Differentiate your Objective

VLC = ∭ Vision Learning Control dVdLdC

Differentiable Programming

How does pre-university calculus relate to AI and the future of computer programming?

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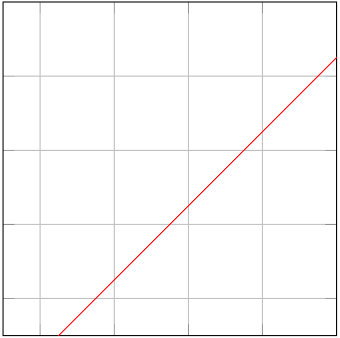
Differentiation

Recap: what is the derivative of a function of one variable?

The derivative in 1D

- Recall that the gradient of a straight line is $\frac{\mathrm{d}y}{\mathrm{d}x}$.



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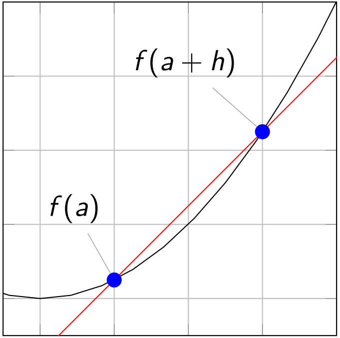
- For an arbitrary real-valued function, $f(a)$, we can approximate the derivative, $f'(a)$ using the gradient of the *secant line* defined by $(a, f(a))$ and a point a small distance, h, away $(a+h, f(a+h))$: $f'(a) \approx \frac{f(a+h)-f(a)}{h}$.



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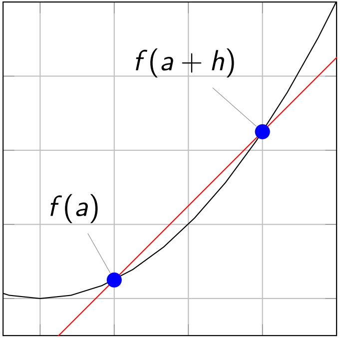

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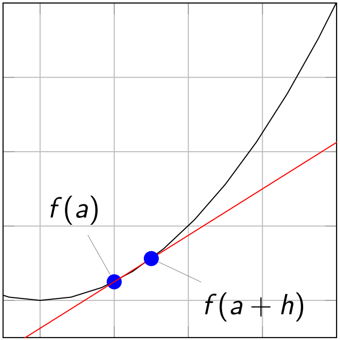

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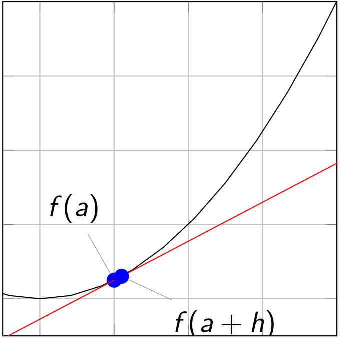
  - As h becomes smaller, the approximated derivative becomes more accurate.
  - If we take the limit as $h \to 0$, then we have an exact expression for the derivative: $\frac{df}{da} = f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.



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The derivative of $y = x^2$ from first principles

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Intuition: What does the gradient dy/dx tell us

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- The 'rate of change' of y with respect to x.
- By how much does y change if I make a small change to the x.

Why should we care?

Solving a simple problem with differentiation

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- Assume initial velocity $u = 28\,\mathrm{m\,s^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$
- Choose to ignore launch height as it is negligable compared to distance travelled.

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- Assume initial velocity $u = 28\,\mathrm{m\,s^{-1}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$
- Choose to ignore launch height as it is negligable compared to distance travelled.
- Kinematics equations:

$$x = ut\cos(\theta) = 28t\cos(\theta)$$
$$y = ut\sin(\theta) - 0.5gt^2 = 28t\sin(\theta) - 4.9t^2$$

Why should we care?

Solving a simple problem with differentiation

$$x = 28t\cos(\theta)$$
$$y = 28t\sin(\theta) - 4.9t^2$$

- Javelin hits ground when $y = 0$ and we only care about $t > 0$:

$$0 = 28t\sin(\theta) - 4.9t^2$$
$$\implies t = \frac{28}{4.9}\sin(\theta)$$

- Substituting into the horizontal component:

$$x = 28\frac{28}{4.9}\sin(\theta)\cos(\theta) = 80\sin(2\theta)$$

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Irrespective of the initial velocity maximum distance is acheived at 45°.

