AA103
Rocket Performance Metrics

April 18, 2019
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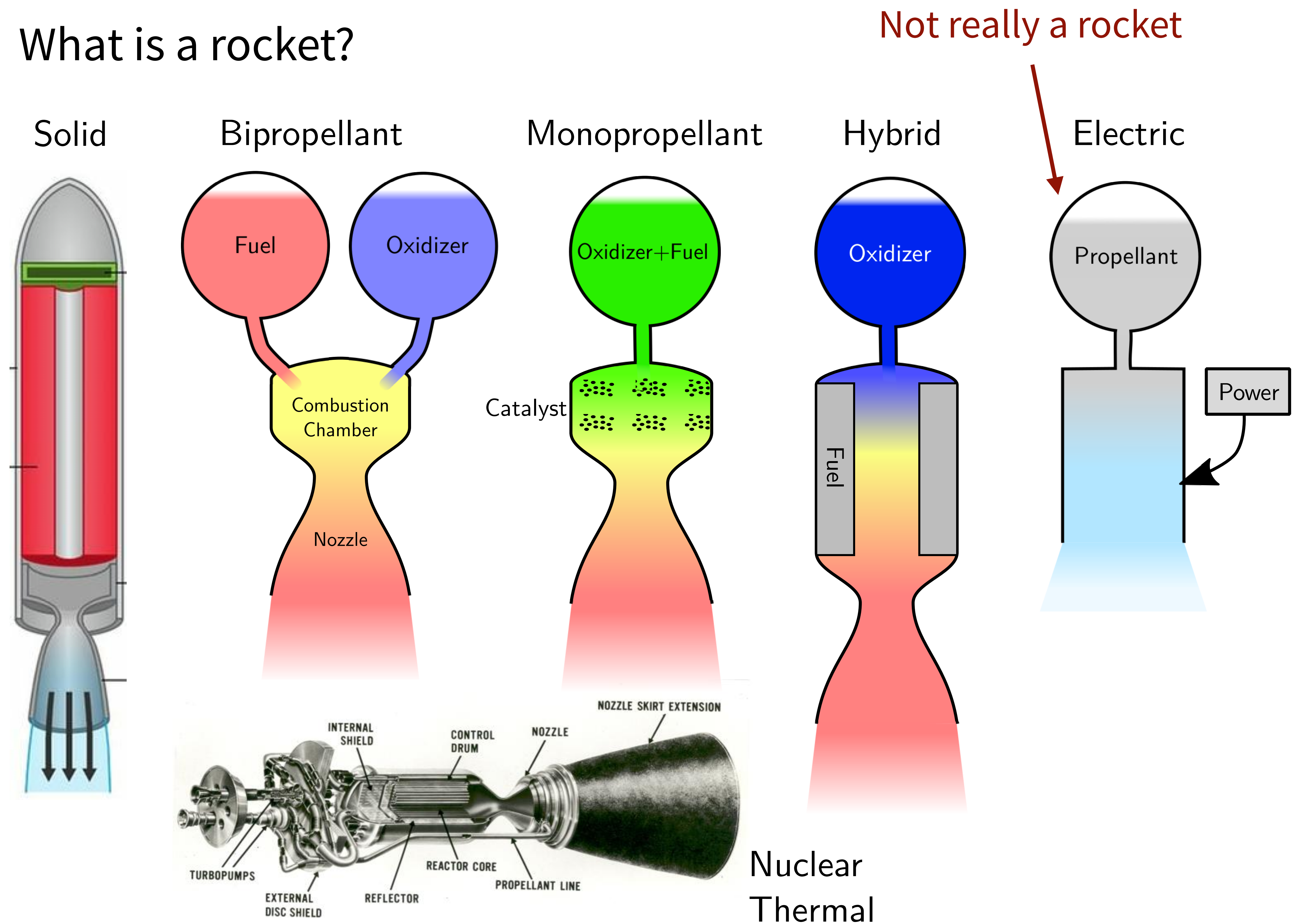
What is a rocket?



