TMA 04

Modelling report 1

Specify the purpose – **3/5 marks**

A concise and non-repetitious specification that is not a direct copy. However deducted marks are for vagueness because the specification should be easy to read and more direct, also the student hasn't mentioned that the no parking area should be optimal meaning it should be long enough to ensure safety but not longer than necessary. While the specification does inform of the problem it should be easier to interpret – see deduction for vagueness.

Create the model

Describe features and outline - **4/5 marks**

A good informative easy to interpret outline which describes the area of Mathematics (geometric argument) of what will be explored. However, an improvement could be what isn't considered this would determine the boundaries of the problem and inform what the model encompasses.

State assumptions – **3/5 marks**

The assumptions can approximate the reality and all of the important assumptions outlined in the mark scheme are covered. What the assumptions don't cover are weather conditions and taking account of going over the speed limit. These assumptions are important is the worst case scenario hasn't fully been addressed. There was no mention of applying the brakes as soon as they see someone – thinking distance.

Variables and parameters – **4/5 marks**

Specific definitions with all variables and parameters used in the report all with SI units. The deducted mark is for two or more characters following each other 'vmax'.

Formulate mathematical relationships – **12/13 marks**

Assumptions 1 to 4 and 8,9 are in the formulation where assumptions 5,6,7 are covered in "do the mathematics section". The formulation provides a very useful diagram and follows from assumptions and described in the outline. A generalised model however it should be notes that w/2 could have been replaced with a ratio parameter for example the widening of the road.

Do the mathematics – **5/10 marks**

The formulas and derivations are easy to understand with providing a text explanation. The student has provided references to assumptions and numerical values are given. However, there are no graphs or dimension analysis so no marks can be given.

Interpret the results – **4/10 marks**

The student provides no references for their chosen values of f and s as well as no references to average width. An accompanied appendix with references to it would bolster the report's reliability. No mention of worst-case scenario of having a narrower road. Could have shown equation with substituted values. Speed limit prediction should really include a 40mph as some drivers may inadvertently go over the speed limit, considering pedestrian safety is an important aspect of the model and report. The values given are given to more than 3 significant figures.

Modelling report 2

Specify the purpose – **4/5 marks**

Easy to read, concise with no repetition but is a little vague. There should be a mention of safe stopping distance in a vehicle in relation to the no parking zone and the significance of the no parking zone.

Create the model

Describe features and outline - **3/5 marks**

The outline is brief and lacks depth – its well under 100 words. The outline raises and important point of worst-case scenario. The outline fails to inform the approach that is used and doesn't mention what is not considered.

State assumptions – **1/5 marks**

Most assumptions are realistic but this is a small list, and has no mention of road width. The student does well to mention the weather conditions but isn't a worst-case scenario since we live in a wet climate this would occur quite often and hence would increase the breaking distance. In addition to this there was no mention of the parked car and its position as well as the road being straight leaving 3 omissions from the mark scheme.

Variables and parameters – **4.5/5 marks**

All variables and parameters used in the report are mentioned and specific. All are useful and correctly labelled and SI units are used. However, like units should be grouped.

Formulate mathematical relationships – **8/13 marks**

This section provides valid references in most cases however, the student mentions that assumption 2 is related to the large car, I believe this is a mistake. Most assumptions are used apart from the Highway code is used but not in assumptions. The weather conditions aren't related to the formulation or the mathematics section. The student hasn't accounted for the front of the car just the position of the driver. The formula looks correct and dimensionally consistent. There could be more information included in the derivations of the formulas.

Do the mathematics – **6.5/10 marks**

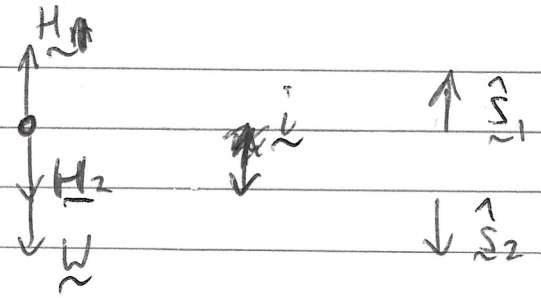
Short explanations but is easy to understand its worth noting I the student says that the equation is a function, this is not the case. Provides the variable of interest in terms of the other variables and parameters. There are no units on graphs which makes reading more difficult to interpret and titles should be also considered. The student should mention that the other variables are kept constant in

relation to the two variables in the plots. The overall presentation of this section is a little crude. Decent attempt at dimension analysis although another method in unit 18 could also be used.