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  - **minimise** the loss with respect to the parameter(s)[1].
- Problems:
  - The loss must be differentiable (or rather you must be able to compute or estimate its gradient somehow).
  - Some loss functions might have many minima; you might have to settle for finding a sub-optimal one (or a saddle-point).
  - The loss function could be arbitrarily complex... you might not be able to analytically compute the solution (or the gradient).

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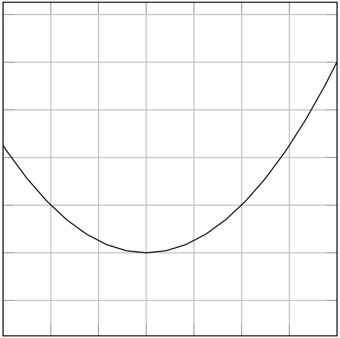
- How can you numerically estimate the value of the parameter that minimises a loss function, ℓ?



A simple algorithm for minimising a function

Gradient Decent

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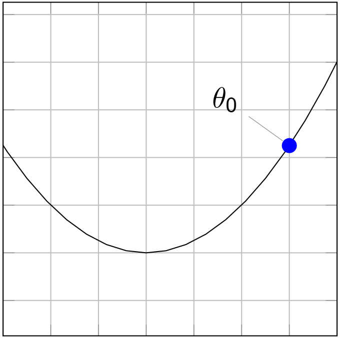
- Really intuitive idea: starting from an initial guess, θ_0, take small steps in the direction of the negative gradient.



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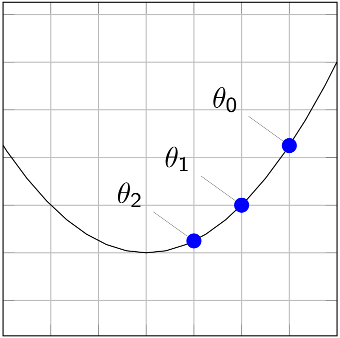
Gradient Descent:

$\theta_{i+1} = \theta_i - \gamma \frac{\mathrm{d}\ell}{\mathrm{d}\theta}$ where γ is the *learning rate*



Javelin throwing again, but with Python code

Derivatives of more general functions

- Almost all complex functions can be broken into simpler parts (often with very simple derivatives).
- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
  - The sum rule, product rule and chain rule tell you how to differentiate these.

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- You can add (or subtract) sub-functions, multiply (or divide) sub-functions and make functions of functions.
  - The sum rule, product rule and chain rule tell you how to differentiate these.
- If you break down functions into their constituent parts computing the derivative becomes very easy
- Example: the sin function can be written in terms of exponentials (Euler's formula) and the derivative of an exponential e^x is just e^x...

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  - In many real applications it can be *millions* of parameters.
- Partial derivatives $\frac{\partial f}{\partial x_i}$ let us compute the gradient of the *i*-th parameter by holding the other parameters constant.
- In general, the partial derivative of a function $f(x_1, \dots, x_n)$ at a point $(a_1, \dots, a_n)$ is given by: $\frac{\partial f}{\partial x_i}(a_1, \dots, a_n) = \lim_{h \to 0} \frac{f(a_1 \dots, a_i + h, \dots, a_n) - f(a_1 \dots, a_i, \dots, a_n)}{h}$.

Back to programming

Programming is really just function composition and control statements

- At the end of the day computer programs are just compositions of really simple functions that computer processors can compute: arithmetic operations (add, multiply, divide, ...), logical operations (and, or, not, comparisons...), operations that move data, etc.

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- Many of these primitive operations have *well defined* gradients with respect to their operands.
- The chain rule tells us how to compute gradients of composite functions.

So, in principle we can find the optimal "parameters" of a computer program designed to solve a specific task by following the gradients to optimise it.

