ECON 899 Problem Set 7

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December 22, 2022

Question 1

Mean:

$x_t = \rho_0 x_{t-1} + \varepsilon_t$

Note that $x_0 = 0$ so clearly $E[x_0] = 0$

$x_1 = 0 + \varepsilon_1 = \varepsilon_1$

And from the distribution given then, $E[x_1] = 0$

So we can see that any further x_t will be a linear combination of ε values and thus the mean will always be zero.

Variance:

$Var[x_t] = \rho_0^2 Var[x_{t-1}] + Var[\varepsilon_t]$

$Var[x_t] = \rho_0^2 Var[x_{t-1}] + \sigma_0^2$

And from the mean argument we know the variance shouldn't change so:

$Var[x_t] = \rho_0^2 Var[x_t] + \sigma_0^2$

$Var[x_t] - \rho_0^2 Var[x_t] = \sigma_0^2$

$Var[x_t](1 - \rho_0^2) = \sigma_0^2$

$Var[x_t] = \frac{\sigma_0^2}{1-\rho_0^2}$

First order autocorrelation:

$Cov(x_{t-1}, \rho_0 x_{t-1} + \varepsilon_t) = Cov(x_{t-1}, \rho_0 x_{t-1}) + Cov(x_{t-1}, \varepsilon_t) = \rho_0 Var[x_{t-1}] = \rho_0 \frac{\sigma_0^2}{1-\rho_0^2}$

Jacobian (dropping the subscript in our notation here):

Clearly the two cells related to the mean will just be zero.

Beyond that —

Derivative of variance w.r.t. σ: $\frac{2\sigma}{1-\rho^2}$

Derivative of variance w.r.t. ρ: $\frac{2\sigma^2\rho}{(1-\rho^2)^2}$

Derivative of autocorrelation w.r.t. σ: $\frac{2\sigma\rho}{1-\rho^2}$

Derivative of autocorrelation w.r.t. ρ: $\frac{\sigma^2(1+\rho)^2}{(1-\rho^2)^2}$

Which make up the second and third rows of the Jacobian.

Clearly the only moment that won't be informative is the unchanging mean.

Questions 2, 3

See attached code.

Question 4

Using the mean and variance will be a problem because the mean does not depend on ρ or σ, so it is uninformative for estimating them. Thus, we will be trying to identify both parameters off the sample variance alone (under-identified).

Part 4(a): Plots and 1-step GMM

Figure 1 plots the objective function $J_{TH}(\rho, \sigma)$ over $(\rho, \sigma) \in [0.35, 0.65] \times [0.8, 1.2]$.

Setting $W = I_2$ and initial guess $(\rho^0, \sigma^0) = (0.4, 0.8)$, we get

$$\hat{b}^1_{TH} \approx (0.9832, 0.2325)$$

This is quite far off from the true values $(\rho_0, \sigma_0) = (0.5, 1.0)$. This makes sense given the discussion above.

Part 4(b): 2-step GMM

Second step of GMM: set $W = \hat{W}^*_{TH}$ as weighting matrix, and use initial guess $= \hat{b}^1_{TH}$.

$$\hat{S}_{TH} = \begin{bmatrix} 6.4078 & -0.2623 \\ -0.2623 & 13.2037 \end{bmatrix}$$

$$\hat{W}^*_{TH} = \hat{S}^{-1}_{TH} = \begin{bmatrix} 0.1562 & 0.0031 \\ 0.0031 & 0.0758 \end{bmatrix}$$

$$\hat{b}^2_{TH} \approx (0.9832, 0.2325)$$

Part 4(c): Estimating VCV

$$\nabla_b g_T(\hat{b}^2_{TH}) = \begin{bmatrix} -5.0286 & -0.5288 \\ -44.7723 & -12.2694 \end{bmatrix}$$

$$\hat{\mathrm{Var}}(\hat{b}^2_{TH}) = \begin{bmatrix} 0.0034 & -0.0124 \\ -0.0124 & 0.0460 \end{bmatrix}$$

$$\mathrm{SE}(\hat{b}^2_{TH}) = (0.058, 0.2144)$$

Part 4(d): J-statistic

$$J = 5.5 \times 10^{-8}$$

Question 5

This should not be a problem because both the variance and autocorrelation depend on ρ and σ, so both are informative about the parameters we're trying to estimate (just-identified).

