# An introduction to diagram categories

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# Chapter 1

# Diagrams

## 1.1 Partition diagrams

The first diagrams we are going to consider are partition diagrams. The reason to start with partition diagrams is that many interesting families of diagrams can be viewed as special types of partition diagrams.

### Set partitions

We define a *(finite) partition* of a set $A$ to be a finite set of mutually disjoint (possibly empty) subsets of $A$ whose union is $A$. The disjoint subsets are called *parts* of the partition. For example, $\{\{1,4\},\{2\},\{3,5\}\}$ is a partition of $\{1,2,3,4,5\}$ into three parts. It is important to note that we allow empty parts. For example, a different partition of $\{1,2,3,4,5\}$ is $\{\varnothing,\varnothing,\{1,4\},\{2\},\{3,5\}\}$, which has five parts (two of which are the empty set $\varnothing = \{\}$). Allowing for empty parts may seem a bit strange. Indeed, the standard definition of a set partition found outside these notes does not permit empty parts. Empty parts, however, play an important role in the theory of diagram categories.

**Example 1.1.** The empty set $\varnothing$ has only one partition that doesn't include empty parts: $\{\}$. This is the only partition (of any set) that has zero parts. All other partitions have at least one part. Here are all the partitions of $\varnothing$:

$$\{\}, \quad \{\varnothing\}, \quad \{\varnothing,\varnothing\}, \quad \{\varnothing,\varnothing,\varnothing\}, \quad \dots$$

**Example 1.2.** A singleton set $\{x\}$ has only one partition that doesn't include empty parts: $\{\{x\}\}$. All other partitions are obtained from that one by adding finitely many empty parts. In other words, the partitions of $\{x\}$ are the following:

$$\{\{x\}\}, \quad \{\varnothing,\{x\}\}, \quad \{\varnothing,\varnothing,\{x\}\}, \quad \{\varnothing,\varnothing,\varnothing,\{x\}\}, \quad \dots$$

**Example 1.3.** A set with two elements $\{a,b\}$ admits exactly two partitions having no empty parts: $\{\{a\},\{b\}\}$ and $\{\{a,b\}\}$. All the other partitions are obtained from these two:

$$\begin{array}{lllll} \{\{a\},\{b\}\}, & \{\varnothing,\{a\},\{b\}\}, & \{\varnothing,\varnothing,\{a\},\{b\}\}, & \{\varnothing,\varnothing,\varnothing,\{a\},\{b\}\}, & \ldots \\ \{\{a,b\}\}, & \{\varnothing,\{a,b\}\}, & \{\varnothing,\varnothing,\{a,b\}\}, & \{\varnothing,\varnothing,\varnothing,\{a,b\}\}, & \ldots \end{array}$$

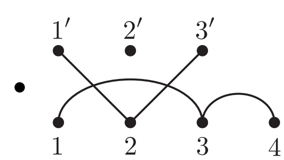
**Exercise 1.4. (Bell numbers)** The number of partitions of an $n$-element set into *non-empty* parts is called the *nth Bell number*. Find the Bell numbers for $n = 0, 1, 2, 3, 4$ by counting the appropriate partitions. You can check your answers by looking at Bell's triangle (similar to Pascal's triangle). Here's the top of Bell's triangle:

$$\begin{array}{lllll} 1 & & & & \\ 1 & 2 & & & \\ 2 & 3 & 5 & & \\ 5 & ? & ? & ? & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

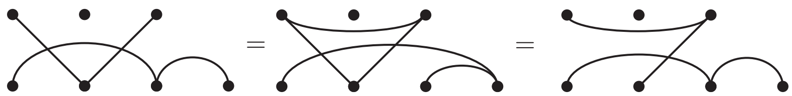
Every entry in the leftmost column is the same as the rightmost entry in the previous row. All other entries are obtained by summing the entry 1 step to the left with the entry 1 step to the left and 1 step up. Construct a couple more rows. The leftmost column in Bell's triangle lists the Bell numbers. Notice that they grow very fast.

## Diagrams of set partitions

Given $n \in \mathbb{Z}_{\geq 0}$, write $[n] = \{i \in \mathbb{N} \ : \ 1 \leq i \leq n\}$ and $[n]' = \{i' \ : \ i \in [n]\}$. For example, $[3] = \{1, 2, 3\}$ and $[4]' = \{1', 2', 3', 4'\}$. Moreover, $[0] = \varnothing = [0]'$. Now, given $m, n \in \mathbb{Z}_{\geq 0}$, a *partition diagram of type* $m \to n$ is a diagrammatic representation of a partition of $[m] \cup [n]'$ obtained by (1) listing $m$ vertices in a bottom row and $n$ vertices in a top row; (2) labeling the bottom row with the elements of $[m]$ and the top row with elements in $[n]'$, both in increasing order from left to right; and (3) connecting vertices with edges in such a way that there is a path between two vertices exactly when those vertices are in the same part of the partition. Finally, for each empty part in the partition, we add a *floating vertex* $\bullet$ to the diagram between the two rows of vertices (usually on the left side). We require that all edges be drawn in between the two rows of vertices. For example, the following diagram is of type $4 \to 3$ and corresponds to the partition $\{\varnothing, \{1, 3, 4\}, \{1', 2, 3'\}, \{2'\}\}$:




Since the vertices' labels are always increasing from left to right, we can draw our diagrams without labeling the vertices without loosing any information. We declare two partition diagrams to be equal if they represent the same partition. For example,



since all of the diagrams correspond to the partition $\{\{1,3,4\},\{1',2,3'\},\{2'\}\}$.

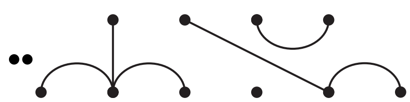
**Exercise 1.5.** Draw the partition diagram of type $4 \to 4$ corresponding to the partition $\{\{1\},\{2,2',3'\},\{3,4'\},\{4,1'\}\}$.

**Exercise 1.6.** Draw all the partition diagrams of types $1 \to 1$ that have fewer than two floating vertices.

**Exercise 1.7.** Draw all the partition diagrams of types $2 \to 1$ and $0 \to 3$ having no floating vertices.

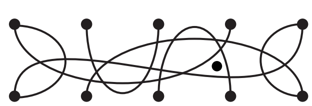
## 1.2 Special types of partition diagrams

A partition diagram is called *non-crossing** if it can be drawn in such a way that no two edges cross (aside from at vertices). Here's an example of a non-crossing partition diagram:



Note that partition diagrams drawn with crossings can be non-crossing since there are multiple ways to draw the same diagram. As long as there is one way to draw the diagram without crossings, the diagram is non-crossing.

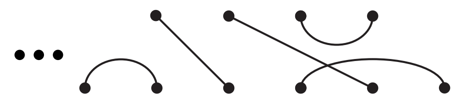
**Exercise 1.8.** Draw a better picture of the following partition diagram to show that it is non-crossing:



**Exercise 1.9.** How many non-crossing partition diagrams are there of type $4 \to 0$ having no floating vertices?
Hint: It might be easier to count the number of non-non-crossing partition diagrams and use Exercise 1.4.

*Non-crossing partition diagrams are also called *planar*.

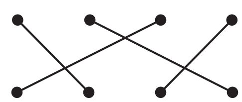
A partition diagram is called a *Brauer diagram* if all of the non-empty parts in the corresponding partition have exactly 2 elements. In other words, a Brauer diagram is a partition diagram where each non-floating vertex is connected to exactly one other (non-floating) vertex. Here's an example:



**Exercise 1.10.** Draw all the Brauer diagrams of type $1 \to 3$ having fewer than two floating vertices. How many Brauer diagrams are there of type $2 \to 3$?

**Exercise 1.11.** How many Brauer diagrams are there of type $m \to n$ having no floating vertices?

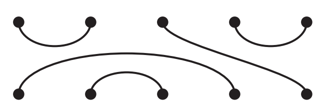
An edge in a Brauer diagram is called a cap if its endpoints are both bottom vertices, a cup if they're both top vertices, and a propagating edge otherwise. A Brauer diagram with only propagating edges (i.e. no caps or cups) and no floating vertices is called a *permutation diagram*. For example, here's a permutation diagram:



**Exercise 1.12.** Draw all the permutation diagrams of type $3 \to 3$.

**Exercise 1.13.** How many permutation diagrams are there of type $m \to n$?

A *Temperley-Lieb diagram* is a non-crossing Brauer diagram. Here's one:



**Exercise 1.14.** Draw all the Temperley-Lieb diagrams of type $4 \to 2$ having no floating vertices.

**Exercise 1.15. (Catalan numbers)** The *nth Catalan number*, denoted $C_n$, are defined recursively by letting $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for $n \geq 0$. Calculate the first few Catalan numbers. Prove that $C_n$ is the number of Temperley-Lieb diagrams of type $2n \to 0$ having no floating vertices.

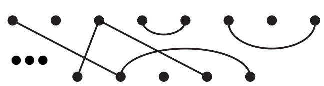
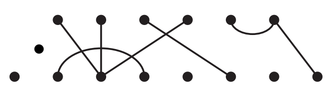
**Exercise 1.16.** Using the following pictures as a guide, describe a bijection between the set of all non-crossing partition diagrams of type $m \to n$ and the set of all Temperley-Lieb diagrams of type $2m \to 2n$.

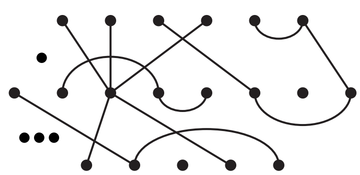
## 1.3 Stacking products

We can "multiply" two diagrams together by stacking them. There are two flavors of multiplication corresponding to the two ways to stack: vertically and horizontally. We will start with vertical stacking. From now on we will write $D : m \to n$ to mean that $D$ is a partition diagram of type $m \to n$.

### Vertical stacking

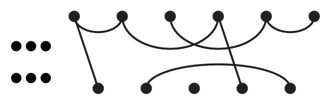
Given partition diagrams $D_1 : l \to m$ and $D_2 : m \to n$, we write $D_2 \star D_1$ for the diagram obtained by stacking $D_2$ on top of $D_1$ (identifying the $m$ middle vertices). For example,

if $D_1 =$  and $D_2 =$ 

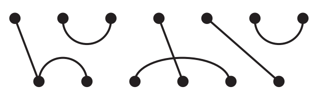
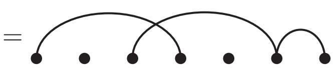
then $D_2 \star D_1 =$  .

We use the term *floating part* of $D_2 \star D_1$ to refer to any connected component that does not involve a vertex in the very top or bottom row of vertices. In other words, a floating part either refers to a floating vertex (from $D_1$ or $D_2$) or a connected component with all vertices in the middle row. For example, the diagram $D_2 \star D_1$ has six floating parts: three floating vertices from $D_1$, one floating vertex from $D_2$, and two connected components involving only middle vertices. These six floating parts are highlighted below:

Finally, we let $D_2 \circ D_1$ denote the partition diagram of type $l \to n$ that has one floating vertex for each floating part in $D_2 \star D_1$, and has is a path between two non-floating vertices in $D_2 \circ D_1$ if and only if there is a path between the corresponding vertices in $D_2 \star D_1$. For instance, in the example above

$D_2 \circ D_1 =$  .

