In this chapter we discuss the notion of a **set** and various terminology associated with it. In many theoretical areas such as mathematics, computer science, and physics, the notion of a set comes up very naturally (in general, it comes up when one is considering any collection or class of objects). In many fields, especially in technical literature, such knowledge about sets and their associated terminology are basic knowledge that everyone should know.

Here are the topics that we will cover:

- First we will have a reasonable discussion about what a set is, and then describe some common set operations. We will discuss some of the relationships between some of these operations.
- After this we will discuss the *function*, which is arguably a more important concept than the sets themselves. We will describe some common properties that functions have and discuss some common operations, such as function composition.
- We will explore relations, which are a way of comparing elements in any given set, and classify certain nice relations called equivalence relations.
- Finally, we will explore the notion of "size" of a set, where we demonstrate that there are different types of "infinity".

1 Sets

1.1 Definition and Notation

Most roughly speaking, a set is a collection of objects. These objects can be numbers, people, or even other sets themselves. In general this collection need not be homogeneous (so for example, we can have a set which contains both a number and a person). For the purposes of this book, we will not even try to discuss a formal logical definition of a set (this gets very complicated). So for the rest of this book we can assume that a set is a collection of objects without any issues whatsoever.

In general, when we are describing sets as a collection of specific objects, we tend to use curly braces {} to enclose these objects while delimiting them by commas.

Example 1. If we wanted to describe the set containing the numbers 1, 2, and 3, we would do so as $\{1, 2, 3\}$. The set $\{1, \{1, 2\}\}$ is the set which contains the number 1, and the set $\{1, 2\}$ containing 1 and 2.

The set $\{1, 1\}$ is the set containing the numbers 1, ... and 1? Since 1 and 1 are in fact the same this set is the same as $\{1\}$. We will make this example more formally clear later in the chapter.

In general describing sets like this gets awkward when the number of elements that the set contains gets very large. For example, it is very awkward to write out

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$$

to describe the set of positive integers between 1 and 15 inclusive. There are a couple of things we can do. For most purposes if we instead wrote this set as

$$\{1, 2, 3, \ldots, 14, 15\}$$

where most people would understand what the set was. Another thing we can do is use a variable, such as A, to denote the set of positive integers between 1 and 15 inclusive. The best convention is to combine these two notions. We can define

$$A = \{1, 2, 3, \ldots, 14, 15\}.$$

In this formulation A is defined as the set of positive integers between 1 and 15 inclusive. Once we have defined A in this way we can freely use just the character "A" instead of $\{1, 2, 3, \ldots, 14, 15\}$, which is a good way to write concisely.

In general, we can refer to any abstract set using a variable such as A, B, etc. (In general most people will use capital letters to refer to sets).

Example 2. The set $\mathbb{N}$, known as the set of **natural numbers**, denotes the set of positive integers. That is,

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

The set $\mathbb{Z}$ (german for *Zahlen*) denotes the set of all integers. That is,

$$\mathbb{Z} = \{\ldots, -2, 1, 0, 1, 2, \ldots\}.$$

In some texts people include 0 in the natural numbers. This is entirely a matter of convention and not for any very deep reason. Sometimes it is convenient to include 0.

The empty set, denoted $\varnothing$ is the set which does not contain any elements.

1.2 Common Set Operations

The first thing we describe in this section is notation to describe whether an object (usually denoted by a lowercase letter) is in a certain set. We use the $\in$ notation in order to do so. That is, we say that

$$x \in A$$

if the object x is in the set A. Alternatively if an object x is not in A we write $x \notin A$ instead.

Example 3. Let $A = \{1, 2, 3, \ldots, 14, 15\}$. Then $6 \in A$.

The empty set $\varnothing$ has the property that $x \notin A$ for any conceivable object x.

The notion of set containment, which is an inquiry about whether any object is contained in a set, leads naturally to a general notion of inquiring whether some set is contained inside another. The definition for this is pretty natural. Roughly speaking, a set A is contained in a set B only if all the objects contained in A are also contained in B. We make this notion more formal now.

