Sorting by Reversals

Jonas Hübotter

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Outline

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where $\pi_i = \pi(i)$, $\pi_0 = 0$, and $\pi_{n+1} = n+1$.

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$\pi \in S_n$ is a permutation.

$id = (0\ 1\ \ \dots\ \ n\ \ n+1) \in S_n$ is the *identity permutation*.

Definition 2

A reversal $\rho(i,j) \in S_n$ is defined as

$$\rho(i,j) = (0\ 1\ \cdots\ i-1\ j\ j-1\ \cdots\ i+1\ i\ j+1\ \cdots\ n\ n+1)$$

for some $i,j \in [1,n]$ with $j \geq i$.

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Example

Let $\pi = (0\ 1\ 3\ 4\ 2\ 5) \in S_4$.
Then

$$\pi \circ \rho(2,4) = (0\ 1\ 2\ 4\ 3\ 5).$$

Definition 3 (reversal distance problem)

Given two permutations $\sigma, \tau \in S_n$ find a sequence of reversals $\rho_1, \ldots, \rho_d \in S_n$ such that

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Observation: The reversal distance between σ and τ is the same as the reversal distance between $\tau^{-1} \circ \sigma$ and id.

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$\implies d(\pi) \leq 2.$

A different perspective: $\pi = (0\ 1 \mid 3\ 4 \mid 2 \mid 5)$

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A pair of consecutive elements π_i and π_j is

- an adjacency if $\pi_i \sim \pi_j$; and
- a breakpoint if $\pi_i \not\sim \pi_j$.

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Corollary 6 (lower bound, Kececioglu et al.)

$d(\pi) \geq \left\lceil \frac{b(\pi)}{2} \right\rceil$ *for all* $\pi \in S_n$.

Definition 7 (breakpoint graph, Bafna et al.)

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- vertices $V = [0, n+1]$ representing the elements of π; and
- edges $E = R \cup B$ with
  - a red edge for every breakpoint in π; and
  - a blue edge for every missing adjacency in π.

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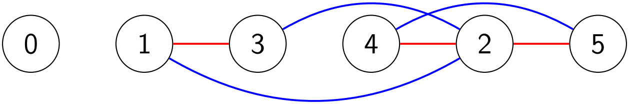

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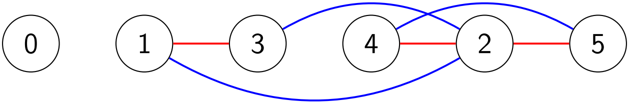

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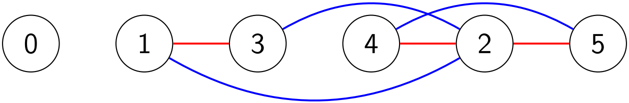

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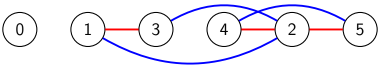
Corollary 8 (Bafna et al.)

$G(\pi)$ can be decomposed into edge-disjoint alternating cycles.



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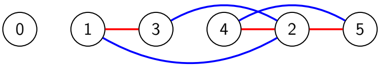
A reversal is called *k-reversal* if it removes k breakpoints.



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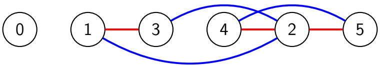
A reversal acts on two red edges of $G(\pi)$ if those two edges represent the breakpoints that are split apart by the reversal.



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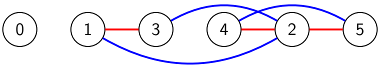

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We call an alternating cycle C in $G(\pi)$ *oriented* if there is a 1- or 2-reversal acting on two constituting red edges of C.

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Let $\pi, \rho \in S_n$ and ρ be a reversal. Then

$$b(\pi) - b(\pi \circ \rho) + c(\pi \circ \rho) - c(\pi) \leq 1.$$

Proof.