**Exercise 1.17.** Compute $D_2 \circ D_1$ where

$D_1 =$  $D_2 =$ 

**Exercise 1.18.** The diagram operation of vertical stacking could be defined purely in terms of set partitions (without reference to diagrams). Indeed, to any partition diagrams $D_1 : l \to m$ and $D_2 : m \to n$ we can associate set partitions $P_1$ of $[l] \cup [m]'$ and $P_2$ of $[m]' \cup [n]''$, respectively. Here $[n]'' = \{i'' \ : \ 1 \leq i \leq n\}$. For instance, to the diagrams $D_1 : 5 \to 8$ and $D_2 : 8 \to 6$ used to illustrate the definition of $D_2 \circ D_1$ prior to Exercise 1.17, we have

$$P_1 = \{\varnothing, \varnothing, \varnothing, \{1, 3', 4\}, \{1', 2, 5\}, \{3\}, \{2'\}, \{4', 5'\}, \{6', 8'\}, \{7'\}\}$$
$$P_2 = \{\varnothing, \{1'\}, \{2', 4'\}, \{1'', 2'', 3', 4''\}, \{5'\}, \{3'', 6'\}, \{7'\}, \{5'', 6'', 8'\}\}$$

Then $D_2 \star D_1$ corresponds to the partition $P_2 \star P_1$ of $[l] \cup [m]' \cup [n]''$ obtained from $P_1$ and $P_2$ by merging any parts that have a common element in $[m]'$. Finally, $D_2 \circ D_1$ corresponds to the partition of $P_2 \circ P_1$ of $[l] \cup [n]''$ obtained from $P_2 \star P_1$ by removing all the elements of $[m]'$ from each part. In particular, all elements in a floating part should be removed, resulting in an empty set (which is why each floating part in $D_2 \star D_1$ becomes a floating vertex in $D_2 \circ D_1$).

(a) Determine the partition $P_2 \star P_1$ for the examples of $P_1$ and $P_2$ above.

(b) Determine the partition $P_2 \circ P_1$ for the examples of $P_1$ and $P_2$ above.

(c) Find $P_1, P_2, P_2 \star P_1$, and $P_2 \circ P_1$ for the diagrams in Exercise 1.17.

**Exercise 1.19.** Suppose $D_1 : l \to m$ and $D_2 : m \to n$. Prove the following:

(a) If $D_1$ and $D_2$ are non-crossing partition diagrams, then $D_2 \circ D_1$ is too.

(b) If $D_1$ and $D_2$ are Brauer diagrams, then $D_2 \circ D_1$ is too.

(c) If $D_1$ and $D_2$ are permutation diagrams, then $D_2 \circ D_1$ is too.

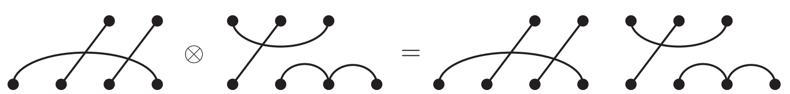
(d) If $D_1$ and $D_2$ are Temperley-Lieb diagrams, then $D_2 \circ D_1$ is too.

**Exercise 1.20. (Identity diagrams)** Find a diagram $1_n : n \to n$ for each $n \in \mathbb{Z}_{\geq 0}$ such that $1_n \circ D = D = D \circ 1_m$ whenever $D : m \to n$.

**Exercise 1.21.** Suppose $D_1 : k \to l, D_2 : l \to m$, and $D_3 : m \to n$. Prove that $(D_3 \circ D_2) \circ D_1 = D_3 \circ (D_2 \circ D_1)$.

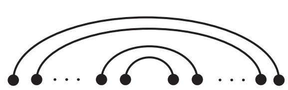
## Horizontal stacking

The *tensor product* of two partition diagrams $D_1 : m_1 \to n_1$ and $D_2 : m_2 \to n_2$, denoted $D_1 \otimes D_2$, is the partition diagram of type $m_1 + m_2 \to n_1 + n_2$ obtained by stacking $D_1$ to the left of $D_2$. For example,



**Exercise 1.22.** Repeat Exercise 1.19 replacing $\circ$'s with $\otimes$'s.

**Exercise 1.23.** Let $\Cap_n : 2n \to 0$ and $\Cup_n : 0 \to 2n$ be the following diagrams:

$\Cap_n =$  $\Cup_n =$ 

Prove that the map $\{D : m \to n\} \to \{D : m + n \to 0\}$ given by

$$D \mapsto \Cap_n \circ (D \otimes 1_n)$$

is a bijection with inverse

$$D \mapsto (D \otimes 1_n) \circ (1_m \otimes \Cup_n).$$

Explain why the bijection is still valid if we require the $D$'s to be any one of the special types: non-crossing, Brauer, Temperley-Lieb. Use this bijection along with Exercises 1.15 and 1.16 to give formulas for the number of Temperley-Lieb diagrams of type $m \to n$ and non-crossing partition diagrams of type $m \to n$ in terms of Catalan numbers. You should restrict to counting diagrams without floating vertices.

## 1.4 Notation for floating vertices: $\delta$

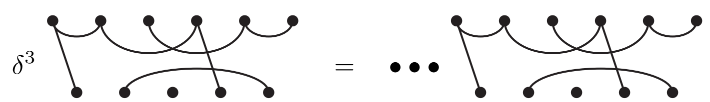
In this section we introduce some new notation for floating vertices. From now on, we let $\delta : 0 \to 0$ denote a single floating vertex $\bullet$. In other words, $\delta$ corresponds to the partition of $\varnothing$ consisting of one (empty) part: $\{\varnothing\}$. More generally, for each integer $k \geq 0$ we write $\delta^k : 0 \to 0$ for the partition diagram consisting of $k$ floating vertices. For example, $\delta^0 = 1_0, \delta^1 = \delta = \bullet, \delta^2 = \bullet\bullet$, and $\delta^3 = \bullet\bullet\bullet$. Note that $\delta^k$ is equal to the (horizontal or vertical) stacking of $k$ floating vertices. Thus, this exponential notation agrees with the usual notation of repeated products:

$$\delta^k = \underbrace{\delta \circ \delta \circ \cdots \circ \delta}_{k \text{ times}} = \underbrace{\delta \otimes \delta \otimes \cdots \otimes \delta}_{k \text{ times}}.$$

Given any partition diagram $D : m \to n$, and any integer $k \geq 0$ we will abbreviate notation and write

$$\delta^k \otimes D = \delta^k D.$$

We will also use the notation $\delta^k$ within drawings of partition diagram. For example,



By moving all floating vertices together on the left it follows that

$$(\delta^{k_2} D_2) \otimes (\delta^{k_1} D_1) = \delta^{k_1+k_2} D_2 \otimes D_1 \tag{1.1}$$

$$\text{and} \quad (\delta^{k_2} D_2) \circ (\delta^{k_1} D_1) = \delta^{k_1+k_2} D_2 \circ D_1. \tag{1.2}$$

for any diagrams $D_1$ and $D_2$ and any integers $k_1, k_2 \geq 0$ (assuming the diagrams are compatible for vertical stacking).

# Chapter 2

# Diagram categories

## 2.1 Definition of a category

We are about to define the term *category*, and the definition can be a lot to swallow. As motivation, consider the following three operations:

1. Vertical stacking of partition diagrams (see §1.3). Given $D_1 : k \to l$ and $D_2 : m \to n$, the product $D_2 \circ D_1$ makes sense if and only if $l = m$.

2. Matrix multiplication. Given an $l \times k$ matrix $A_1$ and an $n \times m$ matrix $A_2$, the matrix product $A_2 A_1$ makes sense if and only if $l = m$.

3. Function composition. Given functions $f_1 : K \to L$ and $f_2 : M \to N$, the composition $f_2 \circ f_1$ makes sense if and only if the sets $L = M$.

There are many applications of category theory, but one good reason to work with categories is they provide a uniform setting for studying operations on stuff where the operation is only defined when the stuff is compatible. In the language of category theory the stuff (diagrams, matrices, functions) are called *morphisms*; the operation (vertical stacking, matrix multiplication, function composition) is called *composition*; and the type of a morphism ($m \to n$, $n \times m$, $M \to N$) is described by a pair of *objects* (nonnegative integers, natural numbers, sets). Keep the three examples above in mind when reading the following:

**Definition 2.1.** A *category* $\mathcal{C}$ consist of the following data:

- A class of *objects* $\mathrm{Ob}\,\mathcal{C}$.

- A class of *morphisms* $\mathrm{Hom}_{\mathcal{C}}(m, n)$ for every pair of objects $(m, n)$. Given $f \in \mathrm{Hom}_{\mathcal{C}}(m, n)$ we will often write $f : m \to n$. We call $m$ the *domain* of $f$ and $n$ the *target* of $f$.

- A *composition map* $\mathrm{Hom}_{\mathcal{C}}(m, n) \times \mathrm{Hom}_{\mathcal{C}}(l, m) \to \mathrm{Hom}_{\mathcal{C}}(l, n)$ for every triple of objects $(l, m, n)$. We will use the notation $f \circ g : l \to n$ for the composition of $f : m \to n$ and $g : l \to m$.

The data above must satisfy the following axioms:

(C1) Every morphism has a unique domain and target.

(C2) (Identity morphisms) For all $n \in \mathrm{Ob}\,\mathcal{C}$ there exists $\mathrm{id}_n \in \mathrm{Hom}_{\mathcal{C}}(n,n)$ such that $f \circ \mathrm{id}_n = f$ and $\mathrm{id}_n \circ g = g$ for all $f : n \to m$ and $g : m \to n$.

(C3) (Composition is associative) $(f \circ g) \circ h = f \circ (g \circ h)$ for all morphisms $f : m \to n, g : l \to m, h : k \to l$.

Here are the formal definitions of the examples of categories discussed above:

**Example 2.2. (Partition diagrams)** Let $\mathcal{P}$ denote the category with

- $\mathrm{Ob}\,\mathcal{P} = \mathbb{Z}_{\geq 0}$.
- $\mathrm{Hom}_{\mathcal{P}}(m,n) = \{\text{partition diagrams of type } m \to n\}$ for each $m, n \in \mathbb{Z}_{\geq 0}$.
- The composition map is given by vertical stacking:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{P}}(m,n) \times \mathrm{Hom}_{\mathcal{P}}(l,m) &\to \mathrm{Hom}_{\mathcal{P}}(l,n) \\ (D_2, D_1) &\mapsto D_2 \circ D_1 \end{aligned}$$

Axiom (C1) follows from the fact that each partition diagram has a unique number of bottom vertices (the domain) and a unique number of top vertices (the target). Axioms (C2) and (C3) follow from Exercises 1.20 and 1.21.1 respectively.

**Example 2.3. (Matrices over $\mathbb{C}$)** Let **Mat** denote the category with

- $\mathrm{Ob}\,\mathbf{Mat} = \mathbb{Z}_{\geq 0}$.
- $\mathrm{Hom}_{\mathbf{Mat}}(m,n) = \{n \times m \text{ matrices with entries in } \mathbb{C}\}$ for all $m, n \in \mathbb{Z}_{\geq 0}$.
- Composition is given by matrix multiplication:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Mat}}(m,n) \times \mathrm{Hom}_{\mathbf{Mat}}(l,m) &\to \mathrm{Hom}_{\mathbf{Mat}}(l,n) \\ (A_2, A_1) &\mapsto A_2 A_1 \end{aligned}$$

**Remark 2.4.** By convention, we declare that there is a unique $0 \times n$ matrix and a unique $n \times 0$ matrix for all $n \in \mathbb{Z}_{\geq 0}$, which we denote by 0 in all cases. Multiplying any matrix by these new zero matrices always results in 0.

