In this chapter we discuss the notion of a **set** and various terminology associated with it. In many theoretical areas such as mathematics, computer science, and physics, the notion of a set comes up very naturally (in general, it comes up when one is considering any collection or class of objects). In many fields, especially in technical literature, such knowledge about sets and their associated terminology are basic knowledge that everyone should know.

Here are the topics that we will cover:

- First we will have a reasonable discussion about what a set is, and then describe some common set operations. We will discuss some of the relationships between some of these operations.

- After this we will discuss the *function*, which is arguably a more important concept than the sets themselves. We will describe some common properties that functions have and discuss some common operations, such as function composition.

- We will explore relations, which are a way of comparing elements in any given set, and classify certain nice relations called equivalence relations.

- Finally, we will explore the notion of "size" of a set, where we demonstrate that there are different types of "infinity".

1 Sets

1.1 Definition and Notation

Most roughly speaking, a set is a collection of objects. These objects can be numbers, people, or even other sets themselves. In general this collection need not be homogeneous (so for example, we can have a set which contains both a number and a person). For the purposes of this book, we will not even try to discuss a formal logical definition of a set (this gets very complicated). So for the rest of this book we can assume that a set is a collection of objects without any issues whatsoever.

In general, when we are describing sets as a collection of specific objects, we tend to use curly braces $\{\}$ to enclose these objects while delimiting them by commas.

Example 1. If we wanted to describe the set containing the numbers 1, 2, and 3, we would do so as $\{1, 2, 3\}$. The set $\{1, \{1, 2\}\}$ is the set which contains the number 1, and the set $\{1, 2\}$ containing 1 and 2.

The set $\{1, 1\}$ is the set containing the numbers 1, ... and 1? Since 1 and 1 are in fact the same this set is the same as $\{1\}$. We will make this example more formally clear later in the chapter.

In general describing sets like this gets awkward when the number of elements that the set contains gets very large. For example, it is very awkward to write out

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

to describe the set of positive integers between 1 and 15 inclusive. There are a couple of things we can do. For most purposes if we instead wrote this set as

$$\{1, 2, 3, \dots, 14, 15\}$$

where most people would understand what the set was. Another thing we can do is use a variable, such as A, to denote the set of positive integers between 1 and 15 inclusive. The best convention is to combine these two notions. We can define

$$A = \{1, 2, 3, \dots, 14, 15\}.$$

In this formulation A is defined as the set of positive integers between 1 and 15 inclusive. Once we have defined A in this way we can freely use just the character "A" instead of $\{1, 2, 3, \dots, 14, 15\}$, which is a good way to write concisely.

In general, we can refer to any abstract set using a variable such as A, B, etc. (In general most people will use capital letters to refer to sets).

Example 2. The set $\mathbb{N}$, known as the set of **natural numbers**, denotes the set of positive integers. That is,

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

The set $\mathbb{Z}$ (german for *Zahlen*) denotes the set of all integers. That is,

$$\mathbb{Z} = \{\dots, -2, 1, 0, 1, 2, \dots\}.$$

In some texts people include 0 in the natural numbers. This is entirely a matter of convention and not for any very deep reason. Sometimes it is convenient to include 0.

The empty set, denoted $\varnothing$ is the set which does not contain any elements.

1.2 Set Inclusion and Set Equality

The first thing we describe in this section is notation to describe whether an object (usually denoted by a lowercase letter) is in a certain set. We use the $\in$ notation in order to do so. That is, we say that

$$x \in A$$

if the object x is in the set A. Alternatively if an object x is not in A we write $x \notin A$ instead.

Example 3. Let $A = \{1, 2, 3, \dots, 14, 15\}$. Then $6 \in A$.

The empty set $\varnothing$ has the property that $x \notin A$ for any conceivable object x.

The notion of set containment, which is an inquiry about whether any object is contained in a set, leads naturally to a general notion of inquiring whether some set is contained inside another. The definition for this is pretty natural. Roughly speaking, a set A is contained in a set B only if all the objects contained in A are also contained in B. We make this notion more formal now.

Definition 1. Suppose A and B are sets. Then we say that $A \subset B$ if we have for every $x \in A$ we have $x \in B$ also.

Example 4. We have
$$\{1, 2\} \subset \{1, 2, 3\}.$$

A closely related concept to set inclusion is set equality. Below we define set equality. Roughly two sets are equal if they contain the same elements. We will define set equality then using the subset definition.

Definition 2. Suppose A and B are sets. Then $A = B$ if $A \subset B$ and $B \subset A$.

This definition roughly encodes the notion of set equality. Since in the definition of subset, $A \subset B$ if all the elements of A were in B. So two sets A and B are equal if all the elements of A are in B, and all the elements of B are in A. This raises some subtle, but not particularly deep observations.

