The Iteration Method

As discussed, being forced to guess a bound for the substitution method is problematic, especially when the level of experience is low. An alternative approach is the iteration method where the recurrence is expanded until a summation can be determined, then the summation is solved. The advantage to this approach is that guessing is not required. The disadvantage is that the algebra required can be messy. The idea is to expand the recurrence and express the resulting sum in terms dependent only on n and the initial conditions.

For example, we will solve the recurrence $T(n) = 3T(\lfloor n/4 \rfloor) + n$ using the iteration method. As a result, we get the following expansion:

$$\begin{aligned} T(n) &= n + 3T(\lfloor n/4 \rfloor) \\ &= n + 3(\lfloor n/4 \rfloor + 3T(\lfloor n/4 \rfloor /4)) \\ &= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/16 \rfloor /4))) \\ &= n + 3\lfloor n/4 \rfloor + 9\lfloor n/16 \rfloor + 27T(\lfloor n/64 \rfloor). \end{aligned}$$

With this, we see a decreasing geometric series. Specifically, we see the series as

$$T(n) \le n + \frac{3}{4}n + \frac{9}{16}n + \frac{27}{64}n + \cdots + 3^{\log_4 n}\Theta(1).$$

[Note: Perhaps the most confusing part of this progression is the last term. We can see we are raising 3 to some power by examining the numerators in the coefficients. The reason for the logarithmic exponent is that we have a similar geometric progression in the denominator, growing as a factor of 4. This will cause the recurrence to "bottom out" on $\log_4 n$ steps.]

This can be rewritten as

$$T(n) \le n \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i + \Theta(n^{\log_4 3}).$$

Observe that we can rewrite $3^{\log_4 n}\Theta(1)$ as $\Theta(n^{\log_4 3})$ because of the identity $a^{\log_b n} = n^{\log_b a}$.

Because of the geometric series, we can solve the summation as

$$T(n) = \frac{n}{1 - \frac{3}{4}} + \Theta(n^{\log_4 3}) = 4n + o(n)$$

since $\log_4 3 < 1$. Thus we have that $T(n) = O(n)$.

A couple of tips to manage the iterations: First, to minimize confusion, focus on only two parameters—the sum of terms arising from the level of iteration and the number of times to recurse to hit the boundary. We saw this in the example above with the summation and the theta term respectively. As an alternative approach, sometimes the iteration method can be used to suggest a reasonable guess to start the substitution method.

So far, we have considered several recurrences with floors and ceilings. Often, it is fine to ignore the floors and ceilings and to focus on exact powers of numbers suggested by the denominator of the recursion. That said, there are times when such simplifications do not work, but they may still suggest a path forward.