Interpret the results – **5/10 marks**

More references are needed in this section for example the student mentions “from the internet”. This is not sufficient for a scientific report, so a source must be include. They also don’t mention a source for the OU speed limit. The student mentions “typical distance” – how was this measured, again a reliable source of information should be used. In the results has included speeds above the speed limit and predictions seem reasonable and are quoted to less than 3 significant figures.

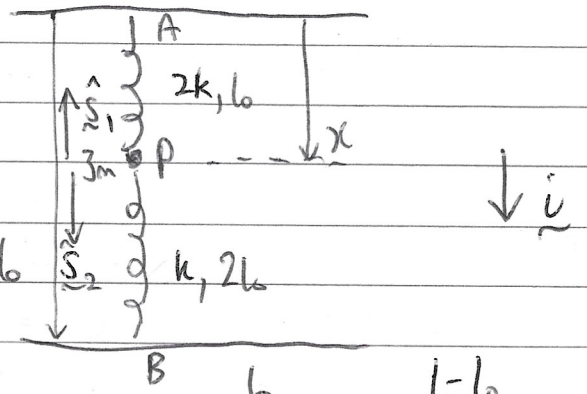
Q2a)



$\underset{\sim}{W}$ is the weight of the particle
$\underset{\sim}{H}_1$ is the spring force in the upper spring
$\underset{\sim}{H}_2$ is the spring force in the lower spring

Choose the x-axis to point vertically ~~upwards~~ downwards. Take the origin at A.

~~Spring~~
~~AP~~
~~BP~~



b)

Spring	Spring length l	Natural length l_0	Extension $l-l_0$	Stiffness	$\hat{\underset{\sim}{s}}$	$\underset{\sim}{H}$
AP	x	l_0	$x-l_0$	$2k$	$-\underset{\sim}{i}$	$2k(x-l_0)(-\underset{\sim}{i})$
BP	$4l_0-x$	$2l_0$	$2l_0-x$	k	$\underset{\sim}{i}$	$k(2l_0-x)\underset{\sim}{i}$

So, $\underset{\sim}{H}_1 = 2k(x-l_0)(-\underset{\sim}{i}) = 2k(l_0-x)\underset{\sim}{i}$
$\underset{\sim}{H}_2 = k(2l_0-x)\underset{\sim}{i}$
$\underset{\sim}{W} = 3mg\,\underset{\sim}{i}$

c) Applying Newton's second law to the particle P gives

$$3m\ddot{\underset{\sim}{r}} = 3m\ddot{x}\,\underset{\sim}{i} = \underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{W}$$

$$= 2k(l_0-x)\underset{\sim}{i} + k(2l_0-x)\underset{\sim}{i} + 3mg\,\underset{\sim}{i}$$

Resolving ~~$= 2kl_0 - 2k$~~ in the $\underset{\sim}{i}$-direction

$$3m\ddot{x} = 2kl_0 - 2kx + 2kl_0 - kx + 3mg$$

$$3(m\ddot{x} + 3kx) = 4kl_0 + 3mg$$

2d) In equilibrium,

$\underset{\sim}{H}_1 + \underset{\sim}{H}_2 + \underset{\sim}{W} = \underset{\sim}{0}$

Resolving in the $\underset{\sim}{i}$-direction

$2k(l_0 - x) + k(2l_0 - x) + 3mg = 0$

$2kl_0 - 2kx + 2kl_0 - kx + 3mg = 0$

In equilibrium $x = x_{eq}$ and $a = 0$ or $\ddot{x} = 0$

$3kx_{eq} = 4kl_0 + 3mg$

$x_{eq} = \dfrac{4kl_0 + 3mg}{3k} = \dfrac{4}{3}l_0 + \dfrac{mg}{k}$

e) $3(m\ddot{x} + kx) = k$, where $k = 4kl_0 + 3mg$

try $x_p = c$, where c is a constant

Then $\dot{x}_p = \ddot{x}_p = 0$ and substituting

$3(m \times 0 + kc) = k$, thus $c = \dfrac{k}{3k}$

So the general solution ~~is the~~ ~~from~~ from c) rearranging

$\ddot{x} + \dfrac{k}{m}x = 0$

Taking $\omega^2 = \sqrt{k/m}$

$x(t) = B\cos\omega t + C\sin\omega t + \dfrac{4l_0}{3} + \dfrac{3mg}{3k}$

$= B\cos\sqrt{\dfrac{k}{m}}\,t + C\sin\sqrt{\dfrac{k}{m}}\,t + \dfrac{4kl_0}{3k} + \dfrac{mg}{k}$, where $\omega = \sqrt{k/m}$

where B and C are arbitrary constants.