To show: $b(\pi) - b(\pi \circ \rho) + c(\pi \circ \rho) - c(\pi) \leq 1$.
We consider each case $b(\pi) - b(\pi \circ \rho) \in [-2, 2]$ separately.

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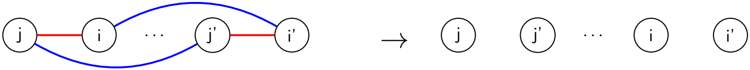

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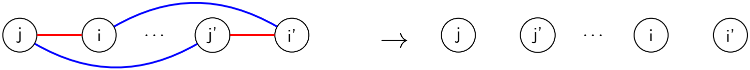

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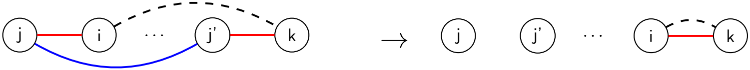

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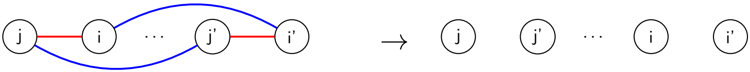

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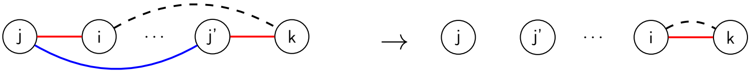

Proof for other cases similar. □

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Let $\pi \in S_n$. *Then*

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$$\begin{aligned} d(\pi_i) &= d(\pi_{i-1}) + 1 \\ &\overset{(10)}{\geq} d(\pi_{i-1}) + b(\pi_i) - b(\pi_{i-1}) + c(\pi_{i-1}) - c(\pi_i) \end{aligned}$$

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Setting $i = t$, proves the theorem. $\square$

Theorem 12 (lower bound with 2-cycles, Christie)

Let $\pi \in S_n$ and $\mathcal{C}$ be a maximum alternating cycle decomposition of $G(\pi)$. Let $c_2(\pi)$ be the minimum number of alternating 2-cycles in any such $\mathcal{C}$. Then

$$d(\pi) \geq \frac{2}{3} b(\pi) - \frac{1}{3} c_2(\pi).$$

$\frac{3}{2}$-approximation

By theorem 12, an algorithm that finds a sorting sequence of reversals of at most length $b(\pi) - \frac{1}{2}c_2(\pi)$ achieves an approximation bound of $\frac{3}{2}$.

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Lastly, we prove the approximation bound.

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Corollary 14 (Christie)

$R(\emptyset)$ consists of $n + 1$ isolated blue vertices.

Example

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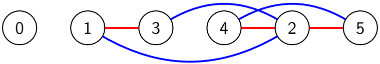
Given the alternating cycle decomposition $\mathcal{C}$ of $G(\pi)$

$$\mathcal{C} = \{(\{1,3\},\{2,3\},\{2,4\},\\ \{4,5\},\{2,5\},\{1,2\})\}$$



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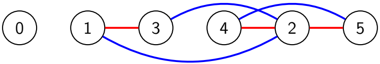
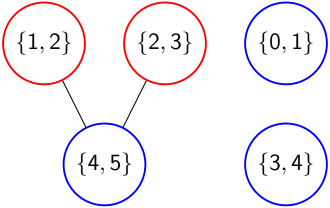


construct $R(\mathcal{C})$.



Example

Let $\pi = (0\ 1\ 3\ 4\ 2\ 5) \in S_4$.
Given the alternating cycle decomposition $\mathcal{C}$ of $G(\pi)$

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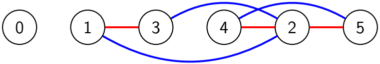

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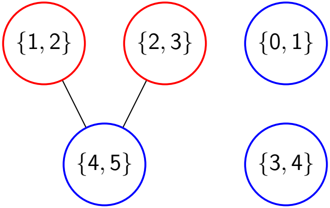

Idea: Each connected component of $R(\mathcal{C})$ can be sorted separately.