**Example 2.5. (The category of sets)** Let **Set** denote the category with

- $\mathrm{Ob}\,\mathbf{Set}$ is the collection of all sets.
- $\mathrm{Hom}_{\mathbf{Set}}(M,N) = \{\text{functions from } M \text{ to } N\}$ for all sets $M$ and $N$.
- Composition is the usual composition of functions:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Set}}(M,N) \times \mathrm{Hom}_{\mathbf{Set}}(L,M) &\to \mathrm{Hom}_{\mathbf{Set}}(L,N) \\ (f_2, f_1) &\mapsto f_2 \circ f_1 \end{aligned}$$

**Exercise 2.6.** Verify that **Mat** and **Set** satisfy axioms (C1)-(C3).

**Exercise 2.7. (The category of relations)** Given two sets $M$ and $N$, a *relation* $R$ from $M$ to $N$ is a subset $R \subseteq M \times N$. We write $R : M \to N$ to indicate that $R$ is a relation from $M$ to $N$. We define the composition of two relations $R_1 \subseteq L \times M$, $R_2 \subseteq M \times N$ to be the subset $R_2 \circ R_1 \subseteq L \times N$ given by $R_2 \circ R_1 = \{(x, z) \in L \times N \ : \ \exists y \in M \text{ with } (x, y) \in R_1 \text{ and } (y, z) \in R_2\}$. Let **Rel** denote the category with

- $\mathrm{Ob}\,\mathbf{Rel}$ is the collection of all sets.
- $\mathrm{Hom}_{\mathbf{Rel}}(M, N) = \{\text{relations from } M \text{ to } N\}$ for all sets $M$ and $N$.
- Composition is the composition of relations defined above:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{Rel}}(M, N) \times \mathrm{Hom}_{\mathbf{Rel}}(L, M) & \to & \mathrm{Hom}_{\mathbf{Rel}}(L, N) \\ (R_2, R_1) & \mapsto & R_2 \circ R_1 \end{array}$$

Note that we declare two relations $R : M \to N$ and $R' : M' \to N'$ to be equal if and only if $M = M'$, $N = N'$, and $R = R'$ (equal as sets). It follows that **Rel** satisfies axiom (C1). Verify that **Rel** satisfies (C2) and (C3).

## 2.2 Subcategories of $\mathcal{P}$

Of the categories introduced in the previous section, the diagram category $\mathcal{P}$ is the main focus of these notes. If we restrict ourselves to the various special types of partition diagram discusses in §1.2, we obtain other interesting diagram categories. These diagram categories are best described as "subcategories" of $\mathcal{P}$. Let's be precise about what that term means:

**Definition 2.8.** Suppose $\mathcal{C}$ is a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ (written $\mathcal{D} \subseteq \mathcal{C}$) consists of the following data:

- A class of objects $\mathrm{Ob}\,\mathcal{D} \subseteq \mathrm{Ob}\,\mathcal{C}$.
- A class of morphisms $\mathrm{Hom}_{\mathcal{D}}(m, n) \subseteq \mathrm{Hom}_{\mathcal{C}}(m, n)$ for each pair of objects $m, n \in \mathrm{Ob}\,D$.

The data above must satisfy the following:

(SC1) ($\mathcal{D}$ has identities) $\mathrm{id}_n \in \mathrm{Hom}_{\mathcal{D}}(n, n)$ for every $n \in \mathrm{Ob}\,\mathcal{D}$.

(SC2) ($\mathcal{D}$ is closed under composition) The composition $f \circ g \in \mathrm{Hom}_{\mathcal{D}}(l, n)$ whenever $f \in \mathrm{Hom}_{\mathcal{D}}(m, n)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(l, m)$.

**Exercise 2.9.** Show that subcategories are indeed categories. More precisely, assume $\mathcal{D}$ is a subcategory of $\mathcal{C}$. Show the composition maps for $\mathcal{C}$ restrict to composition maps for $\mathcal{D}$, and $\mathcal{D}$ satisfies (C1)-(C3).

### The diagram categories $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$

The categories $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$ are defined to be the subcategories of $\mathcal{P}$ with

$$\operatorname{Ob}\mathcal{NC} = \operatorname{Ob}\mathcal{B} = \operatorname{Ob}\mathcal{TL} = \operatorname{Ob}\mathcal{S} = \mathbb{Z}_{\geq 0}$$

and morphisms given by

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{NC}}(m,n) &= \{\text{Non-crossing partition diagrams of type } m \to n\},\\
\operatorname{Hom}_{\mathcal{B}}(m,n) &= \{\text{Brauer diagrams of type } m \to n\},\\
\operatorname{Hom}_{\mathcal{TL}}(m,n) &= \{\text{Temperley-Lieb diagrams of type } m \to n\},\\
\operatorname{Hom}_{\mathcal{S}}(m,n) &= \{\text{permutation diagrams of type } m \to n\}.
\end{aligned}$$

Indeed, each of $\mathcal{NC}, \mathcal{B}, \mathcal{TL}$, and $\mathcal{S}$ has identities (hence (SC1) is satisfied), and by Exercise 1.19 each is closed under composition (hence (SC2) is satisfied).

### Even and odd subcategories of $\mathcal{B}$ and $\mathcal{TL}$

In Exercise 1.11 you ought to have shown that there are no Brauer diagrams (hence no Temperley-Lieb diagrams) of type $m \to n$ unless $m$ and $n$ are both odd or both even. In some sense, this means that the categories $\mathcal{B}$ and $\mathcal{TL}$ each split into an even piece and an odd piece. Let us be more precise:

We start with $\mathcal{B}$. The categories $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are defined as the subcategories of $\mathcal{B}$ with objects

$$\begin{aligned}
\operatorname{Ob}\mathcal{B}^{ev} &= \{2n \ : \ n \in \mathbb{Z}_{\geq 0}\},\\
\operatorname{Ob}\mathcal{B}^{odd} &= \{2n+1 \ : \ n \in \mathbb{Z}_{\geq 0}\}.
\end{aligned}$$

and morphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{B}^{ev}}(2m,2n) &= \operatorname{Hom}_{\mathcal{B}}(2m,2n),\\
\operatorname{Hom}_{\mathcal{B}^{odd}}(2m+1,2n+1) &= \operatorname{Hom}_{\mathcal{B}}(2m+1,2n+1).
\end{aligned}$$

Note that both of these subcategories are obtained from $\mathcal{B}$ by selecting a subset of the objects, and picking *all* the morphisms between the selected objects. Such subcategories get a special name:

**Definition 2.10.** A subcategory $\mathcal{D} \subseteq \mathcal{C}$ is called *full* if $\operatorname{Hom}_{\mathcal{D}}(m,n) = \operatorname{Hom}_{\mathcal{C}}(m,n)$ for all $m,n \in \operatorname{Ob}\mathcal{D}$.

Thus, $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are both full subcategories of $\mathcal{B}$. Not all subcategories are full. For instance, $\mathcal{B}$ is not a full subcategory of $\mathcal{P}$ since there are partition diagrams which are not Brauer diagrams (i.e. there exist $m,n \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Hom}_{\mathcal{B}}(m,n) \neq \operatorname{Hom}_{\mathcal{P}}(m,n)$). One nice property of full subcategories is that they can be defined without checking (SC1) and (SC2) thanks to the following:

**Proposition 2.11.** *Let $\mathcal{C}$ be an arbitrary category. Suppose $\mathcal{D}$ is a collection of objects and morphisms in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{D}}(m,n) = \operatorname{Hom}_{\mathcal{C}}(m,n)$ for all $m,n \in \operatorname{Ob}\mathcal{D}$. Then $\mathcal{D}$ is a full subcategory of $\mathcal{C}$.*

*Proof.* If we can show $\mathcal{D}$ is a subcategory of $\mathcal{C}$, then it will be full by the definition of full. Hence, we must just check that $\mathcal{D}$ satisfies (SC1) and (SC2).

For (SC1), let $n \in \operatorname{Ob}\mathcal{D}$. Then $n \in \operatorname{Ob}\mathcal{C}$ since $\operatorname{Ob}\mathcal{D} \subseteq \operatorname{Ob}\mathcal{C}$. Since $\mathcal{C}$ is a category, $\mathrm{id}_n \in \operatorname{Hom}_{\mathcal{C}}(n,n) = \operatorname{Hom}_{\mathcal{D}}(n,n)$, as desired.

For (SC2), suppose $f \in \operatorname{Hom}_{\mathcal{D}}(m,n)$ and $g \in \operatorname{Hom}_{\mathcal{D}}(l,m)$ for some objects $l,m,n \in \operatorname{Ob}\mathcal{D}$. Then $f \in \operatorname{Hom}_{\mathcal{C}}(m,n)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(l,m)$ by our assumption on $\mathcal{D}$. Since $\mathcal{C}$ is a category, $f \circ g \in \operatorname{Hom}_{\mathcal{C}}(l,n) = \operatorname{Hom}_{\mathcal{D}}(l,n)$, as desired. $\square$

In particular, the previous proposition verifies that $\mathcal{B}^{ev}$ and $\mathcal{B}^{odd}$ are indeed subcategories of $\mathcal{B}$. Moreover, that proposition implies that every subset of objects in a category determines a unique full subcategory. We will exploit this fact in order to define some categories in a new, more concise way. For example, we define $\mathcal{TL}^{ev}$ and $\mathcal{TL}^{odd}$ to be the full subcategories of $\mathcal{TL}$ with objects

$$\operatorname{Ob}\mathcal{TL}^{ev} = \{2n \ : \ n \in \mathbb{Z}_{\geq 0}\},$$
$$\operatorname{Ob}\mathcal{TL}^{odd} = \{2n+1 \ : \ n \in \mathbb{Z}_{\geq 0}\}.$$

## 2.3 Endomorphisms, isomorphisms, and idempotents

In this section we discuss various adjectives for morphisms. The terminology developed in this section be used throughout the rest of the notes. However, the reason we are introducing these new terms now is to get some more practice working with categories.

### Endomorphisms

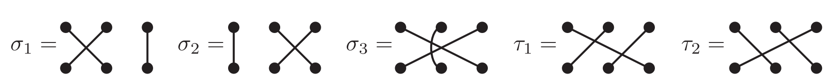
A morphism $f$ in a category is called an *endomorphism* if it has the same domain and target, i.e. $f : n \to n$. We write $\operatorname{End}_{\mathcal{C}}(n) = \operatorname{Hom}_{\mathcal{C}}(n,n)$ for the collection of all *endomorphisms of* $n$. For example:

1. Endomorphisms in $\mathcal{P}$ are diagrams with the same number of top and bottom vertices. For instance, $\operatorname{End}_{\mathcal{P}}(5)$ is the set of all partition diagrams of type $5 \to 5$.

2. Endomorphisms in **Mat** are square matrices. For example $\operatorname{End}_{\mathbf{Mat}}(5)$ is the set of all $5 \times 5$ matrices.