Definition 1. Suppose A and B are sets. Then we say that $A \subset B$ if we have for every $x \in A$ we have $x \in B$ also.

Example 4. We have

$$\{1, 2\} \subset \{1, 2, 3\}.$$

1.3 Exercises

1. It's important to realize the limitations of what kinds of sets we can define in our usual notation. The following example, independently discovered by Russel and Zermelo in the early 20th century illustrates that there cannot be a set of all sets.

   Suppose there was a universal set U, the set of all sets. Then U contains itself (since U is also a set). Consider the subset of U defined by

   $$N = \{S \in U \mid S \notin S\}.$$

   That is, N contains all the sets S which do not contain themselves.

   (a) Show that N is non-empty.

   (b) Suppose $N \in N$. Deduce that $N \notin N$. This is a contradiction, since formally we get the logical expression

   $$(N \in N) \wedge \neg(N \in N) \equiv F.$$

   So then $N \notin N$?

   (c) But now suppose $N \notin N$. Reason that $N \in N$.

   So the subset N cannot exist. The conclusion is that the set U is not well-defined (ie, does not exist).

2. Fill in the table on page 2.

   Here are some definitions we will go over. A and B denote any sets.

   - $x \in A$ means an object x is in the set A.
   - $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.
   - $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.
   - $A \subseteq B \iff ((x \in A) \implies (x \in B))$.
   - $A \subset B \iff (A \neq B) \wedge (A \subseteq B)$.
   - $\mathcal{P}(A) = \{U \mid U \subseteq A\}$. This is the **powerset** of A.
   - $|A|$ simply denotes the number of elements in the set A. For example, $|\{1, 2, 3\}| = 3$.

3. This problem hopefully illustrates that the set cardinality operation is a "depth 1" operation. All that matters when considering cardinality is the number of elements/objects in the first level of the set.

   Determine the numbers for the sets below.

   (a) $|\{3, 6, 9\}|$

   (b) $|\{\{3\}, 6, 9\}|$

(c) $|\{\{3\}, 6, 9, \{9\}\}|$

(d) $|\{\{3\}, 6, 9, \{\{9\}, 9\}\}\}|$

(e) $|\{\mathbb{R}\}|$

4. Just to get you used to the powerset operation. Compute $\mathcal{P}(\{a\})$, $\mathcal{P}(\{a, b\})$, and $\mathcal{P}(\{a, b, c\})$. Compute $\mathcal{P}(\mathcal{P}(\{a, b\}))$.

5. Last question, not really about set theory, but rather about quantifiers. Consider the propositional statement
$$(\forall x \in \varnothing)[x = 5].$$
Is this true? False? Discuss with your tablemates. After this, consider the proposition
$$(\forall x \in \varnothing)[x \in A]$$
where A is any set. (Hopefully, from this we see that $\varnothing \subseteq A$ for all sets A!)

2 Functions and Relations

3 Functions

3.1 Definition and Terminology

In this section, we will answer two questions. Namely, what is a function, and what do functions do?

From prior experience in precalculus and calculus courses most people have an intuitive picture of what functions are. For most purposes, they are "rules" that sends values to other values. In other words:

> A function is a rule that assigns to certain elements in a set, certain elements to another set.

The word rule is not very clear or rigorous. For example, we might consider the real valued functions $f(x) = x^3$ and $g(x) = x^3 + 3x - 3(x + 3) + 9$. These have the same value everywhere, but the rule for evaluating them is slightly different.

It turns out that when defining functions rigorously, the "rule" is a means to an end. All we care about when talking about any specific function is the output given the input.