Example 5. We claim that
$$\{1, 2, 3, 3\} = \{1, 2, 3\}.$$

This can be easily verified using the definition of set equality. However it is true that both sets do not "look" the same.

This subtlety with multiple elements in a set shows that we only need to consider sets where each element of the set is unique in that set, since such a set will always be equal to one with the same elements but maybe some of them duplicated.

1.3 Common Set Operations

In this section we describe common set operations and notations used to manipulate sets and create new sets. First we will explain why this is useful. In practice, when one is using sets, in general one considers sets with certain properties. That is one would like to consider perhaps certain objects with a certain property. In practice one would define these sets using **set builder notation**. The idea of defining such a set S is encapsulated in the notation below.

$$S = \{\text{all the objects } x \text{ of some type} \colon x \text{ has some special property}\}.$$

Let's break down this notation further. With this notation, it is accepted that S is defined to be the set of all objects x such that x has a special property. For example, the set

$$S = \{x \in \mathbb{Z} \colon |x| \leq 5\}$$

is the set of integers with absolute value less than or equal to 5. In other words,

$$S = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}.$$

In general set builder notation is very useful to create sets of objects with certain properties.

Not only do we want to describe objects with a certain property, but much of the time we would like to describe objects with more than one property or objects with at least one of the following few properties. These concepts are encapsulated by set intersection and set union, respectively.

1.3.1 Set Intersection

Suppose E is the set of even numbers and T is the set of multiples of 3. Then if a number n is divisible by 6, then it is both an even number and a multiple of 3. So n is in E and T at the same time. Using notation already presented, one might say that $n \in E$ and $n \in T$. However, this is not the most concise way to write this. In order to express that an object is in two sets and the same time we introduce the notion of *set intersection*.

Definition 3. Let A and B be sets. We define $A \cap B$ (read as "A intersect B") to be the objects n which are contained in both A and B. A little more formally,

$$A \cap B = \{n \colon (n \in A) \wedge (n \in B)\}.$$

In the example described above, the most concise way to write that n is both even and a multiple of three would be $n \in E \cap T$.

Definition 4. Let A and B be sets. We say that A and B are **disjoint** if $A \cap B = \varnothing$. That is, A and B have no elements in common.

1.3.2 Set Union

Similarly to how we would like to describe objects with multiple properties at the same time, in many cases it is useful to describe objects which have one of many properties, but not necessarily all of them at the same time. Consider, for example, the set of all candy C that some person has acquired one Halloween night. Any piece of candy satisfies the following condition. Either it contains chocolate or it does not. These are clearly mutually exclusive possibilities. Let H denote the set of candies in the Halloween loot which contain chocolate, and let J denote the set of candies in the Halloween loot that do not contain chocolate. Then we may observe the following.

- Every candy c in the Halloween loot is either in H or in J, since a piece of candy in the loot either does or does not contain chocolate.
- $H \cap J = \varnothing$, because a piece of candy cannot both contain and not contain chocolate at the same time[1]

As with set intersection, we would like some notation that would indicate that c, a piece of candy, is either in H or in J. This is done using the set union notation, as defined below.

Definition 5. Let A and B be sets. We define $A \cup B$ (read as "A union B") to be the objects n which are contained in at least one of A or B. A little more formally,

$$A \cup B = \{n \colon (n \in A) \vee (n \in B)\}.$$

For instance, in the example above, we could write $c \in H \cup J$ to indicate that each piece of candy is in either H or in J. This is not the most general use of set union because as we observed, the sets H and J were disjoint. As a simple example, one can verify that

$$\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}.$$

[1] Here we are assuming something known as the **law of the excluded middle**.

Definition 6. Suppose that S is a set and A and B are subsets of S. We say that A and B cover S if

$$A \cup B = S.$$

1.3.3 Union and Intersection of More Than Two Sets

In the above two parts we have defined set intersection and set union for two sets A and B. As stated before, these notations are useful for describing objects that have many properties or at least one out of many given properties respectively. However, with our definitions we can only do this when "many" equals 2. So now we define union and intersection for more than two sets. First, we define set union and intersection for arbitrarily many finite sets, and then for an arbitrary index set (which we will define later).

Definition 7. Suppose $n \geq 2$, and $A_1, A_2, \ldots, A_n$ are sets. We define their intersection and union to be

$$A_1 \cap A_2 \cap \cdots \cap A_n = \{x \colon (x \in A_1) \wedge (x \in A_2) \wedge \cdots \wedge (x \in A_n)\}$$

and

$$A_1 \cup A_2 \cup \cdots \cup A_n = \{x \colon (x \in A_1) \vee (x \in A_2) \vee \cdots \vee (x \in A_n)\}$$

We can also write these intersections and unions in big union and intersection notation, which looks like this:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

and

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

One might observe that in the case of chapter 1[2] that the definition of a union (resp. intersection) of more than two sets involves a logical formula that is not fully well defined namely a large connection of logical ands (resp. logical ors). As we have stated then this will not be a problem due to the associativity of the logical operators $\wedge$ and $\vee$. We will resolve this problem fully in Chapter 4.

