Dynamic Models in Biology
Lab 3 Report
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Fixed points
We can verify the fixed points of the system:

$$\dot{S} = -\beta SI + \gamma(N - S - I)$$
$$\dot{I} = \beta SI - \nu I$$
$$where\ N = 2, \beta = \nu = \gamma = 1$$
$$\dot{S} = -SI + 2 - S - I$$
$$\dot{I} = SI - I$$
$$\dot{I} = 0 => SI = I => either\ I = 0\ or\ S = 1$$
$$when\ I = 0, \quad \dot{S} = 0 => S = 2 => s^*, i^* = (2,0)$$
$$when\ S = 1, \quad \dot{S} = 0 => -2I = -1 => I = \frac{1}{2} => s^*, y^* = (1, \frac{1}{2})$$

We can compute the Jacobian based on the partial derivatives of the simplified system:

$$J = \begin{bmatrix} \frac{\partial \dot{s}}{\partial s} & \frac{\partial \dot{s}}{\partial I} \\ \frac{\partial \dot{i}}{\partial s} & \frac{\partial \dot{i}}{\partial I} \end{bmatrix} = \begin{bmatrix} -I - 1 & -S - 1 \\ I & S - 1 \end{bmatrix}$$

For the fixed point (2,0):

$$J(S = 2, I = 0) = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$$

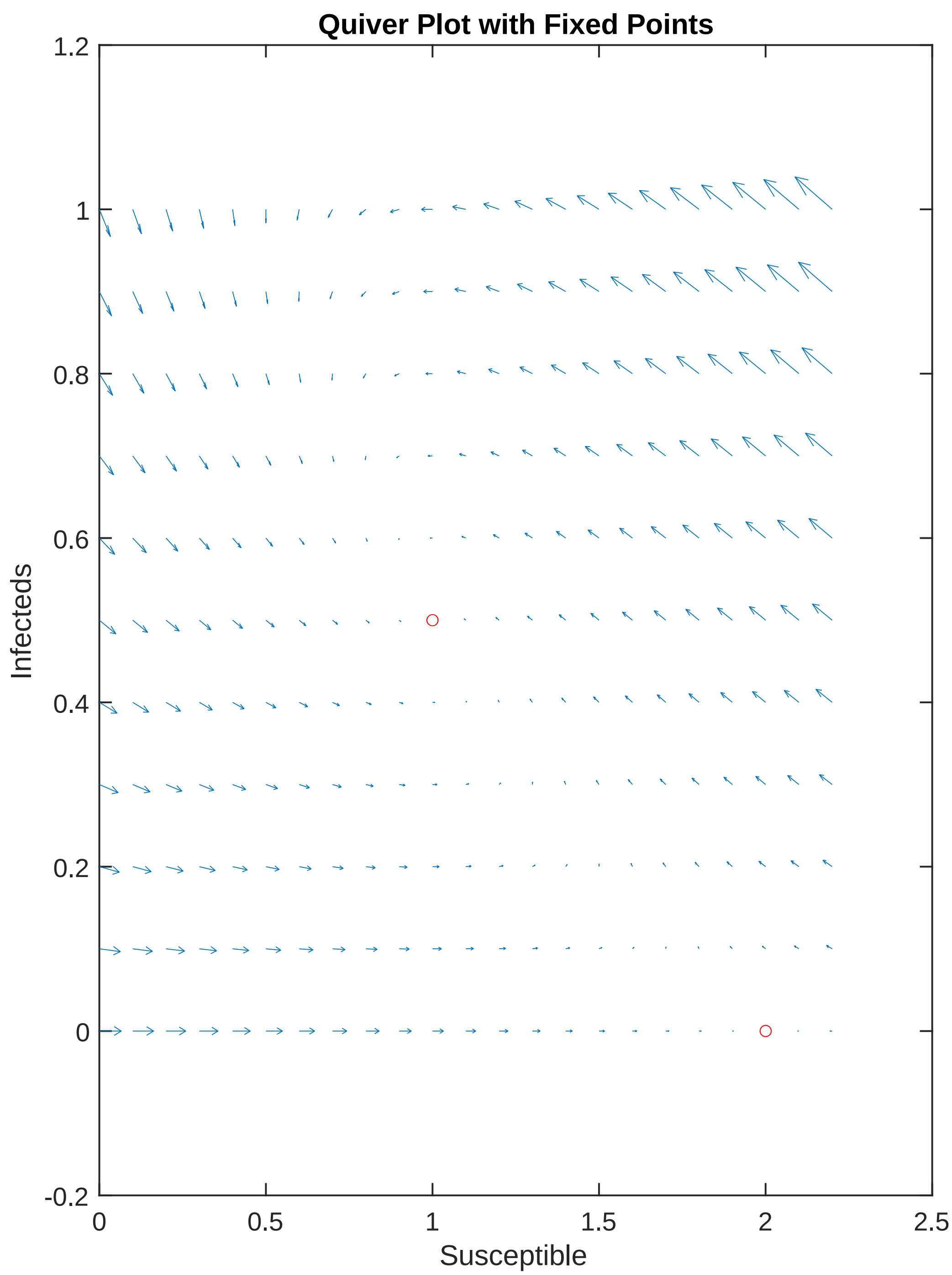
Which has eigenvalues along the diagonal as -1 and 1. When interpreted as a 2D linear system this represents a saddle point, where the first eigensolution grows exponentially, and the second eigensolution decays.

