Stability Problem

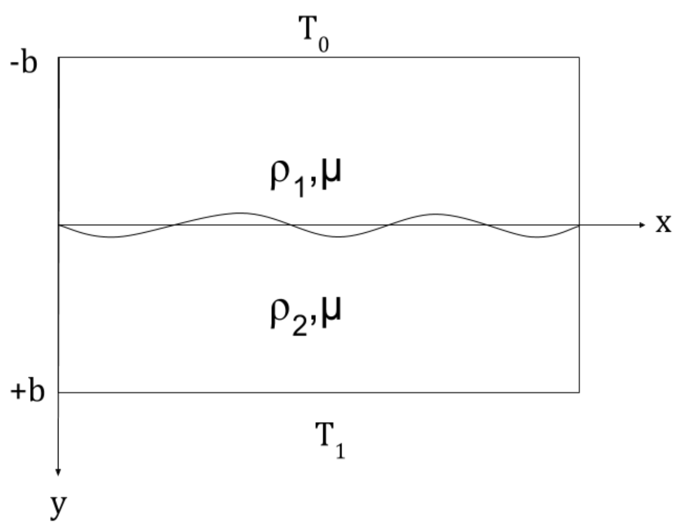
July 26, 2017

1.Introduction and Overview

Based on the derivations in the book *Geodynamics*, we know that both density and temperature differences relates to the instability of a model. According to section 6.12, a model consists of two layers with the bottom layer denser and same temperature always tends to stabilize the initial disturbance while, according to section 6.19, a model with higher temperature at the bottom and same density will not be stable if the Rayleigh number of the model is below certain value.

In this report, we are interested in combining these two problems and fit them into one model. Our model consists of two layers, with the bottom layer hotter and denser. So the density difference tends to stabilize the model while the temperature difference tends to make the model unstable. As the effect of temperature difference depends on Rayleigh numbers, we here investigate the critical Rayleigh below which the model will be stable.

2.Model Setup



Let b be the height of each layer. Let ρ_1, ρ_2 denote the density of the top and bottom layer respectively. Let T_0, T_1 denote the temperature at $y = \pm b$ respectively. The viscosity here is constant, denoted as μ. The initial displacement of the interface is $w = w_0 cos(2\pi x/\lambda)$. Since the difference between ρ_1 and ρ_2 is small compared to their magnitude, we denote $\rho_0 \approx \rho_1 \approx \rho_2$. The Rayleigh number of this model can be expressed as

$$Ra = \frac{\rho_0 g \alpha_v (T_1 - T_0)(2b)^3}{\mu\kappa} \tag{1}$$

and the buoyancy factor

$$B = \frac{\rho_1 - \rho_2}{\rho_0 \alpha_v (T_1 - T_0)} \tag{2}$$

(if $\rho_1 < \rho_2$ and $T_0 < T_1$, $B < 0$)

3.Methods and derivations

3.1 The Rayleigh-Taylor Instability Part

In this model we first ignore the effect of temperature differences. As we intentionally construct our model to be very similar to section 6.12, the method to solve the problem is exactly the same as section 6.12.

To satisfy the in compressibility and force balance of each layer, both stream function of upper and bottom layer must satisfy:

$$0 = \frac{\partial^4 \psi}{\partial x^4} + 2\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} \tag{3}$$

Apply the method of separation of variables and take $\psi = \sin\frac{2\pi x}{\lambda} \times Y(y)$. The general solution will be like:

$$\psi_1 = \sin\frac{2\pi x}{\lambda}(A_1 \cosh\frac{2\pi y}{\lambda} + B_1 \sinh\frac{2\pi y}{\lambda} + C_1 y \cosh\frac{2\pi y}{\lambda} + D_1 y \sinh\frac{2\pi y}{\lambda}) \tag{4}$$

$$\psi_2 = \sin\frac{2\pi x}{\lambda}(A_2 \cosh\frac{2\pi y}{\lambda} + B_2 \sinh\frac{2\pi y}{\lambda} + C_2 y \cosh\frac{2\pi y}{\lambda} + D_2 y \sinh\frac{2\pi y}{\lambda}) \tag{5}$$

, where ψ_1 is the stream function of the upper layer and ψ_2 is the stream function of the bottom layer.