Now we discuss the union of sets over an arbitrary index. Let I be a set, possibly infinite. Suppose we have a collection of sets A_i for each $i \in I$. In many cases we would like to discuss the union (or perhaps intersection) of these sets, denoted by

$$\bigcup_{i \in I} A_i, \bigcap_{i \in I} A_i$$

We will define these unions and intersections similarly to how we did it for finitely many sets.

[2]this is temporary to indicate that i have not mentioned this issue in chapter 1 at all

Definition 8. Let I be any set, and suppose that A_i are sets defined for each $i \in I$. We will define the union and intersection of these sets as follows: We define

$$\bigcup_{i\in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}$$

and

$$\bigcap_{i\in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}.$$

Example 6. Consider the set $I = \mathbb{N}$. We will define the set A_n for each $n \in \mathbb{N}$ to be the set of positive multiples of n. That is,

$$A_n = \{n, 2n, 3n, 4n, \ldots\}.$$

First we will determine

$$\bigcup_{n\in I} A_n.$$

We claim this is just $\mathbb{N}$. Indeed, observe that since $A_n \subset \mathbb{N}$ for all $n \in I$, their union is a subset of $\mathbb{N}$ as well. For the other direction, we only need to observe that for each $n \in \mathbb{N}$, $n \in A_n$. So clearly by the definition of subset we have the other inclusion.

Now we will determine

$$\bigcap_{n\in I} A_n.$$

We claim that this set is empty. Indeed, again we observe since A_n are subsets of $\mathbb{N}$ we observe that their intersection is as well. Now we just need to show that no element of $\mathbb{N}$ is in the intersection (hence the intersection is empty). To see this, we only need to observe that any $n \in \mathbb{N}$ is not contained in A_{n+1}. Hence no $k \in \mathbb{N}$ can be contained in all of the A_n at the same time (which is the requirement for k to be in the intersection.

As a remark, the indexing set $I = \mathbb{N}$ is so common that many people usually use the following notation for a union indexed by the natural numbers. In this case people usually write

$$\bigcup_{i=1}^{\infty} A_i$$

instead of

$$\bigcup_{i\in\mathbb{N}} A_i.$$

Example 7. Consider again the set $I = \mathbb{N}$. We will define a collection of sets A_n for $n \in I$ which have the following properties:

- $$\bigcup_{i=1}^{\infty} A_n = \mathbb{N}$$
- $A_i \cap A_j = \varnothing$ when $i \neq j$,
- A_i are infinite sets for each I.

I will work out this example later because I am lazy.

1.4 Set Difference and Set Complement

So far, our notation for describing objects that live in sets has been very *inclusive*. For example, set intersection describes objects that are contained in many sets all at once, and set union describes objects that are contained in at least one of many sets. Now we develop notation for descriptions which are *exclusive*. More specifically, we will develop notation to describe objects that are *not* contained in some specified set. To begin with we will define the relative complement of two sets.

Definition 9. Let A and B be sets. The **relative complement** of A with respect to B is denoted $B \setminus A$ and is defined as the set

$$B \setminus A = \{x \in B \colon x \notin A\}.$$

The **divided difference** of A and B is denoted $A \Delta B$, and is defined by

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

Note first that $A \setminus B$ is not generally equal to $B \setminus A$. For example, consider the sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$. Then we have

$$\{1, 2, 3\} \setminus \{2, 3, 4\} = \{1\}$$
$$\{2, 3, 4\} \setminus \{1, 2, 3\} = \{4\}.$$

The relative complement is good notation to indicate whether or not some element does not lie in a certain set. Now suppose that we are working within a global set U. For example, U can be the set of natural numbers, integers, or some other "global" set. We will define the global complement now.

Definition 10. Let U be some set fixed in advance and let $A \subset U$. The global complement of A (with respect to U) is the set $U \setminus A$.

Example 8. Suppose I want to describe a set of elements for which each element x is in B but not in A_1 or A_2. We can describe this as follows. Since x is not in A_1 or A_2, we know that $x \notin A_1 \cup A_2$. So the set we are describing is just the set $B \setminus (A_1 \cup A_2)$.

