Master Stability Functions

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1 Introduction

What time is it? It seems pretty obvious: just take a look at your phone or watch or perhaps work out the minutes and hours from the configuration of hands on an analog wall clock. Now, imagine you and a friend decide to grab lunch, say at 12:30 PM? You know that you can look down at your watch to determine the time and so can your friend, but how can you be sure you're both reading off the same time? You can't. Watches, analog and digital, are manmade items. In order to track changes in time, they have to "tick". Whether due to differences in friction, a short circuit, drifting pieces of dirt getting in the gears, or fluctuations at the quantum scale, clocks always manage to tell different times. The difference can range from a minor inconvenience (less than 5 minutes) to getting accidentally stood up (30 minutes). This type of problem extends far beyond our simple example. Everything from stock market trading to public transportation to global shipping require the ability to tell time accurately across different clocks.

So how can we make sure that you and your friend make it to lunch `on time`? We synchronize your clocks. Specifically, we have to devise a mechanism by which the clocks communicate and, based on that communication, correct their times. This is an central paradigm in the field of coupled oscillators, which studies the behavior of a certain class of dynamical systems, known as oscillators, whose states affect each other (i.e. are coupled). These coupled oscillator systems can get quite complicated. Oscillators themselves are nonlinear dynamical systems, prohibiting the use of the methods we learned in EEME6602 to analyze them. On top of that, the couplings themselves can be highly nonlinear.

In this paper we follow some of the work done in `Synchronization Stability in Coupled Oscillator Arrays` [3], which derives the **master stability function** (MSF) and uses it to characterize the synchronization stability of a generalized set of coupled oscillator systems based on the functional form of their coupling. This paper will specifically focus on the ideas behind and conceptualization of the master stability function. We will not go into how one might compute the MSF for a given system.

In *Background*, I introduce a set of concepts that are vital to understanding master stability functions. In the following *Problem* section, I introduce the specific problem within coupled oscillator theory that [3] is attempting to solve. Then in *Master Stability Functions* I build up the MSF as a concept. Next, in *Example*, we apply the MSF to a toy circular motion relaxation oscillator where the "coupling" between our oscillators is linear. Finally, *Limitations* describes some of the shortcomings of the MSF.

2 Background

In this section I will give short synopses of concepts that are important for understanding the MSF.

2.1 Synchronization Manifolds

In EEME 6602 we were introduced to the concept of stability. A system with state x is stable if, given a fixed point x_e and initial value $x(0) = x_0$, $x(t) \to x_e$ as $t \to \infty$. A synchronous system (I made up this term) of coupled oscillators is stable in the sense that, given any two oscillators with states x_i and x_j, $|x_i(t) - x_j(t)| \to 0$ (in the absense of outside interference or internal error). Observe that a synchronous system $x = (x_1, ..., x_N)$ tends towards a **synchronization manifold** defined by $x_1 = x_2 = ... = x_N$ if we have N oscillators. This manifold has interesting geometry (in some cases it's an N-torus), but that is outside of the scope of this paper.

2.2 Invariant Manifolds

An object x is termed as invariant under a map f if $f(x) = x$. Say we have some dynamical system $\dot{x} = F(x)$ where $F : \mathbb{R}^n \to \mathbb{R}^n$. Let f^t denote the map that takes a point r, starts our dynamical system at it (set $x(0) = r$, and follows the dynamical system from time 0 to time t and then returns $x(t)$. Now, let $A \subset \mathbb{R}^n$. If $f^t(A) \subset A$ for all choices of $t > 0$, then A is termed an **invariant set** of our dynamical system. Think of this as the property that $x \in A$ is preserved when f^t is applied over it. An **invariant manifold** is an invariant set with additional constraints, which are outside of the scope of this paper.

