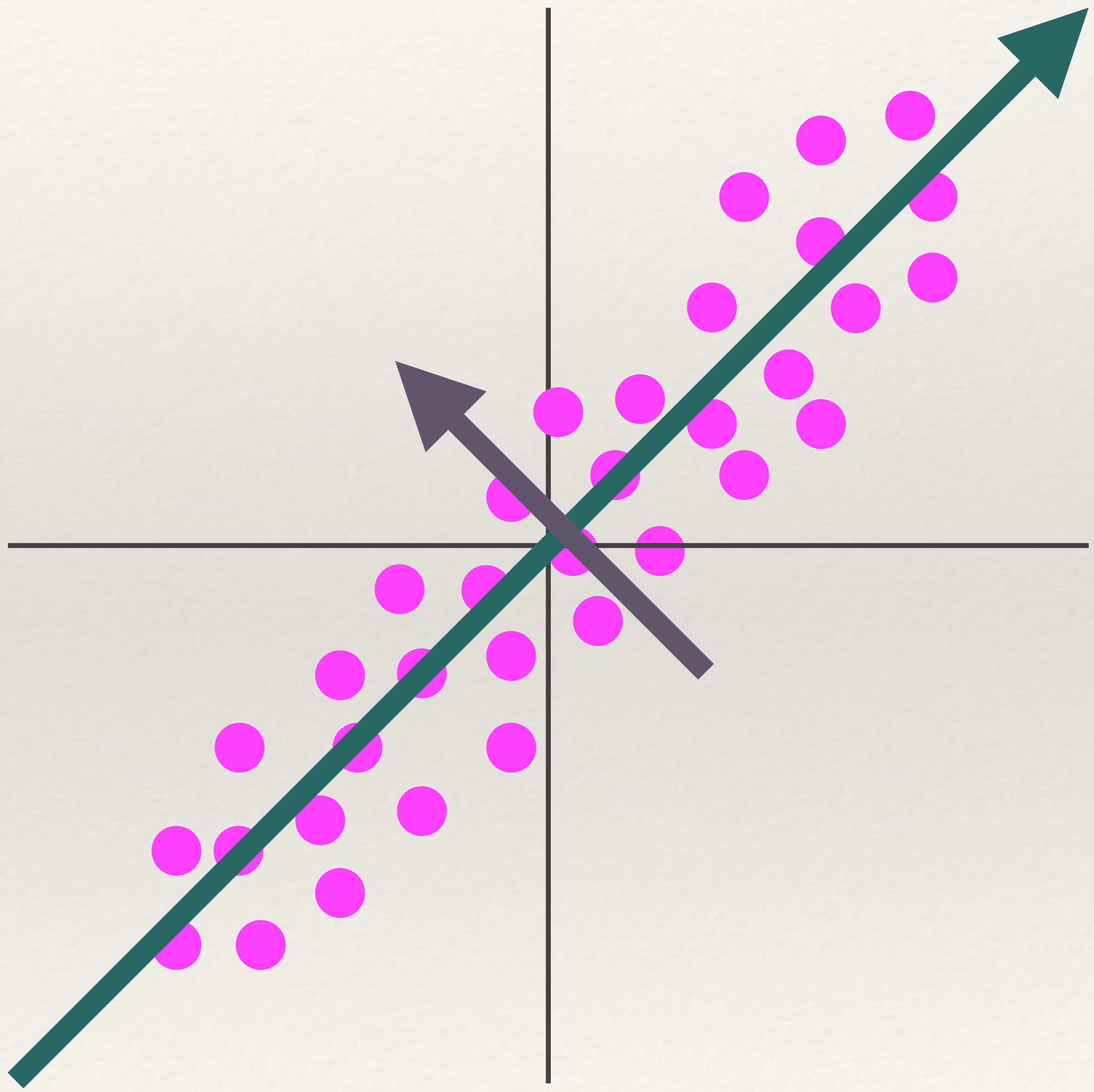
# The first principal axis

- For a given set of *n* dimensional data, the ***first principle axis*** (or just ***principal axis***) is the vector that describes the direction of **greatest** variance.