Q2f) The period, from angular frequency ω

$k_1 = k_A + k_B = 2k + k = 3k$

$m_1 = 3m$

from the SHM equation (p66 handbook)

$\omega^2 = k_1 / m_1$,

Substituting values into this and sqr root

$\omega = \sqrt{\frac{\cancel{3}k}{\cancel{3}m}} = \sqrt{\frac{k}{m}}$

The period is therefore, from p66 handbook

$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi\sqrt{m}}{\sqrt{k}}$ to give the period of oscillations of the particle in motion.

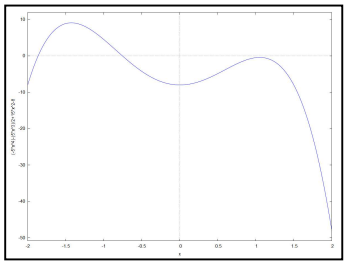
TMA04 - Q3a)b)

1 Part a)

```
U(x):=-5·x^4-5·x^3/2+15·x^2-8;
```

$$U(x):=(-5)\,x^4-\frac{5\,x^3}{2}+15\,x^2-8$$

```
wxplot2d(U(x),[x,-2,2]);
```



Total mechical energy given as: E = mv^2/2+U(x)

```
E:-5;
a:E-U(x);
kill(b);
b:U(x)-E;
z:allroots(a); /*the range or ranges of x-values (accurate to one decimal
    place) that could represent a motion of the system*/
allroots(b);
fpprintprec : 2;

for i:1 thru 2 do disp(z[i]);
for i:3 thru length(z) do disp(z[i]);

/* range is from [-2.0,1.4] */
```

$$-5$$

$$5\,x^4+\frac{5\,x^3}{2}-15\,x^2+3$$

done

$$-5\,x^4-\frac{5\,x^3}{2}+15\,x^2-3$$

$$[x=0.49,x=-0.45,x=1.4,x=-2.0]$$

$$[x=0.49,x=-0.45,x=1.4,x=-2.0]$$

2

$x=0.49$

$x=-0.45$

done

$x=1.4$

$x=-2.0$

done

2 Part b)

```
kill(E);
E:10;
kill(c);
c:E-U(x);
d:allroots(c);
lindex:length(d);
for i:1 thru lindex do disp(d[i]);
```

done

10

done

$5x^4+\frac{5x^3}{2}-15x^2+18$

$[x=0.5637185154442473\ \%i+1.185973063850168, x=1.185973063850168-0.5637185154442473\ \%i, x=0.1605371721026794\ \%i-1.435973063850168, x=-0.1605371721026794\ \%i-1.435973063850168]$

4

$x=0.5637185154442473\ \%i+1.185973063850168$

$x=1.185973063850168-0.5637185154442473\ \%i$

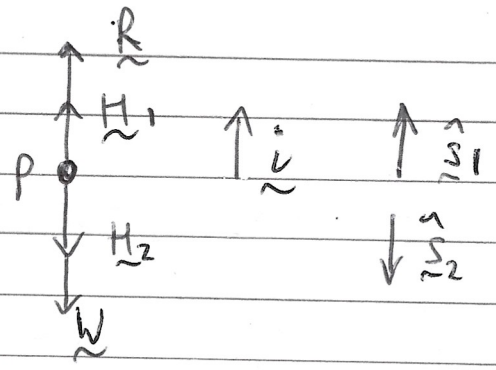
$x=0.1605371721026794\ \%i-1.435973063850168$

$x=-0.1605371721026794\ \%i-1.435973063850168$

done

/* all output values are complex values meaning no possible motions for x, since kinetic energy will be < zero. */

