# Chapter 8: Proving IVT, Max value theorems.

A set $\underline{A}$ of real #s is called

<u>bounded above</u>

if there exists some # $x$ such that

$x \geq a$ for every $a \in A$.

Weaken: $x$
weakly bounded above if there are only finitely many values $a \in A$ such that $x \geq a$.

Uses: I dunno.

Definition: Least upper bound.

Motivation: consider the set $\{x \mid 0 \leq x < 1\}$. Obviously, 138 is an upper bound for this set, so it is bounded above.
We say that 1 is the <u>least upper bound</u> for this set.

Definition: A number $x$ is a least upper bound of $A$ if

(1) $x$ is an upper bound of $A$.

(2) if $y$ is an upper bound of $A$, then $x \leq y$.

bounded above.

Say we have a set of numbers (a box full of numbers).

If we can find a number x such that no matter what number is in the set of numbers, then x is greater than or equal to that number, then thisset is bounded above and x is called an upper bound of this set.

least upper bound.

There is a set of least upper bounds. for a set A.

A "smallest" (least) upper bound would be smaller than or equal two any possible upper bound of A.

The least upper bound is unique. If we have two least upper bounds x and y of a set A, then

$x \leq y$ and $y \leq x \Rightarrow x = y$.

Uniqueness means we can refer to <u>the</u> least upper bound (much like groups). Often we use the term supremum.

The supremum of $A$, $\sup A$, is the least upper bound of $A$.

We can define also a greatest lower bound of $A$, termed $\inf A$, the infimum of $A$.

So which sets have a least upper bound?

There's an axiom for that!

(P13) If $A$ is a set of real numbers such that $A \neq \{\}$, and $A$ is bounded above, then $A$ has a least upper bound.

This is an important property defining the reals.

Note that in $\mathbb{Q}$, $x^2 < 2$ has no least upper bound.

(IVT)

Hypothesis: $f$ is continuous on $[a,b]$

$f(a) < 0 < f(b)$.

Conclusion: $\exists x \in [a,b]$ s.t. $f(x) = 0$.

Proof: If $f(x) < 0$ and $x$ is continuous on $a$,

there is some ~~real~~ interval $\delta$ such that on $(a, a+\delta)$

$f(x) < 0$ for all $x$.

~~Similarly,~~

Define

$A = \{x \mid a \leq x \leq b$ and $f$ is negative on $[a,x]\}$.

~~Also~~ $A \neq \{\}$ since $a$ is in $A$.

$b$ is an upper bound of $A$ since $f(b) > 0$

By ~~theorem~~ (P13)

There exists a number $\alpha$ such that $\alpha = \sup A$.

Then $f(\alpha) = 0$.

Suppose not. Then either $f(\alpha) < 0$, or $f(\alpha) > 0$.

If $f(\alpha) < 0$ there will be an interval $(\alpha-\delta, \alpha+\delta)$ where $f(x) < 0$ as well, this implies that ~~for~~ $\alpha+\delta > A$ and $\alpha+\delta > a$ contradiction.

If $f(\alpha) > 0$ use ↑ theorem similarly. Hence $f(\alpha) > 0$ and.

Proving the max value theorems:

Lemma 1:

Hypothesis: $f$ is continuous at $a$.

Conclusion: there exists some $\delta$ such that $f$ is bounded above on $(a-\delta, a+\delta)$.

Hypothesis $\rightarrow$

$\forall \varepsilon > 0 \quad \exists \delta > 0$ st

if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Define $A = \{ f(x) \mid x \in (a-\delta, a+\delta) \}$. Then

$f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ and

thus has a least upper bound.

If $f$ is continuous on $[0,b]$, then $f$ is bounded above on $[a,b]$.