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Overview:

Vectors u are nonzero.

Let $T \in L(V)$.

If for some $\boxed{\text{nonzero}}$ $u \in V$ that

$$T(u) = \lambda u \text{ for some } \cancel{u \in V}, \lambda \in F$$

we call that scalar an eigenvalue of T.

we call the vector for which

$T(u) = \lambda u$ an eigen vector.

That is, all vectors $u \in V$ such that

$T(u) = \lambda u$, $\lambda \in F$ are eigen vectors.

Example: Let $T \in L(\mathbb{C}^2)$,

$$T(w, z) = T(-z, w)$$

$$\cancel{z = \lambda w, \quad w = -\lambda z.}$$

$$-z = \lambda w, \quad w = \lambda z$$

$$\Rightarrow -z = \lambda^2 z \quad \text{or} \quad \lambda^2 = -1$$

$$\Rightarrow \lambda = -i, \ \lambda = i.$$

Theorem:

Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of $T \in \mathcal{L}(V)$ and $(v_1, \ldots, v_m)$ are corresponding eigenvectors. Then $(v_1, \ldots, v_m)$ is linearly independent.

Corollary: any operator T has at most $\dim V$ eigenvalues.

Proof: A basis of V is the largest list that can be linearly independent.

Given T, we can apply it many times to itself.

Let $T^m = \underbrace{T \ldots T}_{m \text{ times}}$,

Then $(T^m)^n = T^{mn}$, $T^m T^n = T^{m+n}$.

If $p \in \mathcal{P}(F)$ such that

$p(z) = a_0 + a_1 z + \ldots + a_n z^n$,

$p(T)$ is the operator defined by

$p(T) = a_0 I + a_1 T + \ldots + a_n T^n$.

If we fix $T \in \mathcal{L}(V)$, then $p \to p(T)$ is linear.

Theorem 5.10

Every operator on a complex vector space has at least one eigenvalue.

The matrix T wrt $(v_1, \ldots, v_n)$ given

$$T(v_k) = a_{1,k} v_1 + \cdots + a_{n,k} v_n \quad \text{is}$$

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{1,n} & \cdots & a_{n,n} \end{bmatrix}.$$

Question: Given ~~$T \in L(F^n)$, and no basis is specified,~~ ~~If~~ $T \in L(V)$, does there exist a basis of V such that T's matrix is "simple"?

Def: upper triangular matrix: (square)

A matrix in reduced echelon form.

Suppose $T \in L(V)$ and $S: (v_1, \ldots, v_n)$ is a basis for V.

Proposition 5.12: The following are equivalent:

a) $\text{Mat}(T, S)$ is upper triangular

b) $T(v_k) \in \text{span } S$ for $k = 1, \ldots, n$

c) $\text{span}(v_1, \ldots, v_k)$ is invariant ~~for~~ under T for $k = 1, \ldots, n$.

Theorem: Suppose V is a complex vector space and $T \in L(V)$.
Then, T has an upper-triangular matrix wrt some basis of V.

$\boxed{5.16}$ Suppose $T \in L(V)$ is upper triangular (its matrix).
Then T is invertible iff all the entries on the main diagonal are non-zero.

5.18 Suppose $T \in L(V)$ has an upper-triangular matrix wrt some basis of V.
The eigenvalues of T consist precisely the entries on the diagonal on that UTM.

Diagonal Matrices:
matrix in rref form.

$\boxed{5.19}$ $T \in L(V)$ has $\dim V$ distinct eigenvalues
$\Rightarrow$ T has a diagonal matrix wrt a basis of V.

$\boxed{5.21}$ $\tilde{=}$:

a) T has a diagonal matrix

b) V has a basis cont. eigenvectors of T

c) $\exists$ subspaces (1 dim) $U_1, \ldots, U_n$ of V invariant under T st. $U_1 \oplus \cdots \oplus U_n = V$,

d) $V = \text{null}(T - \lambda_1 I) \oplus \cdots \oplus$ ~~than~~ $\text{null}(T - \lambda_m I)$

e) $\dim V = \dim(\ker(T - \lambda_1 I)) + \cdots + \dim(\ker(T - \lambda_m I))$.

Invariant spaces on Real vector spaces

Thm: Every operator on finite-dimensional vector space in $\mathbb{R}$ has an invariant subspace of dimension 1 or 2.

Vector projection ~~onto~~ btwn two subspaces:

If $V = U \oplus W$, then

$v \in V$ can uniquely be represented as a sum

$$v = u + w, \quad u \in U, \; v \in W.$$

$\boxed{5.26}$

Every operator on an odd dimensional vector space over $\mathbb{R}$ has an eigenvalue.