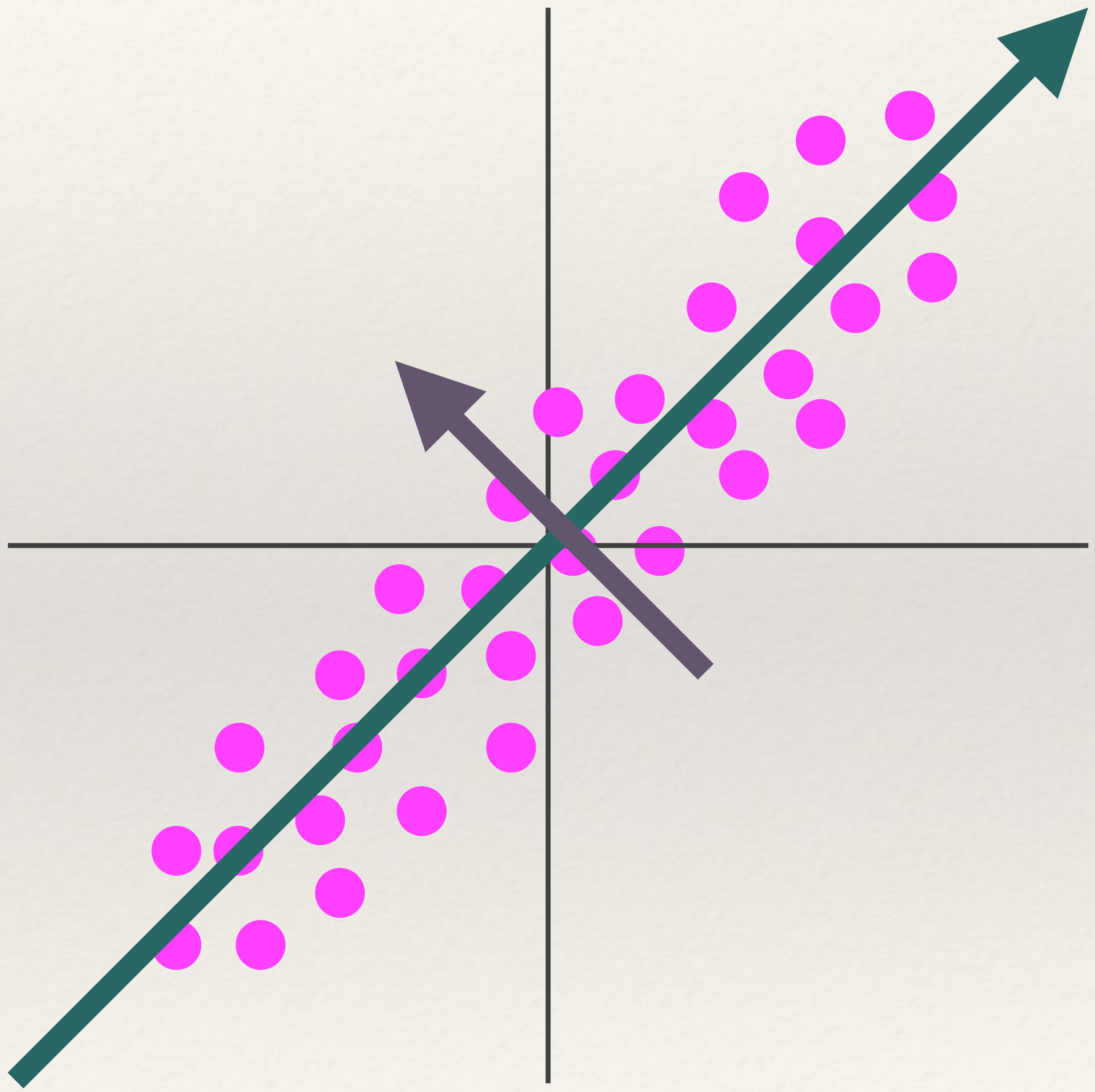
# The second principal axis

- The **second principal axis** is a vector in the direction of greatest variance orthogonal (perpendicular) to the first major axis.



