SOME NOTES ON $\mathfrak{sosp}(m|n)$ FOR JELLYFISH BRAUER CATEGORIES

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1. Definition of $\mathfrak{sosp}(m|n)$

For this document we fix an integer $m \geq 1$ and an even integer $n \geq 0$. We let $V = V_0 \oplus V_1$ denote the superspace with even (resp. odd) part V_0 (resp. V_1) having dimension m (resp. n).

1.1. **Even basis vectors.** The following table gives a basis for the even part V_0 for various values of m:

m	basis vectors
1	$\|0\rangle$
2	$\|\pm 2\rangle$
3	$\|0\rangle, \|\pm 2\rangle$
4	$\|\pm 2\rangle, \|\pm 4\rangle$
5	$\|0\rangle, \|\pm 2\rangle, \|\pm 4\rangle$
6	$\|\pm 2\rangle, \|\pm 4\rangle, \|\pm 6\rangle$
7	$\|0\rangle, \|\pm 2\rangle, \|\pm 4\rangle, \|\pm 6\rangle$

More generally, if m is even the basis consists of all $|k\rangle$ where k is a nonzero even integer with $|k| \leq m$; if m is odd the basis consists of all $|k\rangle$ where k is an even integer with $|k| < m$. In either case, there are precisely m even basis vectors.

1.2. **Odd basis vectors.** The following table gives a basis for the odd part V_1 for various values of n:

n	basis vectors
2	$\|\pm 1\rangle$
4	$\|\pm 1\rangle, \|\pm 3\rangle$
6	$\|\pm 1\rangle, \|\pm 3\rangle, \|\pm 5\rangle$
8	$\|\pm 1\rangle, \|\pm 3\rangle, \|\pm 5\rangle, \|\pm 7\rangle$

More generally, the basis consists of all $|k\rangle$ where k is an odd integer with $|k| < n$. Note that there are precisely n odd basis vectors.

1.3. **Indices.** We will write I_0 for the set of all even integers that correspond to basis vectors of V_0. In other words, when m is even we have

$$I_0 = \{k \in \mathbb{Z} \ : \ k \text{ is even and } 0 < |k| \leq m\},$$

and when m is odd we have

$$I_0 = \{k \in \mathbb{Z} \ : \ k \text{ is even and } |k| < m\}.$$

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We will write I_1 for the set of all odd integers that correspond to basis vectors of V_1. In other words,

$$I_1 = \{k \in \mathbb{Z} \ : \ k \text{ is odd and } |k| < n\}.$$

Finally, we write $I = I_0 \cup I_1$ for the set of all integers that correspond to (even or odd) basis vectors of V.

1.4. **Operators in $\mathfrak{sosp}(m|n)$.** Given $i, j \in I$. Let $e_{i,j} : V \to V$ denote the operator that maps $|j\rangle \mapsto |i\rangle$ and $|k\rangle \mapsto 0$ whenever $k \neq j$.

Generally, these $e_{i,j}$'s are not in $\mathfrak{sosp}(m|n)$, but below we will use the $e_{i,j}$'s to define the (more complicated) operators $E_{i,j}$, which are in $\mathfrak{sosp}(m|n)$. The definition also involves the factors $(j > 0)$ and $(i > 0)$, which should be interpreted as the boolean values 0 (false) or 1 (true). In other words, we write

$$(j > 0) = \begin{cases} 1, & \text{if } j > 0; \\ 0, & \text{if } j \leq 0. \end{cases}$$

The *orthosymplectic Lie superalgebra* $\mathfrak{sosp}(m|n)$ is the span of the following operators, which are defined for each $i, j \in I$:

$$E_{i,j} = e_{i,j} - (-1)^{j+ij+j(j>0)+i(i>0)} e_{-j,-i}$$

2. Examples

2.1. $\mathfrak{sosp}(1|2)$**.** When $m = 1$ and $n = 2$ we have $I = \{-1, 0, 1\}$. The following table shows the operators $E_{i,j}$ for all $i, j \in I$, along with the names we called these operators during our Spring 2024 independent study.

