H and K notes on eigenstuff

Jonathan Lin

July 4, 2019

So the goal of eigen analysis and all that of linear operators is that we would like to analyze a linear operator and "decompose it" into some sort of simpler structure. The canonical example are diagonalizable matrices: if we can diagonalize a matrix we have definitely simplified the problem greatly.

1 All Preliminary Definitions and basic results

Definition 1. Let V be a vector space over the field F and let T be a linear operator on V. An *eigenvalue* of T is a scalar c of F such that there is a non-zero vector v in V such that $T(v) = cv$. If c is an eigenvalue of T then any such vector where $T(v) = cv$ is called an *eigenvector* of T associated with c and the collection of all v such that $T(v) = cv$ is called the *eigenspace* associated with c.

It is clear that the eigenspace associated with c is a subspace of V.

The eigenspace associated with c is the null space of the linear transformation $T - cI$ (Forward and converse directions are very easy). Then an equivalent definition would be that c is an eigenvalue of T if $T - cI$ is not the zero subspace. In summary:

Theorem 1. *Let T be a linear operator on a finite dimensional vector space V and let c be a scalar. The following are equivalent:*

1. *c is a characteristic value of T*
2. *$T - cI$ is not an invertible linear transformation*
3. *$\det(T - cI) = 0$.*

We define the characteristic polynomial of T to be $\det(T-cI)$. In the next proposition we show that similar matrices have the same characteristic polynomial, so that the characteristic polynomial of a linear operator can be defined unambiguously.

Proposition 1. *Similar matrices have the same characteristic polyomial.*

Proof. If $B = PAP^{-1}$, then

$$\begin{aligned} \det(B - cI) &= \det(PAP^{-1} - cI) \\ &= \det(P(A - cI)P^{-1}) \\ &= \det(A - cI) \end{aligned}$$

so we are done. □

If V is an n dimensional vector space and $T \in \mathcal{L}(V, V)$ there are 4 things that could happen:

- T has no characteristic values.
- T has exactly n characteristic values.
- T has less than n characteristic values, but a collection of eigenvalues spans V.
- T has less than n characteristic values and no collection of eigenvalues spans V.

Definition 2. A linear operator T is diagonalizable if there is a basis for V where each vector in the basis is an eigenvector of T.

Suppose that T is diagonalizable and $c_1, \ldots, c_n$ the distinct eigenvalues of T. Then there is an ordered basis B where T is represented by a diagonal matrix where the scalars c_i are on the main diagonal repeated a certain number of times. We deduce that the characteristic polynomial for T is the product of linear factors $(\lambda - c_i)$ and hence the characteristic polynomial will have the form

$$(x - c_1)^{d_1} \cdots (x - c_k)^{d_k}.$$

Now we show that eigenvectors with different eigenvalues are mutually linearly independent.

Theorem 2. *Let T be a linear operator on the finite-dimensional vector space V. Let $c_1, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the eigenspace of c_i. If $W = W_1 + \cdots + W_k$, then*

$$\dim W = \dim W_1 + \cdots + \dim W_k.$$

In fact, if we let B_i be a basis for W_i (for each i) then we have that $B = (B_1, \ldots, B_k)$ is basis for W.

Proof. It suffices to show that W is a direct sum of the W_i, that is, if $w_i \in W_i$ and $w_1 + \cdots + w_k = 0$ then all the w_i are 0.

Suppose $w_i \in W_i$ and that $w_1 + \cdots + w_k = 0$. For any polynomial f we have that

$$\begin{aligned} 0 = f(T)0 &= f(T)w_1 + \cdots + f(T)w_k \\ &= f(c_1)w_1 + \cdots = f(c_k)w_k. \end{aligned}$$

If we choose a polynomial f_i such that $f_i(c_j) = \delta_{ij}$ then it follows that $0 = f_i(c_i)w_i = w_i$. So we are done. For the last statement this just follows from properties of linear independence and the invariance of dimension. □

Corollary 1. *The following statements are all equivalent:*

1. *T is diagonalizable.*

2. The characteristic polynomial for T is

$$(x-c_1)^{d_1}\cdots(x-c_k)^{d_k}.$$

and $dim W_i = d_i$ for i from 1 to k.