# The third principal axis

- In a space with 3 or more dimensions, the third principal axis is the direction of greatest variance orthogonal to both the first and second principal axes.
  - The forth… … and so on…
  - The set of $n$ **principal axes** of an $n$ dimensional space are a **basis**



# Eigenvectors, Eigenvalues and Eigendecomposition

# Eigenvectors and Eigenvalues

**Very important equation!**

$$\mathbf{A}v = \lambda v$$

# Eigenvectors and Eigenvalues

a $n \times n$ square matrix

a scalar value,
known as an
**eigenvalue**

$$\boxed{\mathbf{A}}\boxed{v} = \boxed{\lambda}\boxed{v}$$

an $n$ dimensional vector,
known as an **eigenvector**

# Eigenvectors and Eigenvalues

$$\mathbf{A}v = \lambda v$$

There are at most $n$ eigenvector-eigenvalue pairs

If $\mathbf{A}$ is **symmetric**, then the set of eigenvectors is **orthogonal**

*Can you see where this is going?*

# Eigenvectors and Eigenvalues

$$\mathbf{A}v = \lambda v$$

If **A** is a **covariance matrix**, then the eigenvectors are the **principal axes**.

The eigenvalues are proportional to the **variance** of the data along each eigenvector.

The eigenvector corresponding to the **largest** eigenvalue is the first P.C.

# Finding the EVecs and EVals

- For small matrices ($n \leq 4$) there are algebraic solutions to finding all the eigenvector-eigenvalue pairs
- For larger matrices, numerical solutions to the **Eigendecomposition** must be sought.

# Eigendecomposition

columns of **Q** are the eigenvectors

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$$

diagonal eigenvalue matrix ($\boldsymbol{\Lambda}_{ii} = \lambda_i$)

# Eigendecomposition

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$$

If **A** is *real symmetric* (i.e. a covariance matrix), then $\mathbf{Q}^{-1} = \mathbf{Q}^{T}$ (i.e. eigenvectors are orthogonal), so:

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{T}$$

# Eigendecomposition

In summary, the Eigendecomposition of a covariance matrix $\mathbf{A}$:

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{\mathrm{T}}$$

Gives you the principal axes and their relative magnitudes

# Ordering

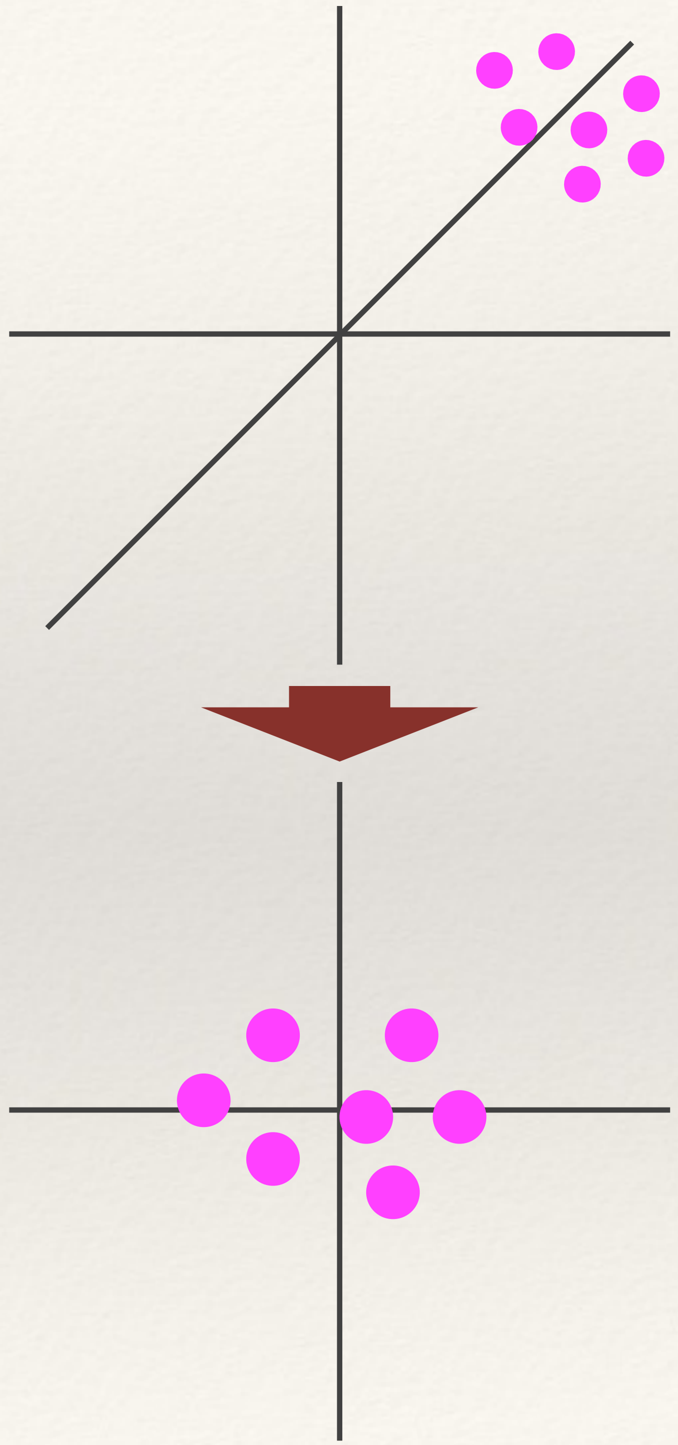
- Standard Eigendecomposition implementations will order the eigenvectors (columns of $\mathbf{Q}$) such that the eigenvalues (in the diagonal of $\mathbf{\Lambda}$) are sorted in order of decreasing value.
    - Some solvers are optimised to only find the top $k$ eigenvalues and corresponding eigenvectors, rather than all of them.

