Homework 6

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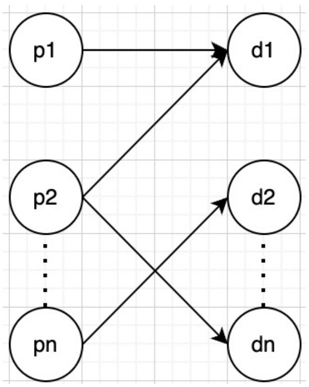
Statement of Integrity: I, Joni Vrapi, attempted to answer each question honestly and to the best of my abilities. I cited any, and all, help that I received in completing this assignment.

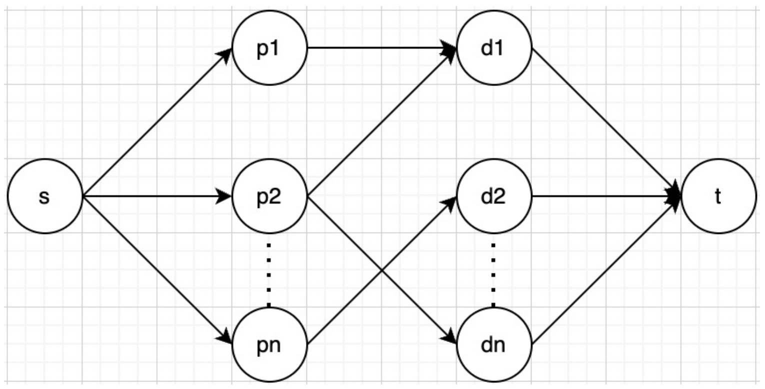
Problem 1. A tree, by definition [1], is a type of graph in which any two vertices are connected by exactly one path and in which there are no cycles. In other words, a connected acyclic graph. We know that the edges in E are undirected. We also know that both bfs and dfs produce the same tree T which we will assume $G \neq T$.

If we therefore assume that G is connected but not a tree, then it must contain a cycle C. If C has n nodes $u_1, u_2, ..., u_n$, then C represents the cycle $u_1 \rightarrow u_2 \rightarrow ... \rightarrow u_n \rightarrow u_1$. If we assume that bfs and dfs encounter the cycle at node u_i, then dfs will add edges $(u_1, u_2), ..., (u_{i-2}, u_{i-1})$ while bfs will start by adding edges $(u_1, u_2), (u_1, (u_i))$. Since we can see that this contradicts the assumption that bfs and dfs have equal trees, we can see that bfs and dfs have equal trees iff the input graph is also a tree. Therefore, $G = T$.

Problem 2a. If we have a bipartite graph G where p_n signifies a person, and d_n signifies a cooking day, then a directed edge (p_i, d_j) between them signifies if person p_i is able to cook on day d_j.



In order to find out whether this bipartite graph has a perfect matching, we can use the Maximum Flow Algorithm [2]. In order to set this up, we must add 2 more nodes (a source node s, and a sink node t) to our bipartite graph like so:

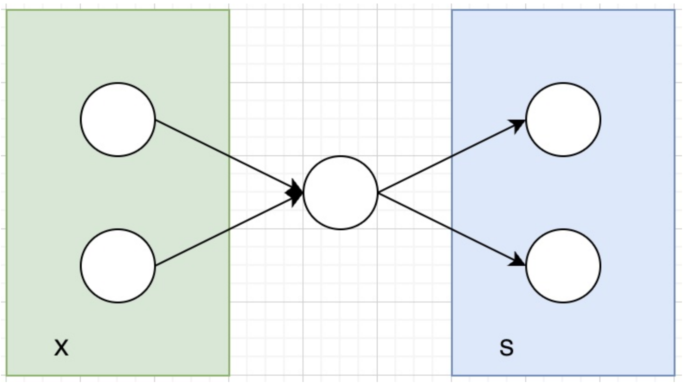


If we set each edge's weight to be 1, then the schedule will be feasible iff the maximum flow is n and the number of edges in G is also n.

Problem 2b. First, we would construct the bipartite graph G from (a). It takes $O(1)$ time to decide if $(p_i, d_j))$ should be an edge so the total time to build the graph would be $O(n^2)$. If we let R denote the set of edges given to us by Renee, we can delete the edge (p_j, d_k) to obtain R' which would have a size of $n-1$ and have no person or day appear more than once. From [3] we need to find an augmenting path with respect to R' in $O(|E|) = O(n^2)$ time. If we find an augmenting path, then this gives us a perfect matching and would correspond to a feasible dinner. If we do not find an augmenting path, then it would hold that R' is a maximum matching in G and there would be no perfect matching – and therefore no feasible dinner schedule.

Problem 4a. This problem can be modeled as a maximum flow network problem, much like problem 1. If we constructed a flow network with edges of weight 1 from the source, through X and all the way into S, then the edges from S into the sink would be of weight $|X|$. From source to X, as well as from S to sink, there should be $|X|$ edges. From here we can use Ford-Fulkerson [4] to compute the maximal flow in $O(V * E^2)$ time. If the maximal flow is equal to $|X|$, then there exists an evacuation route.

Problem 4b. We could solve for this extra condition by splitting up each vertex v in G into a set of 2 vertices v_i and v_j. All in-edges of v will point to v_i while all out-edges of v will leave from v_j, and a single out-edge will leave from v_i and point to v_j. We will weight these edges the same as in problem (4a) and re-run Ford-Fulkerson, with the condition that no two paths can share an edge now meaning that no two paths can share the internal edge between v_i and v_j. We can then shrink the (v_i, v_j) pairs back to produce a set of node-disjoint paths. The most simple example where the answer is "yes" to (a) and "no" to (b) is the following:



References

[1] “Tree (graph theory).” https://en.wikipedia.org/wiki/Tree_(graph_theory). Accessed on 2022-11-25.

[2] “Max flow problem introduction.” https://www.geeksforgeeks.org/max-flow-problem-introduction/. Accessed on 2022-11-25.

[3] “Algorithm solutions.” https://github.com/mathiasuy/Soluciones-Klenberg/blob/master. Accessed on 2022-11-25.

[4] “Ford–fulkerson algorithm.” https://en.wikipedia.org/wiki/Ford%E2%80%93Fulkerson_algorithm. Accessed on 2022-11-25.