Definition 2. A function $f : A \to B$ (read as "f from A to B") is defined as a set of ordered pairs (a, b) where $a \in A$ and $b \in B$ (more specifically, a subset of the set of ordered pairs $A \times B$). f has the following properties:

1. For all $a \in A$, there exists some $b \in B$ such that $(a, b) \in f$.

2. For all $a \in A$, $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.

In other words, every value $a \in A$ will have some value b such that (a, b) is in f. Moreover, this value is **unique**. We define $f(a)$ as the unique value in B such that $(a, f(a)) \in f$.

Here is some more terminology.

Definition 3. Let $f : A \to B$ be a function. We say A is the **domain** of the function f and B is the **codomain** of f. We say that a is mapped to b or f maps a to b if $(a, b) \in f$. You can also write $a \mapsto b$ as long as it is clear what f is.

Using these definitions, we can say the following: Given any function $f : A \to B$, every value in the domain will be mapped to exactly one value in the codomain.

3.2 Injectivity, Surjectivity, Bijectivity

Here are the relevant definitions:

- A function $f : A \to B$ is injective (or one to one) if $f(a) = f(b) \implies a = b$. So two different values in A do not map to the same value in B.
- A function $f : A \to B$ is surjective (or onto) if for all $b \in B$ we can find a value in A such that $f(a) = b$.
- A function f is bijective if it is both injective and surjective.

Here is an interesting observation: if $f : A \to B$ is a bijection, then there is an inverse function $g : B \to A$. For each $b \in B$, we can define $g(b)$ as the unique value $a \in A$ such that $f(a) = b$. We know such a value exists because f is surjective, and we know that this value is unique because f is injective.

4 Relations

Let A be a set. Traditionally, we have lots of different notation for when we might want to *compare* two objects in A. For example, the $=$ notation "compares" for equality, and the $\leq$ symbol compares two objects for magnitude. Relations generalize these comparison operators.

Definition 4. Let A be a set. A relation R on A is a subset of $A \times A$. That is, R is any subset of ordered pairs (a, b) where $a \in A$ and $b \in A$.

This definition can used as a comparison property as we demonstrate using the following terminology. We say that an element $a \in A$ is related to an element $b \in A$ if $(a, b) \in R$. So in fact it is beneficial to just think of R as the set of all possible relations.

Definition 5. Let A be a set and R be a relation on $A \times A$. A relation is

- *reflexive* if $(a, a) \in R$ for every $a \in A$,
- *symmetric*, if for all $a, b \in A$, $(a, b) \in R \implies (b, a) \in R$.
- *transitive* if for all $a, b, c \in A$, $(a, b) \in R, (b, c) \in R \implies (a, c) \in R$.

Usually, given a relation R, we will write aRb if $(a, b) \in R$. This will save use space and ink.

4.1 Equivalence Relations

In this (sub)section we will explore a special kind of relation called an **equivalence relation**. Suppose A is a set. Roughly speaking, there are two useful ways of viewing an equivalence relation A.

- We can view the relation as a classifier. That is, the relation divides the elements of A into groups and classifies them as being the same if they are in the same group.
- Alternatively (and equivalently), the relation compares two elements a and b and determines whether the two elements are "the same" or not the same.

In this section we will see how these two views describe the same thing. In short this relationship can be more formally described as how we can use an equivalence relation to partition A into several *equivalence classes*. We will fully define these terms more formally later. First we begin with the definition of an equivalence relation.

Definition 6. Let A be a set. An equivalence relation $\sim \subset A \times A$ is a relation with the following properties:

1. For all $a \in A$, $a \sim a$.
2. If $a \sim b$ for $a, b \in A$, then $b \sim a$.
3. If $a \sim b$ and $b \sim c$, then $a \sim c$.

Moreover, these conditions have specific names.

1. Any relation satisfying the first condition is called *reflexive*.
2. Any relation satisfying the second condition is called *symmetric*.
3. Any relation satisfying the third condition is called *transitive*.

Let us unpack the criterion for an equivalence relation $\sim$ on A. The first condition simply states that every element in a is related to itself. The second condition models the condition that if a is related to b, then b should be related to a as well. The third condition asserts the *transitivity* of the relation $\sim$. That is, if a is related to b, and b is related to c, then a should be related to c; that is, we can chain relations to get other relations.

