Invariant Manifolds (Word count: 1489)

Jonathan Marty (jm4851), MATHGU4081

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1 Introduction

Faced with the unyielding complexity of dynamical systems, mathematicians have long sought out ways of "simplifying" them, either decreasing the number of "dimensions" or deriving more intuitive versions of the same equations. One technique that has aided in both goals is the method of invariant manifolds. In order to get a sense for this method, take a look at Figure 1.

Observe that if you pick a point in the diagram and follow the vector field from that point, you inevitably end up asymptotically close to the pink "region." This "region" strongly resembles a submanifold of the space our vector field is defined over. Since our dynamical system ends up asymptotically close to this "manifold," just considering the system on the manifold could greatly reduce its complexity while accounting for most of the activity it encodes. The method of invariant manifolds attempts to find this "invariant" manifold. In this paper, we will introduce some of the theory behind the method of invariant manifolds and calculate an invariant manifold under specific conditions. We start by building up some key definitions.

2 Invariant Manifolds

A (first order) differential equation on a manifold M can be uniquely identified with a vector field $V : M \to TM$. For a given $p \in M$, it is possible to find a neighborhood U of p such that V is uniquely identified with a local flow $\phi : (-\epsilon, \epsilon) \times W \to U$ such that $\phi_0(p) = p$ and $\frac{d}{dt}\phi_t(p) = V(\phi_t(p))$ (by convention, we write $\phi(t, x)$ as $\phi_t(x)$), where W is an open subset of U containing p. This follows from the analysis proof that first order differential equations have locally smooth solutions.

However, if our manifold is connected and compact and our vector field is smooth, we can extend our local flow ϕ to a global flow $\psi : \mathbb{R} \times M \to M$. This global flow has the same properties as our local flow, so $\forall t, s \in \mathbb{R}$, $\psi_t \circ \psi_s = \psi_{t+s}$. As such, this global flow is now a group action.

Now, we previously brought up the term "invariant manifold" to describe the manifold we're trying to find. One might ask: what is this manifold invariant under? The answer is that it is invariant under the action of the flow. More technically, if N is our invariant manifold, then for all $t \in \mathbb{R}$ and $p \in N$, $\psi_t(p) \in N$. More elegantly, N contains the orbit of

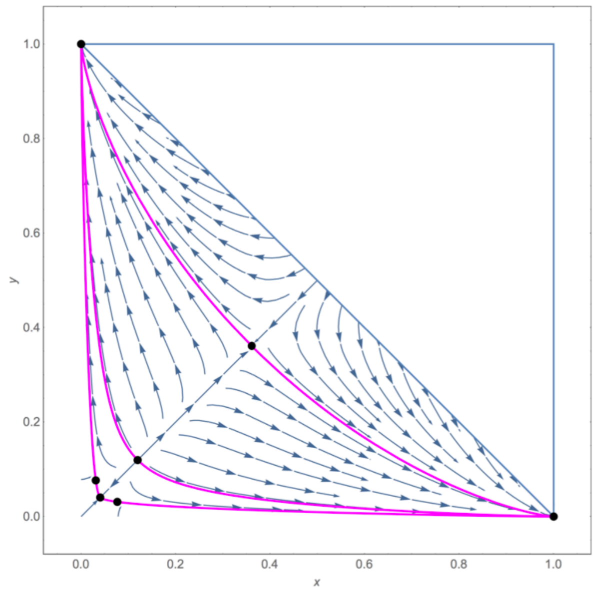

Figure 1: Example of an invariant manifold (from [5])

each of its points. In some sense, the activity on the invariant manifold is isolated from that of the rest of the system.

3 Linear Dynamical Systems

There are numerous ways of calculating invariant manifolds, both locally and globally. They all extend off of the idea of stable, unstable, and center manifolds. Consider the linear differential equation $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. The behaviour of the system depends solely on the eigenspace of A. We can partition this eigenspace into the eigenspace associated with eigenvalues with positive real components, with negative real components, and with no real components. These are, respectively, the unstable (E^u), stable (E^s), and center (E^c) manifolds for our linear system. Observe that if $x \in E^u$, $|x(t)| \to \infty$. Similarly, if $x \in E^s$, $x(t) \to 0$. Finally, if $x \in E^c$, the magnitude of x doesn't change over time. For the sake of simplicity, we will assume that E^c is trivial (the alternative is too complicated to cover here). Let's initialize our dynamical system at a point p. If p is in one of our two eigenspaces, then it never leaves that eigenspace. In the alternative case, both $\pi_s p, \pi_u p \neq 0$. Proceeding forward in time, the stable component of p vanishes and $\phi_t(p)$ approaches E^u. Proceeding backward in time, the unstable component vanishes and $\phi_t(p)$ approaches E^s. Hence, both E^s and E^u are invariant manifolds of $\mathbb{R}^n$.

