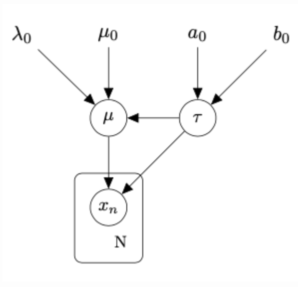
Assignment 1.3 - CAVI

Consider the model defined by Equation (10.21)-(10-23) in Bishop, for which DGM is presented below:



Question 1.3.12:

Implement a function that generates data points for the given model.

```
import numpy as np
def generate_data(mu, tau, N):
  '''
    Generate Gaussian distributed datasets of size N with
    mean mu and precision tau
  '''
  sigma = 1/np.sqrt(tau)
  X = np.random.normal(mu, sigma, N)
  return X
```

Set μ = 1, τ = 0.5 and generate datasets with size N=10,100,1000. Plot the histogram for each of 3 datasets you generated.

```
import matplotlib.pyplot as plt

mu = 1
tau = 0.5

dataset_1 = generate_data(mu, tau, 10)
dataset_2 = generate_data(mu, tau, 100)
dataset_3 = generate_data(mu, tau, 1000)

# Visulaize the datasets via histograms
def PlotHistogram(X):
  plt.hist(X, bins='auto', edgecolor='black')
  plt.xlabel('Value')
  plt.ylabel('Frequency')
  plt.title('Histogram of Dataset')

  plt.show()

PlotHistogram(dataset_1)
PlotHistogram(dataset_2)
PlotHistogram(dataset_3)
```

Question 1.3.13:

Find ML estimates of the variables μ and τ

```
def ML_est(data):
  mu_ml = np.mean(data)
  N = len(data)
  sum = 0
  for x in data:
    sum += (x-mu)**2
  sample_variance = sum / N
  tau_ml = 1/sample_variance

  return mu_ml, tau_ml

print(ML_est(dataset_1))
print(ML_est(dataset_2))
print(ML_est(dataset_3))
#mu_ml is mean of data
#tau_ml is reciprocal of sample variance
```

```
(1.2609804727425804, 0.375813674262337)
(1.077129400862253, 0.4750131798758166)
(1.0300946330314913, 0.4604777327481644)
```

Question 1.3.14:

You will implement the VI algorithm for the variational distribution in Equation (10.24) in Bishop. Start with introducing the prior parameters:

```
# prior parameters
mu_0 = 1.5
lambda_0 = 1
a_0 = 1
b_0 = 2
```

Continue with a helper function that computes ELBO:

```
# from scipy.stats import gamma, norm
import math
from scipy.special import digamma #, gamma
from math import gamma, log
def compute_elbo (D, a_0, b_0, mu_0, lambda_0, a_N , b_N, mu_N, lambda_
  # given the prior and posterior parameters together with the data,
  # compute ELBO here
  E_ln_tau = digamma(a_N) - np.log(b_N)
  E_tau = a_N / b_N
  E_mu = mu_N
  E_mu2 = mu_N**2 + 1 / lambda_N
  N = len(D)

  term1 = E_ln_tau * (N/2 + 1/2 + a_0 - 1)
  sum = 0
  for x in D:
    sum += x**2 - 2*x*E_mu + E_mu2

  term2 = -E_tau * (0.5 * sum + 0.5 * (E_mu2 - 2*mu_0*E_mu * mu_0**2) +

  term3 = -N/2 * np.log(2*math.pi) + a_0 * np.log(b_0) - log(gamma(a_0)

  term4 = a_N - np.log(b_N) + np.log(gamma(a_N)) + (1 - a_N) * digamma(

  elbo = term1 + term2 + term3 + term4

  return elbo
```

```
import numpy as np
from scipy.stats import norm, gamma

def compute_elbo2(D, a_0, b_0, mu_0, lambda_0, a_N , b_N, mu_N, lambda_

  prior_tau = gamma(a=a_0, scale=1/b_0)

  # Number of Monte Carlo samples
  num_samples = 100

  # Monte Carlo estimation of ELBO
  elbo_samples = []
  for _ in range(num_samples):
    log_likelihood = 0
    # Sample from the variational distribution
    sampled_mu = np.random.normal(mu_N, 1.0 /np.sqrt(lambda_N))
    sampled_tau = np.random.gamma(a_N, scale=1/b_N)
    prior_mu = norm(mu_0, 1/np.sqrt(lambda_0*sampled_tau))

    # Log-likelihood
    for x in D:
      log_likelihood += norm.logpdf(x, loc=sampled_mu, scale=np.sqrt(1.

    # Log-prior for mu and tau
    log_prior = prior_mu.logpdf(sampled_mu) + prior_tau.logpdf(sampled_

    # Log-variational posterior for mu and tau
    log_q = norm.logpdf(sampled_mu, loc=mu_N, scale=1.0 / np.sqrt(lambc

    # ELBO for this sample
    elbo_samples.append(log_likelihood + log_prior - log_q)

  # Average ELBO over samples
  elbo_samples = np.array(elbo_samples)
  return np.mean(elbo_samples)

  # print("Estimated ELBO:", elbo_estimate)
```