3. Endomorphisms in **Set** are functions whose domain and target are equal.

Notice that endomorphism sets are closed under composition. In other words, if $f, g \in \operatorname{End}_{\mathcal{C}}(n)$ then $f \circ g \in \operatorname{End}_{\mathcal{C}}(n)$. A set equipped with a single associative operation and an identity element is called a *monoid*. Thus, all endomorphism sets are monoids. For example, $\operatorname{End}_{\mathcal{S}}(3) = \{1_3, \sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2\}$ where



Here is the multiplication table for $\mathrm{End}_{\mathcal{S}}(3)$ where the entry in the row labelled by $x$ and column labelled by $y$ is $x \circ y$:

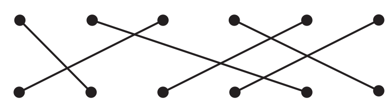
| | $1_3$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\tau_1$ | $\tau_2$ |
|---|---|---|---|---|---|---|
| $1_3$ | $1_3$ | $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\tau_1$ | $\tau_2$ |
| $\sigma_1$ | $\sigma_1$ | $1_3$ | $\tau_1$ | $\tau_2$ | $\sigma_2$ | $\sigma_3$ |
| $\sigma_2$ | $\sigma_2$ | $\tau_2$ | $1_3$ | $\tau_1$ | $\sigma_3$ | $\sigma_1$ |
| $\sigma_3$ | $\sigma_3$ | $\tau_1$ | $\tau_2$ | $1_3$ | $\sigma_1$ | $\sigma_2$ |
| $\tau_1$ | $\tau_1$ | $\sigma_3$ | $\sigma_1$ | $\sigma_2$ | $\tau_2$ | $1_3$ |
| $\tau_2$ | $\tau_2$ | $\sigma_2$ | $\sigma_3$ | $\sigma_1$ | $1_3$ | $\tau_1$ |

**Exercise 2.12.** Construct multiplication tables for the monoids $\mathrm{End}_{\mathcal{P}}(1)$, $\mathrm{End}_{\mathcal{TL}}(2)$, $\mathrm{End}_{\mathcal{S}}(2)$, $\mathrm{End}_{\mathcal{B}}(2)$, and $\mathrm{End}_{\mathcal{TL}}(3)$. Your tables only need to show $x \circ y$ for all diagrams $x$ and $y$ having no floating vertices. Composition of diagrams having floating vertices is determined by these tables along with (1.1).

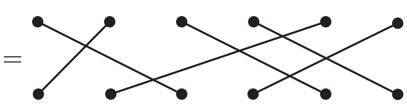
## Isomorphisms

A morphism $f : m \to n$ is called an *isomorphism* if there exists a morphism $g : n \to m$ such that $f \circ g = \mathrm{id}_n$ and $g \circ f = \mathrm{id}_m$. Such a $g$ is called an *inverse* of $f$ and we write $g = f^{-1}$.

For example, the partition diagram

$D =$ 

is an isomorphism in the category $\mathcal{P}$ with

$D^{-1} =$  .

**Exercise 2.13.** Prove that if $D$ is a permutation diagram, then $D$ is an isomorphism in the category $\mathcal{P}$. In particular, given a permutation diagram $D$, explain how to draw $D^{-1}$.

**Example 2.14.** Isomorphisms in **Mat** are invertible matrices. For example, the matrix $A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is an isomorphism in **Mat** with $A^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$. More generally, any square matrix with nonzero determinant is an isomorphism in **Mat**. The inverse of such a matrix is given by Cramer's formula, and in practice can be computed using elementary row operations.

**Example 2.15.** Isomorphisms in **Set** are invertible functions. A standard result in set theory is that a function is invertible if and only if it is a bijection (i.e. one-to-one and onto). Thus, a function is an isomorphism in **Set** exactly when it is a bijection.

**Exercise 2.16.** (a) Show that all identity morphisms are isomorphisms.

(b) Given an isomorphism $f : m \to n$, prove that $f^{-1}$ is also an isomorphism with $(f^{-1})^{-1} = f$.

(c) Given isomorphisms $f : l \to m$ and $g : m \to n$, prove that $g \circ f$ is also an isomorphism with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

### Automorphism groups

Given any object $n$ in any category $\mathcal{C}$, we let

$$\mathrm{Aut}_{\mathcal{C}}(n) = \{f \in End_{\mathcal{C}}(n) \ : \ f \text{ is an isomorphism}\}.$$

It follows from Exercise 2.16 that $\mathrm{Aut}_{\mathcal{C}}(n)$ contains an identity morphism, and is closed under both composition and taking inverses. A monoid that contains all the inverses of its elements is called a *group*. The study of groups (Group Theory) is a common topic for a first course in abstract algebra. We call $\mathrm{Aut}_{\mathcal{C}}(n)$ the *automorphism group* of $n$. Many familiar groups arise as automorphism groups inside of a larger category.

**Example 2.17.** It follows from Exercise 2.13 that $\mathrm{Aut}_{\mathcal{S}}(n) = \mathrm{End}_{\mathcal{S}}(n)$. This group is usually called the *symmetric group* and denoted $S_n$.

**Example 2.18.** $\mathrm{Aut}_{\mathbf{Mat}}(n)$ is the group of all invertible $n \times n$ matrices. This group is usually called the *general linear group* and denoted $GL_n(\mathbb{C})$.

**Example 2.19.** Given a set $X$, $\mathrm{Aut}_{\mathbf{Set}}(X)$ is the group of all bijections of the form $f : X \to X$.

**Exercise 2.20.** Determine the groups $\mathrm{Aut}_{\mathcal{P}}(1)$, $\mathrm{Aut}_{\mathcal{TL}}(2)$, $\mathrm{Aut}_{\mathcal{S}}(2)$, $\mathrm{Aut}_{\mathcal{B}}(2)$, and $\mathrm{Aut}_{\mathcal{TL}}(3)$. This exercise should be quick if you have already done Exercise 2.12. Just find all the diagrams in the table with inverses.

### A closer look at isomorphisms in $\mathcal{P}$

Our next goal is to prove the following classification of isomorphisms in $\mathcal{P}$:

**Proposition 2.21.** *$D$ is an isomorphism in $\mathcal{P}$ if and only if $D$ is a permutation diagram.*

Note that proving one of the implications in the previous proposition is Exercise 2.13. Hence, we are required to show that the only isomorphisms in $\mathcal{P}$ are permutation diagrams. To do so, let us first examine the analogous proposition in **Mat**:

**Proposition 2.22.** *$A$ is an isomorphism in **Mat** if and only if $A$ is a square matrix with nonzero determinant.*

*Proof.* The fact that square matrices with nonzero determinants are invertible (hence isomorphisms) is discussed in Example 2.14. Here's one way to prove the other direction: The *rank* of a matrix $A$, denoted $\mathrm{rk}(A)$, is equal to the number of linearly independent columns in $A$. Here are a couple well-known properties of rank:

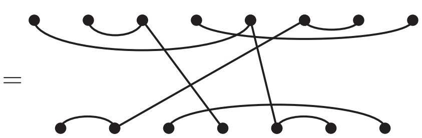
(RK1) $\mathrm{rk}(A) \leq \min\{m, n\}$ whenever $A$ is an $n \times m$ matrix.

(RK2) $\mathrm{rk}(A_2 A_1) \leq \min\{\mathrm{rk}(A_1), \mathrm{rk}(A_2)\}$ for any compatible matrices $A_1, A_2$.

(RK3) Suppose $A$ is an $n \times n$ matrix. $\det(A) \neq 0$ if and only if $\mathrm{rk}(A) = n$.

Now, suppose $A$ is an $n \times m$ invertible matrix. Then we know there exists an $m \times n$ matrix $A^{-1}$ with $AA^{-1} = I_n$ and $A^{-1}A = I_m$. Thus, using (RK2) we have $n = \mathrm{rk}(I_n) \leq \mathrm{rk}(A)$ and $m = \mathrm{rk}(I_m) \leq \mathrm{rk}(A)$. However, by (RK1) we know $\mathrm{rk}(A) \leq n$ and $\mathrm{rk}(A) \leq m$. It follows that $n = \mathrm{rk}(A) = m$. In particular, $A$ is a square matrix. Moreover, by (RK3) we have $\det(A) \neq 0$. □

We can prove Proposition 2.21 in a similar manner, but we need a gadget to play the role of rank: The *core* of a partition diagram $D : m \to n$, written $\mathrm{core}(D)$, is the number of connected components in $D$ that contain both a top and a bottom vertex. For example, $\mathrm{core}(D) = 3$ where

$D =$ 

The following exercise requests proofs for the core analogs of (RK1)-(RK3):

**Exercise 2.23.** Prove the following:

(CR1) $\mathrm{core}(D) \leq \min\{m, n\}$ whenever $D : m \to n$.

(CR2) $\mathrm{core}(D_2 \circ D_1) \leq \min\{\mathrm{core}(D_1), \mathrm{core}(D_2)\}$ for any compatible partition diagrams $D_1$ and $D_2$.

(CR3) Suppose $D$ is a partition diagram of type $n \to n$. $D$ is a permutation diagram if and only if it has no floating vertices and $\mathrm{core}(D) = n$.

**Exercise 2.24.** Using the previous exercise, mimic the proof of Proposition 2.22 to prove Proposition 2.21.

## Isomorphic objects

Two objects $m, n \in \mathrm{Ob}\,\mathcal{C}$ are called *isomorphic* in $\mathcal{C}$ if there is an isomorphism in $\mathrm{Hom}_{\mathcal{C}}(m, n)$.

**Example 2.25.** It follows from Proposition 2.21 that $m$ and $n$ are isomorphic in $\mathcal{P}$ if and only if $m = n$. Similarly, $m$ and $n$ are isomorphism in **Mat** if and only if $m = n$.

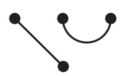
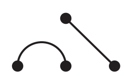
**Exercise 2.26.** Suppose $\mathcal{D} \subseteq \mathcal{C}$ are categories and $m, n \in \operatorname{Ob}\mathcal{D}$. Prove that if $m$ is isomorphic to $n$ in $\mathcal{D}$, then $m$ is isomorphic to $n$ in $\mathcal{C}$.

**Exercise 2.27.** Suppose $m, n \in \mathbb{Z}_{\geq 0}$ and $\mathcal{D}$ is a subcategory of $\mathcal{P}$. Use Example 2.25 and Exercise 2.26 to show that $m$ and $n$ are isomorphic in $\mathcal{D}$ if and only if $m = n$.

**Example 2.28.** Since isomorphisms in **Set** are bijections, it follows that two sets are isomorphic in **Set** if and only if they have the same cardinality.

## Idempotents

Given a morphism $e$ in any category, the composition $e^2 = e \circ e$ is defined if and only if $e$ is an endomorphism. An *idempotent* is a morphism $e$ such that $e^2 = e$. For example, all identity morphisms are idempotents. However, not all idempotents are identity morphisms. For example, both morphisms in $\operatorname{End}_{\mathcal{P}}(1)$ are idempotents (see Exercise 2.12). Here's another example: Let $B : 1 \to 3$ and $D : 3 \to 1$ denote the following diagrams:

$B =$  $D =$ 

Then $D \circ B = 1_1$, and one might be tempted to conclude that $B$ and $D$ are inverse isomorphisms, contradicting the fact that 1 and 3 are not isomorphic objects in $\mathcal{P}$ (Example 2.25). Of course, $B$ and $D$ are not inverse isomorphisms since $B \circ D \neq 1_3$. However, you can check that $B \circ D$ is an idempotent.

**Exercise 2.29.** Suppose $f : m \to n$ and $g : n \to m$ are two morphisms in a category such that $g \circ f = \mathrm{id}_m$. Show that $f \circ g$ is an idempotent.

**Exercise 2.30.** Show that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent in **Mat**. More generally, show that any square matrix whose entries are all 0's except some 1's on the main diagonal is an idempotent in **Mat**. These are not the only idempotents in **Mat**; for instance the matrix $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is an idempotent. In fact any $2 \times 2$ matrix with determinant 0 and trace 1 is an idempotent.

**Exercise 2.31.** Show that constant functions are idempotents in **Set**. Given an example of a function $e : \{1, 2, 3\} \to \{1, 2, 3\}$ that is not a constant function, and not the identity function, but is an idempotent.

