Probabilistic Social Choice
(Brandl, Brandt, and Seedig 2016)

Jonas Hübotter

November 23, 2021

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So where to go?

Aggregating the preferences of individuals
is the problem of social choice.

Framework of Social Choice

Given

- an electorate (Ava, Ben, and Chloe)

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find a social choice function $f : \mathcal{R}_A{}^1 \to A$ that returns a single alternative

[1] $\mathcal{R}_A = \Delta(\mathcal{L}(A))$ is the set of all preference profiles over A. Here, $\mathcal{L}(A)$ is the set of all linear preference relations over A and $\Delta(S)$ is the set of all probability distributions over S.

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find a social choice function $f : \mathcal{R}_A$[1] $\to A$ that returns a single alternative,

i.e., identifies a place for lunch that is in A.

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$2/3$	$1/3$
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$$R(K \succ M) = 1$$
$$R(K \succ F) = 2/3$$

$\rightsquigarrow$ K is the Condorcet winner

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$K \succ M$ ———

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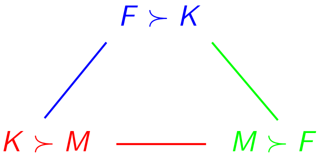

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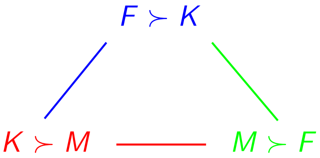

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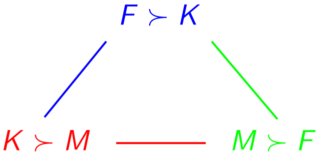

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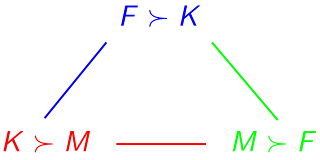

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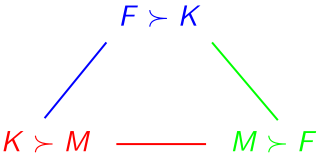

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$\rightsquigarrow$ deterministic social choice functions are problematic

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Instead of returning a single alternative, a probabilistic social choice function returns probability distributions (lotteries) over alternatives.

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Example (random dictatorship)

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$\rightsquigarrow RD(R) = \{1/3F + 1/3K + 1/3M\}$

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$\rightsquigarrow$ deterministic social choice functions are a special case of probabilistic social choice functions which only return *degenerate* lotteries (i.e., lotteries that put all probability mass on a single alternative).

Consistency Axioms

1/2	1/2
K	M
M	F
F	K

R'

1/2	1/2
K	M
F	F
M	K

R''

When $1/2K + 1/2M$ is chosen by R' and R''

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R''

1/4	1/4	1/2
K	K	M
M	F	F
F	M	K

$1/2 R' + 1/2 R'' = R$

When $1/2 K + 1/2 M$ is chosen by R' and R'', then it must also be chosen by R.

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Population Consistency

When two separate electorates R' and R'' agree on a lottery, this lottery must also be chosen by a combination of the electorates.
$\rightsquigarrow$ consistency with respect to a variable electorate

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$R|_{A'}$
$A' = \{F, M1, K\}$

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Cloning Consistency

We should assign the same probability to M independently of the number of clones and the internal relationship between these clones.
$\rightsquigarrow$ consistency with respect to components of *similar* alternatives

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Composition Consistency (extending cloning consistency)

The probability of clones must be directly proportional to the probabilities that f assigns to these alternatives for $R|_B$.

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$\rightsquigarrow$ maximal lotteries settle a dispute between the founding fathers of social choice theory, Chevalier de Borda and Marquis de Condorcet.

Random aggregation does not entail decisions must lack rationale, but rather that results may be random in the absence of a clear winner and as such the procedure is harder to exploit.

"Sometimes, in fact, we want to exclude reasons from decision-making entirely. This is what lotteries can do." (Stone 2011)

Acceptability of Lotteries

Amidst which circumstances are random choices useful?

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- depending on the risk aversion of voters, and
- the effective degree of randomness.

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In the Athenian democracy, lotteries were a principal component.[2] The process of selecting candidates by a lottery for a political office is also known as *sortition* or *selection by lot*.

[2] Special allotment machines (*kleroteria*) were used to select candidates.

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It can be shown that Condorcet's view of equivocality and the randomness of maximal lotteries coincide.[3]

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Empirically, there often exists a Condorcet winner. If there exists no Condorcet winner, the support of maximal lotteries is typically small compared to the number of alternatives.

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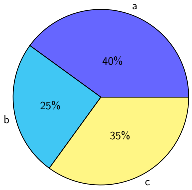

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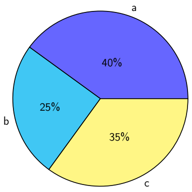

Probabilities can equivalently be interpreted as fractions of a divisible resource such as money or time.

Thank you for your attention

How often do preferences have to be (repeatedly) aggregated until lotteries become acceptable?

Is randomness in decision making at all acceptable?
Is it desirable?

References

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The Inefficiency of Majority Rule without Lotteries

The 101 Club must choose a single form of entertainment for all club members. The membership rolls contain fifty football fanatics, fifty ballet aficionados, and a single lover of musical comedy. For the footballers the musical is almost as bad as the ballet. For the ballet enthusiasts the musical is little better than football. By majority rule, using pairwise comparisons without lotteries, the musical will be chosen. If lotteries were permitted, a fifty-fifty football-ballet lottery would defeat the musical by any required plurality up to 100 out of 101. (Zeckhauser 1969)

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In general, majority rule without lotteries corrects towards the median choice.
This can be considered inefficient or desirable. The judgement may depend on the setting.

Maximal Lotteries

For a set of alternatives A and preference profile $R \in \mathcal{R}_A$, let M_R be the majority margin matrix defined by

$$M_R(x, y) = R(x \succ y) - R(y \succ x).$$

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Any $x \in A$ such that $M_R(x, y) \geq 0$ for all $y \in A$ is a *(weak) Condorcet winner*.

Maximal Lotteries

For a set of alternatives A and preference profile $R \in \mathcal{R}_A$, let M_R be the majority margin matrix defined by

$$M_R(x, y) = R(x \succ y) - R(y \succ x).$$

$M_R(x, y)$ denotes the difference between the fraction of voters preferring x to y and the fraction of voters preferring y to x.
$\rightsquigarrow$ M_R is skew-symmetric

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Any $x \in A$ such that $M_R(x, y) > 0$ for all $y \in A \setminus \{x\}$ is a *strict Condorcet winner*.

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$\rightsquigarrow$ efficient computation using linear programming.

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Example

$1/2$	$1/3$	$1/6$
a	a	b
b	c	c
c	b	a

R

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Clearly, a is the only Nash equilibrium of the symmetric zero-sum game, so $ML(R) = \{a\}$ (a is also the Condorcet winner).

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Note that a random dictatorship returns the lottery $5/6 a + 1/6 b$.

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Let $A = \{x, y\}$ and $R \in \mathcal{R}_A$.

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$\rightsquigarrow$ in a way, maximal lotteries have the smallest reasonable degree of randomness while random dictatorships are the most random.