Thrust Equation

$$T = (P_e - P_0)A_e + \dot{m}_p U_e$$

Rocket Equation

$$\Delta V = C \ln \frac{M_i}{M_f}$$

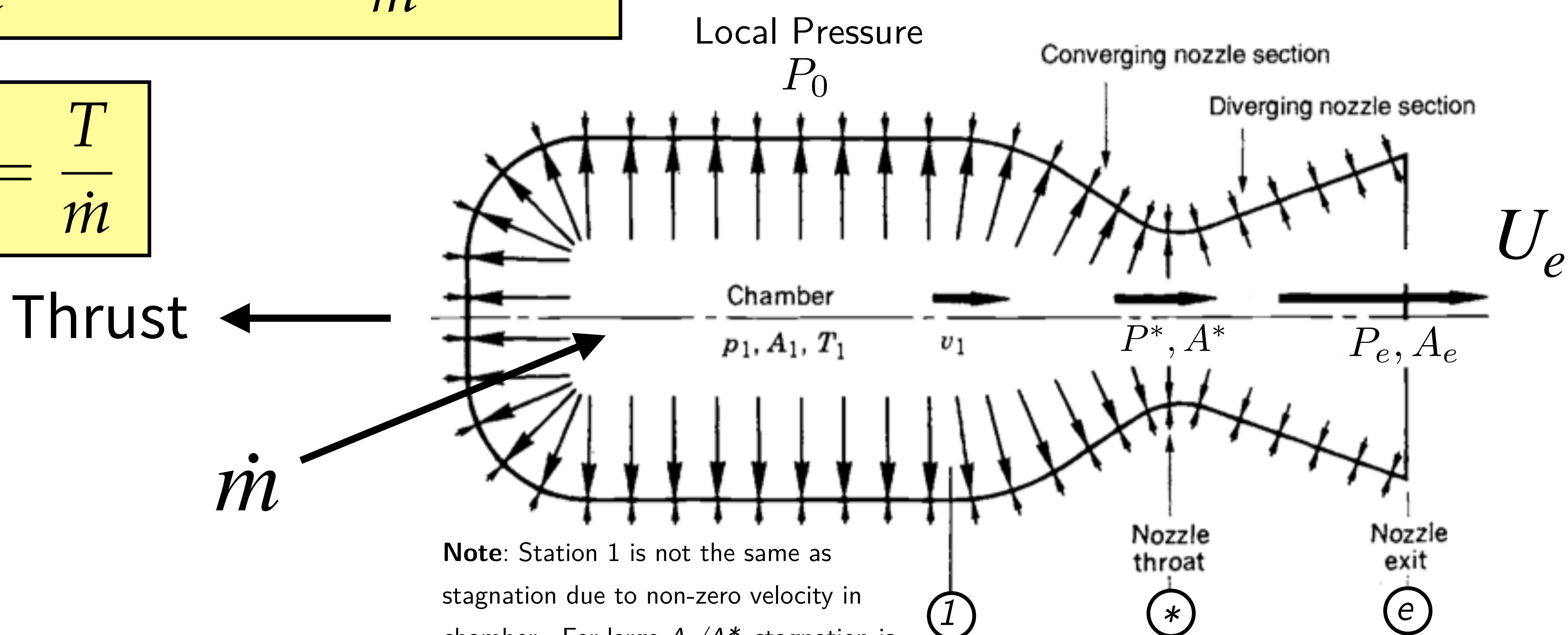
Effective Exhaust Velocity

$$\frac{T}{\dot{m}} = (P_e - P_0)\frac{A_e}{\dot{m}} + U_e$$

$$C = \frac{T}{\dot{m}}$$

Specific Impulse

$$I_{sp} = \frac{C}{g_0} = \frac{T}{g_0 \dot{m}}$$



Note: Station 1 is not the same as stagnation due to non-zero velocity in chamber. For large A_1/A^*, stagnation is still a good approximation.

Any guesses as to what $C = \frac{T}{\dot{m}}$ depends on?

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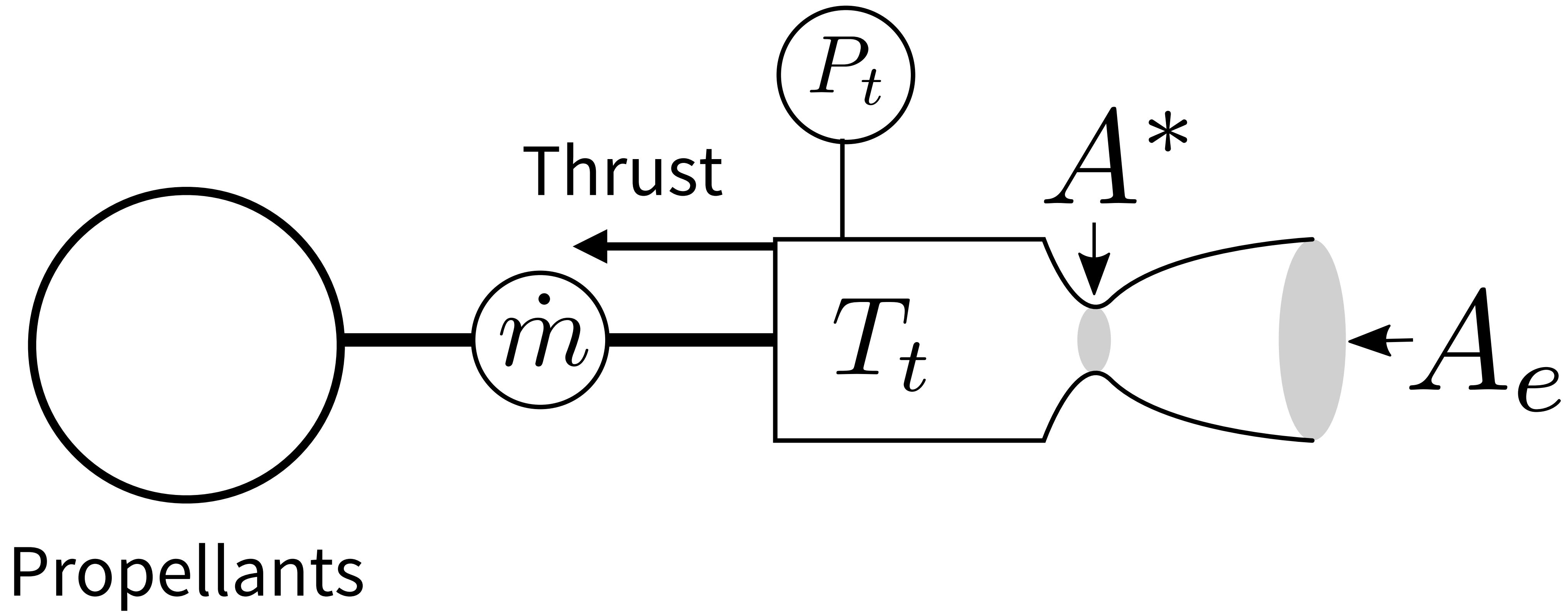
- Propellants
- Nozzle geometry
- Combustion mixing, completeness
- Fluid dynamics - friction, shocks, etc
- Heat transfer
- External environment - pressure
- …