Applying the free-slip boundary condition:

$$v_1 = \frac{\partial u_1}{\partial y} = 0 \qquad \text{on y=-b} \tag{6}$$

$$v_2 = \frac{\partial u_2}{\partial y} = 0 \qquad \text{on y=+b} \tag{7}$$

$$u_1 = u_2, v_1 = v_2 \qquad \text{on y=0} \tag{8}$$

$$\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} = \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial y} \qquad \text{on y=0} \tag{9}$$

We get:

$$A_1 = A_2, B_1 = -B_2, C_1 = -C_2, D_1 = D_2 \tag{10}$$

$$A_1 = b * C_1[\operatorname{sech}^2 \frac{2\pi b}{\lambda} - \frac{\lambda}{2\pi b} * \tanh \frac{\lambda}{2\pi b}] \tag{11}$$

$$B_1 = -\frac{\lambda}{2\pi} C_1 \tag{12}$$

$$D_1 = C_1 \tanh \frac{\lambda}{2\pi b} \tag{13}$$

Solving:

$$0 = \frac{\partial p}{\partial x} + \mu(\frac{\partial^3 \psi_i}{\partial x^2 \partial y} + \frac{\partial^3 \psi_i}{\partial y^3})(6.72) \text{ for i=1,2} \tag{14}$$

$$\frac{\partial w}{\partial t} = v_{y=0}(6.149) \tag{15}$$

$$(\rho_1 - \rho_2) g w = (P_2 - P_1)_{y=0}(6.151) \tag{16}$$

We get: $w = w_0 e^{\alpha_c t}$, where w_0 is the initial amplitude of the displacement wave and

$$\alpha_c = \frac{(\rho_1 - \rho_2) g b}{4\mu} \times (\frac{\lambda}{2\pi b} tanh(\frac{2\pi b}{\lambda}) - sech^2(\frac{2\pi b}{\lambda})) \tag{17}$$

For the sake of clarity, we put constants together and denote it as C_c, so $\alpha_c = \frac{(\rho_1 - \rho_2) g b}{\mu} \times C_c$.

3.2 The Rayleigh-Bénard Instability Part

In this part we will estimate the growth rate of instabilities resulted from temperature differences following the method in section 6.19. By considering the density differences caused by the temperature difference and energy conservation, we have the same governing equations as in section 6.19 for this part. To be specific, (6.309) and (6.310) still applies here, only with a small variance of the size of the model, b:

$$\frac{\partial T'}{\partial t} + \frac{1}{2b}(T_1 - T_0)\frac{\partial \psi'}{\partial x} = \kappa(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}) \tag{18}$$

$$0 = \mu(\frac{\partial^4 \psi'}{\partial x^4} + 2\frac{\partial^4 \psi'}{\partial x^2 y^2} + \frac{\partial^4 \psi'}{\partial y^4}) - \rho_0 g \alpha_v \frac{\partial T'}{\partial x} \tag{19}$$

Here the $\kappa = k/(\rho c)$, $T' \equiv T - T_c = T - \frac{T_1+T_0}{2} - \frac{T_1+T_0}{b}y$.

We set the boundary conditions to be free-slip:

$$\frac{\partial u'}{\partial x} = T' = v' = 0 \qquad when \quad y = \pm b \tag{20}$$

By solving this problem following the steps of the original problem, we can express the steam function and temperature function in the following forms:

$$\psi' = \psi_0' cos(\frac{\pi y}{2b}) sin(\frac{2\pi x}{\lambda}) e^{\alpha_T t} \tag{21}$$

$$T' = T_0' cos(\frac{\pi y}{2b}) cos(\frac{2\pi x}{\lambda}) e^{\alpha_T t} \tag{22}$$

Here ψ' denote the small velocity variance driven by the temperature difference. ψ_0' and T' are some constants.

And $\alpha_T = \frac{\kappa}{4b^2}(Ra \times \frac{\frac{\pi^2 b^2}{\lambda^2}}{((\frac{2\pi b}{\lambda})^2 + (\pi/2)^2)^2} - (\pi^2 + \frac{16\pi^2 b^2}{\lambda^2}))$. By (6.149), the interface displacement caused by temperature should be proportional to $e^{\alpha_T t}$ ($w \propto w_0 e^{\alpha_T t}$).

3.3 Two Parts combined

In order for the whole model to be stable, the effect from temperature difference must be countered by the effect of density difference. Therefore, we have:

$$\alpha_c + \alpha_T = 0 \tag{23}$$

which is: $\frac{(\rho_1 - \rho_2) g b}{\mu} \times C_c = -\frac{\kappa}{4b^2}(Ra \times \frac{\frac{\pi^2 b^2}{\lambda^2}}{((\frac{2\pi b}{\lambda})^2 + (\pi/2)^2)^2} - (\pi^2 + \frac{16\pi^2 b^2}{\lambda^2}))$.