4 Nonlinear Dynamical Systems

We now consider the nonlinear case, where $\dot{x} = F(x)$ for some function $F \in C^1(\mathbb{R}^n)$. Ideally we want to capitalize on what we know about linear dynamical systems when considering nonlinear ones. Recall Taylor's theorem.

Theorem 1 *Taylor's Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a r-times differentiable function at $p \in \mathbb{R}^n$. Then there exists $R_p^r : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x) = T_p^r(x-p) + R_p^r(x-p) \tag{1}$$

and

$$\lim_{h\to 0} \frac{R_p^r(h)}{|h|^r} = 0 \tag{2}$$

where T_p^r and R_p^r are the r-th degree Taylor polynomial and remainder of f

Under specific conditions (which we will assume for all functions we consider), there is a nontrivial "region of convergence" $U_p^r \subset \mathbb{R}^n$ containing p on which $T_p^r = f$.

Returning to our nonlinear system, we can "linearize" it at $x \in \mathbb{R}^n$ by taking $A = J(F)(x)$. We can determine the unstable, stable, and center eigenspaces E_x^u, E_x^s, and E_x^c for A, rooting them at x. By Taylor's theorem, $\dot{x}(t) - \dot{x}(0) = Ax$ for $x(t)$ in a local neighborhood around $x(0)$. $x(0)$ is arbitrary. If we select $x(0)$ such that it is a fixed point (i.e. $\dot{x}(0) = 0$), then our system is locally identical to a linear system. Hence, our eigenspaces describe the system's behavior around $x(0)$. Similarly to the linear case, we will assume that the Jacobian A at $x(0)$ has no center eigenspace (i.e. is hyperbolic).

Now, suppose we have some vector field V on $\mathcal{R}^n$ specifying a dynamical system. This vector field has some degree of "smoothness". This is conventionally specified by the number of times that our vector field can be "differentiated". To understand what this means, recall that, using Einstein summation notation, we can write

$$V = \alpha_\mu \frac{d}{dx^\mu} \tag{3}$$

where α_μ corresponds to the μ-component of F if we instead write our system as $\dot{x} = F(x)$.

When we say that our vector field can be "differentiated" r times, we mean

$$\alpha_\mu \in C^r \tag{4}$$

for all μ.

Now, let's pick a point $p \in U$. Assume $\alpha_\mu \in C^r$. Let $T_{\mu,p}^r : M \to \mathbb{R}$ denote the Taylor polynomial of α_μ and $U_{\mu,p}^r$ its region of convergence. Observe that all of our Taylor polynomials are equal to α_μ within $U_p^r = \cap_\mu U_{\mu,p}^r$. Since n is finite, U_p^r is nontrivial. Observe that if our vector field is smooth, $U_p^\infty = \mathbb{R}^n$ for all p, so a smooth vector field can be replaced by its Taylor polynomial everywhere.

5 Example

We will now demonstrate how to calculate invariant manifolds $W^s(p)$ and $W^u(p)$ around a hyperbolic fixed point p of V. These are called the stable and unstable manifolds because they are tangent to E_p^s and E_p^u respectively at p. Suppose $n = 2$, $p = (0,0)$, and α_x and α_y are such that $T_{x,p} = x$ and $T_{y,p} = -y + x^2$. Working within U_p^r, observe that $J(F)(p) = [[1,0],[2x,-1]] = [[1,0],[0,-1]]$. This matrix has eigenvectors $[1,0]$ and $[0,1]$ with eigenvalues 1 and -1. Hence, E_p^u is the x-axis and E_p^s is the y-axis.

For convenience, let's denote $(x(t), y(t)) = \psi_t(p)$ and $(\dot{x}(t), \dot{y}(t)) = \frac{d}{dt}\psi_t(p)$, where ψ_t is the flow associated with V locally. Thus, our system's differential equation is $\dot{x} = x$ and $\dot{y} = -y + x^2$. In this system, x is our unstable coordinate and y is our stable coordinate. We want to derive a function $h(x)$ such that $y = h(x)$ determines $W^u(p)$. $h(x)$ must satisfy 2 conditions

$$h(0) = 0 \tag{5}$$

$$\dot{y} = \dot{h(x)} = \frac{dh}{dx}\dot{x} \tag{6}$$

(5) states that our invariant manifold passes through p. (6) is a little more complicated. Setting $\dot{h(x)} = \dot{y}$ ensures that our manifold is invariant under the flow and specifically setting $\dot{h(0)} = 0$ means that our manifold is tangent to E_p^u at p.