Example 5. The notion of object equality is an equivalence relation. More formally, the $=$ symbol can be seen as a relation on any set A. We check that this satisfies all the properties that an equivalence relation should satisfy:

1. $a = a$ for all $a \in A$.
2. If $a = b$, then $b = a$ as well.
3. If $a = b$ and $b = c$, then $a = c$ as well.

The verifications we made above are true because they are in fact properties of *equality* itself. In fact, the definition of an equivalence relation was formed to encode more general notions of equality. We will describe more examples of equivalence relations later in the section and explore for each relation in what way the elements are the same.

Example 6. Suppose we have an apartment building A. Let S be the set of all residents in A. We will define an equivalence relation $\sim$ on S as follows. For $s, t \in S$ (ie, any two residents of the building), we say that $s \sim t$ if s and t live at the same apartment number. We can verify that this is an equivalence relation by verifying the three properties an equivalence relation needs to satisfy:

1. For any $s \in S$, $s \sim s$ by default because they are the same person and hence they live at the same apartment number.

2. If s and t live in the same apartment number this is the same as saying that t and s live at the same apartment number. So if $s \sim t$, then $t \sim s$.

3. If $s \sim t$ and $t \sim r$, then s, t, and r all live at the same apartment number. It follows that $s \sim r$ as desired.

Example 7. For the set $\mathbb{Z}$ define a relation $\sim$ as follows. We have $a \sim b$ if $a - b$ is even. We can verify that this relation is an equivalence relation below:

1. $a - a = 0$, which is even (it is 2×0), so $a \sim a$.

2. If $a \sim b$, then $a - b$ is even. It follows that $b - a = -(a - b)$ is even as well. For if $a - b = 2m$ for some integer m then $b - a = -(a - b) = -2m = 2(-m)$ can be seen to be an even integer. So $b \sim a$.

3. If $a \sim b$ and $b \sim c$, then $a - b$ and $b - c$ are both even. So $a - b = 2m$ and $b - c = 2n$ for some integers m and n. It can be seen that their sum must be even as well. But we have
$$a - c = (a - b) + (b - c) = 2m + 2n = 2(m + n)$$
so it follows that $a - c$ is even as well. Hence $a \sim c$ and the relation is transitive.

So it follows that $\sim$ is an equivalence relation. We can verify that $a \sim b$ only if a and b are both odd or both even. So in this case, $\sim$ defines whether or not items are the same based on whether or not they have the same remainder when integer dividing by 2.

We will also show that a similar looking relation is not an equivalence relation. Suppose instead we define $a \sim b$ if $a \sim b$ is odd. One can check that this relation is symmetric, but it is neither reflexive nor transitive.

Example 8. We can generalize the previous example as follows. For the set $\mathbb{Z}$ we can define a relation $\sim$ with the property that $a \sim b$ if $b - a$ is divisible by a positive integer m (where m is given in advance). One can check as above that this forms an equivalence relation.

At the beginning of this (sub)section we described two different ways of looking at an equivalence relation. We have made the second way (an equivalence relation as determining whether two elements are the "same" in some sense) above. We now make the first notion (an equivalence relation as a classifier) precise.

Definition 7. Suppose S is a set. Then a partition P of S is a collection of subsets P_n (possibly infinite) where any two P_i and P_j are mutually disjoint and $\bigcup_{j=1}^{n} P_j = S$.

One can use the word "partition" either as a noun indicating a set partition, as defined above, or as a verb describing the action of dividing a set into disjoint subsets. In the example below "partition" is used in its verb form.

Example 9. We can partition the set $S = \{1, 2, 3, 4, 5, 6\}$ into the even elements $E = \{2, 4, 6\}$ and odd elements $O = \{1, 3, 5\}$. One can easily see that E and O are disjoint and $E \cup O = S$.

