BIBB 585-401—Homework 6

Jonathan Levine
jonlevi@sas.upenn.edu

April 3, 2018

Problem 1

$$p(\eta) = \frac{e^{\frac{-\eta^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

$$H(\eta) = -\int_{-\infty}^{\infty} d\eta\ p(\eta)\ log_2(\eta)$$

$$H(\eta) = -\int_{-\infty}^{\infty} d\eta\ p(\eta)\ log_2(\frac{e^{\frac{-\eta^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}})$$

$$H(\eta) = -\int_{-\infty}^{\infty} d\eta\ p(\eta)\ [log_2(e^{\frac{-\eta^2}{2\sigma^2}}) + log_2(2\pi\sigma^2)^{-\frac{1}{2}}]$$

$$H(\eta) = -\int_{-\infty}^{\infty} d\eta\ p(\eta)\ [\ \frac{-\eta^2}{2\sigma^2}log_2(e) - \frac{1}{2}log_2(2\pi\sigma^2)]$$

$$H(\eta) = -\int_{-\infty}^{\infty} d\eta\ p(\eta)\ \frac{-\eta^2}{2\sigma^2}log_2(e) + \int_{-\infty}^{\infty} d\eta\ p(\eta)\ \frac{1}{2}log_2(2\pi\sigma^2)$$

$$H(\eta) = \frac{log_2 e}{2\sigma^2}\int_{-\infty}^{\infty} d\eta\ p(\eta)\ \eta^2 + [\ \frac{1}{2}log_2(2\pi\sigma^2)\int_{-\infty}^{\infty} d\eta\ p(\eta)]$$

$$H(\eta) = [\frac{log_2 e}{2\sigma^2}\int_{-\infty}^{\infty} d\eta\ p(\eta)\ \eta^2] +\ \frac{1}{2}log_2(2\pi\sigma^2)$$

$$H(\eta) = \frac{log_2 e}{2\sigma^2}Var[\eta] +\ \frac{1}{2}log_2(2\pi\sigma^2)$$

$$H(\eta) = \frac{log_2 e}{2\sigma^2}\sigma^2 +\ \frac{1}{2}log_2(2\pi\sigma^2)$$

$$H(\eta) = \frac{1}{2}(log_2 e +\ log_2(2\pi\sigma^2))$$

$$H(\eta) = \frac{1}{2}log_2(2\pi e\sigma^2)$$

Problem 2

Since $\eta_k and \eta_{k+\Delta}$ are independent, we have:

$$Pr[\eta_k, \eta_{k+\Delta}] = Pr[\eta_k]\ Pr[\eta_{k+\Delta}]$$

$$Pr[\eta_k, \eta_{k+\Delta}] = \frac{e^{\frac{-\eta_k{}^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \times \frac{e^{\frac{-(\eta_{k+\Delta}{}^2)}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

$$Pr[\eta_k, \eta_{k+\Delta}] = \frac{1}{2\pi\sigma^2}e^{\frac{-\eta_k{}^2-(\eta_{k+\Delta})^2}{2\sigma^2}}$$

Problem 3

$$I_m = H[X] - H[X|Y]$$

$$I_m = -\int dx\ Pr[x]\ log_2 Pr[x] - \int dy Pr[y] \int dx\ Pr[x|y]\ log_2 Pr[x|y]$$

Using $Pr[x] = \int Pr[x,y]dy$

$$I_m = -\int dx \int dy\ Pr[x,y]\ log_2 Pr[x] + \int dy Pr[y] \int dx\ Pr[x|y]\ log_2 Pr[x|y]$$

