Notes about Differential Geometry

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1 What is a manifold?

Definition 1. A topological space X is called **locally Euclidean** if for every point $x \in X$ there exists a neighborhood U of x such that U is homeomorphic to a subset U' of $\mathbb{R}^n$. We call a pair (U, ϕ) where ϕ is such a homeomorphism a **chart**.

Definition 2. A $\mathcal{C}^k$**-atlas** on a locally Euclidean space X is a set of charts $\{(U_i, \phi_i)\}$ such that $\cup_i U_i = X$ and ϕ_i is a homeomorphism from U_i to an open subset U_i' of $\mathbb{R}^n$, and with the property that for any i and j $\phi_i \circ \phi_j^{-1}$ is in $\mathcal{C}^k(U_i \cap U_j)$.

Definition 3. Two atlases $\mathcal{F}$ and $\}$ are equivalent if $\mathcal{F} \cup \mathcal{G}$ is an atlas.

Proposition 1. *Equivalence between atlases is an equivalence relation.*

Proof. Reflexivity and Symmetry are obvious. For transitivity, suppose that $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are atlases such that $\mathcal{F} \cup \mathcal{G}$ and $\mathcal{G} \cup \mathcal{H}$ are also atlases. Suppose that (U, h) and (W, ℓ) are charts in $\mathcal{F}$ and $\mathcal{H}$, respectively. Then for all points $p \in U \cap W$, there exists an atlast (V_p, k_p) in $\mathbb{G}$ such that $p \in V_p$. Then it follows that

$$(h \circ \ell^{-1}) = (h \circ k_p^{-1}) \circ (k_p \circ \ell^{-1})$$

is $\mathcal{C}^k$ differentiable in a neighborhood around p for all p in $U \cap W$, as desired. $\square$

Definition 4. Suppose that $\mathcal{A}$ is an atlas. Then a **differentiable structure** generated by $\mathcal{A}$ is the union of all atlases such that their union with $\mathcal{A}$ is an atlas.

Definition 5. A $\mathcal{C}^k$-differentiable manifold is a pair $(M, \mathcal{F})$, where M is a Hausdorff second countable locally Euclidean topological space, and $\mathcal{F}$ is a differentiable structure.