Alternatively, we could employ the following derivation using set builder notation:

$$\begin{aligned}
\{x \colon (x \in B) \wedge (x \notin A_1) \wedge (x \notin A_2)\} &= \{x \in B \colon \neg((x \in A_1) \vee (x \in A_2))\} \\
&= \{x \in B \colon \neg(x \in A_1 \cup A_2)\} \\
&= \{x \in B \colon (x \notin A_1 \cup A_2)\} \\
&= B \setminus (A_1 \cup A_2).
\end{aligned}$$

We will make this method of manipulating clauses in set builder notation more clear and general in the next section.

1.5 Set Identities and Manipulation

In this section we will indicate several set identities (relationships involving sets and various operations on sets) and in general we will develop the general technique for deriving and demonstrating that these identities are true. As in the previous section, we demonstrated various manipulations of set builder notation to abstract sets to other sets using logical identities and notation. We outline the general method of set builder notation manipulation below:

1. Recall that set builder notation has the following form:

$$S = \{x \colon x \text{ has some property}\}.$$

   In general, the property that x has can be described in the form of a logical statement. In this sense this means that we can manipulate this statement using the rules of propositional logic.

2. Suppose now we have some notation of the following form:

$$S = \{x \in B \colon x \text{ satisfies some properties}\}.$$

   This set is the same as the set

$$\{x \colon (x \in B) \wedge x \text{ satisfies some properties}\}.$$

   In this way, the set declaration on the left hand side can be seen as another property.

We illustrate this method by showing several important set theoretic identities.

Proposition 1. *For any three sets A, B, and C,*

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \end{aligned}$$

To show, for example, the first claim is true, we provide the following chain of equalities.

$$\begin{aligned} A \cap (B \cup C) &= \{x \colon (x \in A) \wedge (x \in B \cup C)\} \\ &= \{x \colon (x \in A) \wedge ((x \in B) \vee (x \in C))\} \\ &= \{x \colon ((x \in A) \wedge (x \in B)) \vee ((x \in A) \vee (x \in C))\} \\ &= \{x \colon (x \in A \cap B) \vee (x \in A \cap C)\} \\ &= (A \cap B) \cup (A \cap C) \end{aligned}$$

The set of equalities for the second claim is almost the same. In that case we can apply another law of distributivity to reach the conclusion.

1.6 Exercises

1. It's important to realize the limitations of what kinds of sets we can define in our usual notation. The following example, independently discovered by Russel and Zermelo in the early 20th century illustrates that there cannot be a set of all sets.

   Suppose there was a universal set U, the set of all sets. Then U contains itself (since U is also a set). Consider the subset of U defined by

   $$N = \{S \in U \mid S \notin S\}.$$

   That is, N contains all the sets S which do not contain themselves.

   (a) Show that N is non-empty.

   (b) Suppose $N \in N$. Deduce that $N \notin N$. This is a contradiction, since formally we get the logical expression

   $$(N \in N) \wedge \neg(N \in N) \equiv F.$$

   So then $N \notin N$?

   (c) But now suppose $N \notin N$. Reason that $N \in N$.

   So the subset N cannot exist. The conclusion is that the set U is not well-defined (ie, does not exist).

2. Fill in the table on page 2.

   Here are some definitions we will go over. A and B denote any sets.

   - $x \in A$ means an object x is in the set A.
   - $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.
   - $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.
   - $A \subseteq B \iff ((x \in A) \implies (x \in B))$.
   - $A \subset B \iff (A \neq B) \wedge (A \subseteq B)$.
   - $\mathcal{P}(A) = \{U \mid U \subseteq A\}$. This is the **powerset** of A.
   - $|A|$ simply denotes the number of elements in the set A. For example, $|\{1, 2, 3\}| = 3$.

3. This problem hopefully illustrates that the set cardinality operation is a "depth 1" operation. All that matters when considering cardinality is the number of elements/objects in the first level of the set.

   Determine the numbers for the sets below.

   (a) $|\{3, 6, 9\}|$

   (b) $|\{\{3\}, 6, 9\}|$

(c) $|\{\{3\},6,9,\{9\}\}|$

(d) $|\{\{3\},6,9,\{\{9\},9\}\}\}|$

(e) $|\{\mathbb{R}\}|$

4. Just to get you used to the powerset operation. Compute $\mathcal{P}(\{a\})$, $\mathcal{P}(\{a,b\})$, and $\mathcal{P}(\{a,b,c\})$. Compute $\mathcal{P}(\mathcal{P}(\{a,b\}))$.

5. Last question, not really about set theory, but rather about quantifiers. Consider the propositional statement
$$(\forall x \in \varnothing)[x = 5].$$
Is this true? False? Discuss with your tablemates. After this, consider the proposition
$$(\forall x \in \varnothing)[x \in A]$$
where A is any set. (Hopefully, from this we see that $\varnothing \subseteq A$ for all sets A!)

2 Functions and Relations

3 Functions

3.1 Definition and Terminology

In this section, we will answer two questions. Namely, what is a function, and what do functions do?

