Notes on determinants and the classical adjoint

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July 6, 2019

Proposition 1. *For any square matrix A, we have that $\det A = \det A^T$, where A^T is the transpose of A.*

Proof. If we let $A(x, y)$ denote the entry of A in the x-th row and the y-th column, then we have that $A^T(i, \sigma(i)) = A(\sigma(i), i)$ by definition. Hence

$$\begin{aligned}
\det A^T &= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) A(\sigma(1), 1) \cdots A(\sigma(n), n) \\
&= \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma) A(1, \sigma^{-1}(1)) \cdots A(n, \sigma^{-1}(n)) \\
&= \det A
\end{aligned}$$

so we are done. $\square$

Proposition 2. *Row addition does not change the determinant.*

Proof. Use the fact that the determinant is multilinear, then alternating. $\square$

Proposition 3. *Suppose $A \in M_{r \times r}(F)$, $C \in M_{s \times s}(F)$, $B \in M_{r \times s}(F)$. Then*

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A)\det(C).$$

Proof. Define the function $M(A, B, C)$ as the determinant of the above block matrix. By the properties of the determinant, M is alternating and multilinear as a function of the rows of C. Hence $M(A, B, C) = \det(C)M(A, B, I)$. Using row reduction $M(A, B, I) = M(A, 0, I)$. Now $M(A, 0, I)$ is clearly alternating and multilinear as a function of the columns of A. Then $M(A, 0, I) = \det(A)M(I, 0, I) = \det(A)$. We conclude that $M(A, B, C) = \det(A)\det(C)$ and we are done. $\square$

Define the cofactor matrix $A(i|j)$ to be the matrix obtained from A by deleting the ith row and jth column. Let $C_{ij} = (-1)^{i+j}\det(A(i|j))$. It follows that $\det A = \sum_{i=1}^{n} A_{ij}C_{ij}$ from the usual cofactor definition of the determinant.

Definition 1. The classical adjoint of A is defined to be the transpose of the matrix of cofactors. That is,

$$\operatorname{adj}(A)(i, j) = C_{ji}.$$

We have the identity

$$\sum_{i=1}^{n} A_{ik} C_{ij} = \delta_{jk} \det A$$

because the left hand side describes the determinant of the matrix A if the jth column was replaced by the kth column.

Hence we have that $\operatorname{adj}(A)A = \det AI$ for all matrices A. Hence we have $\operatorname{adj}(A^T)A^T = \det AI$ and taking transposes we have that $A \operatorname{adj} A^{T^T} = A \operatorname{adj}(A) = \det(A)I$. So if $\det A$ is non-zero, we have that $\operatorname{adj} A$ is a natural inverse for A.