

# 1 Preliminaries

First we will recall the definition of a ring:

**Definition 1.** A ring is a set  $R$  along with two operations  $+$  and  $\cdot$  with the following properties:

- $(R, +)$  is an abelian group.
- $\cdot$  is distributive over  $+$ , so that  $a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(a + b) \cdot c = a \cdot c + b \cdot c$  for all  $a, b, c \in R$ .
- $\cdot$  is associative.

If  $\cdot$  is commutative we call  $R$  a *commutative* ring. If  $R$  has a multiplicative identity  $1$  (that is, an element such that  $1 \cdot r = r \cdot 1 = r$  for all  $r \in R$ ), we say that  $R$  is a *unital* ring.

From now on we will assume that our rings are commutative (and unital?) unless otherwise stated.

**Definition 2.** Suppose  $R$  is a ring. Then  $I \subset R$  is an **ideal** if  $RI \subseteq I$  or  $IR \subseteq I$  (where the terms  $RI$  and  $IR$  are the usual notation).  $I$  is called *finitely generated* if

$$I = \langle a_1, \dots, a_n \rangle = \{a_1 r_1 + \dots + a_n r_n \mid r_i \in R\}.$$

Here are some basic examples of rings.

**Example 1.** The ring of integers  $\mathbb{Z}$  with usual integer addition and multiplication is a commutative ring with unity. Same with  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$ .

**Example 2.** Let  $R$  be a ring. Then the polynomial ring  $R[x]$  with coefficients in  $R$  with the usual multiplication and addition is a ring.

**Example 3.** The sets  $\mathbb{Z}/n\mathbb{Z}$  can be made into rings with addition and multiplication modulo  $n$ .

# 2 Hilbert's Basis Theorem

**Definition 3.** Let  $R$  be a ring. Then  $R$  is **Noetherian** (after Emmy Noether) if every ideal of  $R$  is finitely generated.

**Proposition 1.**  $R$  is Noetherian if and only if it satisfies the ascending chain condition: For every sequence of ideals

$$I_1 \subset I_2 \subset I_3 \subset \dots$$

there exists some  $n \in \mathbb{N}$  such that

$$I_n = I_{n+1} = I_{n+2} = \dots$$

**Example 4.** Any PID is Noetherian, and so is any field  $F$ .

**Example 5.** Consider the field  $R = F[x_1, x_2, \dots]$  of infinitely many indeterminates where  $F$  is a field. Then we have an increasing sequence of ideals

$$\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \dots$$

where none of the ideals in the chain are equal to one another.

We are ready to state Hilbert's Basis Theorem.

**Theorem 1.** *If  $R$  is a Noetherian ring, then  $R[x]$  is a Noetherian Ring. It also follows that  $R[x_1, \dots, x_n]$  is a Noetherian ring by induction.*

*Proof.* Suppose that  $I$  is an ideal in  $R[x]$ . Denote  $LC(I)$  to be the leading coefficients of elements in  $I$ . It is straightforward to check that  $LC(I)$  is an ideal in  $R$ . Since  $R$  is Noetherian,  $LC(I)$  is finitely generated. This means

$$LC(I) = \langle a_1, \dots, a_n \rangle.$$

By definition,  $a_i$  is the leading coefficient of some polynomial  $f_i(x) = a_i x_i^n + \dots$ . Let  $N$  be the maximum degree of the  $f_i$ . Now define  $LC(I_d)$  to be the leading coefficients of elements in  $I$  that have degree  $d$  (and 0). It is easy to show that  $LC(I_d)$  is also an ideal of  $R$ . So

$$LC(I_d) = \langle b_{d,1}, \dots, b_{d,n_d} \rangle$$

and we can let  $f_{d,i}(x)$  be the polynomials with the  $b_{d,i}$  as leading coefficient.

We claim that

$$I = \langle f_i, f_{d,j} \rangle$$

where  $i$  ranges from 1 to  $n$ ,  $d$  ranges from 1 to  $n$ , and  $j$  ranges from 1 to  $n_d$  (depending on  $d$ ). This will show that  $I$  is finitely generated.

Suppose not. Then take a polynomial  $f \in \langle f_i, f_{d,j} \rangle$  of minimal degree. If  $\deg f \geq N$ , then we can find polynomials  $q_i(x)$  such that  $\sum_{i=1}^n q_i f_i(x)$  has the same leading term as  $f$ . Subtracting we use the minimal degree assumption and we deduce that  $f$  is in  $\langle f_i \rangle$ . Otherwise, if  $\deg f = d < n$  then we can do the same thing but with the  $f_{d,i}$  instead. This shows that  $I$  is in fact  $\langle f_i, f_{d,j} \rangle$  as desired. ■