Fall 2024 MATH33A Midterm 1 Review

Exercise 1. Re-write the following problems as augmented matrices representing linear systems:

(a) Solving the equation

$$-4x_1 + 3x_2 - 5x_4 = 3$$
$$x_1 - x_3 - x_4 = 7$$
$$-2x_1 + 3x_2 - 2x_3 - 7x_4 = 3$$

(b) Finding degree-2 polynomials $p(x) = ax^2 + bx + c$ such that $p(-2) = 9, p'(1) = 5$.

(a) $\left[\begin{array}{cccc|c} -4 & 3 & 0 & -5 & 3 \\ 1 & 0 & -1 & -1 & 7 \\ -2 & 3 & -2 & -7 & 3 \end{array}\right]$

(b) Plugging in the values we are given, we get

$$p(-2) = a(-2)^2 + b(-2) + c = 4a - 2b + c = 9$$
$$p'(1) = 2a \cdot 1 + b = 5$$

If we write the system in variables a, b, c (in that order), the matrix is $\left[\begin{array}{ccc|c} 4 & -2 & 1 & 9 \\ 2 & 1 & 0 & 5 \end{array}\right]$

Exercise 2.

(a) List the rules for a matrix to be in reduced row echelon form.

(b) Use row reduction to solve the linear systems that you created augmented matrices for in Problem 1.

(a)
  (a) **<u>If</u>** a row has any nonzero entries, the **<u>first</u>** nonzero entry is a 1
  (b) If a **<u>column</u>** has a leading 1, all other entries in the column are 0
  (c) If a row contains a **<u>leading 1</u>**, then each row above it also contains a leading 1 and that leading 1 is further to the left. In particular, rows of all 0's are at the bottom of the matrix.

(b) The first matrix row reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -\frac{4}{3} & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

The last row corresponds to the equation $0 = 1$, so the system is inconsistent and has no solutions.

The second matrix row reduces to $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{8} & \frac{19}{8} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array}\right]$ giving solutions

$$\left\{ \begin{bmatrix} \frac{19}{8} - \frac{1}{8}t \\ \frac{1}{4} + \frac{1}{4}t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} \longrightarrow \boxed{p(x) = \left(\frac{19}{8} - \frac{1}{8}t\right) x^2 + \left(\frac{1}{4} + \frac{1}{4}t\right) x + t}$$

Exercise 3.

(a) Define what the rank of a matrix A is.

(b) What is the rank of $A = \begin{bmatrix} a & a^2 \\ a^3 & a^4 \end{bmatrix}$ in terms of a?

(c) What possible ranks can a 2×3 matrix A have? What form will the RREF's have for each possible rank?

(a) The **rank** of a matrix A is the number of leading ones that appear in the reduced row echelon form of A. In other words: to find the rank of a matrix, row reduce it and count the number of leading 1s.

(b) If $a = 0$, the matrix has rank 0. Otherwise, we can divide rows by a and so A row reduces to $\begin{bmatrix} 1 & a \\ 0 & 0 \end{bmatrix}$, which has rank 1.

(c) There are only two rows, so A could have only 0, 1, or 2 as its rank. Referring back to the rules for RREF, if A has rank 0, $\text{rref}(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. If A has rank 1, $\text{rref}(A) = \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. If A has rank 2, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Throughout, $*$ means any real number could go in that spot.

Exercise 4.

(a) Define what a linear transformation is.

(b) If we know what a linear transformation does geometrically, how can we find the matrix of the linear transformation.

(c) Using this table if necessary for this part, find the matrix A representing reflection across the line $y = \frac{4}{3}x$ and the matrix B representing rotation by $\frac{4\pi}{3}$ (240 degrees). Describe A^{11}, B^{11} geometrically and what the entries of each are. Compute ABA algebraically and use your answer to describe what type of geometric transformation it represents.

(a) A **linear transformation** is a function $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A. An equivalent definition is that T is a function which satisfies the two conditions: $T(c\vec{x}) = cT(\vec{x})$ for any scalar $c \in \mathbb{R}$ and vector $\vec{x}$, and that $T(\vec{x}_1 + \vec{x}_2) = T(\vec{x}_1) + T(\vec{x}_2)$ for any two vectors $\vec{x}_1, \vec{x}_2$.