Part 5(a): Plots and 1-Step GMM

Figure 2 plots the GMM objective function over the same parameter grids as in 4(a).

1-step GMM estimates ($W = I_2$):

$$\hat{b}^1_{TH} \approx (0.5746, 0.9849)$$

This is much closer to the true values $(0.5, 1)$. This makes sense, because both the variance and autocorrelation are informative about ρ and σ.

Part 5(b): 2-Step GMM

2-step GMM estimates:

$$\begin{aligned}
\hat{S}_{TH} &= \begin{bmatrix} 15.0053 & 8.9094 \\ 8.9094 & 10.3165 \end{bmatrix} \\
\hat{W}^*_{TH} = \hat{S}^{-1}_{TH} &= \begin{bmatrix} 0.1368 & -0.1181 \\ -0.1181 & 0.1989 \end{bmatrix} \\
\hat{b}^2_{TH} &= (0.5746, 0.9848)
\end{aligned}$$

Part 5(c): Estimating VCV

$$\begin{aligned}
\nabla_b g_T(\hat{b}^2_{TH}) &= \begin{bmatrix} -2.3997 & -2.896 \\ -2.7846 & -1.6479 \end{bmatrix} \\
\hat{\text{Var}}(\hat{b}^2_{TH}) &= \begin{bmatrix} 0.0125 & -0.0099 \\ -0.0099 & 0.0168 \end{bmatrix} \\
\text{SE}(\hat{b}^2_{TH}) &= (0.1118, 0.1295)
\end{aligned}$$

Part 5(d): J-statistic

$$J = 7.6 \times 10^{-8}$$

Question 6

The results below are not bootstrapped.

Part 6(a): Plots and 1-Step GMM

Figure 3 plots the GMM objective function over the same parameter grids.

1-step GMM estimates ($W = I_2$):

$$\hat{b}^1_{TH} = (0.5763, 0.9833)$$

Part 6(b): 2-Step GMM

Results:

$$\hat{S}_{TH} = \begin{bmatrix} 5.3784 & -0.0768 & 0.0357 \\ -0.0768 & 14.9989 & 8.9291 \\ 0.0357 & 8.9291 & 10.3249 \end{bmatrix}$$

$$\hat{W}^*_{TH} = \hat{S}^{-1}_{TH} = \begin{bmatrix} 0.1860 & 0.0028 & -0.0030 \\ 0.0028 & 0.1375 & -0.1189 \\ -0.0030 & -0.1189 & 0.1997 \end{bmatrix}$$

$$\hat{b}^2_{TH} = (0.5750, 0.9857)$$

Part 6(c): Estimating VCV

$$\nabla_b g_T(\hat{b}^2_{TH}) = \begin{bmatrix} -0.0462 & -0.02 \\ -2.4128 & -2.8996 \\ -2.7957 & -1.6547 \end{bmatrix}$$

$$\hat{\text{Var}}(\hat{b}^2_{TH}) = \begin{bmatrix} 0.0125 & -0.0099 \\ -0.0099 & 0.0168 \end{bmatrix}$$