Combining into double integrals and using $Pr[x,y] = Pr[x|y]Pr[y]$:

$$I_m = -\int\int dx\ dy\ Pr[x|y]\ Pr[y]\ log_2 Pr[x] + \int\int dx\ dy\ Pr[x|y]\ Pr[y]\ log_2 Pr[x|y]$$

$$I_m = -\int\int dx\ dy\ Pr[x|y]\ Pr[y]\ log_2 Pr[x] +\ Pr[x|y]\ Pr[y]\ log_2 Pr[x|y]$$

$$I_m = \int\int dx\ dy\ Pr[x|y]\ Pr[y]\ (-log_2 Pr[x] +\ log_2 Pr[x|y])$$

$$I_m = \int\int dx\ dy\ Pr[x,y]\ log_2 \frac{P[x|y]}{Pr[x]}$$

Multiplying the fraction in the log_2 by $\frac{Pr[y]}{Pr[y]}$:

$$I_m = \int\int dx\ dy\ Pr[x,y]\ log_2 \frac{P[y]P[x|y]}{Pr[x]Pr[y]}$$

$$I_m = \int\int dx\ dy\ Pr[x,y]\ log_2 \frac{Pr[x,y]}{Pr[x]Pr[y]}$$

Problem 4

The mutual information should be 0. Since they are independent random variables, no information given by one of them has any information about the other. Mathematically, this is true as follows:

$$I_m = \int\int d\eta_k\ d\eta_{k+\Delta}\ Pr[\eta_k, \eta_{k+\Delta}]\ log_2 \frac{Pr[\eta_k, \eta_{k+\Delta}]}{Pr[\eta_k]Pr[\eta_{k+\Delta}]}$$

For independent random variables:

$$Pr[\eta_k, \eta_{k+\Delta}] = Pr[\eta_k]\ Pr[\eta_{k+\Delta}]$$

$$\frac{Pr[\eta_k, \eta_{k+\Delta}]}{Pr[\eta_k]Pr[\eta_{k+\Delta}]} = 1$$

$$log_2\ \frac{Pr[\eta_k, \eta_{k+\Delta}]}{Pr[\eta_k]Pr[\eta_{k+\Delta}]} = 0$$

$$I_m = 0$$

Problem 5

Since $n_k = 0$, we have:

$$V_1 = \gamma V_0$$
$$V_2 = \gamma V_1 = \gamma\gamma V_0$$
$$V_3 = \gamma V_2 = \gamma\gamma\gamma V_0$$
$$V_k = \gamma^k V_0$$

It is clear from this that for V_l to go to zero as k goes to ∞, it must be that $\gamma < 1$. For values of $\gamma > 1$, V_k goes to ∞, and for $\gamma = 1$ $V_k = V_0$

Problem 6

$$Pr(V_k; V_{k+1}) = Pr(V_k)Pr(V_{k+1}|V_k)$$
$$Pr(V_k; V_{k+1}) = Pr(V_k)\int d\eta_k \; Pr(\eta_k) \; Pr(V_{k+1}|\eta_k, V_k)$$
$$Pr(V_k; V_{k+1}) = Pr(V_k)\int d\eta_k \; Pr(\eta_k) \; \delta(\eta_k - (V_{k+1} - \gamma V_k)$$

Integrating the delta function, we find the function $Pr(\eta_k)$ where the argument of the delta function is 0, namely where $\eta_k = V_{k+1} - \gamma V_k$

$$Pr(V_k; V_{k+1}) = Pr(V_k) \; Pr(\eta_k = V_{k+1} - \gamma V_k)$$
$$Pr(V_k; V_{k+1}) = Pr(V_k) \; \frac{e^{\frac{-(V_{k+1}-\gamma V_k)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$
$$Pr(V_k; V_{k+1}) = \frac{e^{\dfrac{-V_k^2}{2\sigma_v^2}}}{\sqrt{2\pi\sigma_v^2}} \; \frac{e^{\dfrac{-(V_{k+1}-\gamma V_k)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$
$$Pr(V_k; V_{k+1}) = \frac{e^{\dfrac{-1}{2}(\dfrac{V_k^2}{\sigma_v^2} + \dfrac{(V_{k+1}-\gamma V_k)^2}{\sigma^2})}}{2\pi\sigma_v\sigma}$$