From prior experience in precalculus and calculus courses most people have an intuitive picture of what functions are. For most purposes, they are "rules" that sends values to other values. In other words:

> A function is a rule that assigns to certain elements in a set, certain elements to another set.

The word rule is not very clear or rigorous. For example, we might consider the real valued functions $f(x) = x^3$ and $g(x) = x^3 + 3x - 3(x+3) + 9$. These have the same value everywhere, but the rule for evaluating them is slightly different.

It turns out that when defining functions rigorously, the "rule" is a means to an end. All we care about when talking about any specific function is the output given the input.

Definition 11. A function $f : A \to B$ (read as "f from A to B") is defined as a set of ordered pairs (a, b) where $a \in A$ and $b \in B$ (more specifically, a subset of the set of ordered pairs $A \times B$). f has the following properties:

1. For all $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$.

2. For all $a \in A$, $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

In other words, every value $a \in A$ will have some value b such that (a, b) is in f. Moreover, this value is **unique**. We define $f(a)$ as the unique value in B such that $(a, f(a)) \in f$.

Here is some more terminology.

Definition 12. Let $f : A \to B$ be a function. We say A is the **domain** of the function f and B is the **codomain** of f. We say that a is mapped to b or f maps a to b if $(a, b) \in f$. You can also write $a \mapsto b$ as long as it is clear what f is.

Using these definitions, we can say the following: Given any function $f : A \to B$, every value in the domain will be mapped to exactly one value in the codomain.

3.2 Injectivity, Surjectivity, Bijectivity

Here are the relevant definitions:

- A function $f : A \to B$ is injective (or one to one) if $f(a) = f(b) \implies a = b$. So two different values in A do not map to the same value in B.
- A function $f : A \to B$ is surjective (or onto) if for all $b \in B$ we can find a value in A such that $f(a) = b$.
- A function f is bijective if it is both injective and surjective.

Here is an interesting observation: if $f : A \to B$ is a bijection, then there is an inverse function $g : B \to A$. For each $b \in B$, we can define $g(b)$ as the unique value $a \in A$ such that $f(a) = b$. We know such a value exists because f is surjective, and we know that this value is unique because f is injective.

4 Relations

Let A be a set. Traditionally, we have lots of different notation for when we might want to *compare* two objects in A. For example, the $=$ notation "compares" for equality, and the $\leq$ symbol compares two objects for magnitude. Relations generalize these comparison operators.

Definition 13. Let A be a set. A relation R on A is a subset of $A \times A$. That is, R is any subset of ordered pairs (a, b) where $a \in A$ and $b \in A$.

This definition can used as a comparison property as we demonstrate using the following terminology. We say that an element $a \in A$ is related to an element $b \in A$ if $(a, b) \in R$. So in fact it is beneficial to just think of R as the set of all possible relations.

Definition 14. Let A be a set and R be a relation on $A \times A$. A relation is

- *reflexive* if $(a, a) \in R$ for every $a \in A$,
- *symmetric*, if for all $a, b \in A$, $(a, b) \in R \implies (b, a) \in R$.
- *transitive* if for all $a, b, c \in A$, $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

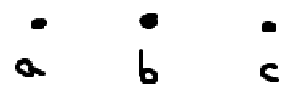
Usually, given a relation R, we will write aRb if $(a, b) \in R$. This will save use space and ink.

4.1 The Graph of a Relation

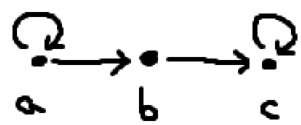
One way of visualizing some relations is by using a directed graph, which we define below.

Definition 15. A **directed graph** consists of two sets V and $E \subset V \times V$. The set V is called the set of vertices and the set E is called the set of (directed) edges.

We can visualize a directed graph as follows. For visual purposes and a concrete example, we will consider the directed graph $V = \{a, b, c\}$ and $E = \{(a, a), (a, b), (b, c), (c, c)\}$. First, we imagine the objects of V as dots.

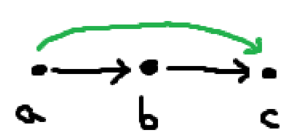


We can imagine each object (x, y) of E as a "directed arrow" (ie an arrow) where the back end of the arrow is x and the front end of the arrow is y. For the example graph pictured this is visualized as follows.



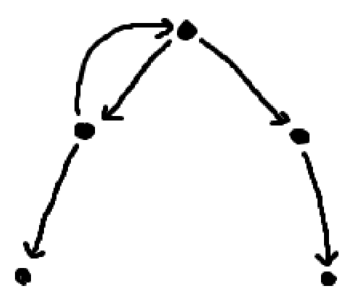
In this way a graph can be visualized. One minor thing to observe is that any edge (x, x) from an object x to itself is drawn as a "loop" from the vertex starting at x and pointing back at itself. The final thing to be observed is that any relation R on a set A can be seen as a graph. In this case, we take the vertices to be the set A and we take the set of edges to be the set $R \subset A \times A$. We call this graph the **associated graph** of the relation R.