Need some practical tools to measure and evaluate performance

A rocket test



What parameters are practically observable to us?

c* ("c star")

Choked mass flow rate

$$\dot{m} = \rho^* a^* A^* = \rho_t \left[\frac{\rho^*}{\rho_t}\right] a_t \left[\frac{a^*}{a_t}\right] A^* = \frac{P_t A^*}{\sqrt{\frac{R T_t}{\gamma}}} \left[\frac{\gamma+1}{2}\right]^{\frac{2(\gamma-1)}{\gamma+1}}$$

$$c^* = \overbrace{\frac{P_t A^*}{\dot{m}}}^{\text{Measure in the Lab}} = \overbrace{\sqrt{\frac{R T_t}{\gamma}} \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}}}^{\text{Compute from properties \& chemistry}}$$

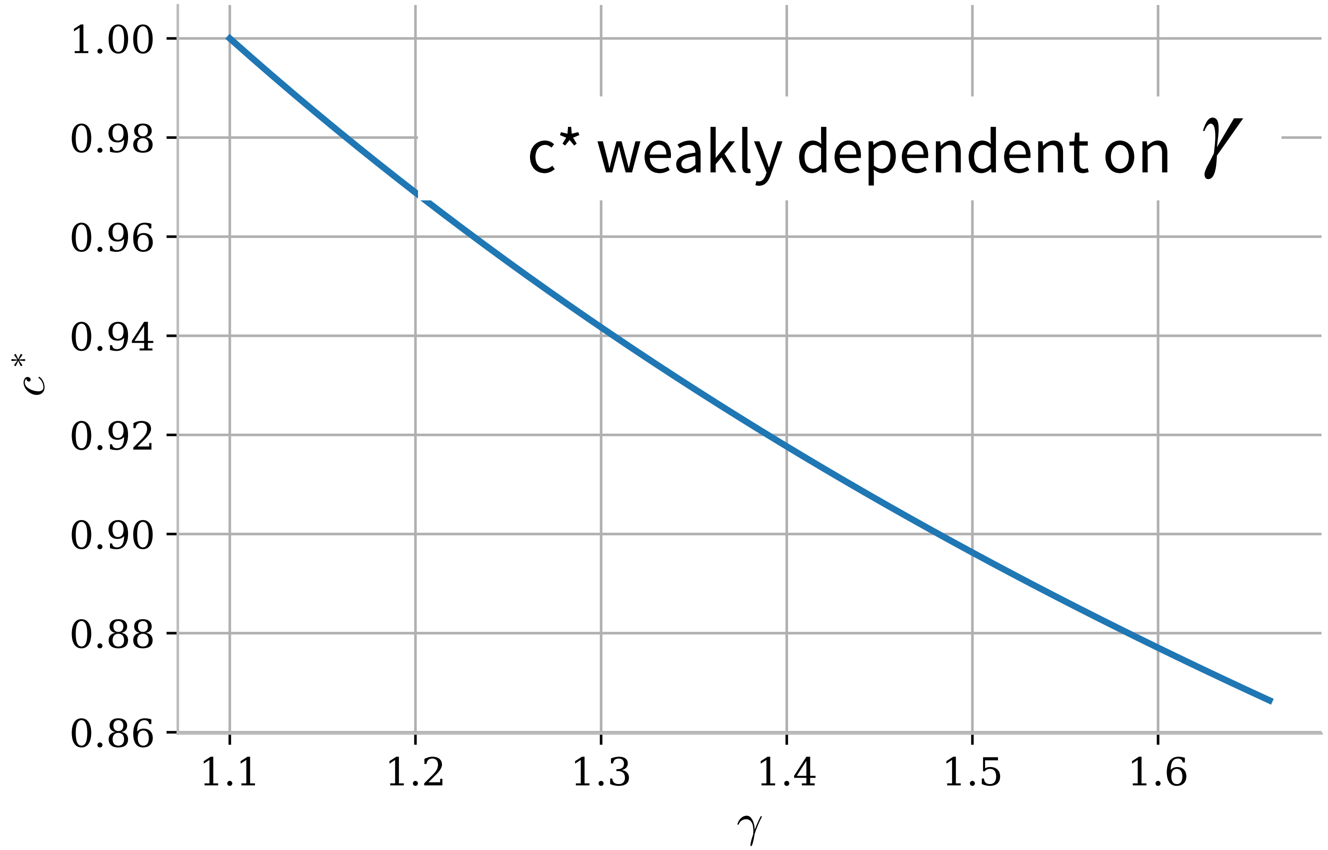
c* depends on…