For the fixed point (1,0.5):

$$J(S = 1, I = 0.5) = \begin{bmatrix} -1.5 & -2 \\ 0.5 & 0 \end{bmatrix}$$

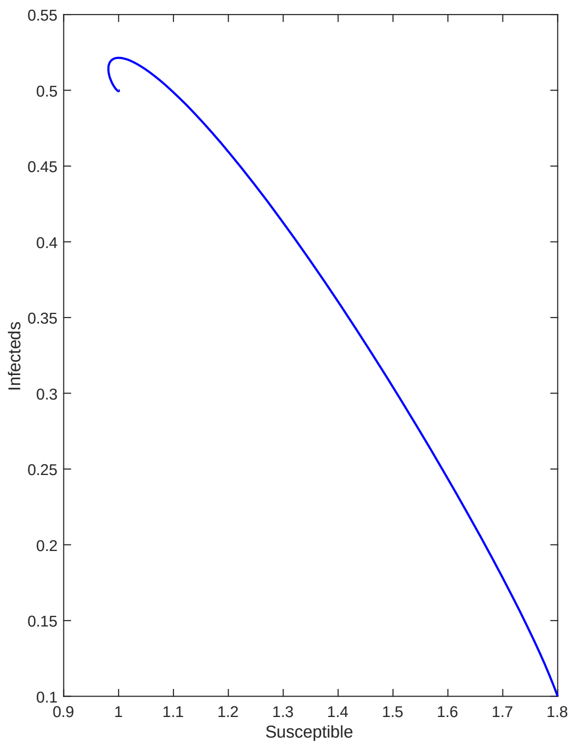
Which has complex eigenvalues (computed in matlab) -0.75 +/- .6614i. When interpreted as a 2D linear system this represents a stable spiral.

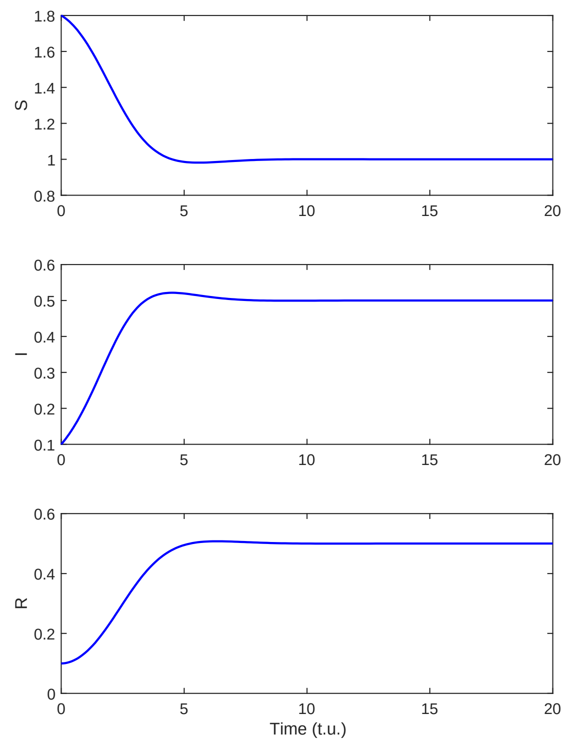
Quiver Plot



Quiver Plot with Fixed Points
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
Infecteds
0
0.5
1
1.5
2
2.5
Susceptible