This is essentially saying that the joint probability of V_k and V_{k+1} is the product of the probability distribution of V_k and the probability distribution of the recurrence relation being exactly the right amount to map V_k to V_{k+1}

Problem 7

$$Pr(V_{k+1}) = \int Pr(V_{k+1}, V_k) \; dV_k$$
$$Pr(V_{k+1}) = \int \frac{e^{\dfrac{-1}{2}(\dfrac{V_k^2}{\sigma_v^2} + \dfrac{(V_{k+1}-\gamma V_k)^2}{\sigma^2})}}{2\pi\sigma_v\sigma} \; dV_k$$
$$Pr(V_{k+1}) = \frac{1}{2\pi\sigma_v\sigma}\int e^{\dfrac{-1}{2}(\dfrac{V_k^2}{\sigma_v^2} + \dfrac{(V_{k+1}-\gamma V_k)^2}{\sigma^2})} \; dV_k$$

$$Pr(V_{k+1}) = \frac{1}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{V_k^2}{\sigma_v^2} + \frac{\gamma^2 V_k^2}{\sigma^2} - \frac{2V_{k+1}V_k\gamma}{\sigma^2} + \frac{V_{k+1}^2}{\sigma^2}} dV_k$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}(\frac{V_{k+1}^2}{\sigma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{V_k^2}{\sigma_v^2} + \frac{\gamma^2 V_k^2}{\sigma^2} - \frac{2V_{k+1}V_k\gamma}{\sigma^2}} dV_k$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}(\frac{V_{k+1}^2}{\sigma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(V_k^2(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2}) - V_k(\frac{2V_{k+1}\gamma}{\sigma^2})} dV_k$$

Let $\alpha = \dfrac{\gamma V_{k+1}}{\sigma^2(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2})}$

$\alpha^2 = \dfrac{\gamma^2\ V_{k+1}^2}{\sigma^4(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2})^2}$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}(\frac{V_{k+1}^2}{\sigma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2})[V_k^2 - 2V_k\alpha]} dV_k$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}(\frac{V_{k+1}^2}{\sigma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2})[V_k^2 - 2V_k\alpha + \alpha^2 - \alpha^2]} dV_k$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}(\frac{V_{k+1}^2}{\sigma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{1}{\sigma_v^2} + \frac{\gamma^2}{\sigma^2})[(V_k-\alpha)^2 - \alpha^2]} dV_k$$

Factoring out the constants from the integrand:

$$Pr(V_{k+1}) = \frac{e^{\frac{-V_{k+1}^2}{2\sigma^2}(1 - \frac{\gamma^2}{\frac{\sigma^2}{\sigma_v^2} + \gamma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2}(\frac{\sigma^2 + \gamma^2\sigma_v^2}{\sigma_v^2\sigma^2})(V_k-\alpha)^2} dV_k$$

Let $b^2 = \dfrac{\sigma_v^2\sigma^2}{\sigma^2 + \gamma^2\sigma_v^2}$

$$Pr(V_{k+1}) = \frac{e^{\frac{-V_{k+1}^2}{2\sigma^2}(1 - \frac{\gamma^2}{\frac{\sigma^2}{\sigma_v^2} + \gamma^2})}}{2\pi\sigma_v\sigma}\int e^{\frac{-1}{2b^2}(V_k-\alpha)^2} dV_k$$

Multiply by $\dfrac{\sqrt{2\pi b^2}}{\sqrt{2\pi b^2}}$

$$Pr(V_{k+1}) = \frac{\sqrt{2\pi b^2}\ e^{\frac{-V_{k+1}^2}{2\sigma^2}(1 - \frac{\gamma^2}{\frac{\sigma^2}{\sigma_v^2} + \gamma^2})}}{2\pi\sigma_v\sigma}\int \frac{e^{\frac{-1}{2b^2}(V_k-\alpha)^2}}{\sqrt{2\pi b^2}} dV_k$$