$$c^* = f(T_t, R, \gamma)$$

$$R = \frac{R_u}{M_w}$$

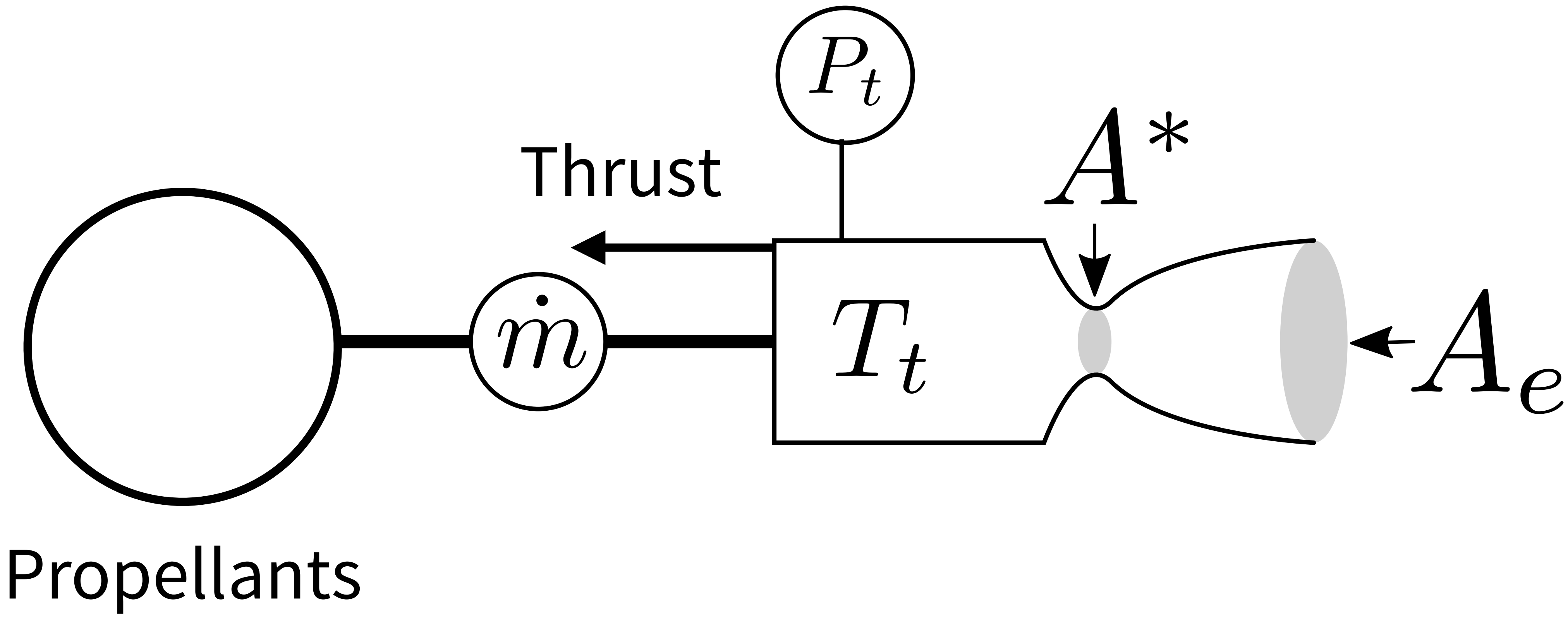
$$c^* = \sqrt{\frac{R_u T_t}{M_w} f(\gamma)}$$

$$\Delta h_t = \int_{T_0}^{T_t} C_p dT$$



$$c^* \propto \sqrt{\frac{T_t}{M_w}} \sim \sqrt{\frac{\Delta h_t}{C_p M_w}}$$

A rocket test



$$\eta_{c^*} = \frac{c^*_{measured}}{c^*_{ideal}} = \frac{P_t A^*}{\dot{m}_p c^*_{ideal}}$$

c* and c* efficiency are very useful parameters for static characterization

Thrust Coefficient, C_f

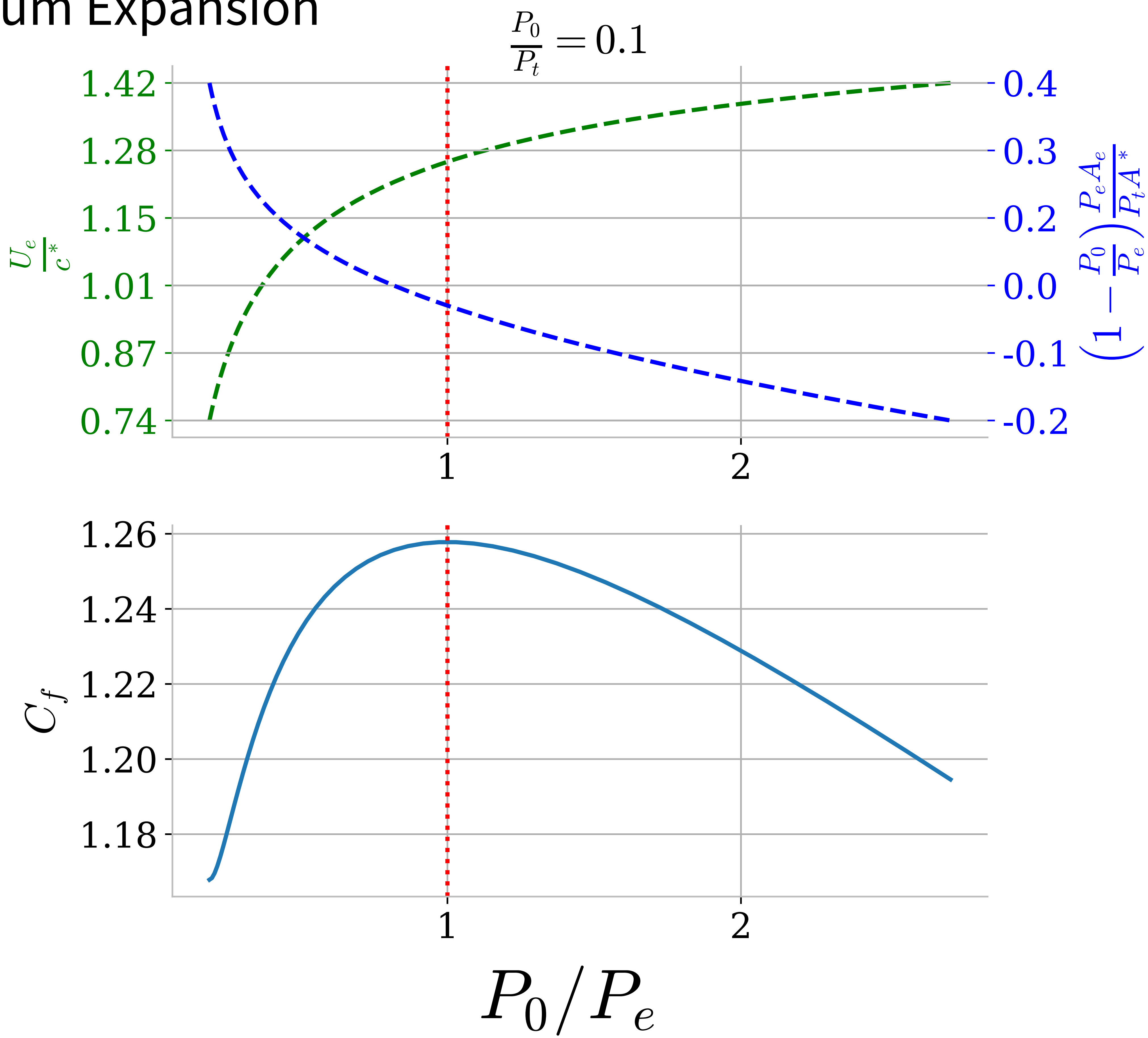
$$C_f = \frac{C}{c^*} = \frac{(P_e - P_0)\frac{A_e}{\dot{m}} + U_e}{\frac{P_t A^*}{\dot{m}}} = \frac{P_e - P_0}{P_t}\frac{A_e}{A^*} + \frac{U_e}{c^*}$$