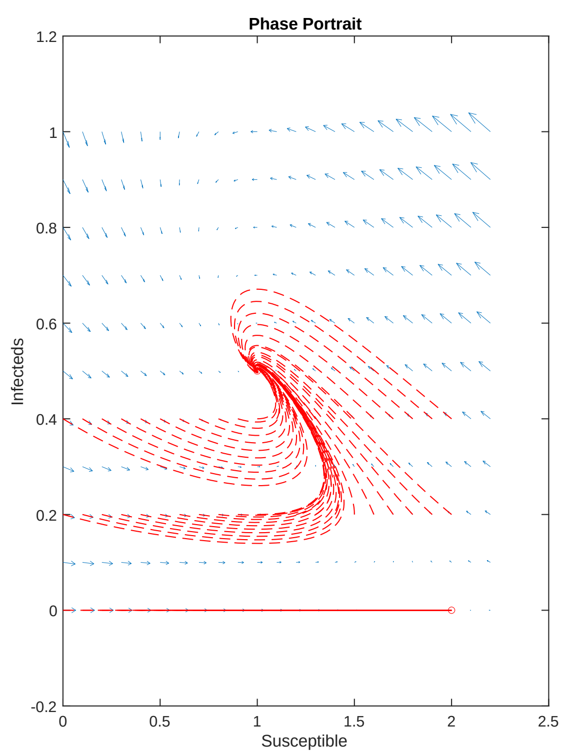
Numerical simulation





Running the given simulation for y0 = [1.8; 0.1], with the phase trajectory plot shown on the left, and the time courses for the individual state variables shown on the right. These dynamics reveal that the number of infections never really rises to levels that would be considered epidemic levels, and quickly (over time) spirals to the stable fixed point at (1,0.5).

With various initial conditions, we can fill out a full phase portrait:

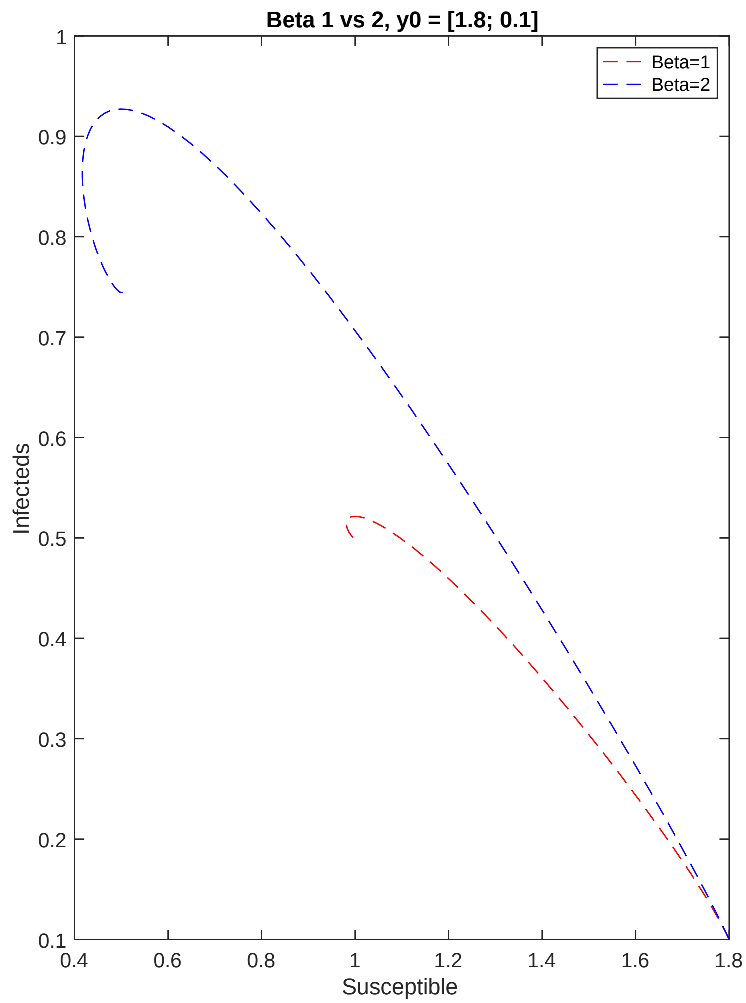


This matches our linearization which showed unstable spiral dynamics at (1,0.5) and an unstable saddle at (2,0) which can only be reached when *I* starts at 0