The integral converges to 1

$$Pr(V_{k+1}) = \frac{\sqrt{2\pi b^2}\ e^{\frac{-V_{k+1}^2}{2\sigma^2}(1 - \frac{\gamma^2}{\frac{\sigma^2}{\sigma_v^2} + \gamma^2})}}{2\pi\sigma_v\sigma}$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-V_{k+1}^2}{2\sigma^2}(1 - \frac{\gamma^2}{\frac{\sigma^2}{\sigma_v^2}+\gamma^2})}}{\sqrt{2\pi(\sigma^2+\gamma^2\sigma_v^2)}}$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-V_{k+1}^2}{2\sigma^2}(\frac{\frac{\sigma^2}{\sigma_v^2}+\gamma^2-\gamma^2}{\frac{\sigma^2}{\sigma_v^2}+\gamma^2})}}{\sqrt{2\pi(\sigma^2+\gamma^2\sigma_v^2)}}$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-V_{k+1}^2}{2}(\frac{\frac{1}{\sigma_v^2}}{\frac{\sigma^2}{\sigma_v^2}+\gamma^2})}}{\sqrt{2\pi(\sigma^2+\gamma^2\sigma_v^2)}}$$

$$Pr(V_{k+1}) = \frac{e^{\frac{-1}{2}\frac{-V_{k+1}^2}{\sigma^2+\gamma^2\sigma_v^2})}}{\sqrt{2\pi(\sigma^2+\gamma^2\sigma_v^2)}}$$

This is a new Gaussian with mean 0, variance $\sigma^2+\gamma^2\sigma_v^2$

Problem 8

$$\sigma^2+\gamma^2\sigma_v^2 = \sigma_v^2$$

$$\sigma_v^2 = \frac{\sigma^2}{1-\gamma^2}$$

$$\sigma_v = \sqrt{\frac{\sigma^2}{1-\gamma^2}}$$

Problem 9

From problem number 1, we know that for a Gaussian random variable with $Var[X] = \sigma^2$:

$$H[X] = \frac{1}{2}log_2(2\pi e\sigma^2)$$

Based on the variance just calculated in #8:

$$H[V_{k+1}] = \frac{1}{2}log_2(2\pi e(\sigma^2+\gamma^2\sigma_v^2))$$

From #6: $Pr[V_{k+1}|V_k] = \frac{e^{\frac{-(V_{k+1}-\gamma V_k)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$

$$H[V_{k+1}|V_k] = \frac{1}{2}log_2(2\pi e\sigma^2)$$

$$I_m = H[V_{k+1}] - H[V_{k+1}|V_k]$$

$$\frac{1}{2}log_2(2\pi e\sigma^2)$$

$$I_m = H[V_{k+1}] = \frac{1}{2}log_2(2\pi e(\sigma^2 + \gamma^2\sigma_v^2)) - \frac{1}{2}log_2(2\pi e\sigma^2)$$

$$I_m = H[V_{k+1}] = \frac{1}{2}log_2\frac{\sigma^2 + \gamma^2\sigma_v^2}{\sigma^2}$$

When $\sigma_v = \sqrt{\frac{\sigma^2}{1-\gamma^2}}$:

$$I_m = H[V_{k+1}] = \frac{1}{2}log_2\frac{\sigma_v^2}{\sigma^2}$$

Problem 10

Supposed you were to flip two coins regularly. The two flips are independent, and there is no mutual information. Suppose now that we do the same thing but with another rule: whatever the first coin was (heads or tails) the second coin had to be exactly the opposite. We can say that not given the first coin, the second coin has the same distribution as the first coin (half tails half heads). However, once the first coin is flipped, the second coin is predetermined and has no entropy. The mutual information between the two would be 1 bit. Here too, just because the probability distributions match, since there is a dependency of V_{k+1} on V_k as given by the recurrence relation, there is some mutual information given by it. Whereas before their independence showed that their was no mutual information, here the dependence shows that some information about one provides information about the other.