2.3 Variational Equations

Recall that Taylor's theorem allows us to locally approximate a nonlinear function $F : \mathbb{R}^n \to \mathbb{R}^m$ around $x_0 \in \mathbb{R}^n$ as $F(x) = A(x - x_0)$ (under specific conditions, which we will assume work out), where $A \in \mathbb{R}^{m \times n}$ is the Jacobian of F at x_0. A variation of F is a vector ζ such that $x_0 + \zeta$ is within the region of convergence of our Taylor polynomial. As such, $F(x_0 + \zeta) = A\zeta$. Observe that if we define a system by $\dot{x} = F(x)$, the previous equation gives us $\dot{x_0} + \dot{\zeta} = \dot{\zeta} = A\zeta$. As such, the variations about a point x_0 of our nonlinear system form a linear system.

2.4 Maximum Lyapunov Exponent

Suppose we start our dynamical system at two points x and y close to each other in $\mathbb{R}^n$. Let $\delta Z(t) = x(t) - y(t)$. The equation for the maximum Lyapunov exponent (MLE) is

$$\lambda_{\max} = \lim_{t \to \infty} \lim_{\delta Z_0 \to 0} \frac{1}{t} \ln \frac{|\delta Z(t)|}{|\delta Z(0)|} \tag{1}$$

The MLE is meant to characterize whether the two paths converge, diverge, or stay at the same distance. It posits that (for large t and small δZ)

$$|\delta Z(t)| \approx e^{\lambda t} |\delta Z(0)| \tag{2}$$

The MLE simply picks up the largest possible λ depending on the orientation of $\delta Z(0)$. A positive MLE indicates that the system exhibits chaotic behavior since paths diverge, whereas a negative MLE indicates that all paths locally contract towards each other. Verifying the above is outside the scope of this paper, but having an intuitive understanding of the MLE is valuable to understand [3].

3 Problem

Suppose we have a system with N coupled oscillators $x = (x_1, ..., x_N)$ where $x_i \in \mathbb{R}^m$.

Say that we can account for this system by

$$\dot{x}_i = \mathbf{F^i}(x_i, \mathbf{H}(x)) \tag{3}$$

where $\mathbf{H} : \mathbb{R}^m \to \mathbb{R}^m$ determines the "signal" sent from each oscillator based on its state (specifically, $\mathbf{H}(x) = (\mathbf{H}(x_1), ..., \mathbf{H}(x_N)))$ and $\mathbf{F^i}$ determines how to alter the state of x_i based on its current state and the signals sent by the other oscillators. Thus, this system of coupled oscillators allows the individual systems to communicate with each other and process that communication.

3.1 Constraints

On top of the system above, [3] assumes the following constraints:

1. The coupled oscillators are identical
2. The synchronization manifold is an invariant manifold
3. The same function of the state of each oscillator is used as an output to couple to other oscillators
4. The nodes are coupled in an arbitrary fashion which is well-approximated near the synchronous state by a constant scale

Constraints 1 and 2 guarantee that a synchronization manifold with a certain shape (a hyperplane) exists for our system. Constraint 3 makes it so that the stability function we end up deriving is uniquely determined by our choice of oscillators and output function. We have already accounted for it by the form of $\mathbf{H}$ above. Finally, constraint 4 is a condition used in many other studies of coupled oscillators, so [3] follows suit by assuming the same.

4 Master Stability Functions

The MSF was originally proposed in [4]. It attempts to measure the **synchronization stability** of a coupled oscillator based on the functional form of the coupling (how the oscillators are connected).

In order to understand synchronization stability, consider the behavior of the system near S, our synchronization manifold. Specifically, if we start our dynamical system at two distinct points, $x \in S$ and $\bar{x}$ near x, how do these two systems evolve?

In their analysis, [3] make the assumption that S is an invariant manifold, so the path $x(t) \in S$ for all $t \geq 0$. We can assume the same for $\bar{x}(t)$ if $\bar{x} \in S$. However, suppose that $\bar{x}(t) \notin S$. Note that we could get to $\bar{x}$ by being "perturbed" from x. Hence, if $\bar{x}(t)$ and $\bar{x}(t)$ diverge from each other, our synchronization can't be said to very stable since paths not on it diverge from it. Additionally, the distance between $x(t)$ and $\bar{x}(t)$ should not stay the same, since this would indicate that when the system is knocked out of a synchronous state it doesn't return to the synchronization manifold.