**Exercise 2.32.** Show that the only idempotents which are also isomorphisms are the identity morphisms.

# Chapter 3

# Functors

## 3.1 Motivating examples: permutations

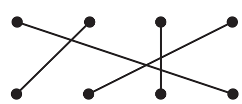
We already know what permutation diagrams are. In this section we will explore permutations in **Mat** (permutation matrices) and in **Set** (set permutations). The connection between permutation diagrams and the latter permutations will serve as motivation for the definition of a functor in §3.2.

### Permutation matrices

A *permutation matrix* is a square matrix whose entries are all 0's and 1's such that there is a unique 1 in each row and each column. For example,

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

To each permutation diagram $D : n \to n$, we associate an $n \times n$ permutation matrix $A = F(D)$ as follows: The $i, j$-entry* of $A$ is 1 if and only if the $i$th top vertex of $D$ is connected to the $j$th bottom vertex (reading vertices left to right). For example, the matrix above is associated to the following permutation diagram:



We have just described a map $F : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathbf{Mat}}(n)$ for all $n \in \mathbb{Z}_{\geq 0}$. This map has the property $F(1_n) = I_n$ for all $n$. Furthermore, it respects the composition in the two categories:

**Exercise 3.1.** Suppose $D$ and $D'$ are permutation diagrams of type $n \to n$ with associated permutation matrices $A = F(D)$ and $A' = F(D')$. Verify

*The $i, j$-entry of a matrix is the one in the $i$th row and $j$th column.

that $F(D \circ D') = AA'$. Thus, multiplication of permutation matrices can be accomplished by vertically stacking the corresponding permutation diagrams.

One can also read off properties of a permutation matrix directly from the corresponding permutation diagram. For instance, the *parity* of a permutation diagram $D$ is the parity of the number of crossings when the diagram is drawn in *generic position* (so that exactly two edges meed at each crossing, and edges are not tangent at crossings). For example, the permutation diagram pictured above is even since it has 4 crossings. It turns out that the determinant of a permutation matrix relies only on the parity of the corresponding diagram:

$$\det(F(D)) = (-1)^D = \begin{cases} 0, & \text{if } D \text{ is even;} \\ 1, & \text{if } D \text{ is odd.} \end{cases} \tag{3.1}$$

### Set permutations

Given a set $X$, a *set permutation on* $X$ is a bijection $X \to X$. To each permutation diagram $D : n \to n$ we associate a set permutation $F(D)$ on $[n] = \{1, \ldots, n\}$ as follows: Let $F(D)$ be the function $[n] \to [n]$ defined by mapping $j \mapsto i$ whenever the $i$th top vertex of $D$ is connected to the $j$th bottom vertex (reading vertices left to right). For example, if we let $D$ denote the permutation drawn above, then

$$\begin{array}{rccc} F(D): & 1 & \mapsto & 2 \\ & 2 & \mapsto & 4 \\ & 3 & \mapsto & 3 \\ & 4 & \mapsto & 1. \end{array}$$

In this case we have a map $F : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathbf{Set}}([n])$ for all $n \in \mathbb{Z}_{\geq 0}$. This map has some nice properties, similar to the properties we observed in the case of permutation matrices. For instance, $F(1_n) = \mathrm{id}_{[n]}$ for all $n$.

**Exercise 3.2.** Suppose $D$ and $D'$ are permutation diagrams of type $n \to n$. Verify that $F(D \circ D') = F(D) \circ F(D')$. Thus, composition of set permutations reduces to vertically stacking the corresponding permutation diagrams.

## 3.2 Definition of a functor

In the previous section we saw two examples where morphisms in one category ($\mathcal{S}$) were mapped to morphisms in another category (**Mat** or **Set**), and these mappings sent identity morphisms to identity morphisms and preserved composition. Although it was not emphasized, those maps were consistent on objects in the sense that if $D$ and $D'$ were two diagrams of the same type, then $F(D)$ and $F(D')$ also had the same type (i.e. domain and target). These are the main ingredients in the following definition:

**Definition 3.3.** Given two categories $\mathcal{C}, \mathcal{D}$, a *functor* $F$ from $\mathcal{C}$ to $\mathcal{D}$, written $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- A map $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ written $n \mapsto F(n)$.
- A map $\mathrm{Hom}_{\mathcal{C}}(m,n) \to \mathrm{Hom}_{\mathcal{D}}(F(m), F(n))$, written $g \mapsto F(g)$, for every $m, n \in \mathrm{Ob}\,\mathcal{C}$.

The data above must satisfy the following properties:

(F1) $F(\mathrm{id}_n) = \mathrm{id}_{F(n)}$ for every $n \in \mathrm{Ob}\,\mathcal{C}$.

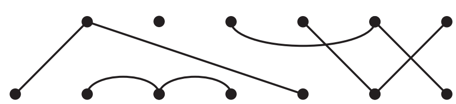
(F2) $F(f \circ g) = F(f) \circ F(g)$ for all morphisms $g : l \to m$ and $f : m \to n$ in $\mathcal{C}$.

**Example 3.4.** The discussion of permutation matrices in the previous section describes a functor $F : \mathcal{S} \to \mathbf{Mat}$. The functor is given on objects by $F(n) = n$ for all $n \in \mathbb{Z}_{\geq 0}$, and on morphisms by letting $F(D)$ denote the $n \times n$ permutation matrix associated to the permutation diagram $D : n \to n$.

**Example 3.5.** The discussion of set permutations in the previous section describes a functor $F : \mathcal{S} \to \mathbf{Set}$ given on objects by $F(n) = [n] = \{1, \ldots, n\}$ for all $n \in \mathbb{Z}_{\geq 0}$, and on morphisms by letting $F(D)$ denote the set permutation $[n] \to [n]$ associated to the permutation diagram $D : n \to n$.

**Example 3.6. (Identity functors)** Suppose $\mathcal{C}$ is an arbitrary category. The functor $\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined to be the identity map on both objects and morphisms. More precisely, $\mathrm{Id}_{\mathcal{C}}(n) = n$ for all $n \in \mathrm{Ob}\,\mathcal{C}$ and $\mathrm{Id}_{\mathcal{C}}(g) = g$ for all morphisms $g : m \to n$ in $\mathcal{C}$.

**Exercise 3.7.** The goal of this exercise is to describe a functor $F : \mathcal{P} \to \mathbf{Rel}$. On objects, this functor is given by $F(n) = [n] = \{1, \ldots, n\}$. To define $F$ on morphisms, let $D$ denote a partition diagram of type $m \to n$. We let $F(D) \subseteq [m] \times [n]$ denote the relation where $(j, i) \in F(D)$ if and only if the $i$th top vertex of $D$ is connected to the $j$th bottom vertex (reading vertices left to right). For example, if

$D =$ 

then $F(D) = \{(1,1), (5,1), (6,4), (6,6), (7,3), (7,5)\} \subseteq [7] \times [6]$. Prove that $F$ is a functor.

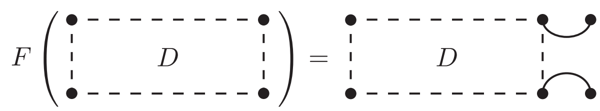
**Exercise 3.8.** Define $F : \mathcal{NC} \to \mathcal{TL}^{ev}$ on objects by $F(n) = 2n$. For morphisms, $F$ is the bijection

$$\mathrm{Hom}_{\mathcal{NC}}(m,n) \to \mathrm{Hom}_{\mathcal{TL}^{ev}}(2m, 2n)$$

prescribed by Exercise 1.16. Verify that this defines a functor $F : \mathcal{NC} \to \mathcal{TL}^{ev}$.

**Exercise 3.9.** Verify that the following maps define a functor $F : \mathcal{B}^{ev} \to \mathcal{B}^{odd}$. On objects, $F(n) = n + 1$. For morphisms $F(D) = D \otimes 1_1$ (i.e. add a single vertical edge on the right).

**Exercise 3.10.** Consider the following candidate for a functor $F : \mathcal{P} \to \mathcal{P}$. On objects, $F(n) = n + 1$. For morphisms, $F(D)$ is the diagram obtained from $D$ by adding a pair of vertices to the right of $D$, and connecting them to their respective neighbors as illustrated below:




Does $F$ satisfy (F1)? How about (F2)?

## 3.3 Isomorphic categories

Recall that two sets are called isomorphic if and only if there is a bijection between them (Examples 2.15 and 2.28). This essentially means that isomorphic sets are the "same" set up to relabeling their elements (the bijection prescribes the relabeling). In this section we use functors to define what it means for two categories to be isomorphic, and this definition will mimic that of isomorphic sets. Indeed, roughly speaking, two categories are isomorphic if they are the "same" category up to relabeling all the objects and morphisms. In order to properly discuss isomorphic categories, we first need to generalize the operation of function composition to functors.

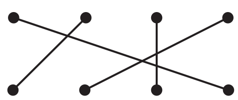
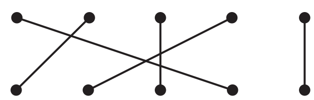
### Composition of functors

Since a functor consists of maps (of objects and morphisms) and we can compose maps (of objects and morphisms), we can compose functors. More precisely, given functors $F : \mathcal{D} \to \mathcal{E}$ and $G : \mathcal{C} \to \mathcal{D}$ we define their composition $F \circ G : \mathcal{C} \to \mathcal{E}$ by setting $F \circ G(n) = F(G(n))$ for all objects $n \in \operatorname{Ob}\mathcal{C}$ and $F \circ G(f) = F(G(f))$ for all morphisms $f$ in $\mathcal{C}$. The following computations verify that $F \circ G$ satisfies (F1) and (F2), whence $F \circ G$ is a functor:

$$
\begin{aligned}
F \circ G(\mathrm{id}_n) &= F(G(\mathrm{id}_n)) && \text{(definition of } F \circ G) \\
&= F(\mathrm{id}_{G(n)}) && (G \text{ is a functor}) \\
&= \mathrm{id}_{F(G(n))} && (F \text{ is a functor}) \\
&= \mathrm{id}_{F \circ G(n)} && \text{(definition of } F \circ G),
\end{aligned}
$$

$$
\begin{aligned}
F \circ G(f \circ g) &= F(G(f \circ g)) && \text{(definition of } F \circ G) \\
&= F(G(f) \circ G(g)) && (G \text{ is a functor}) \\
&= F(G(f)) \circ F(G(g)) && (F \text{ is a functor}) \\
&= (F \circ G(f)) \circ (F \circ G(g)) && \text{(definition of } F \circ G).
\end{aligned}
$$

**Example 3.11.** Let $F : \mathcal{S} \to \mathbf{Mat}$ be the functor from Example 3.4 and let $G : \mathcal{S} \to \mathcal{S}$ be the functor given on objects by $G(n) = n+1$ and on morphisms by $G(D) = D \otimes 1_1$ (add a single vertical edge on the right, as in Example 3.9). The functor $F \circ G : \mathcal{S} \to \mathbf{Mat}$ maps a permutation diagram $D : n \to n$ to $F(G(D)) = F(D \otimes 1_1)$, which is the permutation matrix associated to $D \otimes 1_1$. Here's an explicit example:


$\mapsto$


$$\mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In general, $F \circ G$ maps $D$ to the $(n+1) \times (n+1)$ block matrix $\begin{pmatrix} F(D) & 0 \\ 0 & 1 \end{pmatrix}$.