Example 9. Suppose that a relation R is transitive. Recall that a relation R is transitive if for all $a, b, c \in A$, $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$. What does this mean for its associated graph? Suppose then we have $a, b, c \in A$ with $(a, b) \in R, (b, c) \in R$. Then $(a, c) \in R$. A consequence of this is that if we have a chain of arrows from a to c then by transitivity there must be an arrow from a to c itself. This is illustrated in the picture below.



4.1.1 Exercises

1. Following the discussion of Example 9, interpret what it means for the associated graph of a relation R if R is
   - reflexive.
   - symmetric.
2. Suppose Ryan presents you with the following graph associated with a relation R. He is very sad because the graph pictured is neither symmetric nor transitive. Add in the minimum number of necessary edges to make the relation...
   - reflexive.
   - symmetric.
   - transitive.



4.2 Equivalence Relations

In this (sub)section we will explore a special kind of relation called an **equivalence relation**. Suppose A is a set. Roughly speaking, there are two useful ways of viewing an equivalence relation A.

- We can view the relation as a classifier. That is, the relation divides the elements of A into groups and classifies them as being the same if they are in the same group.
- Alternatively (and equivalently), the relation compares two elements a and b and determines whether the two elements are "the same" or not the same.

In this section we will see how these two views describe the same thing. In short this relationship can be more formally described as how we can use an equivalence relation to partition A into several *equivalence classes*. We will fully define these terms more formally later. First we begin with the definition of an equivalence relation.

Definition 16. Let A be a set. An equivalence relation $\sim \subset A \times A$ is a relation with the following properties:

1. For all $a \in A$, $a \sim a$.
2. If $a \sim b$ for $a, b \in A$, then $b \sim a$.
3. If $a \sim b$ and $b \sim c$, then $a \sim c$.

Moreover, these conditions have specific names.

1. Any relation satisfying the first condition is called *reflexive*.
2. Any relation satisfying the second condition is called *symmetric*.
3. Any relation satisfying the third condition is called *transitive*.

Let us unpack the criterion for an equivalence relation $\sim$ on A. The first condition simply states that every element in a is related to itself. The second condition models the condition that if a is related to b, then b should be related to a as well. The third condition asserts the *transitivity* of the relation $\sim$. That is, if a is related to b, and b is related to c, then a should be related to c; that is, we can chain relations to get other relations.

Example 10. The notion of object equality is an equivalence relation. More formally, the $=$ symbol can be seen as a relation on any set A. We check that this satisfies all the properties that an equivalence relation should satisfy:

1. $a = a$ for all $a \in A$.
2. If $a = b$, then $b = a$ as well.
3. If $a = b$ and $b = c$, then $a = c$ as well.

The verifications we made above are true because they are in fact properties of *equality* itself. In fact, the definition of an equivalence relation was formed to encode more general notions of equality. We will describe more examples of equivalence relations later in the section and explore for each relation in what way the elements are the same.

Example 11. Suppose we have an apartment building A. Let S be the set of all residents in A. We will define an equivalence relation $\sim$ on S as follows. For $s, t \in S$ (ie, any two residents of the building), we say that $s \sim t$ if s and t live at the same apartment number. We can verify that this is an equivalence relation by verifying the three properties an equivalence relation needs to satisfy:

1. For any $s \in S$, $s \sim s$ by default because they are the same person and hence they live at the same apartment number.

2. If s and t live in the same apartment number this is the same as saying that t and s live at the same apartment number. So if $s \sim t$, then $t \sim s$.

3. If $s \sim t$ and $t \sim r$, then s, t, and r all live at the same apartment number. It follows that $s \sim r$ as desired.

Example 12. For the set $\mathbb{Z}$ define a relation $\sim$ as follows. We have $a \sim b$ if $a - b$ is even. We can verify that this relation is an equivalence relation below:

1. $a - a = 0$, which is even (it is 2×0), so $a \sim a$.

2. If $a \sim b$, then $a - b$ is even. It follows that $b - a = -(a - b)$ is even as well. For if $a - b = 2m$ for some integer m then $b - a = -(a - b) = -2m = 2(-m)$ can be seen to be an even integer. So $b \sim a$.

3. If $a \sim b$ and $b \sim c$, then $a - b$ and $b - c$ are both even. So $a - b = 2m$ and $b - c = 2n$ for some integers m and n. It can be seen that their sum must be even as well. But we have
$$a - c = (a - b) + (b - c) = 2m + 2n = 2(m + n)$$
so it follows that $a - c$ is even as well. Hence $a \sim c$ and the relation is transitive.