$$\Rightarrow \underbrace{\left(1 - \frac{P_0}{P_e}\right)\frac{P_e A_e}{P_t A^*}}_{\text{Exhaust Pressure}} + \underbrace{\frac{U_e}{c^*}}_{\text{Exhaust Momentum}}$$

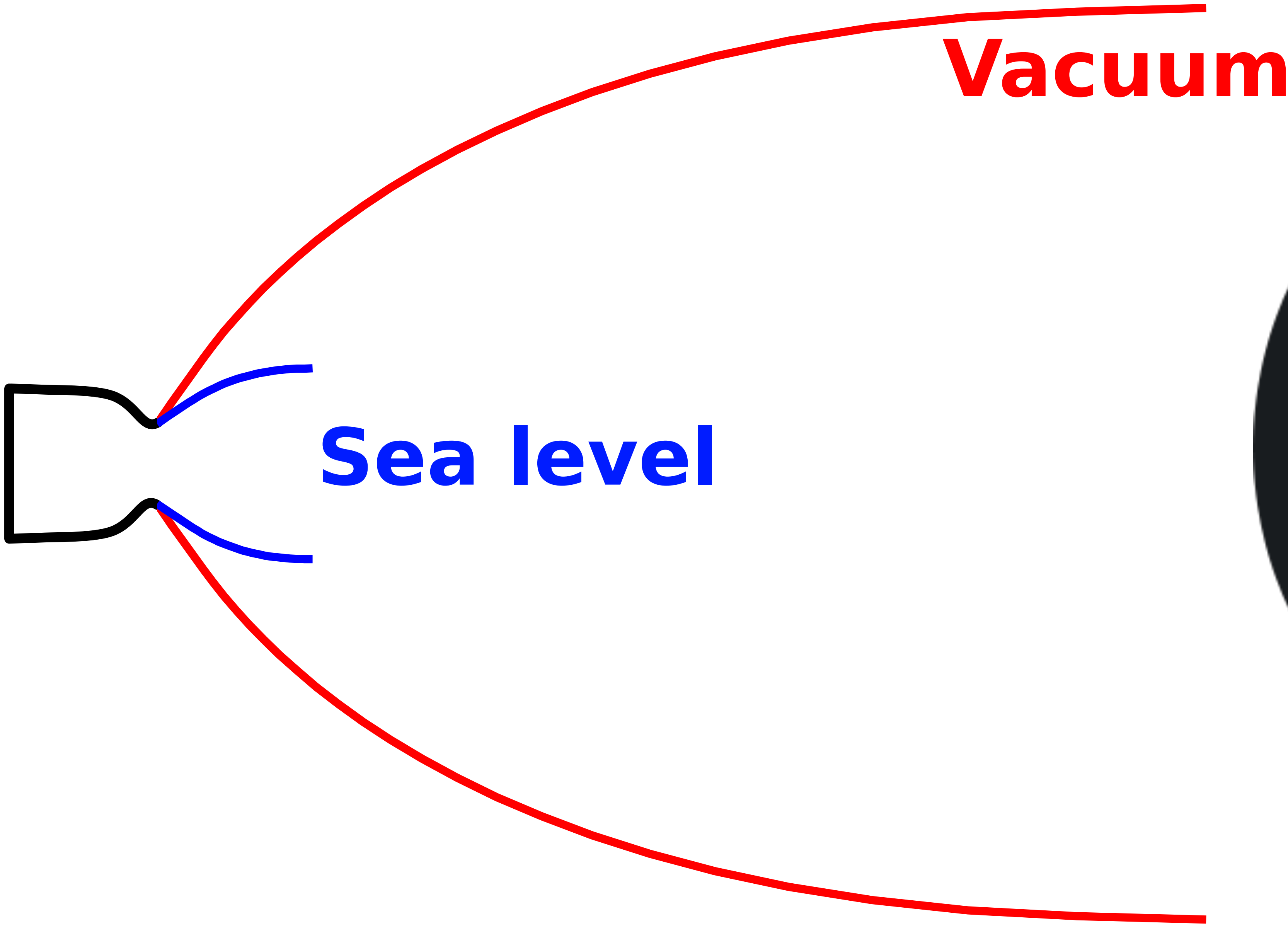
Thrust Coefficient, C_f

$$C_f = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{-2(\gamma-1)}}}{M_e\sqrt{1+\frac{\gamma-1}{2}M_e^2}}\left[\underbrace{\gamma M_e^2 + 1}_{\text{Exhaust Momentum}} \underbrace{- \frac{P_0}{P_e}}_{\text{Exhaust Pressure}}\right]$$