**Exercise 3.12.** Let $G : \mathcal{S} \to \mathcal{S}$ be as in Example 3.11. Now, for each $k \in \mathbb{N}$ we define the functor $G^k : \mathcal{S} \to \mathcal{S}$ recursively by setting $G^1 = G$ and $G^k = G \circ G^{k-1}$ for all $k > 1$ (in other words, $G^k$ is obtained by composing $G$ with itself $k$ times). What is $G^k(n)$ for any object $n \in \operatorname{Ob} \mathcal{S}$? Describe $G^k(D)$ for any permutation diagram $D$ as a tensor product of two diagrams. Give a description of $F \circ G^k(D)$ in terms of a block matrix.

**Exercise 3.13.** Let $F : \mathcal{S} \to \mathbf{Set}$ be as in Example 3.5 and let $G^k : \mathcal{S} \to \mathcal{S}$ be as in Exercise 3.12. Give a detailed description of the functor $F \circ G^k : \mathcal{S} \to \mathbf{Set}$ on both objects and morphisms.

### Equality of functors

As we will soon see, to prove two categories are isomorphic amounts to verifying a couple equalities of functors. Let us first be precise about what is required to prove functors are equal. Suppose $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{C}' \to \mathcal{D}'$ are functors. $F = G$ if and only if each of the following hold:

- $\mathcal{C} = \mathcal{C}'$,
- $\mathcal{D} = \mathcal{D}'$,
- $F(n) = F'(n)$ for all $n \in \operatorname{Ob} \mathcal{C}$,
- $F(g) = F'(g)$ for all morphisms $g$ in $\mathcal{C}$.

**Example 3.14.** For any functor $F : \mathcal{C} \to \mathcal{D}$ we have $F \circ \mathrm{Id}_{\mathcal{C}} = F$. Indeed, both functors are from $\mathcal{C} \to \mathcal{D}$. Moreover, for all $n \in \operatorname{Ob} \mathcal{C}$:

$$\begin{aligned} F \circ \mathrm{Id}_{\mathcal{C}}(n) &= F(\mathrm{Id}_{\mathcal{C}}(n)) && \text{(definition of composition)} \\ &= F(n) && \text{(definition of } \mathrm{Id}_{\mathcal{C}}\text{).} \end{aligned}$$

Finally, for all morphisms $g$ in $\mathcal{C}$:

$$\begin{aligned} F \circ \mathrm{Id}_{\mathcal{C}}(g) &= F(\mathrm{Id}_{\mathcal{C}}(g)) && \text{(definition of composition)} \\ &= F(g) && \text{(definition of } \mathrm{Id}_{\mathcal{C}}). \end{aligned}$$

Similarly, $\mathrm{Id}_{\mathcal{D}} \circ F = F$.

**Exercise 3.15.** Let $\mathbf{Rel}_n$ denote the full subcategory of $\mathbf{Rel}$ with

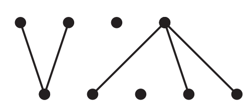
$$\mathrm{Ob}\,\mathbf{Rel}_n = \{[n] \ : \ n \in \mathbb{Z}_{\geq 0}\}.$$

Let $F : \mathcal{P} \to \mathbf{Rel}_n$ denote the functor defined on objects by $F(n) = [n]$. To define $F$ on morphisms, let $D$ denote a partition diagram of type $m \to n$. We let $F(D) \subseteq [m] \times [n]$ denote the relation with $(j, i) \in F(D)$ if and only if the $i$th top vertex of $D$ is connected to the $j$th bottom vertex (reading left to right). Explain why $F$ is not equal to the similar functor defined in Example 3.7.

Next, let $G : \mathbf{Rel}_n \to \mathcal{P}$ be the following functor. On objects, $G([n]) = n$. To define $G$ on morphisms, let $R \subseteq [m] \times [n]$ be a relation. Let $G(R)$ denote the partition diagram obtained by drawing an edge connecting the $i$th top vertex to the $j$th bottom vertex whenever $(j, i) \in R$. For example, if

$$R = \{(1,1), (1,2), (2,4), (4,4), (5,4)\} \subseteq [5] \times [4],$$

then $G(R)$ is the following diagram:



Prove that $F \circ G = \mathrm{Id}_{\mathbf{Rel}_n}$ but $G \circ F \neq \mathrm{Id}_{\mathcal{P}}$.

## Isomorphic categories

Compare the following definitions with the definitions of isomorphisms and isomorphic objects in §2.3†. A functor is $F : \mathcal{C} \to \mathcal{D}$ is called an *isomorphism of categories* if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $F \circ G = \mathrm{Id}_{\mathcal{D}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$. In this case, we write $G = F^{-1}$. Moreover, we say $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic categories* and write $\mathcal{C} \cong \mathcal{D}$.

**Exercise 3.16.** Prove the following hold for all categories $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$:

- $\mathcal{C} \cong \mathcal{C}$.
- If $\mathcal{C} \cong \mathcal{D}$, then $\mathcal{D} \cong \mathcal{C}$.
- If $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{D} \cong \mathcal{E}$, then $\mathcal{C} \cong \mathcal{E}$.

† There is a category **Cat** whose objects are categories and morphisms are functors.

**Proposition 3.17.** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories whose objects and morphisms form sets*[‡]*. A functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories if and only if the corresponding maps on objects $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and on morphisms $\mathrm{Hom}_{\mathcal{C}}(m,n) \to \mathrm{Hom}_{\mathcal{D}}(F(m),F(n))$ are all bijections.*

*Proof.* If $F$ is an isomorphism of categories, then the corresponding maps on objects and morphisms are invertible, hence bijective.

On the other hand, if $F$ induces bijective (hence invertible) maps on objects and morphisms, then the collection of inverse maps prescribes a map $G : \mathcal{D} \to \mathcal{C}$ which is well defined on both objects and morphisms such that $F \circ G = \mathrm{Id}_{\mathcal{D}}$ and $G \circ F = \mathrm{Id}_{\mathcal{C}}$. To complete the proof we need to verify that $G$ is a functor. To do so, notice that

$$\begin{aligned} G(\mathrm{id}_n) &= G(\mathrm{id}_{F\circ G(n)}) && (\text{since } F\circ G = \mathrm{Id}_{\mathcal{D}}) \\ &= G(F(\mathrm{id}_{G(n)})) && (\text{since } F \text{ is a functor}) \\ &= \mathrm{id}_{G(n)} && (\text{since } G\circ F = \mathrm{Id}_{\mathcal{C}}) \end{aligned}$$

whence $G$ satisfies (F1). The following verifies that $G$ satisfies (F2):

$$\begin{aligned} G(f\circ g) &= G(F(G(f))\circ F(G(g))) && (\text{since } F\circ G = \mathrm{Id}_{\mathcal{D}}) \\ &= G(F(G(f)\circ G(g))) && (\text{since } F \text{ is a functor}) \\ &= G(f)\circ G(g) && (\text{since } G\circ F = \mathrm{Id}_{\mathcal{C}}) \end{aligned}$$

□

**Example 3.18.** Let $\mathbf{Bij}_n$ denote the subcategory of **Set** defined by

$$\mathrm{Ob}\,\mathbf{Bij}_n = \{[n] \ : \ n \in \mathbb{Z}_{\geq 0}\},$$

$$\mathrm{Hom}_{\mathbf{Bij}_n}([m],[n]) = \{\text{bijections from } [m] \to [n]\}.$$

Since there's a bijection from $[m]$ to $[n]$ if and only if $m = n$, it follows that $\mathrm{Hom}_{\mathbf{Bij}_n}([m],[n]) = \varnothing$ whenever $m \neq n$. Therefore, morphisms in $\mathbf{Bij}_n$ are exactly set permutations. Let $F : \mathcal{S} \to \mathbf{Bij}_n$ be the functor which agrees with the functor of the same name from Example 3.5 on everything except their target categories. Since the assignment $D \mapsto F(D)$ gives a bijection between diagram permutations of type $n \to n$ and set permutations on $[n]$, it follows from Proposition 3.17 that $F$ is an isomorphism of categories. Hence $\mathcal{S} \cong \mathbf{Bij}_n$.

**Exercise 3.19.** Let **PMat** denote the subcategory of **Mat** defined by setting $\mathrm{Ob}\,\mathbf{PMat} = \mathrm{Ob}\,\mathbf{Mat}$ and $\mathrm{Hom}_{\mathbf{PMat}}(m,n) = \{n \times m \text{ permutation matrices}\}$ for all $m, n \in \mathbb{Z}_{\geq 0}$. Prove that $\mathcal{S} \cong \mathbf{PMat}$.

[‡]Such categories are called *small*. All of the diagram categories we will consider are small, but there are plenty of *large* (not small) categories. For example **Set** is large since the collection of all sets does not form a set – just ask Bertrand Russell. I expect that the proposition is true without the assumption that $\mathcal{C}$ and $\mathcal{D}$ are small, but I don't care enough to check the details. In any case, I'd rather only talk about bijections between sets as opposed to ones between proper classes.

**Exercise 3.20.** Prove that $\mathcal{NC} \cong \mathcal{TL}^{ev}$.

**Exercise 3.21.** Why is the functor $F : \mathcal{P} \to \mathbf{Rel}_n$ in Exercise 3.15 is not an isomorphism. Explain why knowing $F$ is not an isomorphism is not sufficient to conclude that $\mathcal{P}$ and $\mathbf{Rel}_n$ are not isomorphic. Now, count the number of elements (relations) in the set $\mathrm{Hom}_{\mathbf{Rel}_n}([2],[1])$. Use that count along with Exercise 1.4 and Proposition 3.17 to prove that $\mathcal{P}$ and $\mathbf{Rel}_n$ are not isomorphic.

# Chapter 4

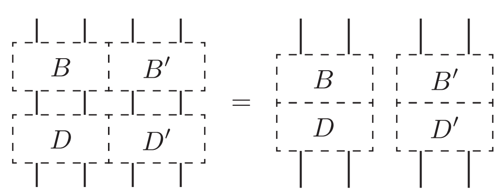
# Strict monoidal categories

## 4.1 Motivating examples: other matrix and function operations

Up to this point, when discussing a category we have focussed on a single operation (vertical stacking of diagram, matrix multiplication, function composition, etc.). In this chapter we develop the categorical framework which allows for a pair of operations. The main example to keep in mind is a diagram category with the two operations of vertical and horizontal stacking. There are a couple important properties of this pair of operations. First, horizontally stacking identity diagrams yields another identity diagram:

$$1_n \otimes 1_{n'} = 1_{n+n'}. \tag{4.1}$$

Also, let us make the following diagrammatic observation:




In other words,

$$(B \otimes B') \circ (D \otimes D') = (B \circ D) \otimes (B' \circ D') \tag{4.2}$$

for all diagrams $B : m \to n$, $B' : m' \to n'$, $D : l \to m$, $D' : l' \to m'$. In this section we look at a few more examples of operations in categories which satisfy formulae similar to (4.1) and (4.2).

**Example 4.1. (Direct sums of matrices)** Given an $n \times m$ matrix $A$ and a $n' \times m'$ matrix $A'$, their *direct sum* is the $(n + n') \times (m + m')$ block diagonal matrix

$$A \oplus A' = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}.$$

The direct sum of identity matrices is another identity matrix: $I_n \oplus I_{n'} = I_{n+n'}$. Moreover, given matrices $A$ $(n \times m)$, $A'$ $(n' \times m')$, $B$ $(m \times l)$, and $B'$ $(m' \times l')$ we have

$$(A \oplus A')(B \oplus B') = \begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & A'B' \end{pmatrix} = (AB) \oplus (A'B').$$

**Example 4.2. (Kronecker products of matrices)** Given an $n \times m$ matrix $A$ and a $n' \times m'$ matrix $A'$, their *Kronecker product* is the $(nn') \times (mm')$ matrix which has the following block form:

$$A \otimes A' = \begin{pmatrix} a_{1,1}A' & \cdots & a_{1,m}A' \\ \vdots & & \vdots \\ a_{n,1}A' & \cdots & a_{n,m}A' \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}.$$

For example,

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 4 & 2 \\ -1 & 0 & 2 & 0 \\ -3 & 2 & 6 & -4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 \end{pmatrix}. \tag{4.3}$$

Kronecker products of identity matrices are identity matrices: $I_n \otimes I_{n'} = I_{nn'}$.