# *Demo: Covariance, Eigendecomposition and principal axes*

# Principal Component Analysis

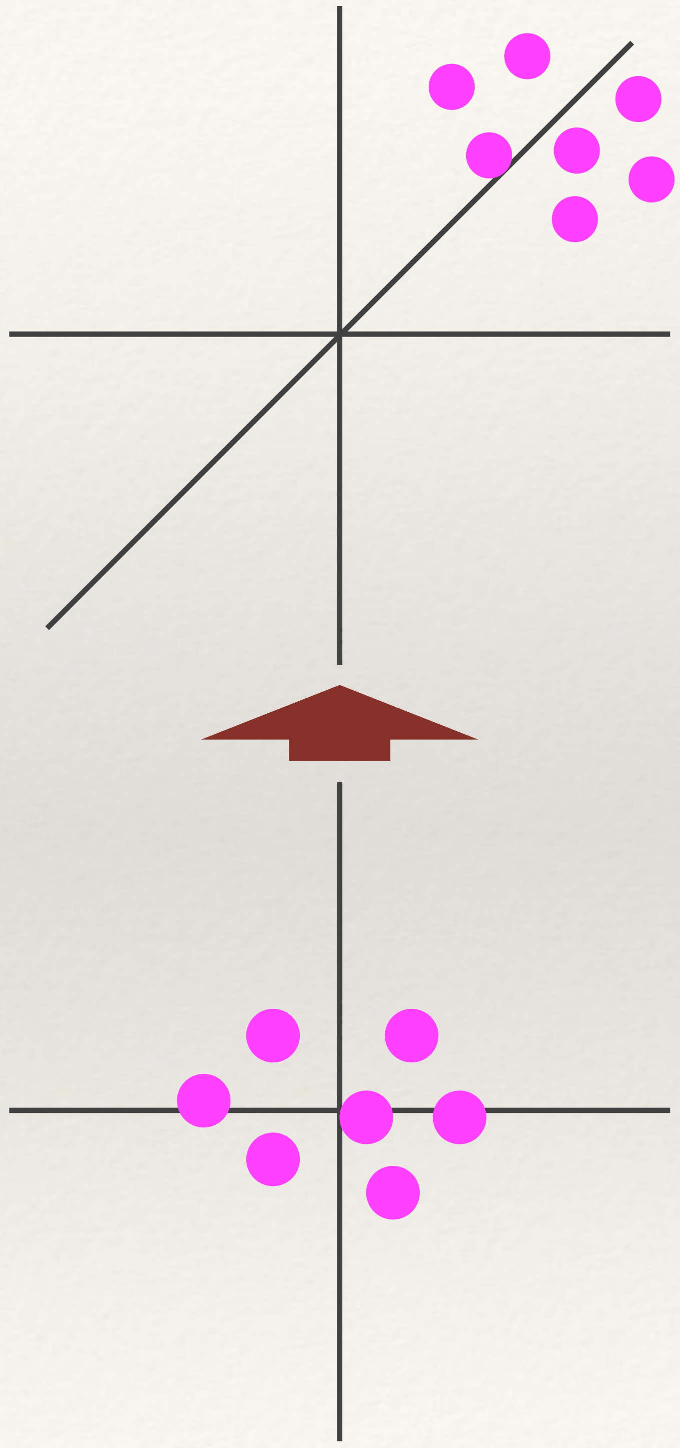
# Linear Transform

- A linear transform **W** projects data from one space into another:
  - **T** = **ZW**
    - Original data stored in the rows of **Z**
  - **T** can have fewer dimensions than **Z**.



# Linear Transforms

- The effects of a linear transform can be reversed if **W** is **invertible**:
  - $\mathbf{Z} = \mathbf{T}\mathbf{W}^{-1}$
    - A lossy process if the dimensionality of the spaces is different



# PCA

- PCA is an **Orthogonal Linear Transform** that maps data from its original space to a space defined by the principal axes of the data.
  - The transform matrix **W** is just the eigenvector matrix **Q** from the Eigendecomposition of the covariance matrix of the data.
  - Dimensionality reduction can be achieved by removing the eigenvectors with low eigenvalues from **Q** (i.e. keeping the first $L$ columns of **Q** assuming the eigenvectors are sorted by decreasing eigenvalue).