3.

$$\dim W = \dim W_1 + \cdots + \dim W_k.$$

Proof. This is pretty straightforward once you consider the previous lemma. □

Here is a matrix analogue of the previous theorem. Let A be an $n \times n$ matrix with entries in a field F and let $c_1, \ldots, c_k$ be the distinct eigenvalues of A in F. For each i let W_i be the null space os $A - c_i I$ and let B_i be an ordered basis for W_i. Then we can string the bases to form the columns of a matrix P. Then the previous theorem implies that A is similar to a diagonal matrix if and only if P is a square matrix. When P is square, P is invertible and $P^{-1}AP$ is diagonal.

2 Polynomials which annihilate a linear operator T

Suppose T is a linear operator on V. If p is a polynomial in $F[x]$, then $p(T)$ is a linear operator on V. We say that p annihilates T if we have that $p(T) = 0$. It is easy to see that the collection of polynomials p which annihilate T are an ideal in $F[x]$.

If V is finite-dimensional, then it is not true that this ideal is the zero ideal. To see why, note that $\dim(\mathcal{L}(V,V)) = n^2$, so that the n^2+1 vectors $I, T, \ldots, T^{n^2}$ are linearly dependent. So there are non-zero scalars such that

$$a_0 I + \cdots + a_{n^2} T^{n^2} = 0.$$

We say the minimal polynomial for T is the unique monic generator of the annihilators of T.

Suppose that $A \in M_{n\times n}(F)$ and suppose that $F_1 \supset F$ is an extension field (or whatever correct term to use) of F. Then it is true that $A \in M_{n\times n}(F_1)$. We claim that the minimal polynomial is the same in both cases. To see why, we note that over F the minimal polynomial is

$$f(x) = x^k + \sum_{j=0}^{k-1} a_j x^j.$$

Expanding any polynomial at A we obtain a system of n^2 homogeneous linear equations for $a_0, a_1, \ldots, a_{k-1}$. The coefficients for these lie in F and it is clear that if a solution exists in F_1 a solution must also exist for F. So the two minimal polynomials are actually the same.

Here is the important trick used here: Suppose A and b are a matrix and vector with entries of a field F and suppose $Ax = b$ has some non-trivial solutions in some extension field F_1. Then it is clear that row-reduction will find a solution, all of whose entries are in F.

Theorem 3. *Suppose T is a linear operator on a finite dimensional vector space V. Then the characteristic polynomial and the minimal polynomial for T have the same roots disregarding multiplicities.*

Proof. It suffices to show that $p(c) = 0$ if and only if c is an eigenvalue of T.

First, suppose $p(c) = 0$. Then there is a unique polynomial q such that

$$p(x) = (x - c)q(x).$$

Choose any vector v such that $q(T)v \neq 0$ (this is possible because p is the minimal polyonmial). Then we have that

$$0 = p(T)v = (T - cI)q(T)v$$

which implies that $q(T)v$ is an eigenvector of T.

Conversely, suppose that c is an eigenvalue of T (that is, $T(v) = cv$ for some vector v). Then we have that $0 = p(T)v = p(c)v$. Since we assume v to be non-zero this implies that $p(c) = 0$, as desired. □

We observe that p must be the product of the linear polynomials which have eigenvalues as their roots.

Here is an important theorem about the relationship between the minimal polynomial and the characteristic polynomial.

Theorem 4. *The minimal polynomial divides the characteristic polynomial. That is, if p is the characteristic polynomial of some linear operator T then $p(T) = 0$.*

There are a couple of proofs of this theorem.

Proof. □