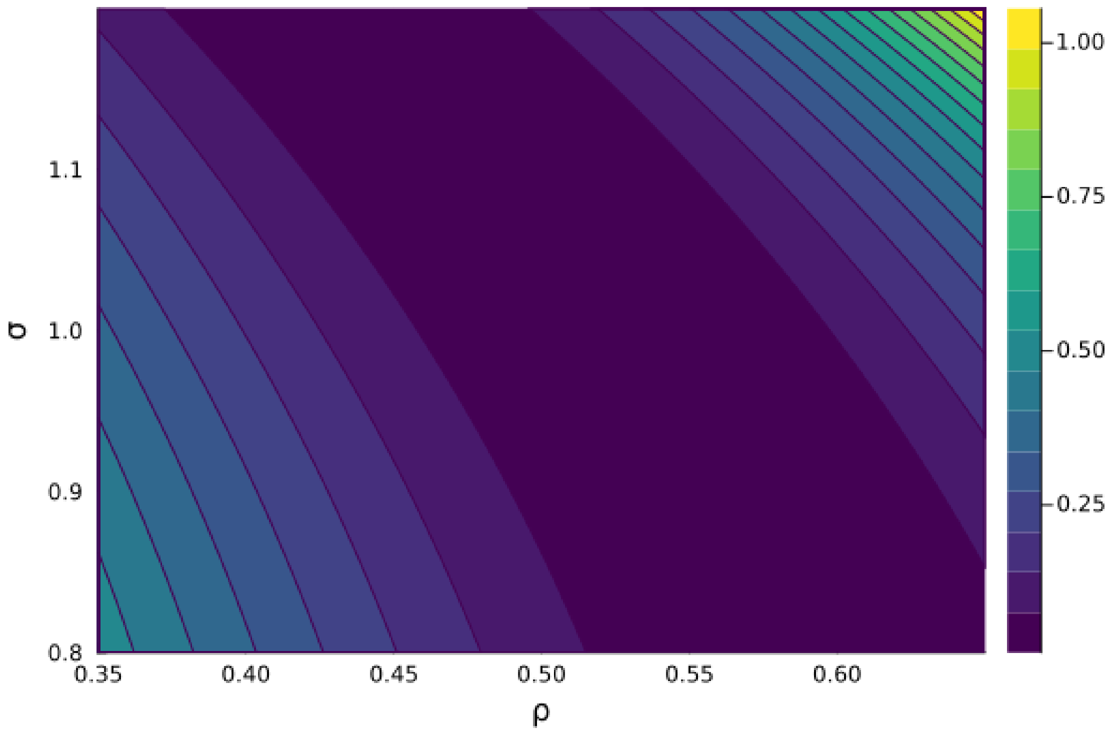
$$\text{SE}(\hat{b}^2_{TH}) = (0.1116, 0.1296)$$

Part 6(d): J-statistic

$$J = 0.3605$$

Figure 1: Plots of $J_{TH}(\rho, \sigma)$ (using mean and variance)