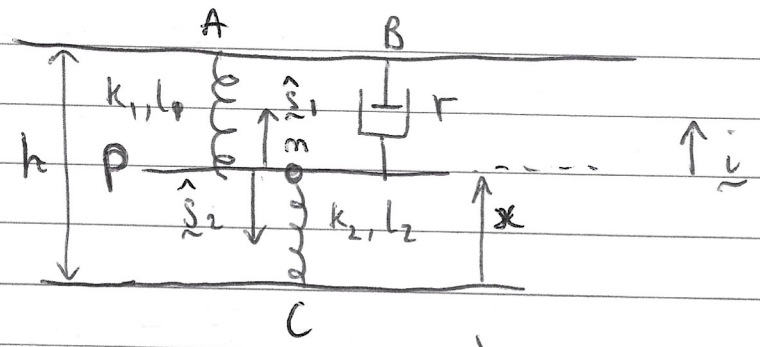
Q4a)



$\cancel{*}\, k_1(h-x-l_1)\underset{\sim}{i}$

b)

Spring	Spring length	Natural length	Extension ($l - l_0$)	Stiffness	$\hat{s}$	H
AP	$h-x$	l_1	$h-x-l_0$	k_1	$\underset{\sim}{i}$	$\cancel{k_1(h-x-l_0)}$
CP	x	l_2	$x-l_2$	k_2	$-\underset{\sim}{i}$	$-k_2(x-l_2)\underset{\sim}{i}$



$$\underset{\sim}{H_1} = k_1(h-x-l_1)\underset{\sim}{i}$$
$$\underset{\sim}{H_2} = -k_2(x-l_2)\underset{\sim}{i} = k_2(l_2-x)\underset{\sim}{i}$$

c)

Damper length	Rate of change of length	Damping constant	$\hat{s}$	R
$h-x$	$\dot{x}$	r	$\underset{\sim}{i}$	$r(\dot{x})\underset{\sim}{i}$

$\underset{\sim}{R} = r(\dot{x})\underset{\sim}{i} = r\dot{x}\underset{\sim}{i}$, since it is assumed the spring is lengthening

d) Using Newton's second law,

$$m\ddot{x}\underset{\sim}{i} = \underset{\sim}{W} + \underset{\sim}{H_1} + H_2 + \underset{\sim}{R}$$

$$= -mg\underset{\sim}{i} + k_1(h-x-l_1) + k_2(l_2-x) + r\dot{x}\underset{\sim}{i}$$

4d) Resolving in the $\underset{\sim}{i}$ direction:

$$m\ddot{x} = -mg + k_1(h - x - l_1) + k_2(l_2 - x) + r\dot{x}$$

$$m\ddot{x} - r\dot{x} = k_1 h - k_1 x - k_1 l_1 + k_2 l_2 - k_2 x - mg$$

$$m\ddot{x} - r\dot{x} + k_1 x + k_2 h = k_1 h - k_1 l_1 - k_2 l_2 - mg$$

$$m\ddot{x} - r\dot{x} + x(k_1 + k_2) = k_1(h - l_1) - k_2 l_2 - mg$$

e) Given $m = 8$, $k_1 = 60$, $k_2 = 40$, $r = 25$, $h = 2$, $l_1 = 0.75$, $l_2 = 0.75$

Substituting given values into part d)

$$8\ddot{x} - 25\dot{x} + x(60 + 40) = 60(2 - 0.75) - \cancel{(40 \times 0.75)}\,(40 \times 0.75) - (8 \times 9.81)$$

$$8\ddot{x} - 25\dot{x} + 100x = 75 - 30 - 78.48 = -33.48$$

L.H.S

The natural angular frequency is the undamped angular frequency, given as (p70 handbook)

$$\omega = \sqrt{k/m} = \sqrt{100/8} = 5\sqrt{2}/2 = 3.54 \text{ (2dp)}$$

Damping ratio given as (p69 handbook)

$$\alpha = \frac{r}{2\sqrt{mk}} = \frac{25}{2\sqrt{8 \times 100}} = \frac{5\sqrt{2}}{16} = 0.44 \text{ (2dp)}$$

4e) Since $\alpha < 1$, this ~~represents~~ is indicative of weak damping. This is suitable for a baby bouncer, as the amplitude decreases gradually with time as opposed to strong damping which exerts a strong opposing force which would make motion difficult for the baby, possibly with little or no motion at all.

