Dynamic Models in Biology
HW 5
Jonathan Levine
Fall 2023

Problem 1

Supercritical pitchfork bifurcation

$\dot{x} = \mu x - x^3 \quad \dot{y} = -y$

Fixed points

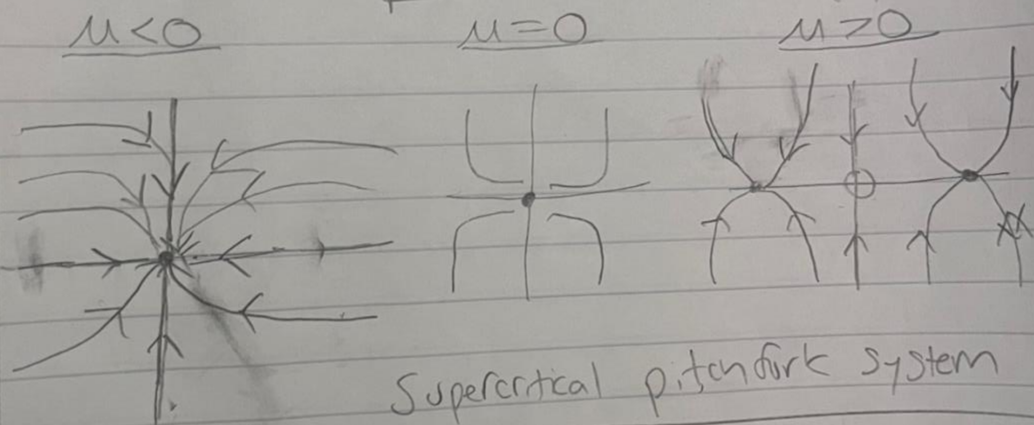
$\dot{y} = 0 \Rightarrow y = 0 \qquad \dot{x} = 0 \Rightarrow x = 0$ or $x^2 = \mu \Rightarrow x = \pm\sqrt{\mu}$

fixed points: $(0,0)$ $(\sqrt{\mu},0)$ $(-\sqrt{\mu},0)$

$$J = \begin{bmatrix} \mu - 3x^2 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{diagonal} \Rightarrow \lambda = \{-1, \mu - 3x^2\}$$

Point	λ values	$\mu < 0$	$\mu > 0$
$(0,0)$	$(\mu, -1)$	stable node	saddle
$(\sqrt{\mu},0)$	$(-2\mu, -1)$	does not exist	stable node
$(-\sqrt{\mu},0)$	$(-2\mu, -1)$	does not exist	stable node

this is a pitchfork bifurcation (supercritical)



Supercritical pitchfork system

Problem 2
saddle-node bifurcation

$\dot{x} = y - 2x$, $\dot{y} = \mu + x^2 - y$

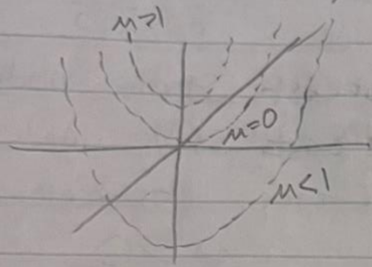
fixed points

$y = 2x$ and $y = x^2 + \mu \Rightarrow x^2 - 2x + \mu = 0$

$x = \frac{2 \pm \sqrt{4 - 4\mu}}{2} = 1 \pm \sqrt{1-\mu}$ $y = 2(1 \pm \sqrt{1-\mu})$

x nullcline $\Rightarrow y = 2x$

y nullcline $\Rightarrow y = x^2 + \mu$



$\mu > 1 \Rightarrow$ no fixed points

$\mu < 1$ $J = \begin{bmatrix} -2 & 1 \\ 2x & -1 \end{bmatrix}$

$x = 1 + \sqrt{1-\mu} \Rightarrow \begin{bmatrix} -2 & 1 \\ z & -1 \end{bmatrix}$ for positive z

