Network Flow

Suppose we wish to model the transport of materials from one location to another. Suppose also that the medium of transport allows for alternative paths from the "source" to the "sink" (i.e., destination). Finally, suppose that there are capacity constraints imposed along the pathway. Given this, we wish to determine the maximum amount of the material that can be transported between the source and the sink at any given point in time.

For example, all of the following can be modeled as network flow problems:

1. Shipping or transporting goods from a factory to distribution centers.
2. Carrying parts along an assembly line.
3. Tracking the flow of liquids through a network of pipes.
4. Flowing current through an electrical circuit.
5. Transferring information through a communications network.

Formally, we will represent these and similar problems using a graph-theoretic model called a flow network.

Definition: A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a non-negative capacity $c(u, v) \geq 0$. If $(u, v) \notin E$, then $c(u, v) = 0$.

In a flow network, we identify two special vertices, denoted s and t for the source and sink respectively. In addition, we require that every vertex $v \in V$ lie on at least one path $s \to v \to t$.

Definition: Let $G = (V, E)$ be a flow network with capacity function c, source s, and sink t. Then the flow in G is a real-valued function satisfying the following properties:

- Capacity constraints: $\forall u, v \in V,\ f(u, v) \leq c(u, v)$
- Skew symmetry: $\forall u, v \in V,\ f(u, v) = -f(v, u)$
- Flow conservation: $\forall u \in V - \{s, t\},\ \sum_{v \in V} f(u, v) = 0$