(a) Contour Plot



(b) 3D Surface Plot

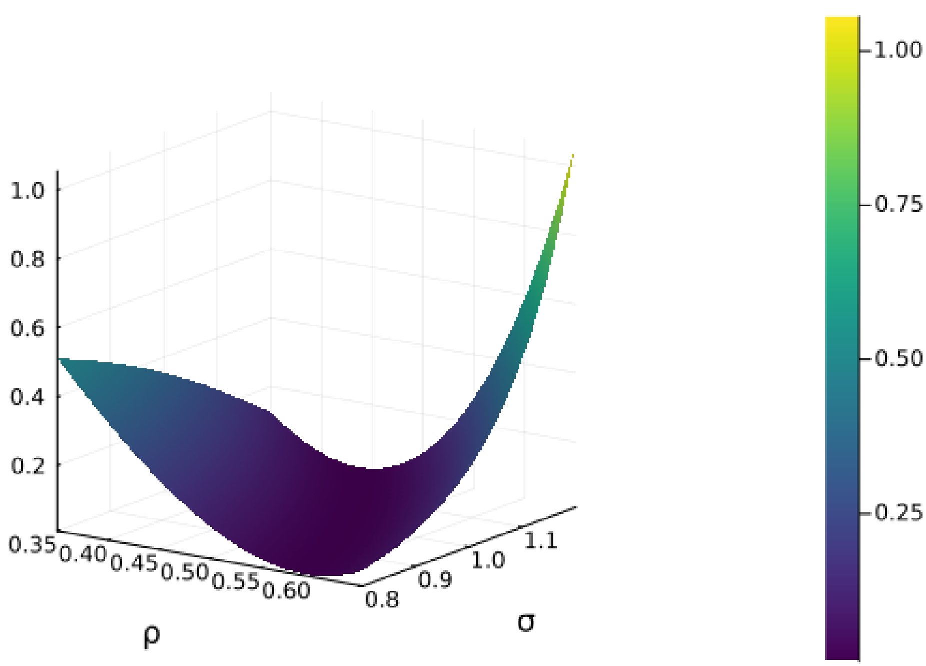
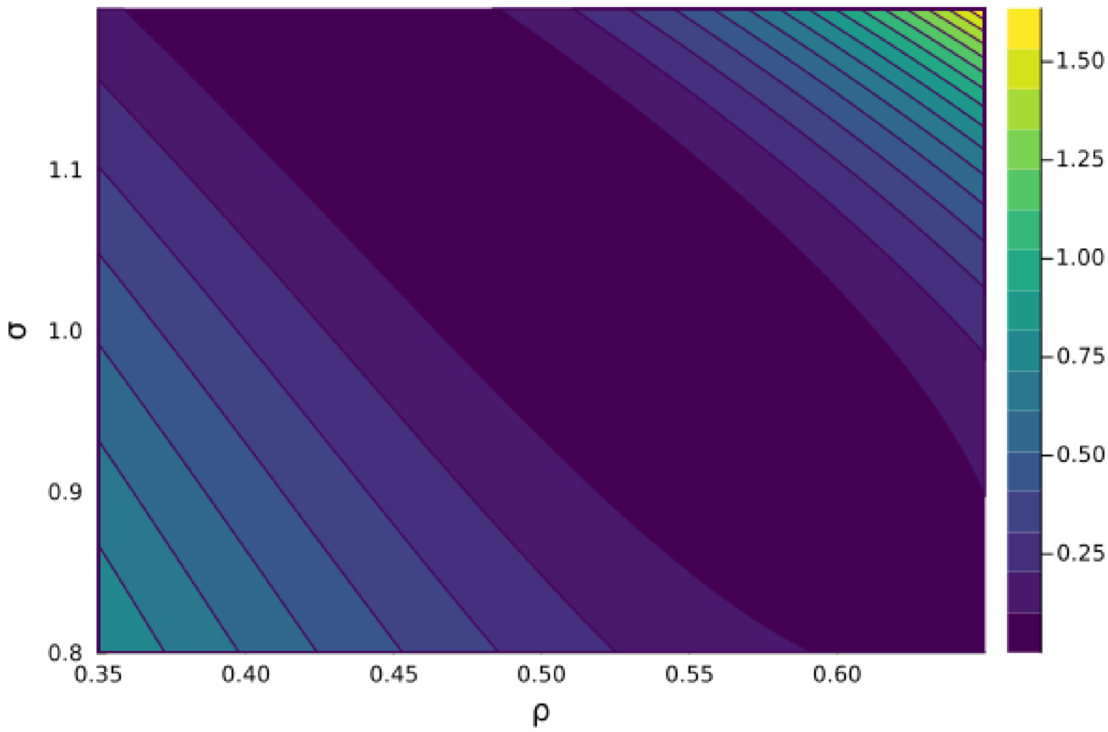


Figure 2: Plots of $J_{TH}(\rho,\sigma)$ (using variance and autocorrelation)

