Dynamic Models in Biology
Homework 2
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Strogatz 5.2.1

The steps are
- Find eigenvalues (and classify fixed point from them)
- Find eigenvectors using eigenvector/eigenvalue equation
- Solution is linear combination of eigenvectors and exponentials, use initial conditions to get specific solution

5.2.1 $\quad \dot{x} = 4x - y \quad \dot{y} = 2x + y$

a) $\dot{\vec{x}} = A\vec{x}$ where $\dot{\vec{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{bmatrix} = (4-\lambda)(1-\lambda) + 2 = \boxed{\lambda^2 - 5\lambda + 6}$$

$$\lambda^2 - 5\lambda + 6 \doteq 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \implies \lambda = 2, 3$$

$\lambda = 2$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2v_1 - v_2 = 0$

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any vector where $v_2 = 2v_1$ like $[1, 2]$

$\lambda = 3$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 - v_2 = 0$

$2v_1 - 2v_2 = 0$

any vector where $v_1 = v_2$ like $[1, 1]$

b) general solution $x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$

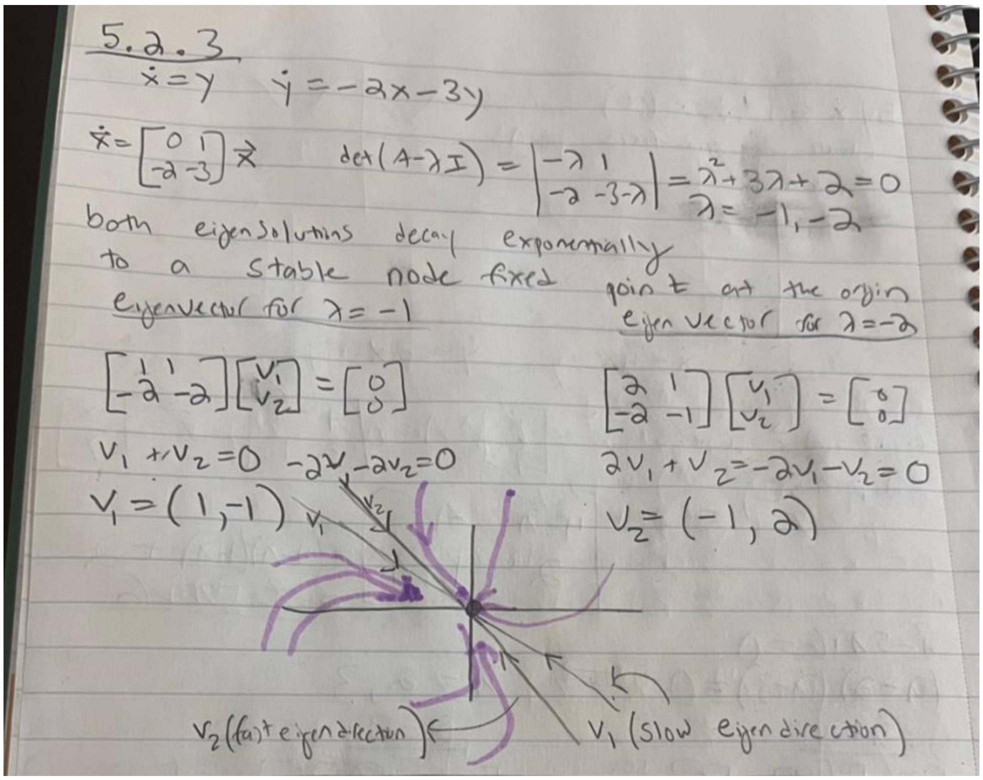
c) $\lambda_2 > \lambda_1 > 0 \implies$ unstable node

at $t=0$

d) $\begin{bmatrix} 3 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} 3 = c_1 + c_2 \\ 4 = 2c_1 + c_2 \\ c_1 = 1 \quad c_2 = 2 \end{matrix} \Rightarrow \boxed{X(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}}$

Strogatz 5.2.3

I am not very good at sketching, but the idea for the sketch is that the trajectories should travel more quickly along the fast eigendirection (corresponding to eigenvector for eigenvalue with larger absolute value), and then travel tangent to the slower eigendirection (corresponding to eigenvector for eigenvalue with smaller absolute value).



5.2.3

$\dot{x} = y \quad \dot{y} = -2x - 3y$

$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \vec{x} \qquad \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0$

$\lambda = -1, -2$

both eigensolutions decay exponentially to a stable node fixed point at the origin

eigenvector for $\lambda = -1$

$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 + v_2 = 0 \quad -2v_1 - 2v_2 = 0$

$v_1 = (1, -1)$

eigenvector for $\lambda = -2$

$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2v_1 + v_2 = -2v_1 - v_2 = 0$

$v_2 = (-1, 2)$

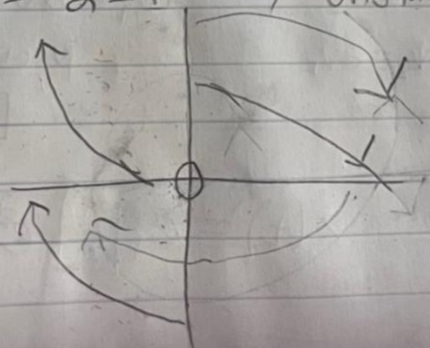
Strogatz 5.2.4

This is a counter-clockwise spiral, with an unstable fixed point leading to growing oscillations that get exponentially larger (although my sketch doesn't do a great job of showing that the oscillations get larger as you go farther from the origin)

5.2.4 $\quad \dot{x} = 5x + 10y \quad \dot{y} = -x - y \quad \dot{\vec{x}} = \begin{bmatrix} 5 & -1 \\ 10 & -1 \end{bmatrix} \vec{x}$

$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & -1 \\ 10 & -1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5 = 0$

$\frac{4 \pm \sqrt{16-20}}{2} \Rightarrow \lambda = 2 \pm i \Rightarrow$ unstable spiral



Problem 4

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$$\dot{x} = -k_1 x + k_2 (y-x) \qquad \dot{y} = k_2(x-y)$$

for $K_1 = 2/hr$ and $k_2 = 0.5/hr$

this becomes

a)

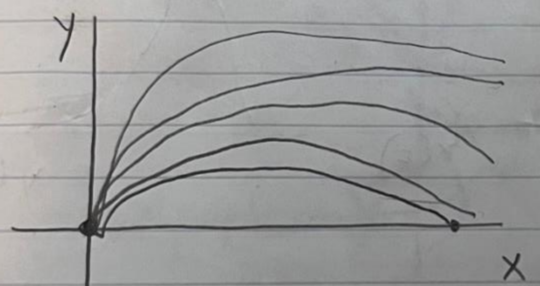
$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} -\frac{5}{2}-\lambda & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}-\lambda \end{vmatrix} = \lambda^2 + 3\lambda + \frac{5}{4} - \frac{1}{4} = \lambda^2 + 3\lambda + 1$$

$$\lambda = \frac{1}{2}\left(-3 \pm \sqrt{5}\right) \Longleftarrow \lambda = \frac{1}{2}\left(-3 \pm \sqrt{9-4}\right)$$

$\lambda_1 < \lambda_2 < 0 \Longrightarrow$ Stable node at $(0,0)$

b)



c) $x(0) = 250 \quad y(0) = 0$