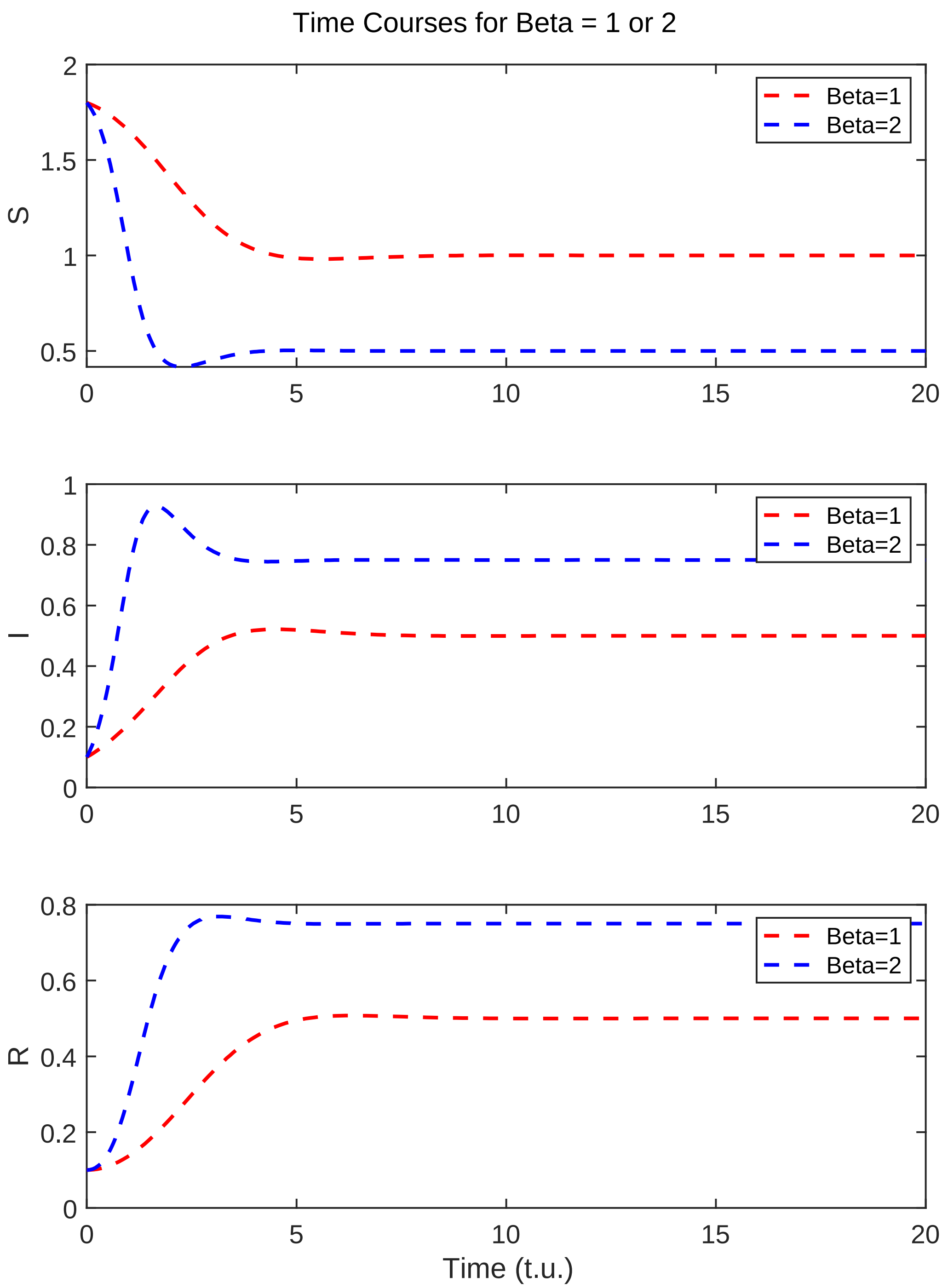
Increased transmission rate

If we were to increase the transmission rate, the saddle point would not change (although the unstable eigendirection would grow more unstable), but the stable spiral point would shift significantly (but with the same eigenvalues/spiral pattern). This would probably lead to spirals that could lead to epidemic level infections (depending on initial conditions) before decaying to the stable fixed point.