We can encapsulate these requirements by evaluating the maximum Lyapunov exponent of our system around points $x \in S$. If it is negative, the synchronization is stable. If it is positive or 0, the synchronization is not stable.

However, just evaluating the MLE at points on the synchronization manifold does not allow us to analyze the coupling configuration. For that, we will need to look at the variational equations for (3). Observe that, near S, constraints 1 and 4 give us

$$\dot{x}_i \approx \tilde{\mathbf{F}}(x_i) + \sum_{j=1}^{N} G_{ij} H(x_j) \tag{4}$$

where $\mathbf{G} \in \mathbb{R}^{N \times N}$ defines our coupling and $\tilde{\mathbf{F}}$ describes the behaviour of the system in isolation, which is the same for all oscillators. As such, the variational equation for our system about $x = (x_1, ..., x_N) \in S$ for variation $\zeta = (\zeta_1, ..., \zeta_N)$ is

$$\dot{\zeta_i} = \mathbf{J} \cdot \zeta_i + \sum_{j=1}^{N} G_{ij} D\mathbf{H} \cdot \zeta_j \tag{5}$$

where J is the Jacobian of $\tilde{\mathbf{F}}$ evaluated at x and $D\mathbf{H}$ is evaluated at $x_1 = ... = x_N$. We can write our variational equation as

$$\dot{\zeta} = [\mathbf{1_N} \otimes \mathbf{J} + \mathbf{G} \otimes D\mathbf{H}]\zeta \tag{6}$$

where $\mathbf{A} \otimes \mathbf{B} = (A_{ij}\mathbf{B})_{ij}$ is a direct product of matrices yielding a block matrix and ζ is the concatenation of the vectors of each ζ_i.

At this point we want to be able to "break down" our expression. We can do so by assuming that G is diagonalizable $\iff \exists \mathbf{P} \in GL_n$ s.t. $\mathbf{PGP}^{-1}$ diagonal. We can use the "diagonalizing" matrix P to "rotate" our perturbation such that ζ_1 represents the components of the variation along the synchronization manifold and $\zeta_2, \zeta_3, ..., \zeta_N$ represent directions transverse to S. This transforms (6) to

$$\begin{aligned} \dot{\zeta} &= [(\mathbf{P}\mathbf{1_N}\mathbf{P}^{-1}) \otimes \mathbf{J} + (\mathbf{PGP}^{-1}) \otimes D\mathbf{H}]\zeta \\ &= [\mathbf{1_N} \otimes \mathbf{J} + \lambda \otimes D\mathbf{H}]\zeta \end{aligned} \tag{7}$$

where λ is a diagonal matrix. (the methodology used here is outlined in [4]). Our full matrix $[\mathbf{1_N} \otimes \mathbf{J} + \lambda \otimes D\mathbf{H}]$ is 0 except for the blocks along the diagonal. As such, we've broken our system into N different systems of the form

$$\dot{\zeta_k} = [\mathbf{J} + \gamma_k D\mathbf{H}]\zeta_k \tag{8}$$

where γ_k is the k-th value along the diagonal of λ, which is simply the k-th eigenvalue of G. Observe that we've managed to break (5) into N isolated equations which cover how the variations along directions tangent and transverse to the synchronization manifold evolve in time.

We're concerned with how the variations transverse to the synchronization manifold evolve in time. If any of them diverge from the synchronization manifold, our synchronization is not stable. In order to evaluate this, we just calculate the MLE of our k-th system transverse to the synchronization manifold. If it is negative we have synchronization stability and if not we don't.

Now, let's consider the inverse problem. Suppose we have some system with a "signal" function $\mathbf{H}$ and $\mathbf{F}$ describing each oscillator's evolution in isolation. We want to evaluate which coupling matrices $\mathbf{G}$ yield synchronization stability.