(b) If T is a linear transformation and A is the matrix of T, the kth column of A is given by $T(\vec{e}_k)$, where $\vec{e}_k$ is the column vector with a 1 in position n and 0 everywhere else. That's because

$$T\left(\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}\right) = A\begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} = \vec{a}_k$$

(c) The line $y = \frac{4}{3}x$ is spanned by unit vector $\begin{bmatrix}3/5\\ 4/5\end{bmatrix}$, so

$$\mathrm{ref}_\ell(\vec{x}) = 2(\vec{x}\cdot\begin{bmatrix}3/5\\ 4/5\end{bmatrix})\begin{bmatrix}3/5\\ 4/5\end{bmatrix} - \vec{x} \implies A = \begin{bmatrix}-7/25 & 24/25\\ 24/25 & 7/25\end{bmatrix}$$

$$B = \begin{bmatrix}\cos(\frac{4\pi}{3}) & -\sin(\frac{4\pi}{3})\\ \sin(\frac{4\pi}{3}) & \cos(\frac{4\pi}{3}))\end{bmatrix} = \begin{bmatrix}-1/2 & \sqrt{3}/2\\ -\sqrt{3}/2 & -1/2\end{bmatrix}$$

For any reflection, $A^2 = I$, so $A^{11} = (A^2)^5 A = I^5 A = A$. On the other hand, B^{11} is rotation by $\frac{4\pi}{3}$ 11 times, so it will just have the matrix for rotation by $\frac{11\cdot 4\pi}{3} = \frac{44\pi}{3}$, which is the same as rotation by $\frac{2\pi}{3}$ (remember that rotation by 2π is the identity). So $B^{11} = \begin{bmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{bmatrix}$.

Just by multiplying together, we see

$$ABA = \begin{bmatrix}-1/2 & -\sqrt{3}/2\\ \sqrt{3}/2 & -1/2\end{bmatrix}$$

Therefore (by looking at the provided table, for instance) ABA is a rotation. Specifically, it is rotation by $\frac{2\pi}{3}$.

Exercise 5.

(a) Let A be an $m \times n$ matrix and B be a $p \times q$ matrix. What conditions are needed on m, n, p, q for the product AB to be defined? If the product is defined, what will be the dimensions of the matrix AB?

(b) If $T : \mathbb{R}^p \to \mathbb{R}^s$ is a linear transformation, how many rows and columns does the matrix A representing T have?

(c) Recall that square matrices A and B commute if $AB = BA$ (this is not true in general for square matrices). Turn "the set of matrices commuting with $\begin{bmatrix}1 & \pi\\ 0 & 1\end{bmatrix}$" into a system of linear equations.

(a) The product AB is defined if $n = p$. The resulting matrix will be an $m \times q$ matrix.

(b) The matrix of T will be an $s \times p$ matrix.

(c) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix and we want to check if A commutes with the given matrix.

$$A \begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & \pi a + b \\ c & \pi c + d \end{bmatrix}$$

$$\begin{bmatrix} 1 & \pi \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} a + \pi c & b + \pi d \\ c & d \end{bmatrix}$$

So the system of equations is

$$\begin{aligned} a &= a + \pi c \\ c &= c \\ \pi a + b &= b + \pi d \\ \pi c + d &= d \end{aligned}$$

These equations reduce to

$$\begin{aligned} c &= 0 \\ a &= d \end{aligned}$$

Exercise 6.

(a) What does it mean for an $m \times n$ matrix A to be invertible?

(b) How might you check that A is invertible and produce its inverse?

(c) If $n \times n$ matrices A, T satisfy $AT = I_n$, is A invertible? Is T invertible?

(d) For what values of k is $A = \begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ invertible? Give its inverse for those values. (Adapted from a textbook problem)

(a) It means that for any $\vec{b}$ in $\mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution and that solution is unique. This also forces m to equal n.

(b) If $m \neq n$ then A is not invertible. If $m = n$, then A is invertible exactly when $\text{rref}(A) = I_n$. In that case, the inverse of A will be the matrix you see on the right when row-reducing the augmentation of A by the identity: $\begin{bmatrix} A \mid I \end{bmatrix} \to \begin{bmatrix} I \mid A^{-1} \end{bmatrix}$.

(c) In this case, both A and T are invertible and they are each other's inverses ($A^{-1} = T, T^{-1} = A$). This is proved at the top of Page 93 of the textbook.