Let u be a vertex of $R(\mathcal{C})$ representing reversal $\rho(u)$.

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Definition 15

Denote by $\mathcal{C}_u$ the alternating cycle decomposition of $G(\pi \circ \rho(u))$ that is obtained from $\mathcal{C}$.

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Lemma 16 (Christie)

$R(\mathcal{C}_u)$ can be derived from $R(\mathcal{C})$ by making the following changes to $R(\mathcal{C})$:

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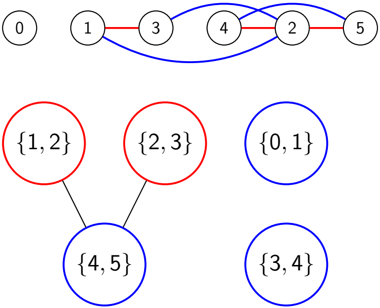
1. *flip the color of every vertex adjacent to u;*
2. *flip the adjacency of every pair of vertices adjacent to u; and*
3. *if u is a red vertex, turn it into an isolated blue vertex.*

Example

Let $\pi = (0\ 1\ 3\ 4\ 2\ 5) \in S_4$ and $u = \{1, 2\}$.

$G(\pi)$ and $R(\mathcal{C})$	$G(\pi \circ \rho(u))$ and $R(\mathcal{C}_u)$
	

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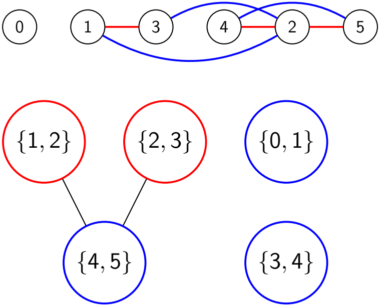

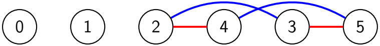

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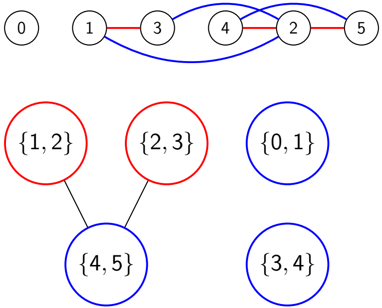	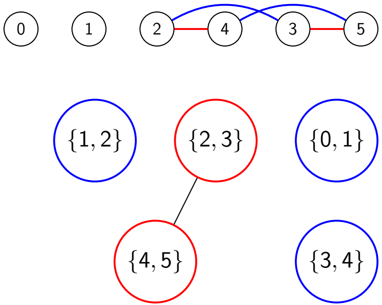

Lemma 17 (Christie)

All vertices arising from the same alternating cycle in $\mathcal{C}$ *are in the same connected component of* $R(\mathcal{C})$*.*

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All vertices arising from the same alternating cycle in $\mathcal{C}$ are in the same connected component of $R(\mathcal{C})$.

Lemma 18 (Christie)

Vertices arising from an unoriented 2-cycle of $\mathcal{C}$ must be in a connected component of $R(\mathcal{C})$ with vertices arising from at least one more alternating cycle of $\mathcal{C}$.

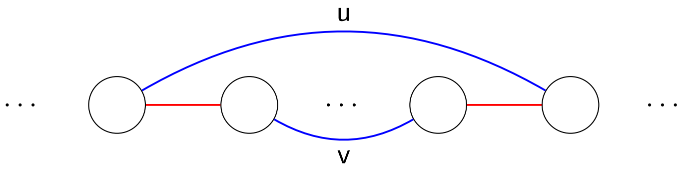

Figure 1: Unoriented 2-cycle

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We call a connected component of $R(\mathcal{C})$ oriented if it contains a red vertex or if it consists solely of an isolated blue vertex.