i	j	$E_{i,j}$	old name
0	0	0	0
1	1	$e_{1,1} - e_{-1,-1}$	H
-1	-1	$e_{-1,-1} - e_{1,1}$	$-H$
1	-1	$2e_{1,-1}$	$2E^+$
-1	1	$2e_{-1,1}$	$2E^-$
0	-1	$e_{0,-1} + e_{1,0}$	F^+
1	0	$e_{1,0} + e_{0,-1}$	F^+
0	1	$e_{0,1} - e_{-1,0}$	F^-
-1	0	$e_{-1,0} - e_{0,1}$	$-F^-$

2.2. $\mathfrak{sosp}(2|2)$**.** When $m = 2$ and $n = 2$ we have $I = \{-2, -1, 1, 2\}$. The following table shows the operators $E_{i,j}$ for all $i, j \in I$, along with the names we called these operators during our Spring 2024 independent study.

i	j	$E_{i,j}$	old name
2	2	$e_{2,2} - e_{-2,-2}$	H_2
-2	-2	$e_{-2,-2} - e_{2,2}$	$-H_2$
1	1	$e_{1,1} - e_{-1,-1}$	H_1
-1	-1	$e_{-1,-1} - e_{1,1}$	$-H_1$
1	-1	$2e_{1,-1}$	$2E^+$
-1	1	$2e_{-1,1}$	$2E^-$
2	-2	0	0
-2	2	0	0
2	-1	$e_{2,-1} + e_{1,-2}$	F_3^+
1	-2	$e_{1,-2} + e_{2,-1}$	F_3^+
2	1	$e_{2,1} - e_{-1,-2}$	F_1^+
-1	-2	$e_{-1,-2} - e_{2,1}$	$-F_1^+$
-2	-1	$e_{-2,-1} + e_{1,2}$	F_1^-
1	2	$e_{1,2} + e_{-2,-1}$	F_1^-
-2	1	$e_{-2,1} - e_{-1,2}$	F_3^-
-1	2	$e_{-1,2} - e_{-2,1}$	$-F_3^-$

3. Goals

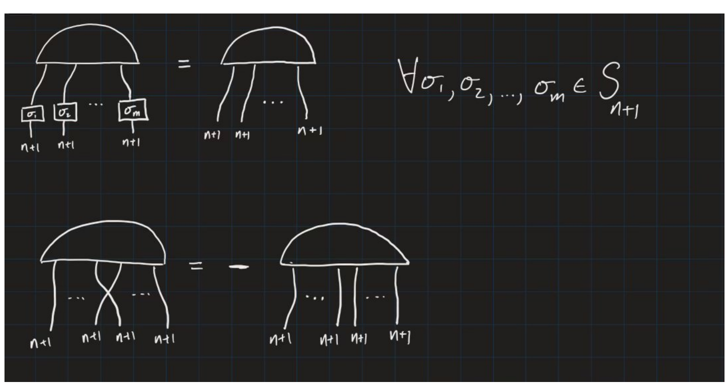
3.1. **Find a rule for the jellyfish operator.** For each m and n, there should be a $\mathfrak{sosp}(m|n)$-map of the form

$$\phi : V^{\otimes m(n+1)} \to \mathbb{C}$$

Recall that being a $\mathfrak{sosp}(m|n)$-map means that $\phi(E_{i,j} \cdot |v\rangle) = 0$ for all $i, j \in I$ and all $|v\rangle \in V^{\otimes m(n+1)}$.

I expect there exists such a ϕ (which is unique up to a scalar multiple) satisfying the following properties:

- ϕ cannot be realized as a linear combination of maps corresponding to (non-jellyfish) Brauer diagrams.
- ϕ satisfies the diagram relations pictured below. In the picture, each strand has thickness $n+1$ (meaning that each strand pictured represents $n+1$ regular strands). The relations say that permutations *within* each of the m thick strands has no effect on the output of ϕ, whereas permuting the m thick strands contributes a sign.



The first goal is to find a rule for ϕ. The rule is given by the determinant in the case $n = 0$. I gave the rule for $\mathfrak{sosp}(1|2)$ at the start of the Spring 2024 independent study, and we found the rule for $\mathfrak{sosp}(2|2)$ at the end of that independent study. What is the rule for arbitrary m and n?

3.2. **Find the jellyfish relation.** Stacking a jellyfish under an upside-down jellyfish gives us a map $V^{\otimes m(n+1)} \to V^{\otimes m(n+1)}$. The rule for this map is completely determined by ϕ. I expect this map can be realized as a linear combination of Brauer diagrams (without jellyfish). The goal here is to explain how to write this map as linear combo of Brauer diagrams.

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