So it follows that $\sim$ is an equivalence relation. We can verify that $a \sim b$ only if a and b are both odd or both even. So in this case, $\sim$ defines whether or not items are the same based on whether or not they have the same remainder when integer dividing by 2.

We will also show that a similar looking relation is not an equivalence relation. Suppose instead we define $a \sim b$ if $a \sim b$ is odd. One can check that this relation is symmetric, but it is neither reflexive nor transitive.

Example 13. We can generalize the previous example as follows. For the set $\mathbb{Z}$ we can define a relation $\sim$ with the property that $a \sim b$ if $b - a$ is divisible by a positive integer m (where m is given in advance). One can check as above that this forms an equivalence relation.

At the beginning of this (sub)section we described two different ways of looking at an equivalence relation. We have made the second way (an equivalence relation as determining whether two elements are the "same" in some sense) above. We now make the first notion (an equivalence relation as a classifier) precise.

Definition 17. Suppose S is a set. Then a partition P of S is a collection of subsets P_n (possibly infinite) where any two P_i and P_j are mutually disjoint and $\bigcup_{j=1}^{n} P_j = S$.

One can use the word "partition" either as a noun indicating a set partition, as defined above, or as a verb describing the action of dividing a set into disjoint subsets. In the example below "partition" is used in its verb form.

Example 14. We can partition the set $S = \{1, 2, 3, 4, 5, 6\}$ into the even elements $E = \{2, 4, 6\}$ and odd elements $O = \{1, 3, 5\}$. One can easily see that E and O are disjoint and $E \cup O = S$.

Now we will describe how given an equivalence relation we can partition a set in a natural way. The idea is to include elements in the same subset if they are related to each other. A typical example is the previous example. If $\sim$ represents the equivalence relation $a \sim b$ if $a-b$ is even, then all the elements in E are equivalent and all the elements in O are equivalent. To generalize this notion to any equivalence relation, we will introduce the notion of an *equivalence class*.

Definition 18. Suppose that S is a set and $\sim$ is an equivalence relation on S. Let $a \in S$ be any element. We define

$$C_a = \{s \in S \mid s \sim a\}.$$

In other words, C_a is the subset of S whose elements are all equivalent to a.

Example 15. **??** In example 12 we defined an equivalence relation on $\mathbb{Z}$ with $a \sim b$ if $a - b$ was an even number. For this example we consider what C_0 is. If $a \sim 0$, then $a - 0 = a$ is an even number, and conversely. So $C_0 = \{\dots, -4, -2, 0, 2, 4, \dots,\}$. Similarly, we see that $C_1 = \{\dots, -3, -1, 1, 3, \dots\}$. We might observe the following:

- C_0 and C_1 are disjoint. Their union is all of $\mathbb{Z}$, so C_0 and C_1 form a partition of the integers.
- We might notice that C_4, C_2, and C_0 are all the same sets. Indeed, we can convince ourselves that in this example, no matter what element i we take from C_0, it will be the case that $C_i = C_0$. So an equivalence class seems to not depend on the *choice of representative*.

Let's further explain what we mean by a *choice of representative*. In defining an equivalence class C_a, we make a choice of element a. But as the above example demonstrates, it seems like we can choose any element from an equivalence class and the resulting equivalence class will turn out to be same. In fact, this happens for any general equivalence relation and we will explain why below. The reasoning follows from a particular claim:

Proposition 2. *Let S be a set, and let $\sim$ be an equivalence relation on S. For any $a, b \in S$, two possibilities happen.*

1. *$C_a \cap C_b = \varnothing$, or the two equivalence classes are disjoint.*

2. *$C_a = C_b$, ie, the two equivalence classes are the same.*

To show that this claim is true, we examine the two subsets C_a and C_b. Maybe it is the case that $C_a \cap C_b = \varnothing$. Then this possibility has happened and there is nothing left to show. The other possibility is that $C_a \cap C_b \neq \varnothing$. Then to show that the claim is true we need to show that in fact $C_a = C_b$.

First we show that $a \sim b$. Since $C_a \cap C_b \neq \varnothing$ there is an element $c \in C_a \cap C_b$. By definition of set intersection, $c \in C_a$, and also $c \in C_b$. By definition of C_a and C_b, we have that

$$c \sim a, c \sim b.$$

By symmetry of the equivalence relation, $a \sim c$, and hence by transitivity we have $a \sim b$.