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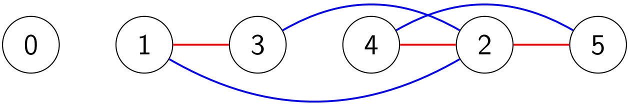
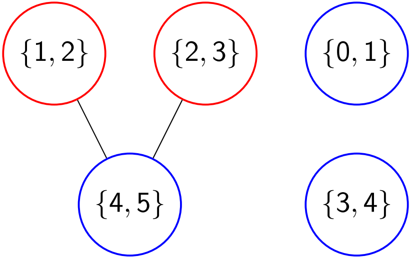
Lemma 20 (Christie)

If a connected component A of $R(\mathcal{C})$ is oriented and not an isolated blue vertex, it contains a red vertex u such that A_u is still oriented.

Example (elimination sequence)

Let $\pi = (0\ 1\ 3\ 4\ 2\ 5) \in S_4$





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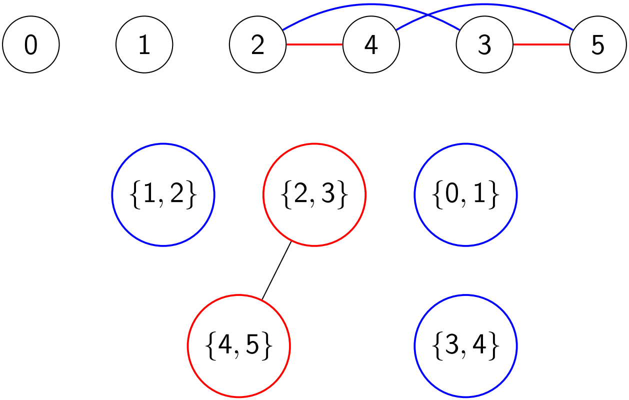

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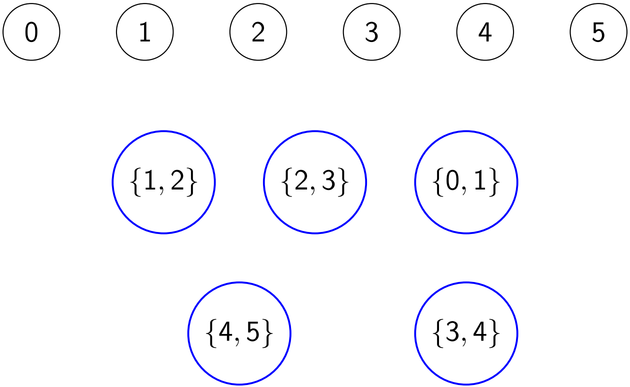

Observations:

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- An unoriented connected component requires one initial 0-reversal.

Theorem 21 (Christie)

Let $\pi \in S_n$ and $\mathcal{C}$ be an alternating cycle decomposition of $G(\pi)$. Then

$$d(\pi) \leq b(\pi) - |\mathcal{C}| + u(\mathcal{C})$$

where $u(\mathcal{C})$ is the number of unoriented components in $R(\mathcal{C})$.

Goal: Find a cycle decomposition of $G(\pi)$ that has a large number of 2-cycles.

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Idea 22

1. Construct a matching graph $F(\pi)$ where vertices represent red edges in $G(\pi)$ and vertices u, v are adjacent if they form a 2-cycle in $G(\pi)$.

Goal: Find a cycle decomposition of $G(\pi)$ that has a large number of 2-cycles.

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1. Construct a matching graph $F(\pi)$ where vertices represent red edges in $G(\pi)$ and vertices u, v are adjacent if they form a 2-cycle in $G(\pi)$.
2. Find maximum cardinality matching M of $F(\pi)$.

Goal: Find a cycle decomposition of $G(\pi)$ that has a large number of 2-cycles.

Idea 22

1. Construct a matching graph $F(\pi)$ where vertices represent red edges in $G(\pi)$ and vertices u, v are adjacent if they form a 2-cycle in $G(\pi)$.
2. Find maximum cardinality matching M of $F(\pi)$.
3. Use a ladder graph $L(M)$ with vertices representing 2-cycles in M and form connected components (*ladders*) with 2-cycles sharing a blue edge in $G(\pi)$.