**Exercise 4.3.** Given matrices $A$ $(n \times m)$, $A'$ $(n' \times m')$, $B$ $(m \times l)$, and $B'$ $(m' \times l')$ prove that

$$(A \otimes A')(B \otimes B') = (AB) \otimes (A'B'). \tag{4.4}$$

Hint: We can index the entry $a_{i,j}a'_{i',j'}$ in $A \otimes A'$ by a tuple $(i, i', j, j')$. For example, the $(1, 3, 2, 1)$-entry in (4.3) is 6. Compute the $(i, i', j, j')$-entry in both sides of (4.4).

**Example 4.4. (Cartesian products)** Given functions $f : M \to N$ and $f' : M' \to N'$, their *Cartesian product* is the function

$$\begin{aligned} f \times f' : M \times M' &\to N \times N' \\ (x, x') &\mapsto (f(x), f'(x')). \end{aligned}$$

The Cartesian product of two identity functions is another identity function: $\mathrm{id}_N \times \mathrm{id}_{N'} = \mathrm{id}_{N \times N'}$. Moreover, given functions $f : M \to N$, $f' : M' \to N'$, $g : L \to M$, and $g' : L' \to M'$, we have

$$\begin{aligned} (f \times f') \circ (g \times g')(x, x') &= f \times f'(g \times g'(x, x')) \\ &= f \times f'(g(x), g'(x')) \\ &= (f(g(x)), f'(g'(x'))) \end{aligned}$$

$$\begin{aligned} &= (f \circ g(x), f' \circ g'(x')) \\ &= (f \circ g) \times (f' \circ g')(x, x') \end{aligned}$$

for all $(x, x') \in L \times L'$. Therefore $(f \times f') \circ (g \times g') = (f \circ g) \times (f' \circ g')$.

**Example 4.5. (Disjoint unions)** The *disjoint union of sets* $M$ and $M'$ is the set

$$M \sqcup M' = \{(x, 0) \ : \ x \in M\} \cup \{(x', 1) \ : \ x' \in M'\}.$$

Now, given functions $f : M \to N$ and $f' : M' \to N'$, their *disjoint union* is the function

$$\begin{array}{rcl} f \sqcup f' : M \sqcup M' & \to & N \sqcup N' \\ (x, 0) & \mapsto & (f(x), 0) \\ (x', 1) & \mapsto & (f'(x'), 1) \end{array}$$

Note that the disjoint unions of two identity functions is another identity function: $\mathrm{id}_N \sqcup \mathrm{id}_{N'} = \mathrm{id}_{N \sqcup N'}$.

**Exercise 4.6.** Given functions $f : M \to N$, $f' : M' \to N'$, $g : L \to M$, and $g' : L' \to M'$, verify that $(f \sqcup f') \circ (g \sqcup g') = (f \circ g) \sqcup (f' \circ g')$ by calculating the images of $(x, 0)$ and $(x', 1)$ for arbitrary $x \in L$ and $x' \in L'$.

## 4.2 Bifunctors

Compare the following definition with the operation of horizontal stacking in diagram categories and the other examples of operations in §4.1.

**Definition 4.7.** A *bifunctor* $\odot$ on a category $\mathcal{C}$ consists of the following data:

- A map $\mathrm{Ob}\,\mathcal{C} \times \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$ written $(n, n') \mapsto n \odot n'$.
- A map $\mathrm{Hom}_{\mathcal{C}}(m, n) \times \mathrm{Hom}_{\mathcal{C}}(m', n') \to \mathrm{Hom}_{\mathcal{C}}(n \odot n', m \odot m')$, written $(f, f') \mapsto f \odot f'$, for every $m, n, m', n' \in \mathrm{Ob}\,\mathcal{C}$.

The data above must satisfy the following properties:

(B1) $\mathrm{id}_n \odot \mathrm{id}_{n'} = \mathrm{id}_{n \odot n'}$ for every $n, n' \in \mathrm{Ob}\,\mathcal{C}$.

(B2) $(f \odot f') \circ (g \odot g') = (f \circ g) \odot (f' \circ g')$ for all morphisms $f : m \to n$, $f' : m' \to n'$, $g : l \to m$, $g' : l' \to m'$ in $\mathcal{C}$.

**Example 4.8.** Horizontal stacking gives a bifunctor $\otimes$ on $\mathcal{P}$. We already know how this bifunctor acts on morphisms (stack diagrams horizontally). On objects we set $n \otimes n' = n + n'$. Properties (4.1) and (4.2) verify that $\otimes$ satisfies (B1) and (B2) respectively. Note that $\otimes$ can also be viewed as a bifunctor on $\mathcal{NC}$, $\mathcal{B}$, $\mathcal{TL}$, or $\mathcal{S}$ (see Exercise 1.22).

**Exercise 4.9.** Determine which of the following diagram categories horizontal stacking defines a bifunctor $\otimes$ on: $\mathcal{B}^{ev}$, $\mathcal{B}^{odd}$, $\mathcal{TL}^{ev}$, $\mathcal{TL}^{odd}$.

**Example 4.10.** The direct sum of matrices gives a bifunctor $\oplus$ on **Mat**. On objects we set $n \oplus n' = n + n'$. The definition of $\oplus$ on morphisms (matrices) as well as the verification of (B1) and (B2) can be found in Example 4.1.

**Example 4.11.** The Kronecker product of matrices gives another bifunctor $\otimes$ on **Mat**. On objects we set $n \otimes n' = nn'$. The definition of $\otimes$ on morphisms (matrices) as well as the verification of (B1) and (B2) can be found in Example 4.2 and Exercise 4.3.

**Example 4.12.** It follows from Example 4.4 that the Cartesian product defines a bifunctor $\times$ on **Set**.

**Example 4.13.** It follows from Example 4.5 and Exercise 4.6 that the disjoint union defines a bifunctor $\sqcup$ on **Set**.

## 4.3 Definition of a strict monoidal category

Our main example of a bifunctor (horizontal stacking of diagrams) satisfies some additional nice properties. First off, horizontal stacking is associative:

$$(D_1 \otimes D_2) \otimes D_3 = D_1 \otimes (D_2 \otimes D_3) \tag{4.5}$$

for all partition diagrams $D_1, D_2, D_3$. Moreover, there is a special (empty) diagram that acts as a unit with respect to horizontal stacking:

$$D \otimes 1_0 = D = 1_0 \otimes D \tag{4.6}$$

for any partition diagram $D$. A category equipped with a bifunctor and such nice properties gets a special name:

**Definition 4.14.** A *strict monoidal category* is a triple of data $(\mathcal{C}, \odot, \mathbb{1})$ where:

- $\mathcal{C}$ is a category.
- $\odot$ is a bifunctor on $\mathcal{C}$.
- $\mathbb{1}$ is an object in $\mathcal{C}$ called the *unit object*.

The data above must satisfy the following properties:

(M1) $f \odot \mathrm{id}_{\mathbb{1}} = f = \mathrm{id}_{\mathbb{1}} \odot f$ for every morphism $f$ in $\mathcal{C}$.

(M2) $(f \odot g) \odot h = f \odot (g \odot h)$ for all morphisms $f, g, h$ in $\mathcal{C}$.

**Example 4.15.** Formulae (4.5) and (4.6) verify that $(\mathcal{P}, \otimes, 0)$ is a strict monoidal category. Similarly, $(\mathcal{NC}, \otimes, 0)$, $(\mathcal{B}, \otimes, 0)$, $(\mathcal{TL}, \otimes, 0)$, and $(\mathcal{S}, \otimes, 0)$ are all strict monoidal categories.

**Example 4.16.** Direct sums of matrices (see Example 4.1) give rise to a strict monoidal category $(\mathbf{Mat}, \oplus, 0)$. The following computation verifies (M1):

$$A \oplus 0 = A = 0 \oplus A,$$

where 0 in the line above denotes the unique $0 \times 0$ matrix, which exists by convention (see Remark 2.4). Here's the verification of (M2):

$$(A \oplus B) \oplus C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \oplus C = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} = A \oplus \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} = A \oplus (B \oplus C).$$

**Exercise 4.17.** Show that $(\mathbf{Mat}, \otimes, 1)$ is a strict monoidal category. Take care in verifying (M2).

**Exercise 4.18.** Show that $A \oplus B$ and $A \otimes B$ are permutation matrices whenever $A$ and $B$ are. Conclude that both $(\mathbf{PMat}, \oplus, 0)$ and $(\mathbf{PMat}, \otimes, 1)$ are strict monoidal categories.

**Exercise 4.19.** Although the defining properties of a strict monoidal category are in terms of only morphisms, the analogous properties for objects hold too. Apply axiom (C1) to (M1) and (M2) to prove the following hold in any strict monoidal category:

(a) $n \odot \mathbb{1} = n = \mathbb{1} \odot n$ for all $n \in \operatorname{Ob} \mathcal{C}$.

(b) $(l \odot m) \odot n = l \odot (m \odot n)$ for all $l, m, n \in \operatorname{Ob} \mathcal{C}$.

As a consequence, we have the following:

**Proposition 4.20.** *Unit objects in strict monoidal categories are unique. In fact, if $(\mathcal{C}, \odot, \mathbb{1})$ is a strict monoidal category, and $\mathbb{1}'$ is any object in $\mathcal{C}$ satisfying either $\mathbb{1}' \odot n = n$ or $n \odot \mathbb{1}' = n$ for all $n \in \operatorname{Ob} \mathcal{C}$, then $\mathbb{1}' = \mathbb{1}$.*

*Proof.* By part 1 of Exercise 4.19 we have $\mathbb{1}' \odot \mathbb{1} = \mathbb{1}' = \mathbb{1} \odot \mathbb{1}'$. The assumption on $\mathbb{1}'$ implies that either $\mathbb{1}' \odot \mathbb{1} = \mathbb{1}$ or $\mathbb{1} \odot \mathbb{1}' = \mathbb{1}$. □

## 4.4 Non-strict monoidal categories

Not every bifunctor determines a strict monoidal category. For example, consider the bifunctor $\times$ on **Set**. If we're careful, we can see that this bifunctor is not associative on objects:

$$(L \times M) \times N \neq L \times (M \times N). \tag{4.7}$$

Indeed, elements of the left side have the form $((x, y), z)$ whereas elements of the right look like $(x, (y, z))$. These are close to the same, but they are not equal. It follows from part 2 of Exercise 4.19 that $\times$ cannot be used to give **Set** the structure of a strict monoidal category.

We can also arrive at the same conclusion by looking for unit objects. Suppose $\mathbb{1}$ is a set such that $N \times \mathbb{1} = N$ for any set $N$. Comparing sizes will lead you to the conclusion that $\mathbb{1}$ must be a singleton set: $\mathbb{1} = \{\star\}$. Although $N \times \{\star\}$ is not equal to $N$ on the nose, there is an obvious way to identify their elements: $x \leftrightarrow (x, \star)$. But what singleton set do you pick? Indeed, there is not a unique choice for a unit object. Hence by Proposition 4.20, $\times$ cannot be used to give **Set** the structure of a strict monoidal category.