Now that we know that $a \sim b$ it is pretty easy to show that $C_a \subset C_b$ and vice versa. To show that $C_a \subset C_b$, suppose d is any element of C_a. By definition, $d \sim a$. But since $a \sim b$, by transitivity, $d \sim b$. So $d \in C_b$. It follows that $d \in C_b$. So $C_a \subset C_b$ as desired. To show $C_b \subset C_a$ is the same reasoning except we use the fact that $b \sim a$ (which is true by the symmetry property, since $a \sim b$).

We are almost done. Now we just need to answer the following question: how does this claim help us generate a partition of the set S? To answer this partially we might describe how to partition S using an iterative process:

1. Choose any element $a \in S$. We take C_a.

2. Choose any element $b \in S$, $b \notin C_a$. This ensures that $C_a \cap C_b = \varnothing$ (why?).

3. Choose any element $c \in S$, $c \notin C_a \cup C_b$. This ensures that C_c is disjoint from the first two sets.

4. Keep going like this until S is exhausted of elements.

In more informal terms, we think about S as a big box of elements, and we keep taking out elements of S in bags (which are the equivalence classes), until S is empty. The reason why this is not quite complete is that of course S may be infinite, and the equivalence classes finite, so of course one might never reach a complete partition of S by doing this. But the idea is still the same.

To partition any general set S with an equivalence relation $\sim$ we first observe that the following identity holds:

$$S = \bigcup_{a \in S} C_a.$$

To see why this is true, we first observe that since all the $C_a \subset S$, then the backwards inclusion is true, ie $S \supset \bigcup_{a \in S} C_a$. For the forwards inclusion we observe that for any $c \in S$, $c \in C_c$ because $c \sim c$ by reflexivity. It follows that the forward inclusion holds, and hence the set equality holds.

Now we are tempted to say that the collection $\{C_a\}_{a \in S}$ is the required partition, because by the claim we showed above, the C_a are either the same or disjoint. However, since some of the C_a may be the same this is not generally a partition yet. What we do now is consider all collections $C_a, C_b, \ldots$ which are the same, and discard all but one of them. After this process is completed we have our required partition.

Example 16. In Example 11, we showed that the equivalence relation $p \sim q$ on two apartment building residents if p and q live at the same apartment number was an equivalence relation. The equivalence class C_p consists of the set of all people who are roommates with p. In general, by our reasoning above, the collection of distinct and disjoint equivalence classes partitions the set of apartment residents A. Ignoring the representative of any equivalence class, we see that any subset used in our partition is just the collection of all the people living in a particular apartment number.

If we are to view this equivalence relation as a classifier, then roughly speaking this equivalence relation classifies people based on what apartment number they reside at.

Example 17. In Example **??** we divided the integers $\mathbb{Z}$ based on whether they were odd or even. So in terms of a classifier this equivalence relation classifies integers based on whether or not they are even or odd.

4.2.1 Exercises

1. Let X be any non-empty set, and consider the relation $R = X \times X \subseteq X \times X$. Verify that this relation is an equivalence relation.

2. Give an example of a relation R on a set A that is

   - reflexive and symmetric but not transitive,
   - symmetric and transitive but not reflexive,
   - reflexive and transitive but not symmetric.

3. Suppose Lily presents to you the following argument that a relation having the transitive and symmetric property is in fact reflexive: they explain that since $a \sim b$, then by symmetry, $b \sim a$. But then by transitivity, $a \sim a$. So the relation must be reflexive. Is Lily right? If not, what is wrong with their argument?

4. (The classification of symmetric, transitive relations.) Let A be a non-empty set, and suppose that R is a non-empty relation which is symmetric and transitive. Show there is a non-empty set $B \subseteq A$ for which R is an equivalence relation restricted to $B \times B$. Explicitly describe the set B. (It might help to do the second part of problem 2 first.)

5. (Equivalence relations as surjective functions) Let X be a set, and let $\sim \quad \subset X \times X$ be an equivalence relation.

   (a) Let $C = \{C_a \mid a \in A\}$. That is, C is the set of all the equivalence classes (which are sets, specifically subsets of X). Consider the function $f : X \to C$ given by $f(a) = X_a$. First, justify that f is actually a function (that is, it satisfies all the properties that functions satisfy). Then prove that f is a surjective (this is pretty easy).

(b) Now consider any surjective function

$$f : X \to C,$$

where C is any set. Consider the sets

$$M_z = \{x \in X \mid f(x) = z\}.$$

Show that the relation $\sim \ \subset X \times X$ defined by $x \sim y$ if and only if $f(x) = f(y)$ is an equivalence relation. Show that the equivalence classes of $\sim$ are precisely the sets M_z for $z \in C$.

(c) Let D be the set of equivalence classes of $\sim$ in the previous part. Now define $\tilde{f} : D \to C$ by $\tilde{f}(C_a) = f(a)$. Show that $\tilde{f}$ with this rule defines a valid function and that $\tilde{f}$ is in fact bijective.