$i = \sqrt{-1}$

Q5a) ① $10\ddot{x} + 4\dot{x} + 16x = 3\cos\Omega t = 3\cos 4t$

$10\lambda^2 + 4\lambda + 16 = 0$, the associated homogeneous differential equation has this auxilliary equation

$\overset{a}{5}\lambda^2 + \overset{b}{2}\lambda + \overset{c}{8} = 0$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4\times 5\times 8}}{2\times 5} = \frac{-2 \pm \sqrt{-156}}{10}$$

$$\lambda = \frac{-2 + 2\sqrt{39}\,i}{10} \quad \text{or} \quad \lambda = \frac{-2 - 2\sqrt{39}\,i}{10}$$

$\lambda = -0.2 + 1.249\,i \qquad \lambda = -0.2 - 1.249\,i$

Leads to the complementary function

$$x_c = Ae^{-0.2t}\cos(0.2\sqrt{39}\,t + \phi)$$

~~try $x = p\cos 4t + q\sin 4t$~~

with very large t values this reduces to zero

According to principal of superposition, we need a particular integral

try $x_p = p\cos 4t + q\sin 4t$, where p and q are constants.

$\dot{x} = -4p\sin 4t + 4q\cos 4t$

~~try~~ $\ddot{x} = -16p\cos 4t - 16q\sin 4t$ } differentiating

Substituting into ① yields

$-160p\cos 4t - 160q\sin 4t - 16p\sin 4t + 16q\cos 4t + 16p\cos 4t + 16q\sin 4t$

$\Rightarrow -144p\cos 4t - 144q\sin 4t - 16p\sin 4t + 16q\cos 4t$

Equating coefficients: $-144p + 16q = 3$

$-144q - 16p = 0$

$p = \frac{-27}{1312}$, $q = \frac{3}{1312}$ $\Rightarrow p = -0.020579$

$q = 0.00229$

$$x_p = -0.020579 \cos 4t - 0.00229 \sin 4t$$
$$= -(0.020579 \cos 4t + 0.00229 \sin 4t)$$

Using the identity: $A \cos x + B \sin x = R \cos(x - \alpha)$

where $R = \sqrt{A^2 + B^2}$ and $\alpha = \tan^{-1}\left(\frac{B}{A}\right)$

$A = 0.020579$, $B = 0.00229$

$$R = \sqrt{0.020579^2 + 0.00229^2} = 0.0207$$

$$\alpha = \tan^{-1}\left(\frac{0.00229}{0.020579}\right) = 0.11$$

$$\therefore -(0.020579 \cos 4t + 0.00229 \sin 4t) = \cancel{-(0.021 \cos 4t}$$
$$= -(0.021 \cos(4t - 0.11))$$

So, $x = x_c + x_p = Ae^{-0.2t} \cos(0.2\sqrt{39}\, t + \phi) = 0.021 \cos(4 - 0.1$

The x_c tends to zero after a long time has elapsed, leaving the magnitude to be 0.021 (p149 unit 10, book C)

$$A = 2.1 \times 10^{-2}\,\text{m} \simeq 2\,\text{cm}$$
$$= \underline{\underline{0.021}}\ (2\text{sf})$$

Also checking, $A = \dfrac{P}{\sqrt{(K - m\Omega^2)^2 + r^2\Omega^2}}$ (p72 handbook)

$$A = \frac{3}{\sqrt{(16 - 10 \times 4^2)^2 + 4^2 \times 4^2}} = 0.021 \ (2\text{sf}).$$

Q5b) From ①, we can take; $m = 10$, $r = 4$, $k = 16$

From p69 the damping ratio is,

$$\alpha = \frac{r}{2\sqrt{mk}} = \frac{4}{2\sqrt{10 \times 16}} = \frac{\sqrt{10}}{20}$$

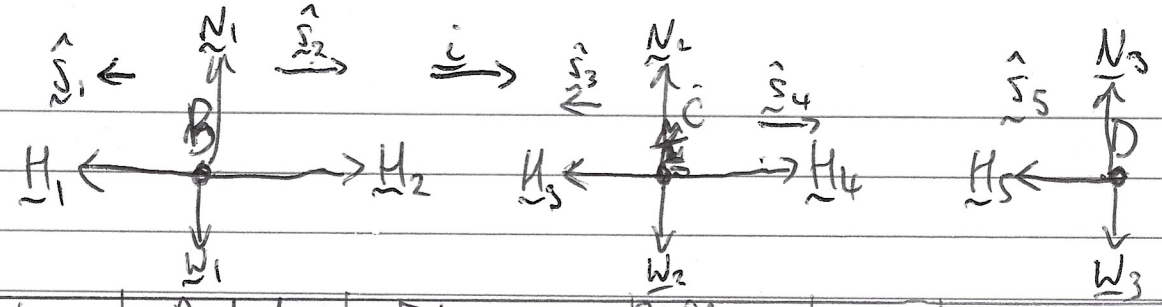
Resonance can occur when and only when:

$$1 - 2\alpha^2 > 0$$

So, $$1 - 2\left(\frac{\sqrt{10}}{20}\right)^2 = 0.95$$

$\therefore$ $0.95 > 0$ so resonance <u>can</u> occur.

