Linear Programming in Game Theory

One application of linear programming is in basic game theory. Game theory is the formal study of conflict and competition. The purpose of studying games is to determine rational decisions by competitors intending to maximize some payoff or minimize some penalty. In this section, we discuss games that can be represented as payoff matrices. Specifically, one method of representing a game (especially a two-person game) is by using a payoff matrix of the form

$$\begin{pmatrix} \rho_1^{11} : \rho_2^{11} & \rho_1^{12} : \rho_2^{12} & \cdots & \rho_1^{1n} : \rho_2^{1n} \\ \rho_1^{21} : \rho_2^{21} & \rho_1^{2,2} : \rho_2^{22} & \cdots & \rho_1^{2n} : \rho_2^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1^{m1} : \rho_2^{m1} & \rho_1^{m2} : \rho_2^{m2} & \cdots & \rho_1^{mn} : \rho_2^{mn} \end{pmatrix}$$

where ρ_1^{jk} indicates the payoff to player 1 when player 1 plays move j and player 2 plays move k. Similarly, ρ_2^{jk} is the payoff to player 2 under the same pair of moves.

Note that the above representation actually corresponds to two matrices, one representing the payoffs to player 1 and the other representing the payoffs to player 2. Thus games represented this way are sometimes called "bimatrix games." Suppose $\rho_1^{jk} = f_1(a_1^{jk}, a_2^{jk})$ for actions (moves) corresponding to players 1 (a_1) and 2 (a_2) respectively. Similarly, $\rho_2^{jk} = f_2(a_1^{jk}, a_2^{jk})$. If we have a situation where $\forall a_1, a_2\ f_1(a_1^{jk}, a_2^{jk}) + f_2(a_1^{jk}, a_2^{jk}) = \kappa$ for some constant κ, then this kind of game is called a "constant sum game." Furthermore, if $k = 0$, we have what is called a "zero-sum game." It turns out that we can find the optimal strategies for any zero-sum game (under a few additional assumptions) using linear programming.

One of the most commonly used concepts in game theory for "solving" a game is something called the *Nash equilibrium point*. We define a Nash equilibrium point in the context of a zero-sum matrix game as follows:

Definition: Let $\mathcal{A} = \{\langle a_1, \ldots, a_p\rangle | a_1 \in \mathcal{A}_1 \wedge \cdots \wedge a_p \in \mathcal{A}_p\}$ be a set of joint strategies over p players.

Definition: For any $p \in \mathcal{P}$, $a'_p \in \mathcal{A}_p$, and $a \in \mathcal{A}$, let $a||a'_p$ denote the member of $\mathcal{A}$ obtained by replacing a_p with a'_p. This is called a *unilateral defection* from the joint strategy a by player p.

Definition: Let $f_p(a)$ denote the payoff to player p given joint strategy $a \in \mathcal{A}$.

Definition: If, for all $a'_p \in \mathcal{A}_p$, we have $f_p(a||a'_p) \leq f_p(a)$, then $a \in \mathcal{A}$ is said to be *admissible*. In other words, the player p has no profitable unilateral defection from joint strategy a.

Definition: Joint strategy $a \in \mathcal{A}$ is a Nash equilibrium point for game Γ if and only if it is admissible for all players $p \in \mathcal{P}$.

Unfortunately, the above definition is not sufficient as is—many games do not have Nash equilibria when considering what are called *pure strategies*. In this case, we need to consider the space of *mixed strategies*. The set of mixed strategies for player p, denoted Σ_p, is defined to be the set of probability distributions over the strategy space $\mathcal{A}_p$.

For example, recall the game Rock-Paper-Scissors. In this game, two players face off and together call out, "One, two, three, shoot." At the point when they say "shoot," they display a hand figure where a balled fist represents a rock, a flat hand represents paper, and two fingers extended in a V represent scissors. The winner of the game is determined based on the following payoff matrix (from the perspective of player $P1$):

	Rock	Paper	Scissors
Rock	0	−1	1
Paper	1	0	−1
Scissors	−1	1	0

Suppose the two players play repeatedly. If $P1$ always makes the same move, then $P2$ will discover this and exploit this behavior to always win. Therefore, $P1$ should mix things up a bit. Indeed, we can show that if $P1$ tends to "prefer" one move over the other two, then $P2$ would be able to detect this in time and exploit this information. Therefore, the best thing for $P1$ to do is pick between the choices randomly using a uniform probability distribution. The same argument can be applied to $P2$, so that player should also pick randomly. These "strategies" of picking between Rock, Paper, and Scissors are mixed strategies because they correspond to selecting a probability mixture (i.e., a probability distribution) over the pure choices.

Given the concept of mixed strategies, we are now in a position where we can find a Nash equilibrium point for any matrix game.

Theorem: Given any n-player non-cooperative game, there exists at least one Nash equilibrium point consisting of mixed strategies.

If we reconsider the concept of a matrix game and limit our discussion to two-person zero-sum games, it has been shown that Nash equilibria with mixed strategies can be found by solving the following linear program. Let f^π be the expected value of the game Γ given in matrix form.

$$\text{minimize } f^\pi$$

subject to

$$\sum_{a_1 \in \mathcal{A}_1} p(a_1) = 1$$

$$\forall a_1 \in \mathcal{A}_1,\ p(a_1) \geq 0$$

$$\forall a_1 \in \mathcal{A}_1,\ \sum_{a_2 \in \mathcal{A}_2} p(a_1) f(a_1, a_2) - f^\pi \leq 0.$$

This will determine an optimal mixed strategy for player $P1$. For player $P2$, the dual linear program must be solved:

$$\text{maximize } f^\pi$$

subject to

$$\sum_{a_2 \in \mathcal{A}_2} p(a_2) = 1$$

$$\forall a_2 \in \mathcal{A}_2,\ p(a_2) \geq 0$$

$$\forall a_2 \in \mathcal{A}_2,\ \sum_{a_1 \in \mathcal{A}_1} p(a_2) f(a_1, a_2) - f^\pi \geq 0.$$

This will find an optimal mixed strategy for player $P2$. In fact, we can also show that the pairing of any optimal mixed strategy for $P1$ with any optimal mixed strategy for $P2$ will maintain optimal play for both players.

Given the calculation of these mixed strategies, we can also determine the expected value for playing the game. In particular, the value of the game with respect to $P1$ is simply

$$f_1^\pi(\Gamma) = \sum_{\forall a_1 \in \mathcal{A}_1} \sum_{\forall a_2 \in \mathcal{A}_2} p(a_1) p(a_2) f(a_1, a_2).$$

Similarly, the value of the game with respect to $P2$ is

$$f_2^\pi(\Gamma) = -\sum_{\forall a_1 \in \mathcal{A}_1} \sum_{\forall a_2 \in \mathcal{A}_2} p(a_1) p(a_2) f(a_1, a_2) = -f_1^\pi(\Gamma).$$

What is interesting is that the optimization problem can be reduced to finding

$$\max_{a_1 \in \mathcal{A}_1} \min_{a_2 \in \mathcal{A}_2} f_1^\pi = \max_{a_2 \in \mathcal{A}_2} \min_{a_1 \in \mathcal{A}_1} f_2^\pi = \min_{a_2 \in \mathcal{A}_2} \max_{a_1 \in \mathcal{A}_1} f_1^\pi.$$

This property corresponds to the *minimax theorem*, and we can see that it is very similar to the *max-flow min-cut theorem*.