Now we will describe how given an equivalence relation we can partition a set in a natural way. The idea is to include elements in the same subset if they are related to each other. A typical example is the previous example. If $\sim$ represents the equivalence relation $a \sim b$ if $a-b$ is even, then all the elements in E are equivalent and all the elements in O are equivalent. To generalize this notion to any equivalence relation, we will introduce the notion of an *equivalence class.*

Definition 8. Suppose that S is a set and $\sim$ is an equivalence relation on S. Let $a \in S$ be any element. We define

$$C_a = \{s \in S \mid s \sim a\}.$$

In other words, C_a is the subset of S whose elements are all equivalent to a.

Example 10. **??** In example 7 we defined an equivalence relation on $\mathbb{Z}$ with $a \sim b$ if $a - b$ was an even number. For this example we consider what C_0 is. If $a \sim 0$, then $a - 0 = a$ is an even number, and conversely. So $C_0 = \{\dots, -4, -2, 0, 2, 4, \dots, \}$. Similarly, we see that $C_1 = \{\dots, -3, -1, 1, 3, \dots\}$. We might observe the following:

- C_0 and C_1 are disjoint. Their union is all of $\mathbb{Z}$, so C_0 and C_1 form a partition of the integers.

- We might notice that C_4, C_2, and C_0 are all the same sets. Indeed, we can convince ourselves that in this example, no matter what element i we take from C_0, it will be the case that $C_i = C_0$. So an equivalence class seems to not depend on the *choice of representative.*

Let's further explain what we mean by a *choice of representative.* In defining an equivalence class C_a, we make a choice of element a. But as the above example demonstrates, it seems like we can choose any element from an equivalence class and the resulting equivalence class will turn out to be same. In fact, this happens for any general equivalence relation and we will explain why below. The reasoning follows from a particular claim:

Proposition 1. *Let S be a set, and let $\sim$ be an equivalence relation on S. For any $a, b \in S$, two possibilities happen.*

1. *$C_a \cap C_b = \varnothing$, or the two equivalence classes are disjoint.*

2. *$C_a = C_b$, ie, the two equivalence classes are the same.*

To show that this claim is true, we examine the two subsets C_a and C_b. Maybe it is the case that $C_a \cap C_b = \varnothing$. Then this possibility has happened and there is nothing left to show. The other possibility is that $C_a \cap C_b \neq \varnothing$. Then to show that the claim is true we need to show that in fact $C_a = C_b$.

First we show that $a \sim b$. Since $C_a \cap C_b \neq \varnothing$ there is an element $c \in C_a \cap C_b$. By definition of set intersection, $c \in C_a$, and also $c \in C_b$. By definition of C_a and C_b, we have that

$$c \sim a, c \sim b.$$

By symmetry of the equivalence relation, $a \sim c$, and hence by transitivity we have $a \sim b$.

Now that we know that $a \sim b$ it is pretty easy to show that $C_a \subset C_b$ and vice versa. To show that $C_a \subset C_b$, suppose d is any element of C_a. By definition, $d \sim a$. But since $a \sim b$, by transitivity, $d \sim b$. So $d \in C_b$. It follows that $d \in C_b$. So $C_a \subset C_b$ as desired. To show $C_b \subset C_a$ is the same reasoning except we use the fact that $b \sim a$ (which is true by the symmetry property, since $a \sim b$).

We are almost done. Now we just need to answer the following question: how does this claim help us generate a partition of the set S? To answer this partially we might describe how to partition S using an iterative process:

1. Choose any element $a \in S$. We take C_a.
2. Choose any element $b \in S$, $b \notin C_a$. This ensures that $C_a \cap C_b = \varnothing$ (why?).
3. Choose any element $c \in S$, $c \notin C_a \cup C_b$. This ensures that C_c is disjoint from the first two sets.
4. Keep going like this until S is exhausted of elements.

