Duality in Linear Programming

Recall from our discussion above that we can use linear programming to solve the maximum flow problem in flow networks. Recall also from our discussion on flow networks that we were able to prove a very interesting theorem—the *max-flow min-cut theorem.* Basically, this theorem states that the size of the maximum flow in a network is equal to the capacity of the smallest (*s*-*t*)-cut in that network. Thus solving the maximum flow problem carries with it the implicit solution to a corresponding minimization problem (i.e., the minimum cut problem).

The maximum-flow problem actually illustrates a more general property of linear programming. Specifically, we can show that every linear maximization problem has a corresponding ***dual*** linear minimization problem. Indeed, we are able to prove that, if a linear program has a bounded optimum, then so does it's dual, and the two values are the same.

Let's consider duality with an example. Specifically, suppose we have the following *primal* linear program (which corresponds to our candy production problem above):

$$\text{maximize } x_1 + 6x_2$$

subject to

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0. \end{aligned}$$

If we were to solve this linear programming using the simplex method, we would find an optimum solution at $x_1 = 100$ and $x_2 = 300$ with corresponding objective value 1900. The question facing us is whether this solution really is optimal.

To test whether or not an optimal solution has been found, suppose we take the first constraint and add it to six times the second constraint. Why would we do this? Well, this will give us an upper bound of what really is possible given these two simple constraints and the objective function. Doing this, we find

$$x_1 + 6x_2 \leq 200 + 6 \cdot 300 = 2000.$$

Thus there is no way any solution to our problem can exceed 2000. We can continue this idea by trying to find ways to "multiply" and "add" constraint together to find additional combinations, thus tightening the bound on the objective function. Said another way, we are trying to find these multipliers that *minimize* the difference between the upper bound and the optimum value. Should we find such multipliers that minimize this difference (which, by the way, is another objective function), then we have solved another corresponding linear programming problem. Indeed, it is this problem that corresponds to the dual.

Continuing with our example, suppose we wish to find multipliers y_1, y_2, and y_3 such that

$$\begin{aligned} y_1 \cdot x_1 + y_2 \cdot x_2 + y_3 \cdot (x_1 + x_2) &\leq y_1 \cdot 200 + y_2 \cdot 300 + y_3 \cdot 400 \\ (y_1 + y_3) \cdot x_1 + (y_2 + y_3) \cdot x_2 &\leq y_1 \cdot 200 + y_2 \cdot 300 + y_3 \cdot 400 \end{aligned}$$

In reality, we are seeking to find values of y_i such that the objective function is bound by the right-hand-side of the above inequality. In other words,

$$x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3.$$

Since we are trying to minimize the difference, we are looking for values of the right hand variables that changes the inequality to equality. This actually corresponds to the following linear program:

$$\text{minimize } 200y_1 + 300y_2 + 400y_3$$

subject to

$$\begin{aligned} y_1 + y_3 &\geq 1 \\ y_2 + y_3 &\geq 6 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

Where do the right-hand-sides come from? Take a look at the original objective function, $x_1 + 6x_2$ and notice that above we wrote the left-hand-side of the inequality as $(y_1 + y_3)x_1 + (y_2 + y_3)x_2$. Thus we have the left-hand-sides in these constraints serving as coefficients in the original objective function. Notice that if we apply the simplex algorithm to this problem, we find $y_1 = 0$, $y_2 = 5$, and $y_3 = 1$ yields an optimal solution with the same objective value of 1900. Because the maximal value in the primal problem equals the minimal value in the dual problem, we know the solutions must be optimal, similar to the maximum flow and minimum cut in the flow network problem.

To generalize, if we have a primal linear programming problem specified in standard form, we can derive the dual linear programming problem as follows:

$$\text{minimize} \sum_{i=1}^{m} b_i y_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \text{ for } j = 1, 2, \ldots, n \\ y_i &\geq 0 \text{ for } i = 1, 2, \ldots, m. \end{aligned}$$

Let's build up to proving that duality exists and that the solutions to the primal and dual problems will be equal. We start with a simple lemma:

Lemma: Let $\mathbf{x}$ be any feasible solution to a primal linear program, specified in standard form, and let $\mathbf{y}$ be any feasible solution to the corresponding dual linear program, also specified in standard form. Then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i.$$

(In other words, the solution to the primal problem is bound from above by any feasible solution to the dual problem.)

Proof: We have

$$\begin{aligned} \sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j \text{ using the inequalities of the dual} \\ &= \sum_{i=}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \text{ by commutativity} \\ &\leq \sum_{i=1}^{n} b_i y_i \text{ using the inequalities of the primal.} \end{aligned}$$

Corollary: Let $\mathbf{x}$ be a feasible solution to a primal linear program, and let $\mathbf{y}$ be a feasible solution to the corresponding dual linear program. If

$$\sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i,$$

then $\mathbf{x}$ and $\mathbf{y}$ are optimal solutions to the primal and dual linear programs respectively.

Proof: By the Lemma, the objective value of any feasible solution to the dual problem bounds any feasible solution to the primal problem from above (and vice-versa). Since the primal problem (in standard form) is a maximization problem and the corresponding dual problem is thereby a minimization problem, the maximum objective value of the primal problem cannot exceed the minimum objective value of the dual problem. Therefore, if the solutions to both problems are equal, neither can be improved. Thus they must be optimal.

These two problems provide a way to say that, if we have equal solution to the primal and dual linear programming problems, then both solutions must be optimal. What is missing is something that such such equal solutions must exist for any linear programming problem. This last result is the duality theorem we used at the beginning of this section. The Cormen text provides a proof of this theorem in the context of the simplex algorithm; however, the result applies regardless of the algorithm used.