Showing the phase trajectory for the initial condition explored earlier, we do see that the spiral point has shifted and is now including a trajectory that could be described as an epidemic:

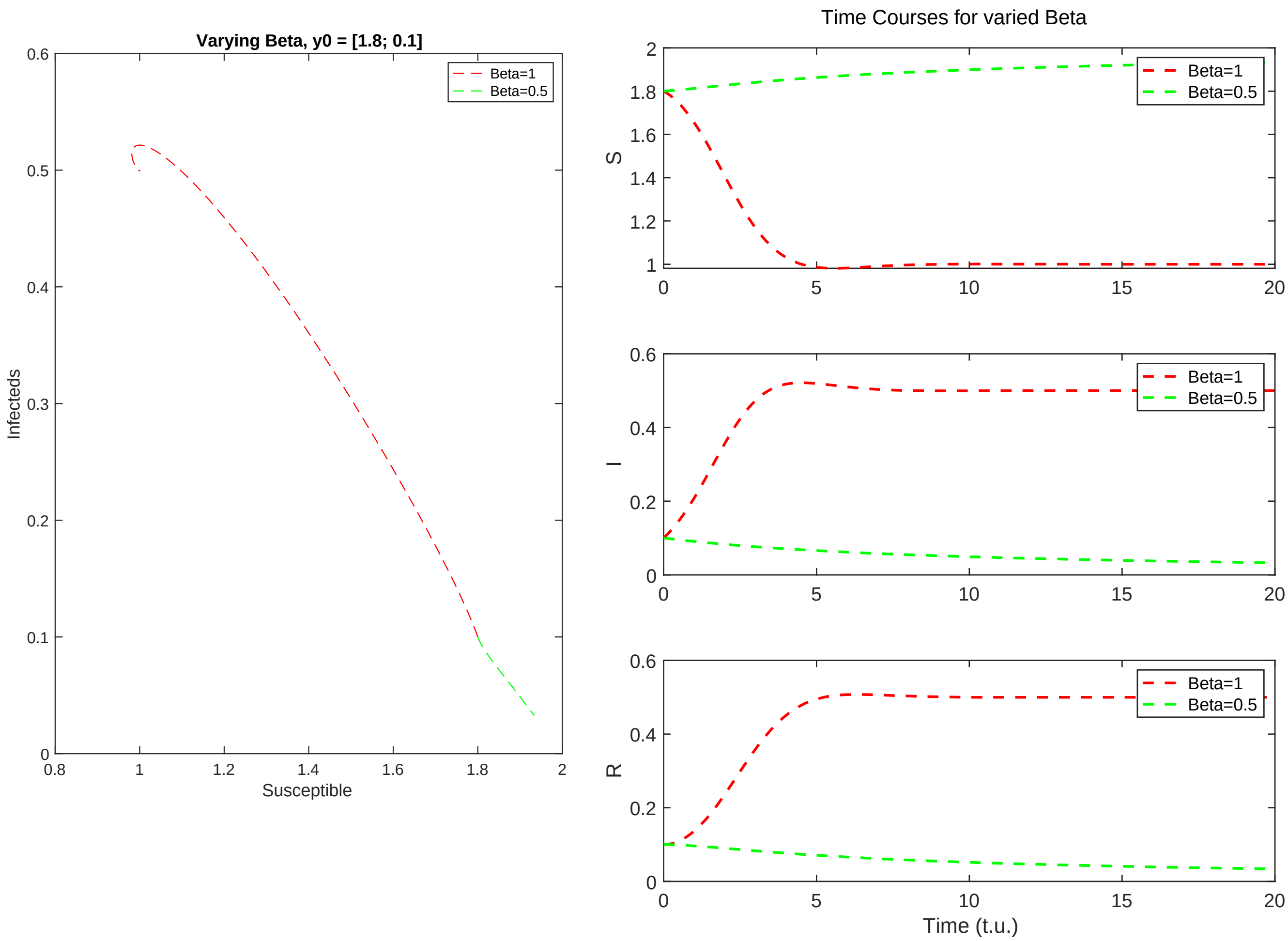


Looking at the individual trajectories also reveals a significantly higher peak infection for beta=2, as well as different stable points at t=infinity for all variables, reflecting the new fixed point.