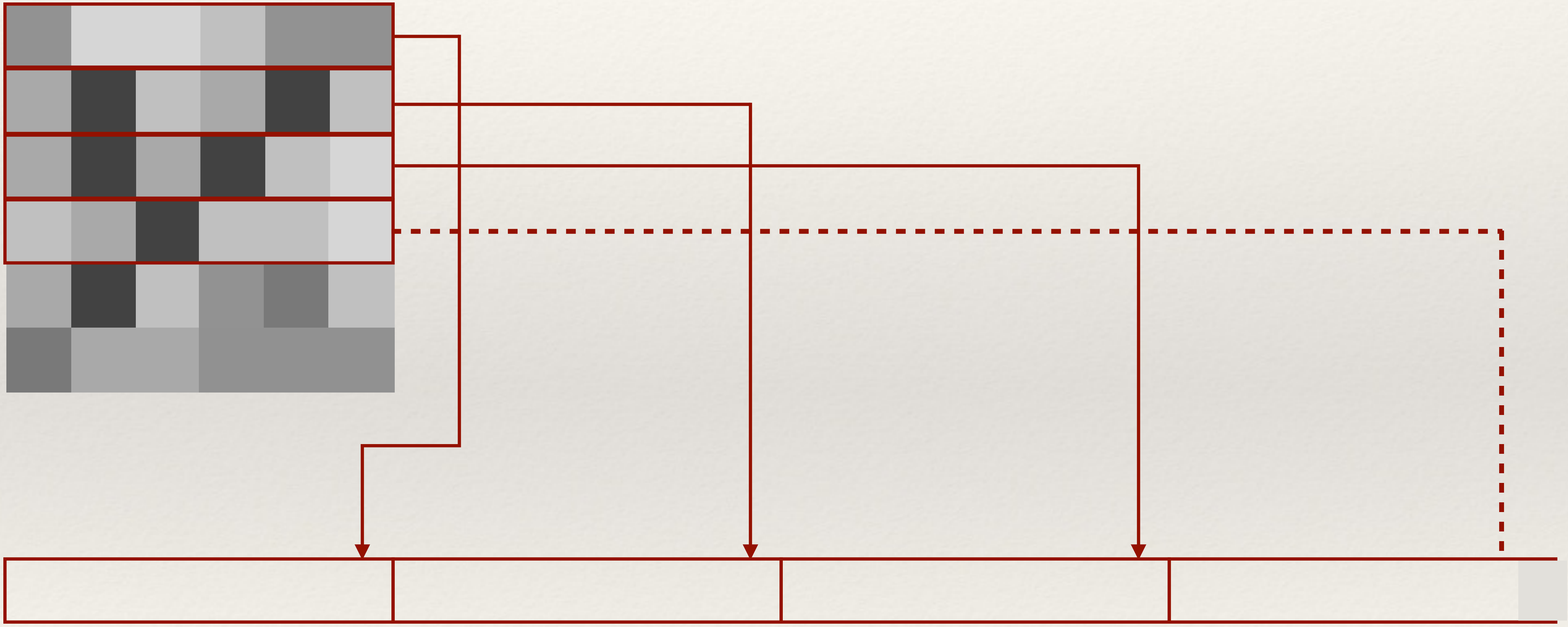
# PCA Algorithm

1. Mean-centre the data vectors
2. Form the vectors into a matrix $\mathbf{Z}$, such that each row corresponds to a vector
3. Perform the Eigendecomposition of the matrix $\mathbf{Z}^\mathsf{T}\mathbf{Z}$, to recover the eigenvector matrix $\mathbf{Q}$ and diagonal eigenvalue matrix $\mathbf{\Lambda}$: $\mathbf{Z}^\mathsf{T}\mathbf{Z} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^\mathsf{T}$
4. Sort the columns of $\mathbf{Q}$ and corresponding diagonal values of $\mathbf{\Lambda}$ so that the eigenvalues are decreasing.
5. Select the $L$ largest eigenvectors of $\mathbf{Q}$ (the first $L$ columns) to create the transform matrix $Q_L$.
6. Project the original vectors into a lower dimensional space, $\mathbf{T}_L$: $\mathbf{T}_L = \mathbf{Z}\mathbf{Q}_L$

# *Demo: PCA*

# Eigenfaces
# (Eigenimages)

# A simple kind of feature…



# … but with some problems …

- Not invariant to:
  - Change in the position/orientation/scale of the object in the image
  - Changes in lighting
  - Size of the image
- Highly susceptible to image noise

# Making it invariant

- Require (almost) the same object pose across images (i.e. full frontal faces)
- Align (rotate, scale and translate) the images so that the a common feature is in the same place (i.e. the eyes in a set of face images)
- Make all the aligned images the same size
- (optional) Normalise (or perhaps histogram equalise) the images so they are invariant to global intensity changes