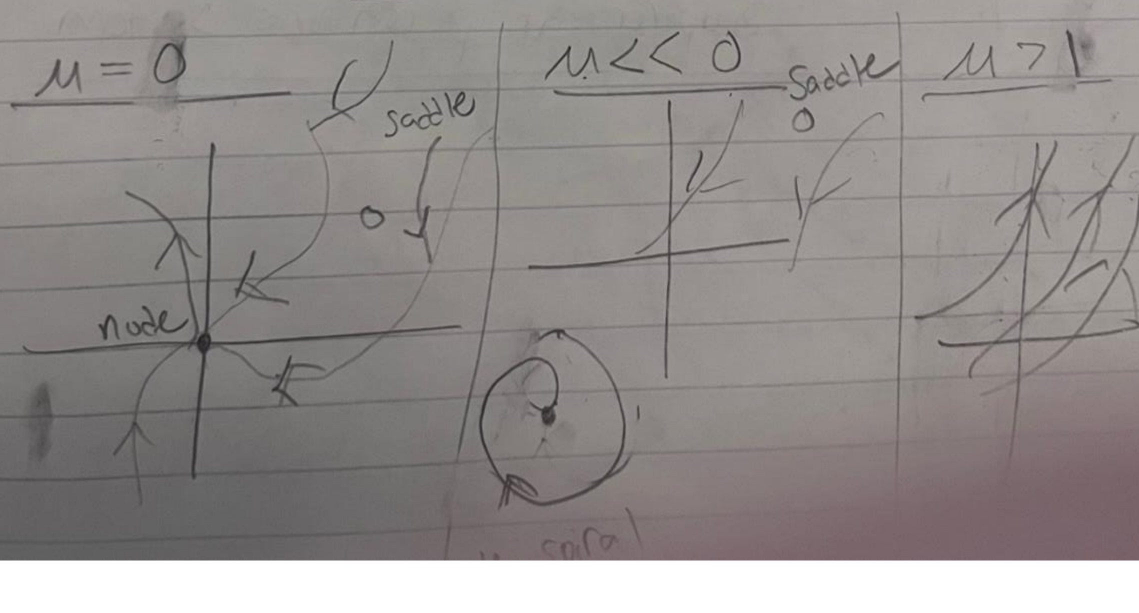
saddle

$x = 1 - \sqrt{1-\mu}$ say $\mu = 0$

$\begin{bmatrix} -2 & 1 \\ 2(1-\sqrt{1-\mu}) & -1 \end{bmatrix} \Rightarrow \lambda = (-2, -1) \Rightarrow$ stable node
(or spiral for smaller μ)

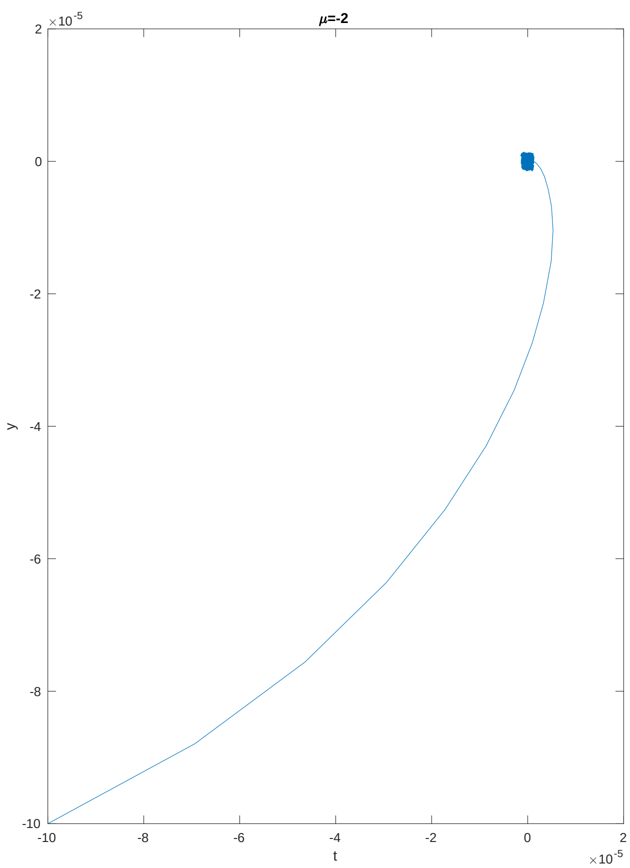
saddle + node $\Longrightarrow$ no fixed points