The triple $(\mathbf{Set}, \times, \{\star\})$ is an example of a *non-strict monoidal category.* I don't want to give the precise definition of that term since our primary concern are diagram categories, which are strict monoidal categories. However, I think it's useful to know that there are plenty of important bifunctors in mathematics which are "almost associative" and "almost" have unit objects. Making these "almost-statements" precise involves plenty of isomorphisms. Indeed, although the sets in (4.7) are not equal, they are isomorphic in **Set** via the function $((x, y), z) \mapsto (x, (y, z))$. Moreover, although there are infinitely many singleton sets, they are all isomorphic in **Set**.

**Exercise 4.21.** Show that the bifunctor $\sqcup$ on **Set** is not associative. Find a unit object $\mathbb{1}$ so that $(\mathbf{Set}, \sqcup, \mathbb{1})$ is a non-strict monoidal category, whatever that means. Is the unit object unique?

### Strictification

One reason I am okay with dodging the definition of a non-strict monoidal category is that it turns out every non-strict monoidal category is equivalent (in some precise way) to a strict one. The process of replacing a non-strict monoidal category with an equivalent strict monoidal category is called *strictification.* The rest of this section is devoted to the strictifications of $(\mathbf{FinSet}, \sqcup, \mathbb{1})$ and $(\mathbf{FinSet}, \times, \{\star\})$, where **FinSet** is the full subcategory of **Set** whose objects are finite sets. To start, let $\mathbf{Set}_n$ denote the full subcategory of **Set** with

$$\operatorname{Ob} \mathbf{Set}_n = \{[n] \; : \; n \in \mathbb{Z}_{\geq 0}\}.$$

**Exercise 4.22.** Show that neither the disjoint union $\sqcup$ nor the Cartesian product $\times$ define a bifunctor on $\mathbf{Set}_n$.

To strictify the disjoint union, we define the bifunctor $\uplus$ on the category $\mathbf{Set}_n$ as follows. On objects we set $[n] \uplus [n'] = [n + n']$. To define $\uplus$ on morphisms, suppose $f : [m] \to [n]$ and $g : [m'] \to [n']$ are two functions. Set

$$\begin{array}{rcll} f \uplus g : [m + m'] & \to & [n + n'] & \\ j & \mapsto & f(j) & (1 \leq j \leq m) \\ m + j & \mapsto & n + g(j) & (1 \leq j \leq m') \end{array}$$

Let's compare the operations $\sqcup$ and $\uplus$ with a specific example. Let $f$ and $g$ denote the following functions:

$$\begin{tikzcd}[row sep=tiny]
f : [3] \arrow[r] & {[2]} \\
1 \arrow[dr] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny]
g : [3] \arrow[r] & {[3]} \\
1 \arrow[dr] & 1 \\
2 \arrow[dr] & 2 \\
3 \arrow[uur] & 3
\end{tikzcd}$$

Applying the operations $\sqcup$ and $\uplus$ gives the following:

$$\begin{tikzcd}[row sep=tiny]
f \sqcup g : [3] \sqcup [3] \arrow[r] & {[2] \sqcup [3]} \\
(1,0) \arrow[dr] & (1,0) \\
(2,0) \arrow[ur] & (2,0) \\
(3,0) \arrow[ur] & \\
(1,1) \arrow[dr] & (1,1) \\
(2,1) \arrow[dr] & (2,1) \\
(3,1) \arrow[uur] & (3,1)
\end{tikzcd}
\qquad
\begin{tikzcd}[row sep=tiny]
f \uplus g : [6] \arrow[r] & {[5]} \\
1 \arrow[dr] & 1 \\
2 \arrow[ur] & 2 \\
3 \arrow[ur] & \\
4 \arrow[dr] & 3 \\
5 \arrow[dr] & 4 \\
6 \arrow[uur] & 5
\end{tikzcd}$$

Both operations are, in some sense, obtained by stacking the rule for $f$ on top of the rule for $g$. The only difference is the labeling of the elements in the domain and target.

**Exercise 4.23.** Show that $\uplus$ satisfies (B1) and (B2). Find the unique unit object $\mathbb{1} \in \operatorname{Ob} \mathbf{Set}_n$ that makes $(\mathbf{Set}_n, \uplus, \mathbb{1})$ a strict monoidal category. Be sure to verify (M1) and (M2).

**Exercise 4.24.** Show that $f \uplus g$ is a bijection whenever $f$ and $g$ are. Conclude that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ is also a strict monoidal category.

To strictify the Cartesian product, we will define a bifunctor $*$ on $\mathbf{Set}_n$. For objects we set $[n] * [n'] = [nn']$. Note that $[n] \times [n']$ and $[n] * [n']$ are isomorphic in $\mathbf{Set}$. My favorite bijection between those sets is the following:

$$\begin{aligned} \phi_{n,n'} : [n] \times [n'] &\to [nn'] \\ (i,j) &\mapsto j + (i-1)n' \end{aligned}$$

For example, the following illustrates $\phi_{3,4} : [3] \times [4] \to [12]$:

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & & 1 & 2 & 3 & 4 \\ (2,1) & (2,2) & (2,3) & (2,4) & \xrightarrow{\phi_{3,4}} & 5 & 6 & 7 & 8 \\ (3,1) & (3,2) & (3,3) & (3,4) & & 9 & 10 & 11 & 12 \end{matrix}$$

Now, we define $*$ on morphisms by setting

$$f * g = \phi_{n,n'} \circ (f \times g) \circ \phi_{m,m'}^{-1} \tag{4.8}$$

whenever $f : [m] \to [n]$ and $g : [m'] \to [n']$. For example, let $f : [3] \to [2]$ and $g : [3] \to [3]$ be the functions defined above. The map $f * g : [3] * [3] \to [2] * [3]$ can be found using the following maps between arrays:

$$\begin{tikzcd}
\begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \\ (3,1) & (3,2) & (3,3) \end{matrix} \arrow[r, "f \times g"] & \begin{matrix} (1,1) & (1,2) & (1,3) \\ (2,1) & (2,2) & (2,3) \end{matrix} \arrow[d, "\phi_{2,3}"] \\
\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \arrow[u, "\phi_{3,3}^{-1}"] \arrow[r, "f * g"] & \begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}
\end{tikzcd}$$

For instance, $f * g(7) = 5$ since $7 \xrightarrow{\phi_{3,3}^{-1}} (3,1) \xrightarrow{f \times g} (2,2) \xrightarrow{\phi_{2,3}} 5$.

**Exercise 4.25.** Use (4.8) to show that $*$ satisfies (B1) and (B2).

**Exercise 4.26.** Given arbitrary functions $f : [m] \to [n]$ and $g : [m'] \to [n']$, find an explicit formula for $f * g(k)$ for any $1 \leq k \leq mm'$.
[Hint: Write $k = j + (i-1)m'$ with $1 \leq j \leq m'$ and $1 \leq i \leq m$.]

**Exercise 4.27.** Prove that $(\mathbf{Set}_n, *, [1])$ is a strict monoidal category.

**Exercise 4.28.** Show that $f * g$ is a bijection whenever $f$ and $g$ are. Conclude that $(\mathbf{Bij}_n, *, [1])$ is also a strict monoidal category.

## 4.5 Strict monoidal functors

In Chapter 3 we saw how functors give us a precise way to compare vertical stacking in diagram categories with operations such as matrix multiplication and function composition. In order to simultaneously compare vertical horizontal stacking of diagram categories with the multiple operations in other strict monoidal categories, our functors need to satisfy some additional properties to be sure that monoidal structure is preserved.

**Definition 4.29.** Given two strict monoidal categories $(\mathcal{C}, \odot, \mathbb{1})$ and $(\mathcal{D}, \odot, \mathbb{1})$, a *strict monoidal functor* from $(\mathcal{C}, \odot, \mathbb{1})$ to $(\mathcal{D}, \odot, \mathbb{1})$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that satisfies the following properties:

(MF1) $F(\mathbb{1}) = \mathbb{1}$.

(MF2) $F(f \odot g) = F(f) \odot F(g)$ for all morphisms $f$ and $g$ in $\mathcal{C}$.

**Example 4.30.** Consider the functor $F : \mathcal{S} \to \mathbf{Mat}$ that maps each permutation diagram to its corresponding permutation matrix (see Example 3.4). We will show $F$ is a strict monoidal functor from $(\mathcal{S}, \otimes, 0)$ to $(\mathbf{Mat}, \oplus, 0)$. Since $F(0) = 0$, it follows that (MF1) holds. You are encouraged to fill in the details in the following verification of (MF2):

$$F(D \otimes D') = \begin{pmatrix} F(D) & 0 \\ 0 & F(D') \end{pmatrix} = F(D) \oplus F(D').$$

**Exercise 4.31.** Show that the functor $F : \mathcal{S} \to \mathbf{Set}_n$ that maps each permutation diagram to its corresponding set permutation is a strict monoidal functor from $(\mathcal{S}, \otimes, 0)$ to $(\mathbf{Set}_n, \uplus, \mathbb{1})$.

**Exercise 4.32.** Explain how to define a bifunctor $\uplus$ on $\mathbf{Rel}_n$ so that functor $F : \mathcal{P} \to \mathbf{Rel}_n$ from Exercise 3.15 is a strict monoidal functor from $(\mathcal{P}, \otimes, 0)$ to $(\mathbf{Rel}_n, \uplus, \mathbb{1})$. Of course, you need to determine the unit object too.

**Exercise 4.33.** Prove that $F \circ G$ is a strict monoidal functor whenever $F$ and $G$ are.

### Isomorphic strict monoidal categories

If there is a strict monoidal functor from $(\mathcal{C}, \odot, \mathbb{1})$ to $(\mathcal{D}, \odot, \mathbb{1})$ which is also an isomorphism of categories, then we say that $(\mathcal{C}, \odot, \mathbb{1})$ and $(\mathcal{D}, \odot, \mathbb{1})$ are *isomorphic*, or that $\mathcal{C}$ and $\mathcal{D}$ are *isomorphic as strict monoidal categories* if the monoidal structure is understood from the context.

**Example 4.34.** The strict monoidal categories $(\mathbf{PMat}, \oplus, 0)$ and $(\mathcal{S}, \otimes, 0)$ are isomorphic. Indeed, arguing as in Example 4.30, the functor $F : \mathcal{S} \to \mathbf{PMat}$ which maps a permutation diagram to its corresponding permutation matrix is a strict monoidal functor. Moreover, $F$ is an isomorphism of categories (see Exercise 3.19).

**Exercise 4.35.** Prove that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ is isomorphic to $(\mathcal{S}, \otimes, 0)$. Use Example 4.34 along with Exercises 3.16 and 4.33 to prove that $(\mathbf{Bij}_n, \uplus, \mathbb{1})$ and $(\mathcal{S}, \otimes, 0)$ are isomorphic.

**Exercise 4.36.** Show $\mathcal{NC}$ and $\mathcal{TL}^{ev}$ are isomorphic as strict monoidal categories.

**Exercise 4.37.** Prove that $(\mathbf{PMat}, \otimes, 1)$ is isomorphic to $(\mathbf{Bij}_n, *, [1])$.

**Exercise 4.38.** Give a diagrammatic description of a bifunctor $\boxtimes$ on $\mathcal{S}$ such that $(\mathcal{S}, \boxtimes, 1)$ is a strict monoidal category which is isomorphic to $(\mathbf{PMat}, \otimes, 1)$ and $(\mathbf{Bij}_n, *, [1])$.