1 Sets

1. It's important to realize the limitations of what kinds of sets we can define in our usual notation. The following example, independently discovered by Russel and Zermelo in the early 20th century illustrates that there cannot be a set of all sets.

   Suppose there was a universal set U, the set of all sets. Then U contains itself (since U is also a set). Consider the subset of U defined by

   $$N = \{S \in U \mid S \notin S\}.$$

   That is, N contains all the sets S which do not contain themselves.

   (a) Show that N is non-empty.

   (b) Suppose $N \in N$. Deduce that $N \notin N$. This is a contradiction, since formally we get the logical expression

   $$(N \in N) \wedge \neg(N \in N) \equiv F.$$

   So then $N \notin N$?

   (c) But now suppose $N \notin N$. Reason that $N \in N$.

   So the subset N cannot exist. The conclusion is that the set U is not well-defined (ie, does not exist).

2. Fill in the table on page 2.

   Here are some definitions we will go over. A and B denote any sets.

   - $x \in A$ means an object x is in the set A.
   - $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.
   - $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.
   - $A \subseteq B \iff ((x \in A) \implies (x \in B))$.
   - $A \subset B \iff (A \neq B) \wedge (A \subseteq B)$.
   - $\mathcal{P}(A) = \{U \mid U \subseteq A\}$. This is the **powerset** of A.
   - $|A|$ simply denotes the number of elements in the set A. For example, $|\{1, 2, 3\}| = 3$.

3. This problem hopefully illustrates that the set cardinality operation is a "depth 1" operation. All that matters when considering cardinality is the number of elements/objects in the first level of the set.

   Determine the numbers for the sets below.

   (a) $|\{3, 6, 9\}|$

   (b) $|\{\{3\}, 6, 9\}|$

(c) $|\{\{3\},6,9,\{9\}\}|$

(d) $|\{\{3\},6,9,\{\{9\},9\}\}\}|$

(e) $|\{\mathbb{R}\}|$

4. Just to get you used to the powerset operation. Compute $\mathcal{P}(\{a\})$, $\mathcal{P}(\{a,b\})$, and $\mathcal{P}(\{a,b,c\})$. Compute $\mathcal{P}(\mathcal{P}(\{a,b\}))$.

5. Last question, not really about set theory, but rather about quantifiers. Consider the propositional statement
$$(\forall x \in \varnothing)[x = 5].$$
Is this true? False? Discuss with your tablemates. After this, consider the proposition
$$(\forall x \in \varnothing)[x \in A]$$
where A is any set. (Hopefully, from this we see that $\varnothing \subseteq A$ for all sets A!)

2 Functions and Relations

3 Functions

3.1 Definition and Terminology

In this section, we will answer two questions. Namely, what is a function, and what do functions do?

From prior experience in precalculus and calculus courses most people have an intuitive picture of what functions are. For most purposes, they are "rules" that sends values to other values. In other words:

> A function is a rule that assigns to certain elements in a set, certain elements to another set.

The word rule is not very clear or rigorous. For example, we might consider the real valued functions $f(x) = x^3$ and $g(x) = x^3 + 3x - 3(x+3) + 9$. These have the same value everywhere, but the rule for evaluating them is slightly different.

It turns out that when defining functions rigorously, the "rule" is a means to an end. All we care about when talking about any specific function is the output given the input.

Definition 1. A function $f : A \to B$ (read as "f from A to B") is defined as a set of ordered pairs (a, b) where $a \in A$ and $b \in B$ (more specifically, a subset of the set of ordered pairs $A \times B$). f has the following properties:

1. For all $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$.

2. For all $a \in A$, $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

In other words, every value $a \in A$ will have some value b such that (a, b) is in f. Moreover, this value is **unique**. We define $f(a)$ as the unique value in B such that $(a, f(a)) \in f$.

Here is some more terminology.

Definition 2. Let $f : A \to B$ be a function. We say A is the **domain** of the function f and B is the **codomain** of f. We say that a is mapped to b or f maps a to b if $(a, b) \in f$. You can also write $a \mapsto b$ as long as it is clear what f is.

Using these definitions, we can say the following: Given any function $f : A \to B$, every value in the domain will be mapped to exactly one value in the codomain.

3.2 Injectivity, Surjectivity, Bijectivity

Here are the relevant definitions:

- A function $f : A \to B$ is injective (or one to one) if $f(a) = f(b) \implies a = b$. So two different values in A do not map to the same value in B.
- A function $f : A \to B$ is surjective (or onto) if for all $b \in B$ we can find a value in A such that $f(a) = b$.
- A function f is bijective if it is both injective and surjective.

Here is an interesting observation: if $f : A \to B$ is a bijection, then there is an inverse function $g : B \to A$. For each $b \in B$, we can define $g(b)$ as the unique value $a \in A$ such that $f(a) = b$. We know such a value exists because f is surjective, and we know that this value is unique because f is injective.