In more informal terms, we think about S as a big box of elements, and we keep taking out elements of S in bags (which are the equivalence classes), until S is empty. The reason why this is not quite complete is that of course S may be infinite, and the equivalence classes finite, so of course one might never reach a complete partition of S by doing this. But the idea is still the same.

To partition any general set S with an equivalence relation $\sim$ we first observe that the following identity holds:

$$S = \bigcup_{a \in S} C_a.$$

To see why this is true, we first observe that since all the $C_a \subset S$, then the backwards inclusion is true, ie $S \supset \bigcup_{a \in S} C_a$. For the forwards inclusion we observe that for any $c \in S$, $c \in C_c$ because $c \sim c$ by reflexivity. It follows that the forward inclusion holds, and hence the set equality holds.

Now we are tempted to say that the collection $\{C_a\}_{a \in S}$ is the required partition, because by the claim we showed above, the C_a are either the same or disjoint. However, since some of the C_a may be the same this is not generally a partition yet. What we do now is consider all collections $C_a, C_b, \ldots$ which are the same, and discard all but one of them. After this process is completed we have our required partition.

Example 11. In Example 6, we showed that the equivalence relation $p \sim q$ on two apartment building residents if p and q live at the same apartment number was an equivalence relation. The equivalence class C_p consists of the set of all people who are roommates with p. In general, by our reasoning above, the collection of distinct and disjoint equivalence classes partitions the set of apartment residents A. Ignoring the representative of any equivalence class, we see that any subset used in our partition is just the collection of all the people living in a particular apartment number.

If we are to view this equivalence relation as a classifier, then roughly speaking this equivalence relation classifies people based on what apartment number they reside at.

Example 12. In Example **??** we divided the integers $\mathbb{Z}$ based on whether they were odd or even. So in terms of a classifier this equivalence relation classifies integers based on whether or not they are even or odd.

4.1.1 Exercises

1. Let X be any non-empty set, and consider the relation $R = X \times X \subseteq X \times X$. Verify that this relation is an equivalence relation.

2. Give an example of a relation R on a set A that is

   - reflexive and symmetric but not transitive,
   - symmetric and transitive but not reflexive,
   - reflexive and transitive but not symmetric.

3. Suppose Lily presents to you the following argument that a relation having the transitive and symmetric property is in fact reflexive: they explain that since $a \sim b$, then by symmetry, $b \sim a$. But then by transitivity, $a \sim a$. So the relation must be reflexive. Is Lily right? If not, what is wrong with their argument?

4. (The classification of symmetric, transitive relations.) Let A be a non-empty set, and suppose that R is a non-empty relation which is symmetric and transitive. Show there is a non-empty set $B \subseteq A$ for which R is an equivalence relation restricted to $B \times B$. Explicitly describe the set B. (It might help to do the second part of problem 2 first.)

5. (Equivalence relations as surjective functions) Let X be a set, and let $\sim \quad \subset X \times X$ be an equivalence relation.

   (a) Let $C = \{C_a \mid a \in A\}$. That is, C is the set of all the equivalence classes (which are sets, specifically subsets of X). Consider the function $f : X \to C$ given by $f(a) = X_a$. First, justify that f is actually a function (that is, it satisfies all the properties that functions satisfy). Then prove that f is a surjective (this is pretty easy).

   (b) Now consider any surjective function

   $$f : X \to C,$$

where C is any set. Consider the sets

$$M_z = \{x \in X \mid f(x) = z\}.$$

Show that the relation $\sim \; \subset X \times X$ defined by $x \sim y$ if and only if $f(x) = f(y)$ is an equivalence relation. Show that the equivalence classes of $\sim$ are precisely the sets M_z for $z \in C$.

(c) Let D be the set of equivalence classes of $\sim$ in the previous part. Now define $\tilde{f} : D \to C$ by $\tilde{f}(C_a) = f(a)$. Show that $\tilde{f}$ with this rule defines a valid function and that $\tilde{f}$ is in fact bijective.