Definition 23

We call a 2-cycle selected if its corresponding edge of $F(\pi)$ is in M.

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Theorem 24 (Christie)

Given a maximum cardinality matching M of $F(\pi)$ it is possible to find an alternating cycle decomposition $\mathcal{C}$ of $G(\pi)$ that contains at least $\left\lceil \frac{y}{2} \right\rceil$ ladder 2-cycles and z independent 2-cycles.

Theorem 25 (Christie)

Let $\pi \in S_n$. Then

$$d(\pi) \leq b(\pi) - \frac{1}{2}c_2(\pi).$$

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Proof.

Using theorem 24, first find an alternating cycle decomposition $\mathcal{C}$ of $G(\pi)$ with at least $\left\lceil \frac{y}{2} \right\rceil$ 2-cycles as part of ladders and z independent 2-cycles.

Proof (cont.)

- Let k be the number of 2-cycles in oriented connected components of $R(\mathcal{C})$.

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- Let k be the number of 2-cycles in oriented connected components of $R(\mathcal{C})$.
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- Let v be the number of remaining unoriented connected components consisting only of vertices representing m independent selected 2-cycles.

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By theorem 21, we can sort π using at least $k + l + u + m$ 2-reversals and only $u + v$ 0-reversals.

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$$\begin{aligned} d(\pi) &\leq b(\pi) - k - l - u - m + u + v \\ &= b(\pi) - k - l - m + v \end{aligned}$$

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$$\begin{aligned} d(\pi) &\leq b(\pi) - k - l - u - m + u + v \\ &= b(\pi) - k - l - m + v \end{aligned}$$

Left to show: $-k - l - m + v \leq -\frac{1}{2} c_2(\pi)$.

Proof (cont.)

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Proof (cont.)

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1. $k + l + m \geq \left\lceil \frac{y}{2} \right\rceil + z$ as $\left\lceil \frac{y}{2} \right\rceil + z$ is the number of selected 2-cycles in $\mathcal{C}$

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3. $|M| = y + z \geq c_2(\pi)$.

Proof (cont.)

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$$k + l + m - v \geq \left\lceil \frac{y}{2} \right\rceil + z - v \tag{1}$$

Proof (cont.)

Left to show: $k + l + m - v \geq \frac{1}{2} c_2(\pi)$. We know that

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$$
\begin{aligned}
k + l + m - v &\geq \left\lceil \frac{y}{2} \right\rceil + z - v && (1) \\
&\geq \left\lceil \frac{y}{2} \right\rceil + z - \left\lfloor \frac{z}{2} \right\rfloor && (2)
\end{aligned}
$$

Proof (cont.)

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$$\begin{aligned} k + l + m - v &\geq \left\lceil \frac{y}{2} \right\rceil + z - v && (1) \\ &\geq \left\lceil \frac{y}{2} \right\rceil + z - \left\lfloor \frac{z}{2} \right\rfloor && (2) \\ &= \left\lceil \frac{y}{2} \right\rceil + \left\lceil \frac{z}{2} \right\rceil \end{aligned}$$

Proof (cont.)

Left to show: $k + l + m - v \geq \frac{1}{2} c_2(\pi)$. We know that

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&\geq \frac{1}{2} c_2(\pi) && (3)
\end{aligned}
$$

$\square$

Run time: $O(n^4)$, can be improved to $O(n^2)$ (Kaplan et al.).

Summary

- the number of alternating cycles in a breakpoint graph $G(\pi)$ is related to $d(\pi)$
- a sorting sequence of reversals can be constructed from an alternating cycle decomposition of $G(\pi)$

Outlook

- there exists a 1.375-approximation (Berman et al.)
- MIN-SBR for signed permutations is in P (Hannenhalli et al.)

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