(8) reveals that we need all he eigenvalues of our candidate G (except the one along the synchronization manifold) be such that the MLE of (8) is negative. The master stability function (MSF) for our system is the MLE of

$$\dot{\zeta_k} = [\mathbf{J} + (\alpha + i\beta) D\mathbf{H}]\zeta_k \tag{9}$$

evaluated for given α and β. Observe that it allows us to probe potential couplings G intuitively based on their eigenvalues.

5 Example

Let's try and get a feel for the MSF. As an example, [3] analyzes stability of Roessler oscillator arrays, playing around with the form of $\mathbf{H}$ to alter the MSF. The Roessler oscillator is a canonical chaotic oscillator with interesting properties. However, it is difficult to intuit. As such, I have decided to analyze a simpler and more intuitive oscillator, which I have termed the circular motion relaxation oscillator. Its differential equations are below

$$\dot{x} = \epsilon y + \delta(1-r)x$$
$$\dot{y} = -\epsilon x + \delta(1-r)y \tag{10}$$

It can also be written as

$$\dot{s} = (\epsilon \mathbf{A} + \delta(1-|s|)\mathbf{I})s = \mathbf{M}s \tag{11}$$

where $\mathbf{A}$ is the rotation matrix $R_{\frac{\pi}{2}}$. The ϵ portion of these differential equations correspond with revolution around the origin (circular motion) with velocity $r\epsilon$. The δ portion causes the system to decay towards the unit circle over time with decay constant δ. As such, the system approaches a "limit cycle" of S^1 while simultaneously revolving around the origin. For convenience, we set $\epsilon, \delta = 1$.

We define our coupled oscillator system (with N oscillators) by

$$\dot{x}_i = \mathbf{F^i}(x_i, \mathbf{H}(x)) = \mathbf{M}x_i + \sum_{j=1}^{N} G_{ij}\mathbf{H}(x_j) \tag{12}$$

where $\mathbf{G} \in \mathbb{R}^{n \times n}$ and $\mathbf{H} : \mathbb{R}^m \to \mathbb{R}^m$. Observe that since $(1-r) \approx 0$ around S^1, our equation is

$$\dot{x}_i = \mathbf{A}x_i + \sum_{j=1}^{N} G_{ij}\mathbf{H}(x_j) \tag{13}$$

near the portion of the synchronization manifold where x_1 is close to S^1. Hence our master stability function evaluated on this subset of the synchronization manifold yields the maximum lyapunov exponent of

$$\dot{\zeta}_j = (\mathbf{A} + (\alpha + i\beta)D\mathbf{H})\zeta_j \tag{14}$$

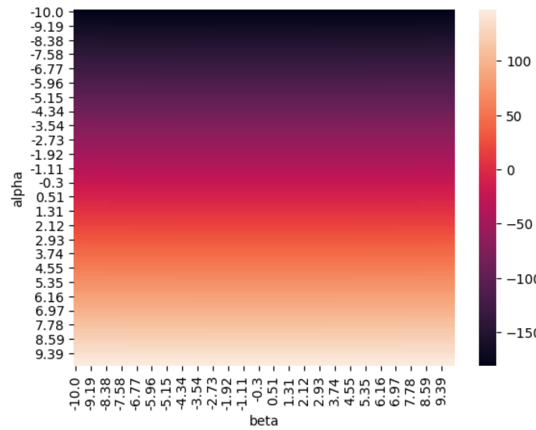
for a given choice of α and β. All that's left is to select a signal function $\mathbf{H}$. For convenience, we'll make $\mathbf{H}$ a linear function. Consider the following choice

$$\mathbf{H} = \mathbf{I} \implies D\mathbf{H} = \mathbf{I} \tag{15}$$

If we have two oscillators x_1 and x_2 near the synchronization manifold, then $x_1 \approx x_2$ and so our system equation is

$$\dot{x}_i = \mathbf{A}x_i + \sum_{j=1}^{N} G_{ij}x_j \approx \mathbf{A}x_i + x_i\sum_{j=1}^{N} G_{ij} \tag{16}$$

Hence, each oscillator pushes other oscillators towards or away from the origin (depending on G_{ij}), transverse to the synchronization manifold. The MSF for this choice of H is shown below:



We can see that eigenvalues of G with negative real parts cause the synchronization of the system to be stable (negative maximum Lyapunov exponent) wheras at a certain point eigenvalues with positive real parts cause the synchronization to become unstable.

