Fall 2024 MATH33A Worksheet 3: Sections 2.3, 2.4

Exercise 1. Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$.

(a) Compute A^{-1}.

(b) Use the inverse to find all solutions to $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, and all solutions to $A\vec{x} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$.

(a) $A^{-1} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix}$.

(b) $A\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ implies $\vec{x} = A^{-1} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$,

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ -29 \\ 9 \end{bmatrix}$$

Thus the only solution is $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Similarly for the second equation, we find that

$$\vec{x} = \begin{bmatrix} 2 & -2 & 3 \\ -3 & 4 & -6 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

is the only solution.

Exercise 2. Show that the following subsets are *not* subspaces of $\mathbb{R}^2$:

(a) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

(b) $V = \left\{ \begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} \mid s \in \mathbb{R} \right\}$

Show that the following subsets *are* subspaces of $\mathbb{R}^2$:

(c) $V = \left\{ \begin{bmatrix} t \\ 3s \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$.

(d) $V = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$

To show a subset is *not* a subspace, we need to find $u, v \in V$ such that $u + v \notin V$ (i.e., V is not *closed under addition*), or some $u \in V$ and a scalar $c \in \mathbb{R}$ such that $c\dot{u} \notin V$ (i.e., V is not *closed under scalar multiplication)*.

(a) Let $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, which are both in V. Then $u + v = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ is not an element of V, so V is not a subspace.

(b) Let $u = \begin{bmatrix} 3+1 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 7 \\ 0 \end{bmatrix} \in V$, so $u + v = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$. Let us show that $u + v$ is not in V, so there does not exist $s \in \mathbb{R}$ such that $\begin{bmatrix} 3s+1 \\ 2-s \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \end{bmatrix}$. In other words, we aim to show that there are no solutions to the linear system

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} s \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

Thus performing augmented row reduction, we have

$$\left(\begin{array}{c|c} 3 & 10 \\ -1 & -1 \end{array} \right) \Rightarrow^{\text{swap and multiply by } -1} \left(\begin{array}{c|c} 1 & 1 \\ 3 & 10 \end{array} \right) \Rightarrow \left(\begin{array}{c|c} 1 & 1 \\ 0 & 7 \end{array} \right)$$

Thus, there are no solutions to $\begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} s \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$, so $\begin{bmatrix} 11 \\ 1 \end{bmatrix}$ is not in V, so V is not a subspace.

To show subsets are subspaces of $\mathbb{R}^2$, we have to show that for **all** $u, v \in V$ that $u+v$ is in V (V is *closed under addition*), and that for all $v \in V, c \in \mathbb{R}$, that $c \cdot v \in V$ (V is *closed under scalar multiplication*).

(c) Let u, v be two elements of V, so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$ and $v = \begin{bmatrix} t_2 \\ 3s_2 \end{bmatrix}$ for some $t_2, s_2 \in \mathbb{R}$. Then, $u + v = \begin{bmatrix} t_1 + t_2 \\ 3s_1 + 3s_2 \end{bmatrix}$. In particular, $u + v = \begin{bmatrix} t \\ 3s \end{bmatrix}$ for $t = t_1 + t_2, s = s_1 + s_2$, so $u + v \in V$. Now let u be an element of V, so $u = \begin{bmatrix} t_1 \\ 3s_1 \end{bmatrix}$ for some $t_1, s_1 \in \mathbb{R}$, and let $c \in \mathbb{R}$ be arbitrary. Then $c \cdot u = \begin{bmatrix} ct_1 \\ 3cs_1 \end{bmatrix} = \begin{bmatrix} t \\ 3s \end{bmatrix}$ for $t = ct_1, s = 3s_1$, so $c \cdot u \in V$.

(d) Let u, v be two elements of V. Since V only has a single vector, we must have $u = v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $u + v = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under addition. Now let $u \in V$, so $u = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and let $c \in \mathbb{R}$ arbitrary. Then $c \cdot u = \begin{bmatrix} c \cdot 0 \\ c \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in V$, so V is closed under scalar multiplication.

Exercise 3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of projection onto the line $y = x$. Is T invertible? Argue both (1) geometrically, and (2) algebraically by finding the matrix representation for T and computing its determinant.

Geometrically: Since T is projection onto the line $y = x$, for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, T cannot have an inverse since it is not one-to-one/injective.

Algebraically: Since T is projection onto $y = x$, for every point (a, b) on the line $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Similarly for every point (a, b) on the line $y = -x$ perpendicular to $y = x$, $T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Using linearity of T, we have:

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{2} T \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Similarly,

$$T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = T\left(\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \frac{1}{2} T \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2} T \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Therefore, T is represented by the matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Since $\det A = 0$, T is not invertible.

Exercise 4. Let A be an $m \times n$ matrix, and let B be an **invertible** $n \times n$ matrix.

(a) Suppose that for all $\vec{b} \in \mathbb{R}^m$, $Ax = \vec{b}$ has a solution. What does this tell you about the image of A?

(b) How many solutions are there to $Bx = \vec{b}$ for any $\vec{b} \in \mathbb{R}^n$?

(c) What is the image of B (hint: does $Bx = \vec{b}$ always have a solution?)

(d) **(Challenge)** If B is invertible, what is $\mathrm{Im}(AB)$ in terms of the images of B, A? If C is an invertible $m \times m$ matrix, can you answer the same question for $\mathrm{Im}(CA)$?

(a) Let $\vec{b} \in \mathbb{R}^m$ be any vector. Since $Ax = \vec{b}$ has a solution, $\vec{b}$ is in the image of A. Therefore, $\mathrm{Im}A$ is all of $\mathbb{R}^m$ since it contains every element of $\mathbb{R}^m$.

(b) There is *exactly one* solution since B is invertible, given by $x = B^{-1}\vec{b}$.

(c) By part (a), (b), $\mathrm{Im}B = \mathbb{R}^n$.

(d) $\mathrm{Im}(AB)$ is the set of all the vectors in $\mathbb{R}^m$ of the form ABx for $x \in \mathbb{R}^n$. $\mathrm{Im}(A)$ is the set of all the vectors in $\mathbb{R}^m$ of the form Ay for $y \in \mathbb{R}^n$. Therefore, every element of $\mathrm{Im}(AB)$ is also an element of $\mathrm{Im}(A)$ since $ABx = Ay$ for $y = Bx \in \mathbb{R}^m$, i.e., $\mathrm{Im}(AB)$ is a subset of $\mathrm{Im}(A)$. Now take any element Ay of $\mathrm{Im}(A)$. Since B is invertible, there is a solution x to $Bx = y$. Thus, $Ay = ABx$, so $Ay \in \mathrm{Im}(AB)$. Therefore, $\mathrm{Im}(A)$ is a subset of $\mathrm{Im}(AB)$, so since both sets are subsets of the other they are equal: $\mathrm{Im}(A) = \mathrm{Im}(AB)$.

There is no characterization of $\mathrm{Im}(CA)$ in terms of $\mathrm{Im}(A)$. For instance, let $A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then if $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\mathrm{Im}(CA) = \mathrm{Im}(A)$ is the span of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. But if $C = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (which is invertible), then $\mathrm{Im}(CA)$ is the span of $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so the image depends on the choice of C.

Exercise 5. Find the inverse of the following matrix

$$A = \begin{bmatrix} 1 & c & c^3 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

in terms of $c \in \mathbb{R}$. Verify your answer with matrix multiplication.

$$A^{-1} = \begin{bmatrix} 1 & -c & c^2 - c^3 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$