Now, implement the CAVI algorithm:

```
import numpy as np
import math
def CAVI(D, a_0, b_0, mu_0, lambda_0):
  delta_stop = 0.0001
  delta = math.inf
  initial_guess_exp_tau = a_0 / b_0
  elbos = []
  N = len(D)
  x_mean = np.mean(D)
  a_N = a_0
  b_N = b_0
  lambda_N = lambda_0

  # CAVI iterations ...
  # save ELBO for each iteration, plot them afterwards to show converge
  while delta > delta_stop:
    E_tau = a_N / b_N
    lambda_N_old = lambda_N
    mu_N = (lambda_0 * mu_0 + N * x_mean) / (lambda_0 + N)
    lambda_N = (lambda_0 + N) * E_tau

    elbos.append(compute_elbo2(D,a_0,b_0,mu_0,lambda_0,a_N,b_N,mu_N,lam

    E_mu = mu_N
    E_mu2 = 1/lambda_N + mu_N**2
    b_N_old = b_N
    a_N = a_0 + N/2
    sum = 0
    for x in D:
      sum += x**2 - 2*x*E_mu + E_mu2

    b_N = b_0 + 0.5 * sum + (lambda_0 / 2) * (E_mu2 - 2*mu_0*E_mu + mu_

    elbos.append(compute_elbo2(D,a_0,b_0,mu_0,lambda_0,a_N,b_N,mu_N,lam

    delta = max(math.fabs(lambda_N - lambda_N_old), math.fabs(b_N - b_N

  return a_N, b_N, mu_N, lambda_N, elbos
```

Question 1.3.15:

What is the exact posterior? First derive it in closed form, and then implement a function that computes it for the given parameters:

```
from scipy.special import gamma
import scipy
import mpmath

def compute_exact_posterior(D, a_0, b_0, mu_0, lambda_0, mu, tau):
  N = len(D)
  beta = b_0 + 0.5 * np.sum(D**2) + 0.5 * lambda_0 * mu_0**2 - ((lambda

  alpha = a_0 + N / 2
  l = N + lambda_0
  m = ((lambda_0 * mu_0) + np.sum(D)) / (N + lambda_0)

  # p = (beta**alpha * np.sqrt(l)) / (scipy.special.gamma(alpha) * np.s
  p = (mpmath.power(beta,alpha) * np.sqrt(l)) / (mpmath.gamma(alpha) *

  return p

def Gaussian(x, mu, var):
  q = 1/(np.sqrt(var * 2*math.pi)) * math.exp(-0.5*((x-mu)/np.sqrt(var)
  return q

def Gamma(x,a,b):
  # q = (x**(a-1)*math.exp(-b*x)*b**a) / scipy.special.gamma(a)
  q = (mpmath.power(x,(a-1))*mpmath.exp(-b*x)*mpmath.power(b,a)) / mpma
  return q

from scipy.stats import norm
import scipy
def computeCAVI(a_N, b_N, mu_N, lambda_N, mu, tau):
  # q = scipy.stats.gamma.pdf(tau, a_N, scale=b_N) * norm.pdf(mu, mu_N,
  q = Gaussian(mu,mu_N,1/lambda_N) * Gamma(tau,a_N, b_N)
  return q

def standardize(dist):
  magn = np.linalg.norm(dist)
  for i in range(dist.shape[0]):
    for j in range(dist.shape[1]):
      dist[i,j] /= magn
  return dist
```

Question 1.3.16:

Run the VI algorithm on the datasets. Compare the inferred variational distribution with the exact posterior and the ML estimate. Visualize the results and discuss your findings.

```
import matplotlib.pyplot as plt
from matplotlib.lines import Line2D

mu_ml, tau_ml = ML_est(dataset_2)
a_N, b_N, mu_N, lambda_N, elbos = CAVI(dataset_2, a_0, b_0, mu_0, lambc
plt.plot(elbos)
plt.xlabel('Iterations')
plt.ylabel('ELBO')
plt.title('ELBO vs Iterations')
plt.show()
mu_vect = np.linspace(0,2,100)
tau_vect = np.linspace(0.1,2,100)

q = np.zeros((100,100))
p = np.zeros((100,100))
for i in range(100):
  for j in range(100):
    q[i,j] = computeCAVI(a_N,b_N,mu_N,lambda_N,mu_vect[j], tau_vect[i])
    p[i,j] = compute_exact_posterior(dataset_2,a_0, b_0, mu_0, lambda_0

# q = standardize(q)
# p = standardize(p)

C1 = plt.contour(mu_vect, tau_vect, q, colors='blue')
C2 = plt.contour(mu_vect, tau_vect, p, colors='red')
plt.scatter(mu_ml, tau_ml, color='green', label='MLE')

h1,_ = C1.legend_elements()
h2,_ = C2.legend_elements()
h3 = Line2D([0], [0], marker='o', color='w', markerfacecolor='green', m
plt.legend([h1[0], h2[0], h3], ['variationally inferred posterior', 'ex

plt.xlabel("mu")
plt.ylabel("tau")
plt.title('Distributions for dataset 2')
plt.show()

print(a_N,b_N)
print(mu_N, lambda_N)
```

```
51.0 108.11122853504462
1.0813162384774782 47.64537479096254
```

```
import matplotlib.pyplot as plt
from matplotlib.lines import Line2D

mu_ml, tau_ml = ML_est(dataset_1)
a_N, b_N, mu_N, lambda_N, elbos = CAVI(dataset_1, a_0, b_0, mu_0, lambc
plt.plot(elbos)
plt.xlabel('Iterations')
plt.ylabel('ELBO')
plt.title('ELBO vs Iterations')
plt.show()
mu_vect = np.linspace(0,2,100)
tau_vect = np.linspace(0.1,2,100)

q = np.zeros((100,100))
p = np.zeros((100,100))
for i in range(100):
  for j in range(100):
    q[i,j] = computeCAVI(a_N,b_N,mu_N,lambda_N,mu_vect[j], tau_vect[i])
    p[i,j] = compute_exact_posterior(dataset_1,a_0, b_0, mu_0, lambda_0

# q = standardize(q)
# p = standardize(p)

C1 = plt.contour(mu_vect, tau_vect, q, colors='blue')
C2 = plt.contour(mu_vect, tau_vect, p, colors='red')
plt.scatter(mu_ml, tau_ml, color='green', label='MLE')

h1,_ = C1.legend_elements()
h2,_ = C2.legend_elements()
h3 = Line2D([0], [0], marker='o', color='w', markerfacecolor='green', m
plt.legend([h1[0], h2[0], h3], ['variationally inferred posterior', 'ex

plt.xlabel("mu")
plt.ylabel("tau")
plt.title('Distributions for dataset 1')
plt.show()

print(a_N,b_N)
print(mu_N, lambda_N)
```

```
6.0 16.35259450089353
1.2827095206750732 4.036060743286496
```

```
import matplotlib.pyplot as plt
from matplotlib.lines import Line2D

mu_ml, tau_ml = ML_est(dataset_3)
a_N, b_N, mu_N, lambda_N, elbos = CAVI(dataset_3, a_0, b_0, mu_0, lambc
plt.plot(elbos)
plt.xlabel('Iterations')
plt.ylabel('ELBO')
plt.title('ELBO vs Iterations')
plt.show()
mu_vect = np.linspace(0,2,100)
tau_vect = np.linspace(0.1,2,100)

q = np.zeros((100,100))
p = np.zeros((100,100))
for i in range(100):
  for j in range(100):
    q[i,j] = computeCAVI(a_N,b_N,mu_N,lambda_N,mu_vect[j], tau_vect[i])
    p[i,j] = compute_exact_posterior(dataset_3,a_0, b_0, mu_0, lambda_0

# q = standardize(q)
# p = standardize(p)

C1 = plt.contour(mu_vect, tau_vect, q, colors='blue')
C2 = plt.contour(mu_vect, tau_vect, p, colors='red')
plt.scatter(mu_ml, tau_ml, color='green', label='MLE')

h1,_ = C1.legend_elements()
h2,_ = C2.legend_elements()
h3 = Line2D([0], [0], marker='o', color='w', markerfacecolor='green', m
plt.legend([h1[0], h2[0], h3], ['variationally inferred posterior', 'ex

plt.xlabel("mu")
plt.ylabel("tau")
plt.title('Distributions for dataset 3')
plt.show()

print(a_N,b_N)
print(mu_N, lambda_N)
```

```
501.0 1088.5726864468677
1.0305640689625286 460.69592438197066
```