Differentiating Branches

Code - *if-else* statement

```
if a > 0.5:
    b = 0
else:
    b = 2 * a
```

Math

$$b(a) = \begin{cases} 0 & \text{if } a > 0.5 \\ 2a & \text{if } a \leq 0.5 \end{cases}$$

$$\frac{\partial b}{\partial a} = \begin{cases} 0 & \text{if } a > 0.5 \\ 2 & \text{if } a \leq 0.5 \end{cases}$$

Differentiating Loops

Code - *for loop* statement

```
b = 1
for i in range(3):
    b = b + b * a
```

Math

$$
\begin{aligned}
b_0 &= 1 \\
b_1 &= b_0 + b_0 a = 1 + a \\
b_2 &= b_1 + b_1 a = 1 + 2a + a^2 \\
b_3 &= b_2 + b_2 a = 1 + 3a + 3a^2 + a^3 \\
\frac{\partial b}{\partial a} &= 3 + 6a + 3a^2
\end{aligned}
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- not every operation or function has *useful* gradients
  - discontinuities, large areas of zero-gradient, ...
- Computer science researchers are actively developing mathematical 'tricks' to circumvent many of these problems.
  - *Relaxations* of functions that behave almost the same, but have well defined gradients.
  - *Reparameterisations* of functions involving randomness.
  - *Approximations* of useable gradients for functions that have ill-posed gradients.

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Automatic Differentiation is key to differentiating programs with millions of parameters, but as the 'programmer' **you** still need to have a really good intuition and understanding of what the implications of composing many functions will be on the gradients of the parameters with respect to the loss for optimisation to work successfully.

What kinds of functional building blocks are common?

- Today, the most common operations with parameters are:
  - *Vector addition*: the input vector to a function is added to a vector of weights)
  - *Vector-Matrix multiplication*: the input vector to the function is multiplied with a matrix of weights
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- The above operations are *linear*, so they are often combined with element-wise nonlinearities; e.g.:
  - $\max(0, x)$ aka ReLU.
  - $\tanh(x)$.
  - $\frac{1}{1+e^{-x}}$ aka *sigmoid* or the *logistic* function.

Real Examples of Differentiable Programming

Playing Games

- You can use differentiable programming to write (and train) 'agents' that can play games.
- It can be hard to get a gradient from a single game involving many moves, but there is a clever trick which allows good estimates of gradients to be created over the average of *many* games.
- This is broadly the area of what is called *reinforcement learning*.

Demo: AlphaStar

Object detection

- Consider a function that takes an image as input and produces an array of *bounding boxes* and corresponding *labels*.
- With enough *training data* we can learn the parameters required to detect objects in images.

Face detection and recognition

- Consider a function that takes an image as input and produces an array of *bounding boxes*.
- With enough *training data* we can learn the parameters required to detect faces in images.
- With some clever training tricks (in the loss function) we can also make the function return a feature for each box that describes the content, such that pictures of the same person have very similar features.

Face detection and recognition
Demo

Drawing

- We could envisage a differentiable function that takes in a set of line coordinates and turns them into an image...
- With such a function we can optimise the line coordinates so they e.g. match a photograph, thus automatically creating a *sketch*.

Drawing

Demo

Drawing

Demo

Where is this all going?

Software 2.0

There is a revolution happening and you're going to be part of it

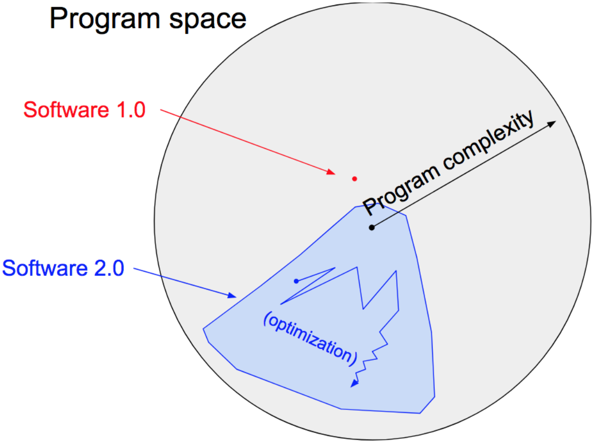

Image credit: Andrei Karpathy

https://karpathy.medium.com/software-2-0-a64152b37c35

Any Questions?