Q6a)



Spring	Particle	Extension	Stiffness	$\hat{s}$	ΔH
AB	B	x_1	k	$-i$	$-kx_1 i$
BC	B	$x_2 - x_1$	$2k$	i	$2k(x_2-x_1)i$
BC	C	$x_2 - x_1$	$2k$	$-i$	$-2k(x_2-x_1)i$
CD	C	$x_3 - x_2$	$3k$	i	$3k(x_3-x_2)i$
CD	D	$x_3 - x_2$	$3k$	$-i$	$-3k(x_3-x_2)i$

c)
$$m_1\ddot{r}_1 = -kx_1 i + 2k(x_2-x_1)i$$
$$m_2\ddot{r}_2 = -2k(x_2-x_1)i + 3k(x_3-x_2)i$$
$$m_3\ddot{r}_3 = \cancel{3kx_3 i} - 3k(x_3-x_2)i$$

$$\ddot{r}_1 = \ddot{x}_1 i, \quad \ddot{r}_2 = \ddot{x}_2 i, \quad \ddot{r}_3 = \ddot{x}_3 i$$

~~Collecting terms, substituting~~

Substituting, resolving in the i-direction, and collecting terms

$$m_1: \; m\ddot{x}_1 = 2kx_2 - 3kx_1$$
$$m_2: \; \tfrac{1}{2}m\ddot{x}_2 = 2kx_1 - 2kx_2 + 3kx_3 - 3kx_2 = 2kx_1 - 5kx_2 + 3kx_3$$
$$m_3: \; \tfrac{1}{2}m\ddot{x}_3 = \cancel{-3kx_3} - 3k(x_3 - x_2)$$

Rearranging...

$$\ddot{x}_1 = \frac{k}{m}(2x_2 - 3x_1) = \frac{k}{m}(-3x_1 + 2x_2)$$
(Multiply by 2) $$\ddot{x}_2 = \frac{k}{m}(4x_1 - 10x_2 + 6x_3) = \frac{k}{m}(4x_1 - 10x_2 + 6x_3)$$
(Multiply by 2) $$\ddot{x}_3 = \cancel{\frac{k}{m}(-6x_3 - 3x_2)} = \frac{k}{m}(6x_2 - 6x_3)$$

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = \frac{k}{m}\begin{pmatrix} -3 & 2 & 0 \\ 4 & -10 & 6 \\ 0 & 6 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Q7a) $\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ eigenvector $= (0.16\ -0.93\ 0.32)^T$

Substituting $(x\ y\ z)^T$ for the eigenvector

$$\begin{pmatrix} -1 & \frac{1}{2} & 0 \\ 1 & -3 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0.16 \\ -0.93 \\ 0.32 \end{pmatrix}$$

$$= \begin{pmatrix} -0.16 - 0.465 + 0 \\ 0.16 + 2.79 + 0.64 \\ 0 - 0.93 - 0.32 \end{pmatrix} = \begin{pmatrix} -0.625 \\ 3.59 \\ -1.25 \end{pmatrix}$$

$$-0.625x = 0.16$$

$$x \approx \frac{-32}{125} \approx -0.256 \approx -0.26 \text{ (2 s.f.)}$$

so the corresponding eigenvalue is $\lambda = \underline{-0.26}$

b) $\omega = \sqrt{-\lambda}$ (p.73 handbook)

$= \sqrt{-0.256} \approx 0.51$ (2dp)

c) $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0.16 \\ -0.93 \\ 0.32 \end{pmatrix}$

d) For $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0.16 \\ -0.93 \\ 0.32 \end{pmatrix}$

~~For~~ x_0 and y_0 are phase opposed due to the difference in signs
x_0 and z_0 are in-phase due to both being positive
y_0 and z_0 are phase opposed due to the difference in signs

~~Since the signs of~~