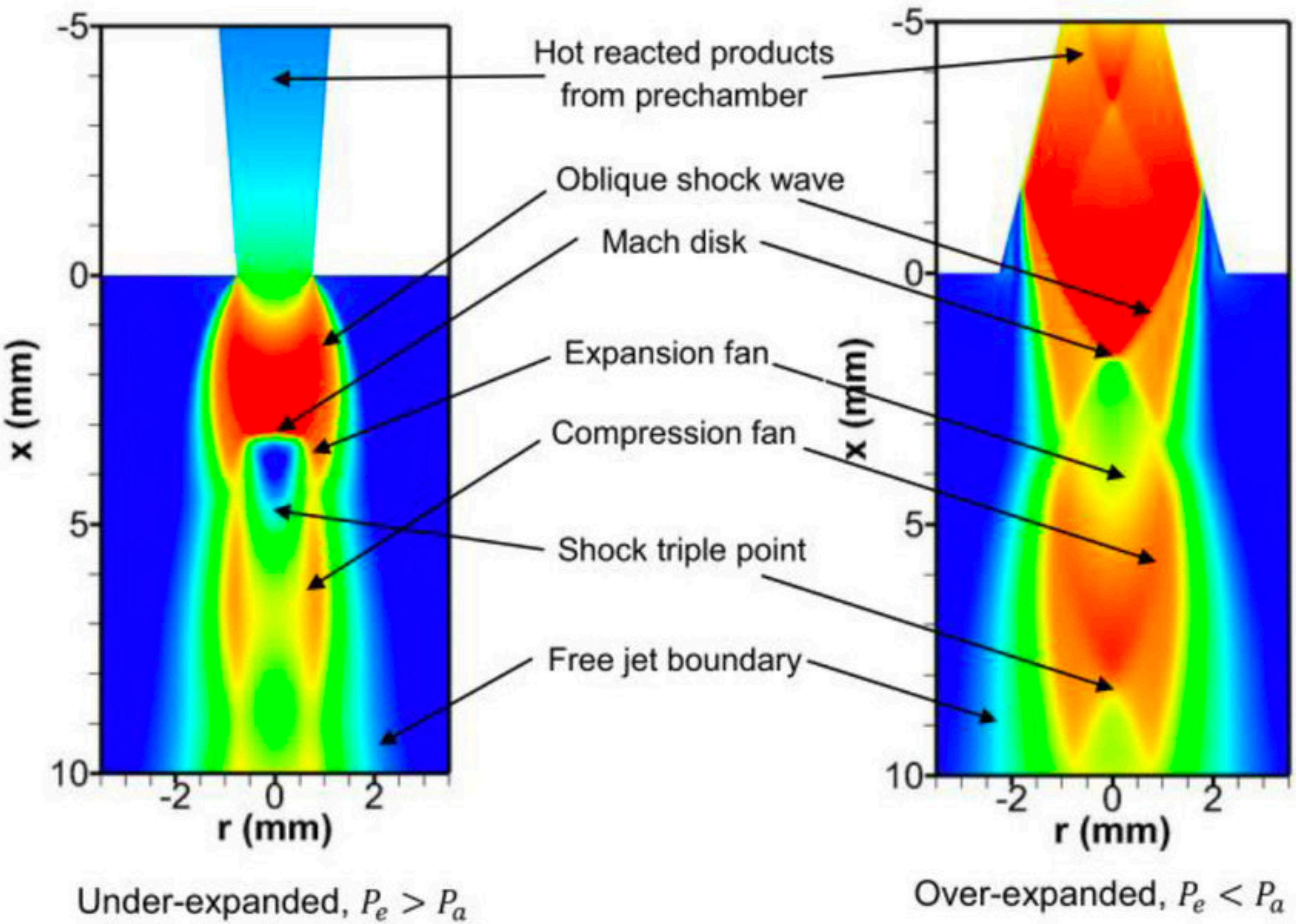
Optimum Expansion



Pressure Effect on Nozzles

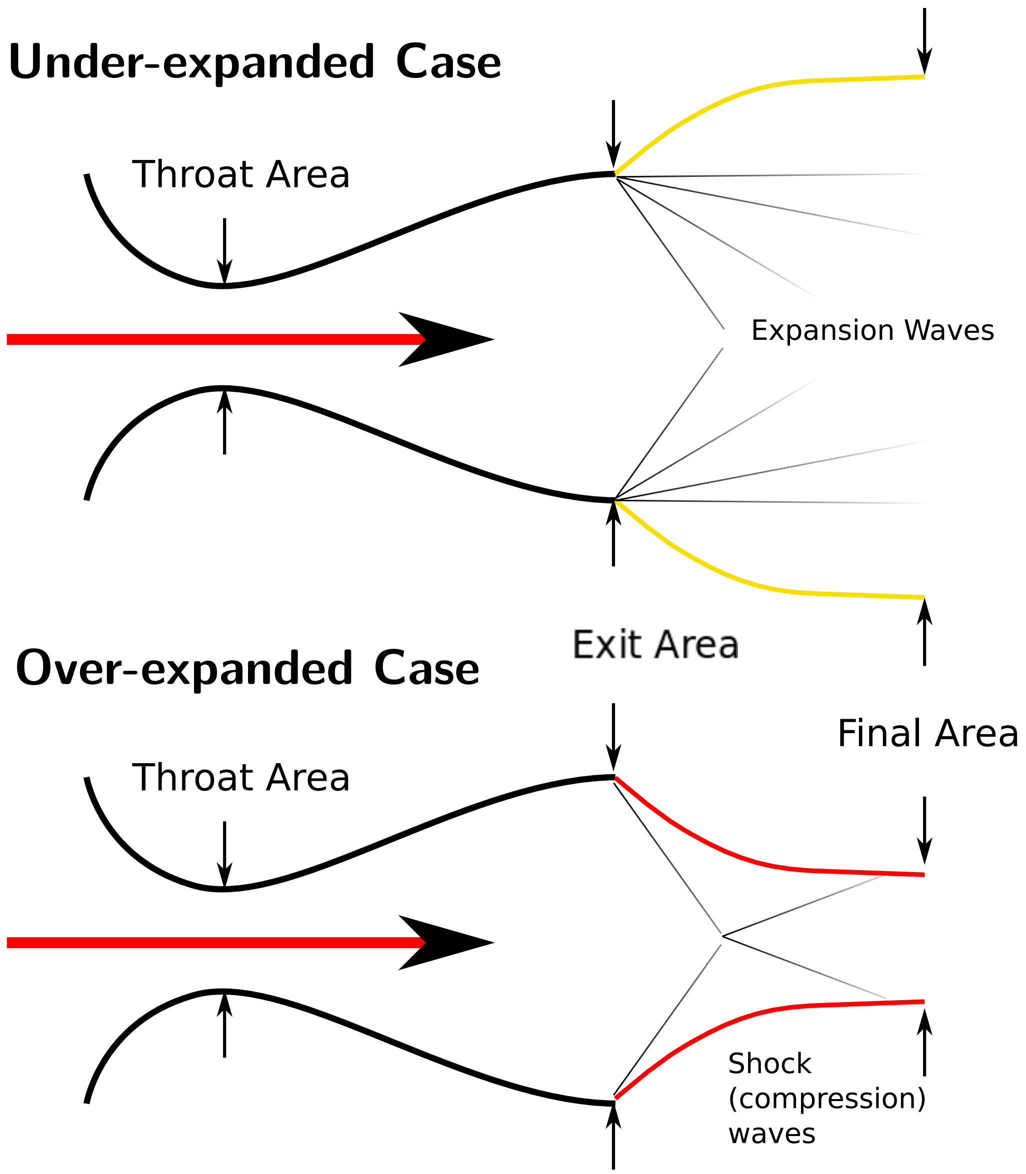