We can rewrite the second condition as

$$\frac{dh}{dx}\dot{x} - \dot{y} = 0 \tag{7}$$

Substituting in our equations for $\dot{x}$ and $\dot{y}$

$$\frac{dh}{dx}x - (-y + x^2) = \frac{dh}{dx}x - (-h(x) + x^2) = 0 \tag{8}$$

Now, let's approximate $h(x)$ as a Taylor polynomial

$$h(x) \approx 0 + ax + bx^2 + cx^3 + ... \tag{9}$$

This gives us

$$\frac{dh}{dx} \approx a + 2bx + 3cx^2 + ... \tag{10}$$

Substituting (9) and (10) into (8), we get

$$(a + 2bx + 3cx^2 + ...)x - (-(ax + bx^2 + cx^3 + ...) + x^2) = 0 \tag{11}$$

With some further manipulation, this yields

$$(2a)x + (3b - 1)x^2 + (4c)x^3 + ... = 0 \tag{12}$$

Thus, $a, c, d, ... = 0$ and $b = \frac{1}{3}$. Hence

$$h(x) = \frac{1}{3}x^2 \tag{13}$$

which determines $W^u(p) = \{(x, h(x)) : x \in \mathbb{R}\}$ uniquely.

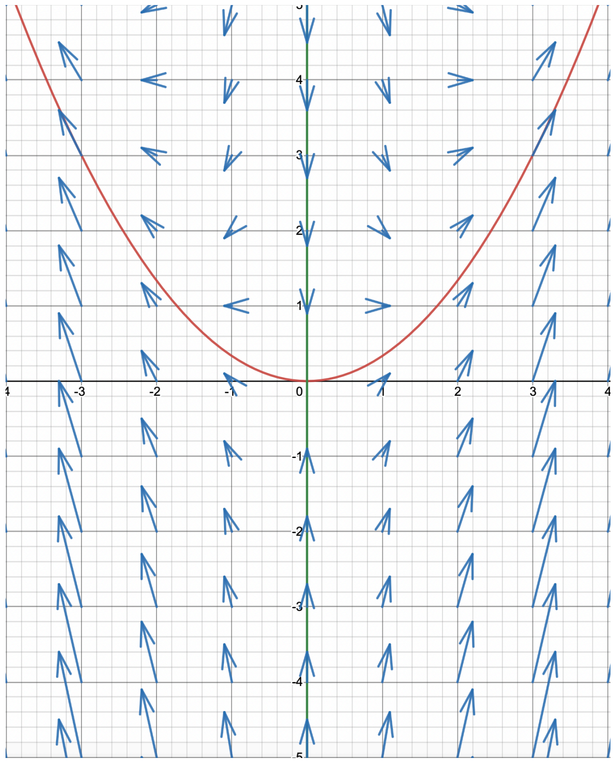

Figure 2: W^u in red, W^s in green

A similar calculation can be used to determine $W^s(p)$, the stable invariant manifold. It yields $W^s(p) = \{x = 0\}$, so the stable manifold is just the y axis. The vector field is visualized in Figure 2, as well as our two invariant manifolds.

6 Discussion

The above technique can be adapted to calculate invariant manifolds for a wide range of nonlinear systems, giving the following result:

Theorem 2 *Local Invariant Manifold Theorem for Hyperbolic Points: If X is a smooth vector field on $\mathbb{R}^n$ and x_e is a hyperbolic fixed point. There is a k-manifold $W^s(x_e)$ and a (n-k)-manifold $W^u(x_e)$ each containing the point x_e such that*

1. *Each of $W^s(x_e)$ and $W^u(x_e)$ are locally invariant under X*
2. *The tangent space of $W^s(x_e)$ at x_e is $E^s_{x_e}$ and that of $W^u(x_e)$ is $E^u_{x_e}$*
3. *If $x \in W^s(x_e)$ then $\phi_t(x) \to x_e$ as $t \to \infty$ and if $x \in W^u(x_e)$ then $\phi_t(x) \to x_e$ as $t \to -\infty$*
4. *$W^s(x_e)$ and $W^u(x_e)$ are locally unique*

However, Theorem 2 is limited to hyperbolic fixed points and does not necessarily give global invariant manifolds. Both cases are far more complicated. For those interested, refer to Smale's global stable manifold theorem [6] and the following paper on center manifold theory [1].

7 Notes

[3], [4], and [2] inspired the style and content of this paper.

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