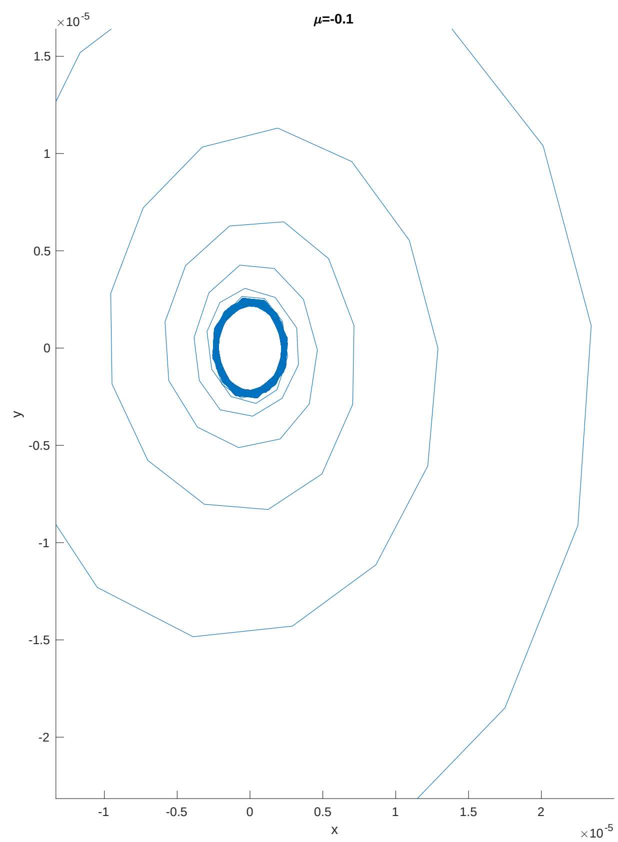
creates saddle node bifurcation at $\mu = 1$