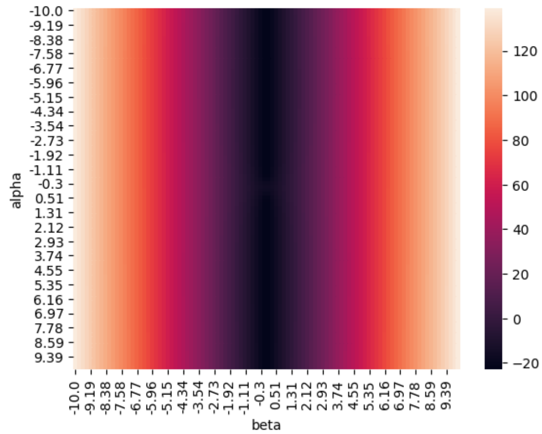
Now, let's consider a different choice of $\mathbf{H}$:

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A \implies D\mathbf{H} = A \tag{17}$$

Recalling that $x_i \approx x_j$ for any 2 oscillators x_i and x_j near the synchronization manifold, observe that our system equation is:

$$\dot{x}_i = \mathbf{A}x_i + \sum_{j=1}^{N} G_{ij}\mathbf{A}x_j \approx \mathbf{A}x_i + \mathbf{A}x_i\sum_{j=1}^{N} G_{ij} \tag{18}$$

Hence, each oscillator pushes other oscillators along their limit cycles or in the reverse direction, tangent to the synchronization manifold. The MSF for this choice of H is shown below:



Because the real component of eigenvalues (α) contributes in the same direction as the limit cycle, it can change the speed at which each oscillator goes around its limit cycle but it can't knock it off of the limit cycle. Alternatively, since **H** has eigenvalues $\pm i$, $\beta i D\mathbf{H}$ scales vectors by ± 1. Hence, for $|\beta| > 1$, our synchronization is unstable.

Observe that $|\beta|$ has to reach a threshold before the MSF crosses 0 and indicates chaotic behavior. This is due to our decay terms.

5.1 Methods

There is no standard code library released with [3]. As such, I utilized a library called `PyMSF` I found on Github, which implements the MSF [2]. I coded the circular motion relaxation oscillator model using cython [1] in the format layed out by [2] and used `PyMSF` to calculate its MSF.

6 Limitations

While the MSF is a powerful tool for analyzing the synchronization of coupled oscillators, the assumptions it makes limit its application to more complex situations. For example, the MSF cannot be applied to analyze coupled oscillators that synchronize with a phase or frequency offset (for the latter, think the hands of a clock). To observe this, note that our synchronization manifold $x_1 = ... = x_N$ specifies a certain type of synchronization. Additionallly, MSF does not account for distinct couplings between oscillators since all couplings are assumed to be identical (using the same **H**) except for their scale and sign. It might be possible to extend the usefulness of this model by considering a more complex synchronization manifold or removing the Constraint 3 requiring that the same function of the state of each oscillator is used as an output to couple to other oscillators.

References

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[2] Dmytro Belevtsoff. Pymsf. https://github.com/belevtsoff/pymsf, 2013.

[3] Lou Pecora, Tom Carroll, Gregg Johnson, Doug Mar, and Kenneth S Fink. Synchronization stability in coupled oscillator arrays: Solution for arbitrary configurations. *International Journal of Bifurcation and Chaos*, 10(02):273–290, 2000.

[4] Louis M Pecora and Thomas L Carroll. Master stability functions for synchronized coupled systems. *Physical review letters*, 80(10):2109, 1998.