(d) If you wanted you can do this by determinants and see that it's invertible when $0 \neq k(k-6) - 5(-2) = k^2 - 6k + 10$. This has only imaginary zeros (use the quadratic formula), so A is always invertible for real k. The inverse is the right side of the row reduction of $\begin{bmatrix} A \mid I_2 \end{bmatrix}$, which is

$$A^{-1} = \begin{bmatrix} (k-6)/(k^2 - 6k + 10) & 2/(k^2 - 6k + 10) \\ (-5)/(k^2 - 6k + 10) & k/(k^2 - 6k + 10) \end{bmatrix}$$

Exercise 7.

(a) Let A be an $m \times n$ matrix. What is the kernel of A? What is the image of A?

(b) Give a 3×2 matrix A whose image includes $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$.

(c) Find the kernel of $A = \begin{bmatrix} a & a^2 \\ a^3 & a^4 \end{bmatrix}$.

(a) The kernel of A is the solutions $\vec{x}$ <u>in $\mathbb{R}^n$</u> to $A\vec{x} = 0$. The image is the $\vec{b}$ <u>in $\mathbb{R}^m$</u> such that <u>there exists some solution to $A\vec{x} = \vec{b}$</u>.

(b) Notice that $\begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, the image of

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 8 & 1 \end{bmatrix}$$

includes all three vectors (other answers work too). In particular,

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}, A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

(c) The kernel is $\{\begin{bmatrix} -at \\ t \end{bmatrix} : t \in \mathbb{R}\}$ if $a \neq 0$ and all of $\mathbb{R}^2$ if $a = 0$.

Exercise 8.

(a) Consider a set of vectors $S = \{\vec{v}_1, \dots, \vec{v}_n\}$. What does it mean for $\vec{b}$ to be in the span of S (or a linear combination of vectors in S) and how can you check that? What does it mean for S to be linearly independent and how can you check that?

(b) What is a subspace of $\mathbb{R}^n$? Show that the set of vectors S in $\mathbb{R}^3$ whose coordinates add to 0 is a subspace and find a linearly independent set of vectors in S that spans S (i.e. a basis of S).

(c) Give a 3×2 matrix A whose image is the plane $5x + 3y - z = 0$. Show that the columns of A are linearly independent but the rows aren't.

(a) A vector $\vec{b}$ is in the span of S (or a linear combination of vectors in S) if there exists some $a_1, \dots, a_n$ such that

$$\vec{b} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

Another way to write that is there is a solution $\vec{a}$ to

$$\vec{b} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{a}$$

Therefore, you can check if $\vec{b}$ is in the span of S by seeing if this system has any solutions for $\vec{a}$ (by row reducing the augmented matrix).

S is linearly independent if there are no redundant vectors, meaning none of the v_i is a linear combination of the rest of the vectors in S. You can check this condition by checking that the only solution to

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix} \vec{x} = \vec{0}$$

is $\vec{x} = 0$ (by row reducing the matrix augmented by $\vec{0}$).

(b) A subset S of $\mathbb{R}^n$ is a subspace if for any $\vec{v}, \vec{w}$ in S we have $\vec{v} + \vec{w}$ is in S, and for any c in $\mathbb{R}$ and $\vec{v}$ in S we have $c\vec{v}$ is in S. The set S described in the question is a subspace because if $\vec{v}, \vec{w} \in S, c \in \mathbb{R}$ then the sum of the coordinates of $\vec{v} + \vec{w}$ is the sum of the sums of their coordinates (which is 0) and the sum of the coordinates of $c\vec{v}$ is c times the sum of the coordinates of $\vec{v}$ (which is 0). You can check that one basis of S is $\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \}$ (there are infinitely-many choices for a basis though).

(c) One choice of matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 8 & 2 \end{bmatrix}$$

This has the correct image because both of the columns are in the plane and because for any $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in the plane we have $z = 5x + 3y$, so the system

$$\left[\begin{array}{cc|c} 1 & 1 & x \\ 1 & -1 & y \\ 8 & 2 & 5x+3y \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & (x+y)/2 \\ 0 & 1 & (x-y)/2 \\ 0 & 0 & 0 \end{array}\right]$$

always has a solution.

We can check the columns of A are linearly independent because

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 8 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies \text{Ker}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

However, the rows are not linearly independent because

$$\left[\begin{array}{ccc|c} 1 & 1 & 8 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right]$$

which has, for instance, $\begin{bmatrix} -5 \\ -3 \\ 1 \end{bmatrix}$ in its kernel.