# …but there is still a bit of a problem…

- The featurevectors are huge!
  - If the images are 100x200 pixels, the vector has 20000 dimensions
  - That's not really practical…
  - Also, the vectors are still highly susceptible to imaging noise and variations due to slight mis-alignments

# Potential solution… Apply PCA

- PCA can be used to reduce the dimensionality
  - smaller number of dimensions allows greater robustness to noise and mis-alignment
    - there are fewer degrees of freedom, so noise/mis-alignment has much less effect
    - and the dominant features are captured
  - Fewer dimensions makes applying machine-learning much more tractable.

# *Demo: a dataset of faces*

# *Demo: mean-face*

# *Demo: mean-centred faces*

# *Demo: principal components*

# *Demo: reconstructed faces*

# *Demo: reconstruction from weights*

# Summary

- Covariance measures the “shape” of data by measuring how different dimensions change together.
- The principle axes are a basis, aligned such they describe most directions of greatest variance.
- The Eigendecomposition of the covariance matrix produces pairs of eigenvectors (corresponding to the principal axes) and eigenvalues (proportional to the variance along the respective axis).
- PCA aligns data with its principal axes, and allows dimensionally reduction by discounting axes with low variance.
- Eigenfaces applies PCA to vectors made from pixel values to make robust low-dimensional image descriptors.