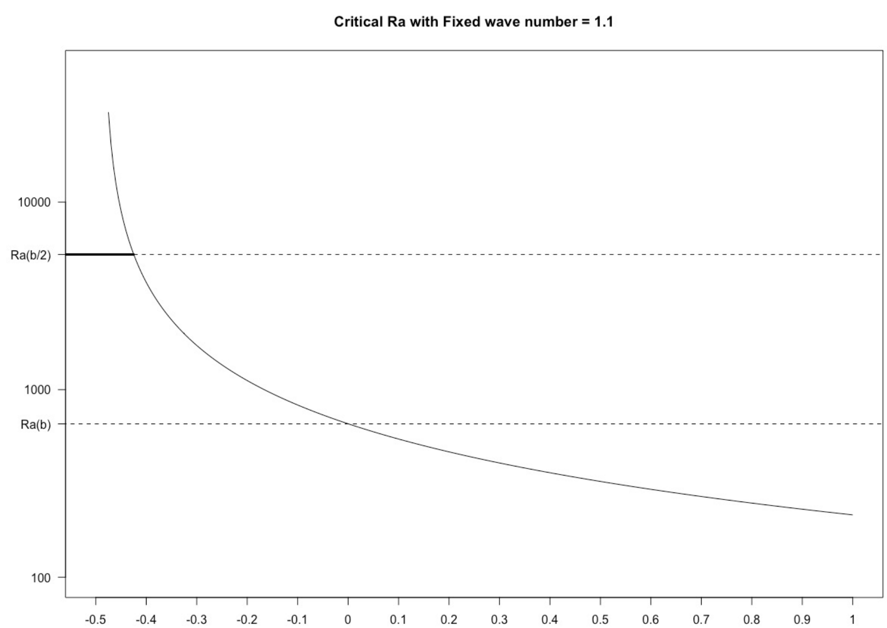
Recalling that the buoyancy factor B satisfies: $\frac{1}{2}RaB = \frac{4(\rho_1-\rho_2)gb^3}{\mu\kappa}$.

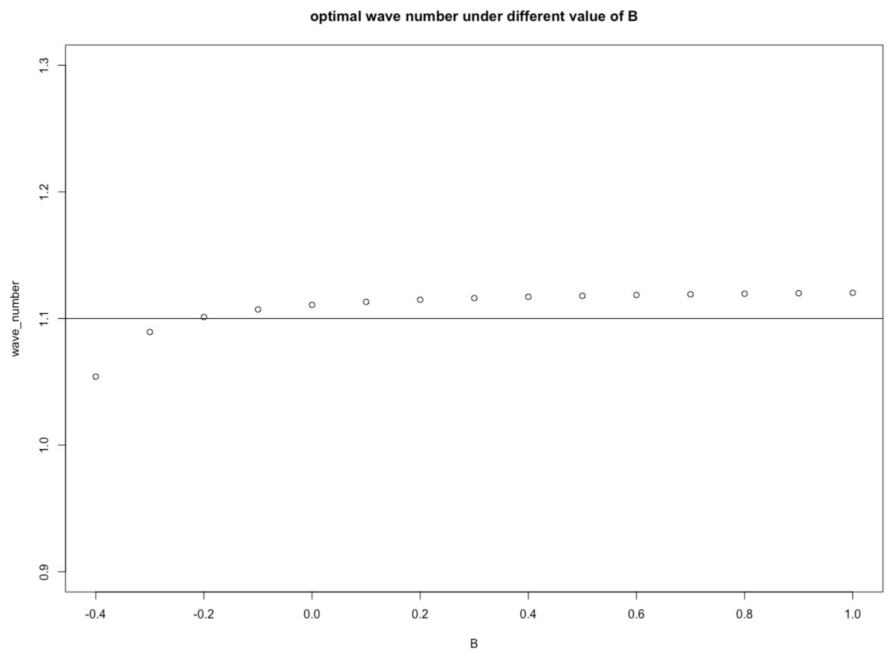
Plug in B, $\frac{1}{2}RaB \times C_c = (\pi^2 + \frac{16\pi^2b^2}{\lambda^2}) - Ra \times \frac{\frac{\pi^2b^2}{\lambda^2}}{((\frac{2\pi b}{\lambda})^2+(\pi/2)^2)^2}$.

Plug in C_c, let $\sigma = \frac{2\pi b}{\lambda}$,

$$Ra = \frac{\pi^2 + 4\sigma^2}{\frac{1}{8}B \times (\frac{tanh(\sigma)}{\sigma} - sech^2(\sigma)) + \frac{\sigma^2/4}{(\sigma^2+(\pi/2)^2)^2}} \tag{24}$$

For each value of B, Ra is a function of x and define Ra_{cr} to be the least positive Ra for some x with such B. Ra_{cr} is a function of B and the following page gives a plot of Ra_{cr} with respect to B.





Unsolved issues

1. Boundary conditions; Now solved

2. May need to explain more about eq.(4)

3. When adding two stream functions together, the temperature might affect the solution.