(a) Contour Plot



(b) 3D Surface Plot

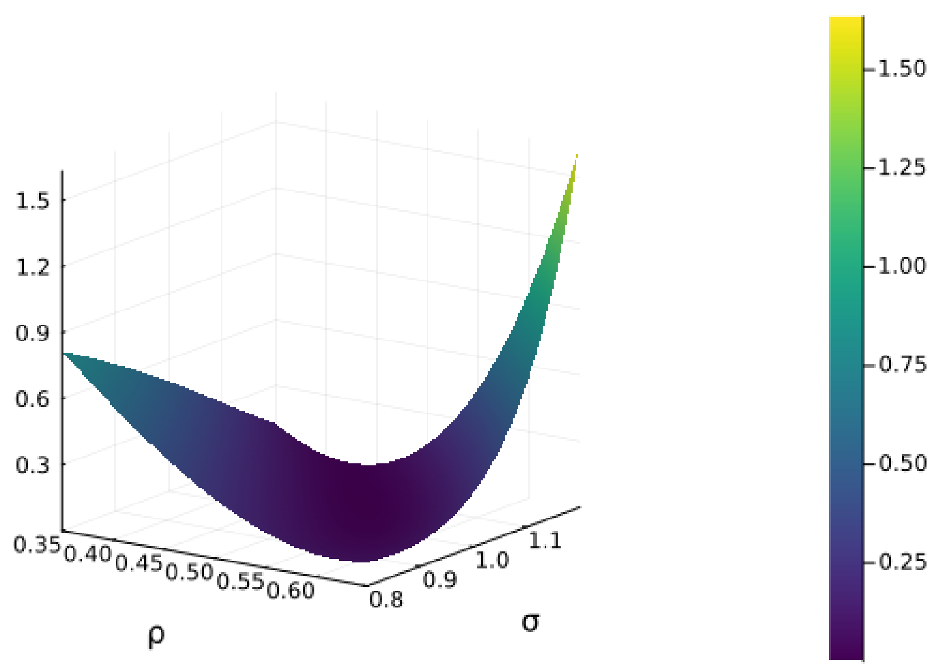
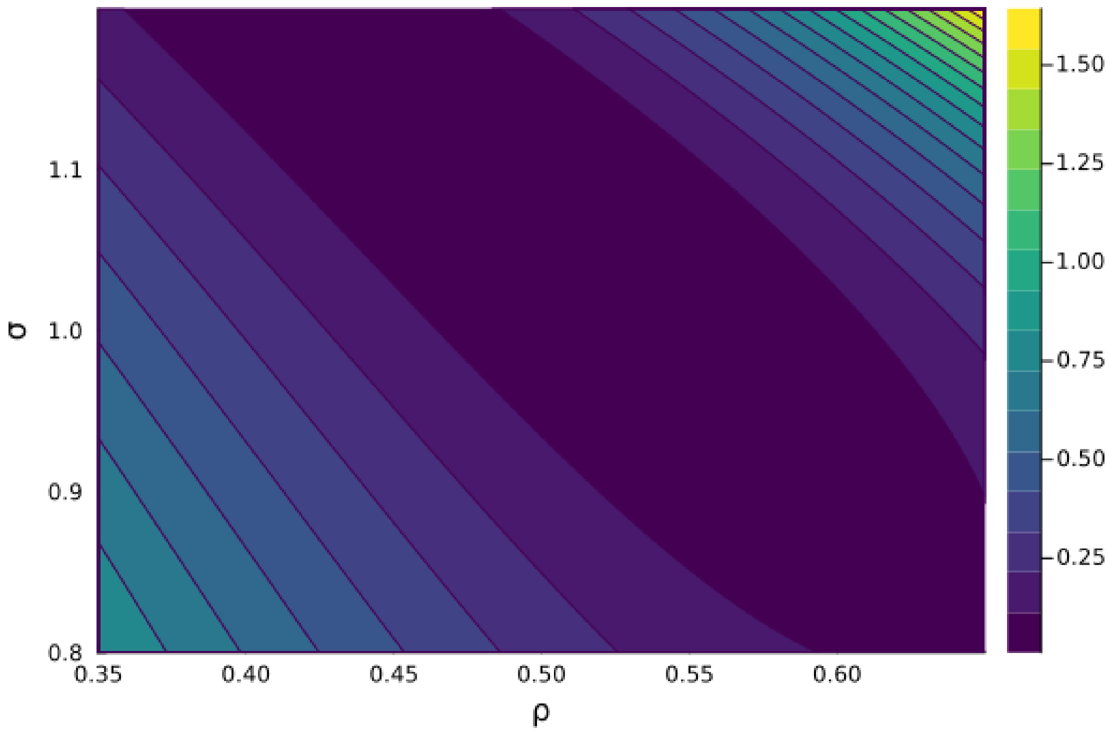


Figure 3: Plots of $J_{TH}(\rho, \sigma)$ (using mean, variance, autocorrelation)

(a) Contour Plot



(b) 3D Surface Plot