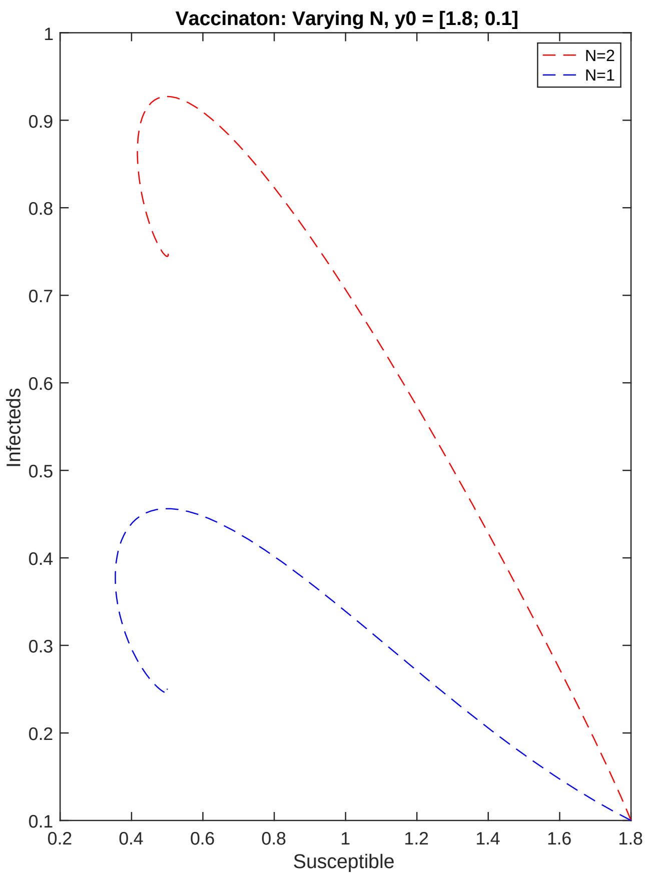
Decreased transmission rate

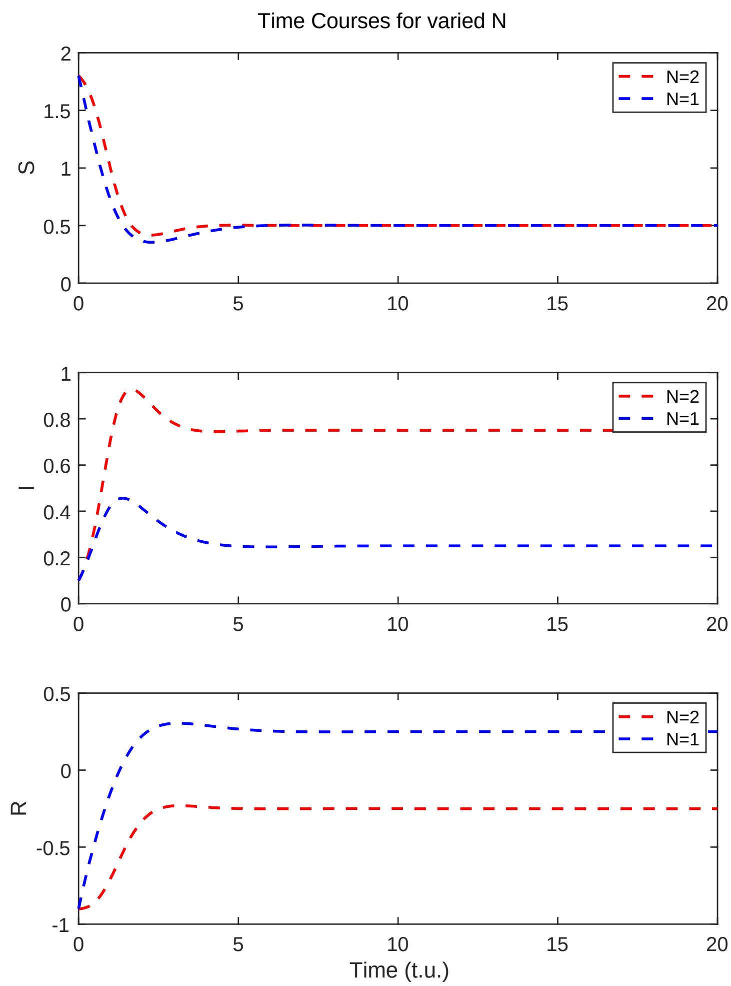
We see the opposite shift in the spiral point for beta = 0.5, which helps to reduce the prospects of an epidemic greatly:



Vaccination (Decreased Effective Population Size)

Decreasing N by a factor of 2, even with the high transmission rate of beta=2, seems to do a very good job at preventing epidemic levels of disease in this model. This has the effect of shifting the spiral point down and keeping infections lower even as there is a high transmission rate.





Testing and Stay at Home (increasing ν)

Inspired by real events, one might imagine that there could be a way to test for being infected even before feeling sick. If we were to widely test people in the population, and if they test positive require them to stay home, this would be equivalent to increasing ν in the simulation. As ν represents the rate of transitioning from the "Infected" to the "Removeds" group, we can essentially increase this rate by making sure individuals know when they are infected, and having them stay home and become "removed".

This does in fact work, as we can see in the simulation: