Defining Linear Programming Problems

In this section, we introduce the basic forms of linear programming problems. As mentioned above, linear programming corresponds to a constrained optimization problem where we seek to minimize or maximize some linear function, subject to a set of linear constraints. More formally, suppose we are given a set of real numbers $a_1, a_2, \ldots, a_n$ and a set of variables $x_1, x_2, \ldots, x_n$. Then we can define a linear function f on those variables such that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i x_i.$$

The optimization problem consists of finding values for $x_1, \ldots, x_n$ (also sometimes called "decision variables") that minimize or maximize the function. To make things more interesting, we also define several constraints, either as equalities or inequalities, that must be satisfied when when instantiate the variables $x_1, \ldots, x_n$. For example, given some real number b, an equality constraint would have the form

$$g(x_1, \ldots, x_n) = b$$

and an inequality constraint would have the form

$$h(x_1, \ldots, x_n) \leq b$$

or

$$h(x_1, \ldots, x_n) \geq b.$$

When solving a linear programming problem, we can partition the set of possible values for $x_1, \ldots, x_n$ into two disjoint subsets, $\mathcal{F}$ and $\mathcal{I}$. Specficially, we define $\mathcal{F}$ to be the set of points $x_1, \ldots, x_n$ such that all of the constraints are satisifed. This is referred to as the ***feasible*** region. We then define $\mathcal{I}$ to be the set of points $x_1, \ldots, x_n$ such that one or more of the constraints has been violated. This is referred to as the ***infeasible*** region. Solving linear programs seeks to find a point in $\mathcal{F}$ such that the objective function is maximized or minimized.

As an example, suppose we are consulting for a producer of fine candies. This company currently offers (among other things) two different types of assortments, labeled *fine* and *exquisite*. The question they are trying to answer is how much to make of each product to maximize their profits. Suppose they plan to make x_1 boxes of the fine assortment and x_2 boxes of the exquisite assortment. Furthermore, assume the fine assortment yields a profit of \$1 per box while the exquisite assortment yields a profit of \$6 per box. Unfortunately, the production resources of the company limit how much can be made in a day. Specifically, suppose at most 200 boxes of the fine candies can be made while no more than 300 boxes of the exquisite candy can be made. Also, assume there is total limit of 400 boxes per day, regardless of the product. Given this, what values of x_1 and x_2 will yield the highest profit?

This question can be answered by setting up and solving a linear programming problem. Specifically, we can specify the objective function be

$$\text{maximize } x_1 + 6x_2$$

subject to the following constraints:

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$