4 Relations

Let A be a set. Traditionally, we have lots of different notation for when we might want to *compare* two objects in A. For example, the $=$ notation "compares" for equality, and the $\leq$ symbol compares two objects for magnitude. Relations generalize these comparison operators.

Definition 3. Let A and B be sets. A relation R on A and B is a subset of $A \times B$. That is, R is any subset of ordered pairs (a, b) where $a \in A$ and $b \in B$.

For example, a function f is a relation.

Most of the time we will consider relations in the special case where $A = B$. This is, as we might observe, the use case for the equality and inequality operators mentioned at the beginning of this section.

Definition 4. Let A be a set and R be a relation on $A \times A$. A relation is

- *reflexive* if $(a, a) \in R$ for every $a \in A$,
- *symmetric*, if for all $a, b \in A$, $(a, b) \in R \implies (b, a) \in R$.

- *transitive* if for all $a, b, c \in A$, $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

We write aRb if $(a, b) \in R$.
A relation is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

5 Exercises

1. Let X be any non-empty set, and consider the relation $R = X \times X \subseteq X \times X$. Verify that this relation is an equivalence relation.

2. Give an example of a relation R on a set A that is
   - reflexive and symmetric but not transitive,
   - symmetric and transitive but not reflexive,
   - reflexive and transitive but not symmetric.

3. Suppose $f : A \to B$ and $g : B \to A$ are functions such that
$$g(f(a)) = a$$
for all $a \in A$. Show that f is injective and that g is surjective.

4. Let f and g be functions as in the last problem. Suppose also that $f(g(b)) = b$ for all $b \in B$. Show that g is the only function with these properties, that is if h has these properties then $h = g$. (Notice that this equality is technically realized as an equality of **sets**.)

5. (The classification of symmetric, transitive relations.) Let A be a non-empty set, and suppose that R is a non-empty relation which is symmetric and transitive. Show there is a non-empty set $B \subseteq A$ for which R is an equivalence relation restricted to $B \times B$. Explicitly describe the set B. (It might help to do the second part of problem 2 first.)

6. (Equivalence relations as surjective functions) Let X be a set, and let $\sim \quad \subset X \times X$ be an equivalence relation. Given any $a \in X$, define
$$X_a = \{b \in X \mid a \sim b\}$$
We call this the equivalence class of a.
   (a) Given any $a, b \in X$, show that either $X_a \cap X_b = \varnothing$ or $X_a = X_b$. So distinct equivalence classes are disjoint.
   (b) Let $C = \{X_a \mid a \in A\}$. That is, C is the set of all the equivalence classes (which are sets, specifically subsets of X). Consider the function $f : X \to C$ given by $f(a) = X_a$. First, justify that f is actually a function (that is, it satisfies all the properties that functions satisfy). Then prove that f is a surjective function (this is pretty easy).

(c) Now consider any surjective function

$$f : X \to C,$$

where C is any set. Consider the sets

$$M_z = \{x \in X \mid f(x) = z\}.$$

Show that the relation $\sim \ \subset X \times X$ defined by $x \sim y$ if and only if $f(x) = f(y)$ is an equivalence relation. Show that the equivalence classes of $\sim$ are precisely the sets M_z for $z \in C$.

(d) Let D be the set of equivalence classes of $\sim$ in the previous part. Now define $\tilde{f} : D \to C$ by $\tilde{f}(X_a) = f(a)$. Show that $\tilde{f}$ with this rule defines a valid function and that $\tilde{f}$ is in fact bijective.

Remark: for the more mathematically inclined the reason why this construction works boils down to fact that the inverse image of a function is well behaved under set union and intersection; see if you can figure out why this fact basically makes the problem work.

Statement	**True**	**False**
$(A \cap B) \subseteq (A \cup B)$	○	○
$(A \subset B) \Rightarrow (A \subseteq B)$	○	○
$(A \subseteq B) \Rightarrow (A \subset B)$	○	○
$0 \in \emptyset$	○	○
$\emptyset \subseteq \emptyset$	○	○
$\emptyset \in \{\{\emptyset\}\}$	○	○
$0 \in \{\{0, \emptyset\}\}$	○	○
$2, 4, 6 \in \mathbb{N}$	○	○
$\{2, 4, 6\} \subseteq \mathbb{N}$	○	○
$\{2, 4, 6\} \in \mathbb{N}$	○	○
$2, 4, 6 \in \mathcal{P}(\mathbb{N})$	○	○
$\{2, 4, 6\} \subseteq \mathcal{P}(\mathbb{N})$	○	○
$\{2, 4, 6\} \in \mathcal{P}(\mathbb{N})$	○	○