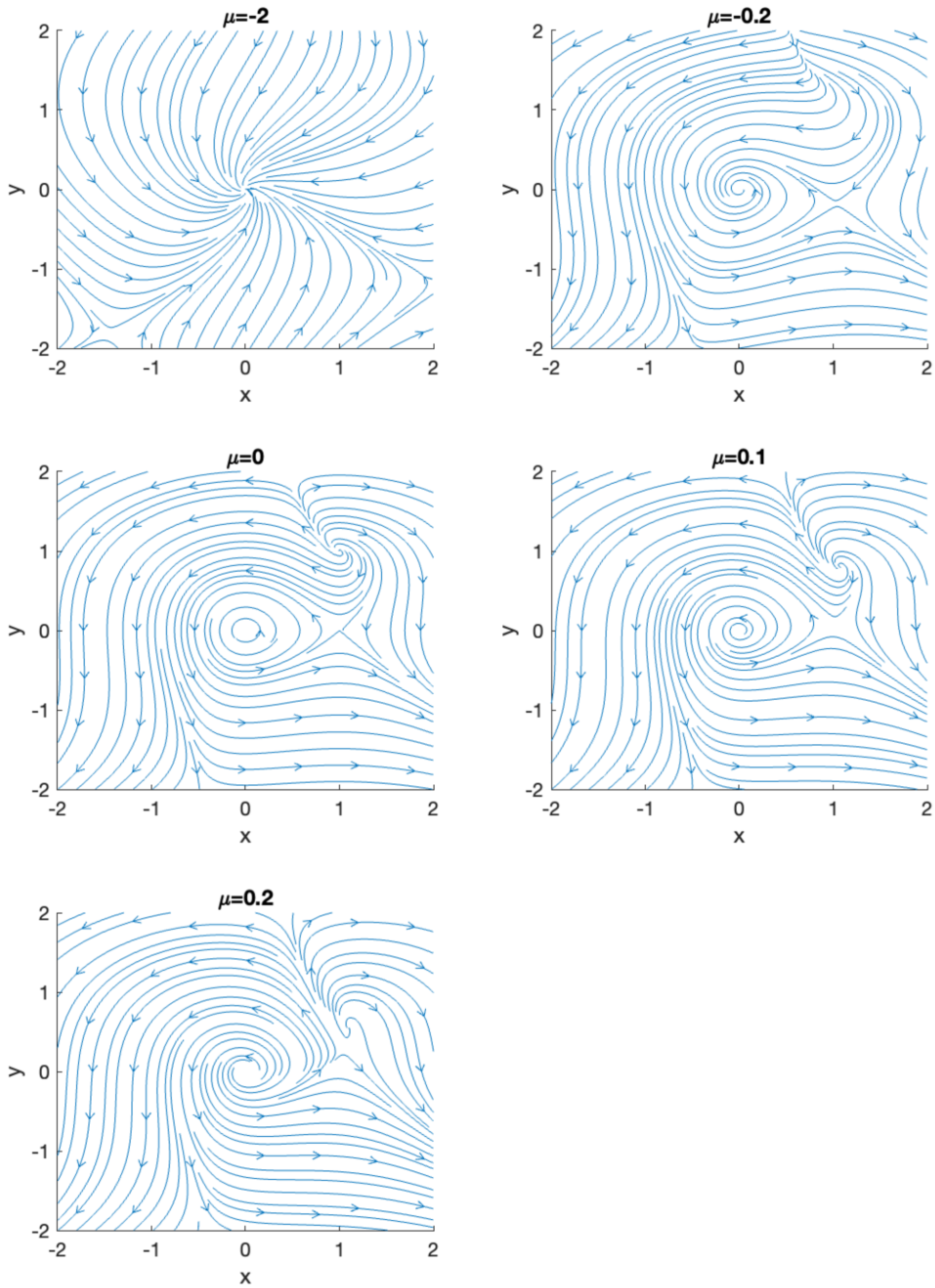


Problem 3

This is a subcritical Hopf bifurcation. This is because for mu less than 0, we see that there is a stable limit cycle for some initial conditions, and a stable node for other initial conditions. This indicates subcritical, because for a supercritical for mu < bifurcation point, there is no point for which there is both a stable point and a limit cycle.






μ=-2
μ=-0.2
μ=0
μ=0.1
μ=0.2
x
y
-2
-1
0
1
2

Problem 4

There is a **transcritical bifurcation** for beta = ½ (assuming N=2, gamma=nu=1). We can see that the 2 fixed points collide at beta=(1/2), where (2,0) and (1/beta, 1-1/2beta) are the same. They then flip stability as you can see from the Jacobian eigenvalues for (2,0) and for the determinant flipping signs for (and (1/beta, 1-1/2beta)

Assuming $N=2, \gamma=\nu=1$

$\dot{S} = -\beta S I + 2 - S - I \qquad \dot{I} = \beta S I - I$

fixed points

$\dot{I}=0 \Rightarrow \beta S I = I \Rightarrow I=0 \quad \text{or} \quad S = \frac{1}{\beta}$

$I=0 \Rightarrow \dot{S}=0 \Rightarrow S=2 \Rightarrow (2,0)$

$S=\frac{1}{\beta} \Rightarrow \dot{S}=0 = -I + 2 - I - \frac{1}{\beta} \Rightarrow 2I = 2 - \frac{1}{\beta}$

$I = 1 - \frac{1}{2\beta} \Rightarrow \left(\frac{1}{\beta}, 1 - \frac{1}{2\beta}\right)$

$$J = \begin{bmatrix} -\beta I - 1 & -\beta S - 1 \\ \beta I & \beta S - 1 \end{bmatrix}$$

$$J(2,0) \Rightarrow \begin{bmatrix} -1 & -2\beta - 1 \\ 0 & 2\beta - 1 \end{bmatrix} \Rightarrow \lambda = \{-1, 2\beta - 1\}$$

Saddle for $\beta > \frac{1}{2}$, node for $\beta < \frac{1}{2}$

$$J\left(\frac{1}{\beta}, 1 - \frac{1}{2\beta}\right) = \begin{bmatrix} -\beta - \frac{1}{2} & -2 \\ \beta - \frac{1}{2} & 0 \end{bmatrix} \qquad \begin{aligned} \Delta &= 2\beta - 1 \\ \tau &= -\beta - \frac{1}{2} \end{aligned}$$

τ is always negative for positive β

Δ goes from positive to negative as β decreases and flips signs at $\beta = \frac{1}{2}$

for $\beta < \frac{1}{2}$ you have saddle & node

for $\beta > \frac{1}{2}$ you have stable spiral point & saddle

for $\beta = \frac{1}{2}$ the fixed points collide & exchange stability. $\beta = \frac{1}{2}$ is transcritical bifurcation