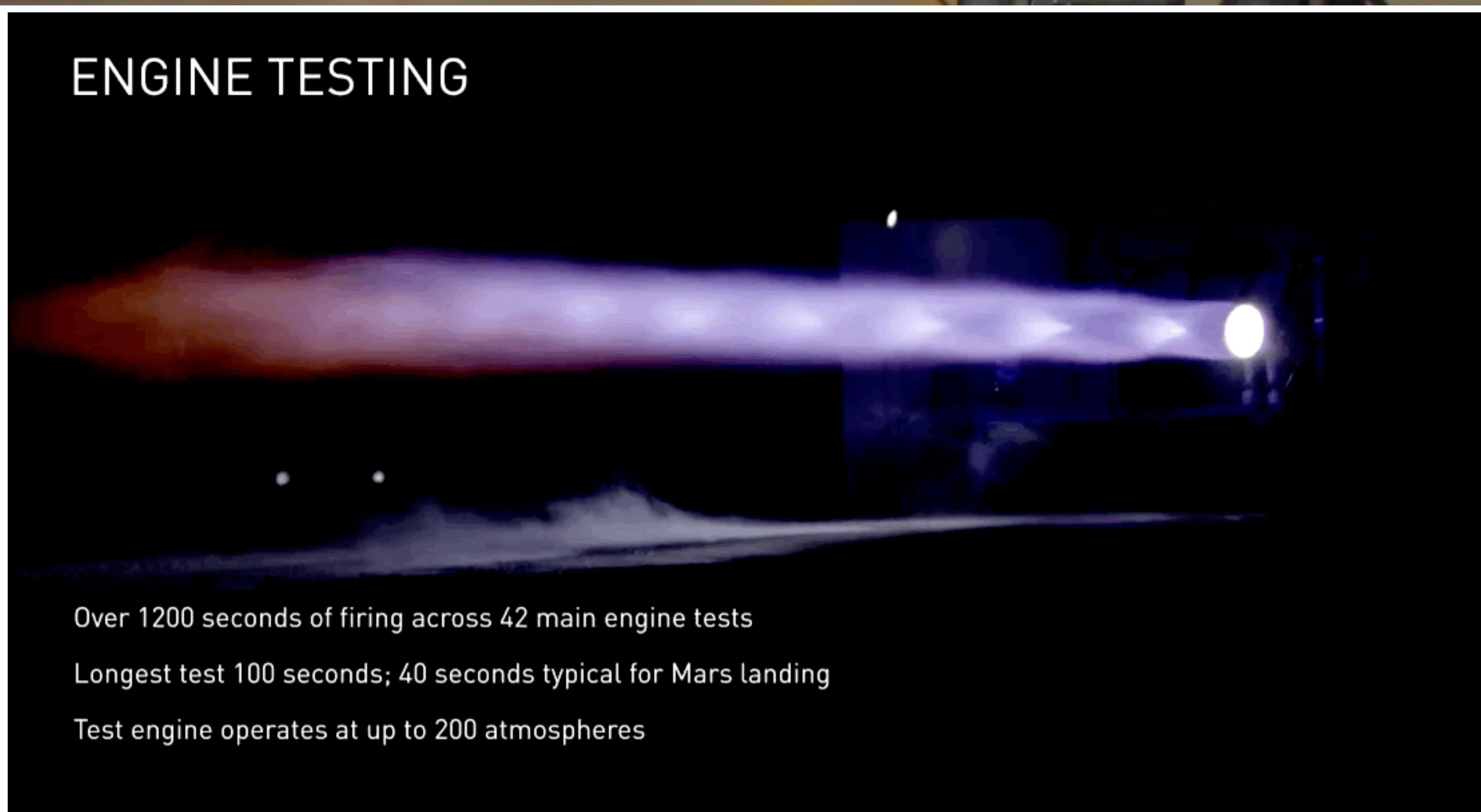
Under- and Over- Expansion



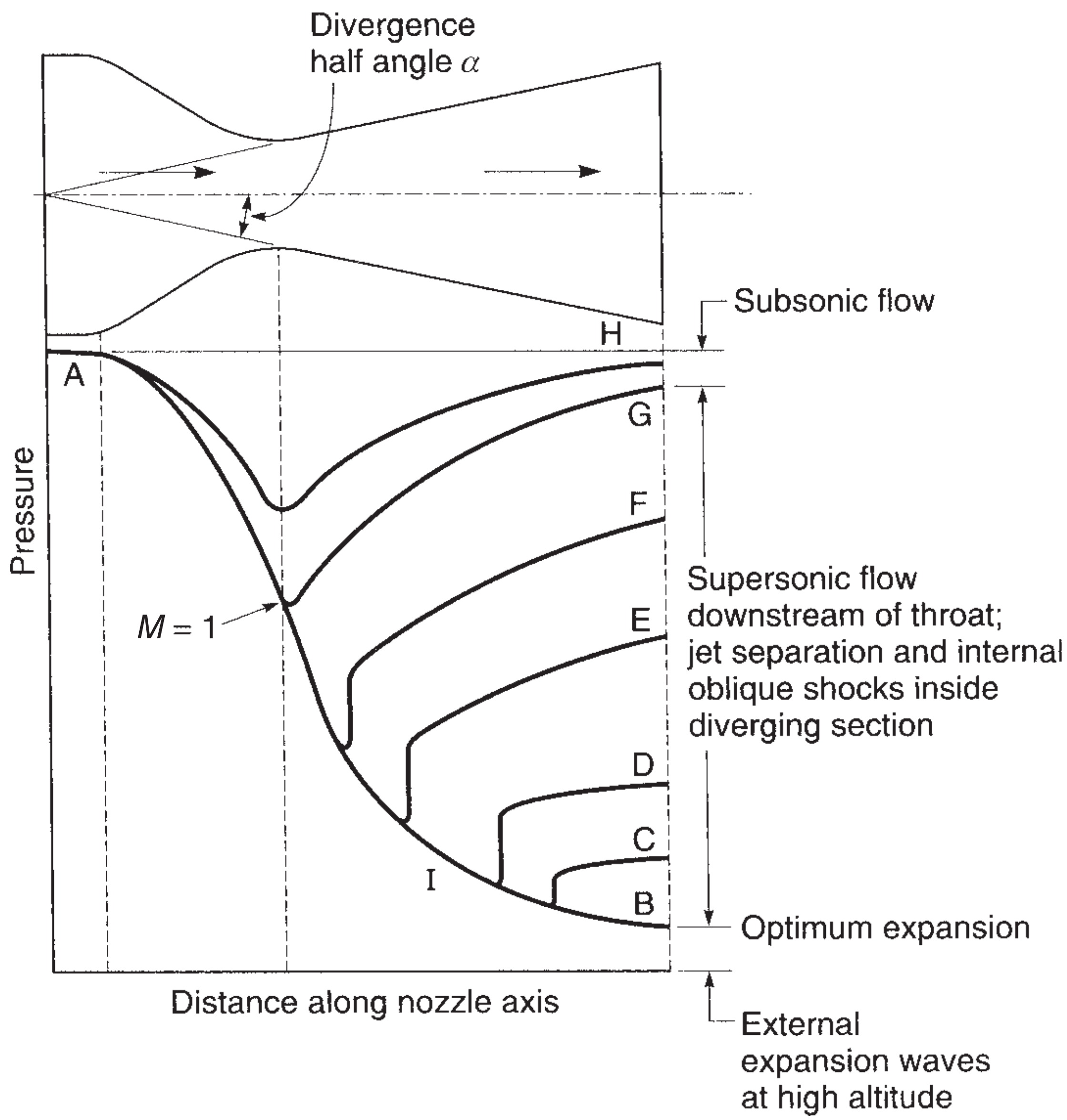
From https://www.researchgate.net/publication/320246686_A_Numerical_Investigation_of_Ignition_of_Ultra-Lean_Premixed_H_2_Air_Mixtures_by_Pre-Chamber_Supersonic_Hot_Jet

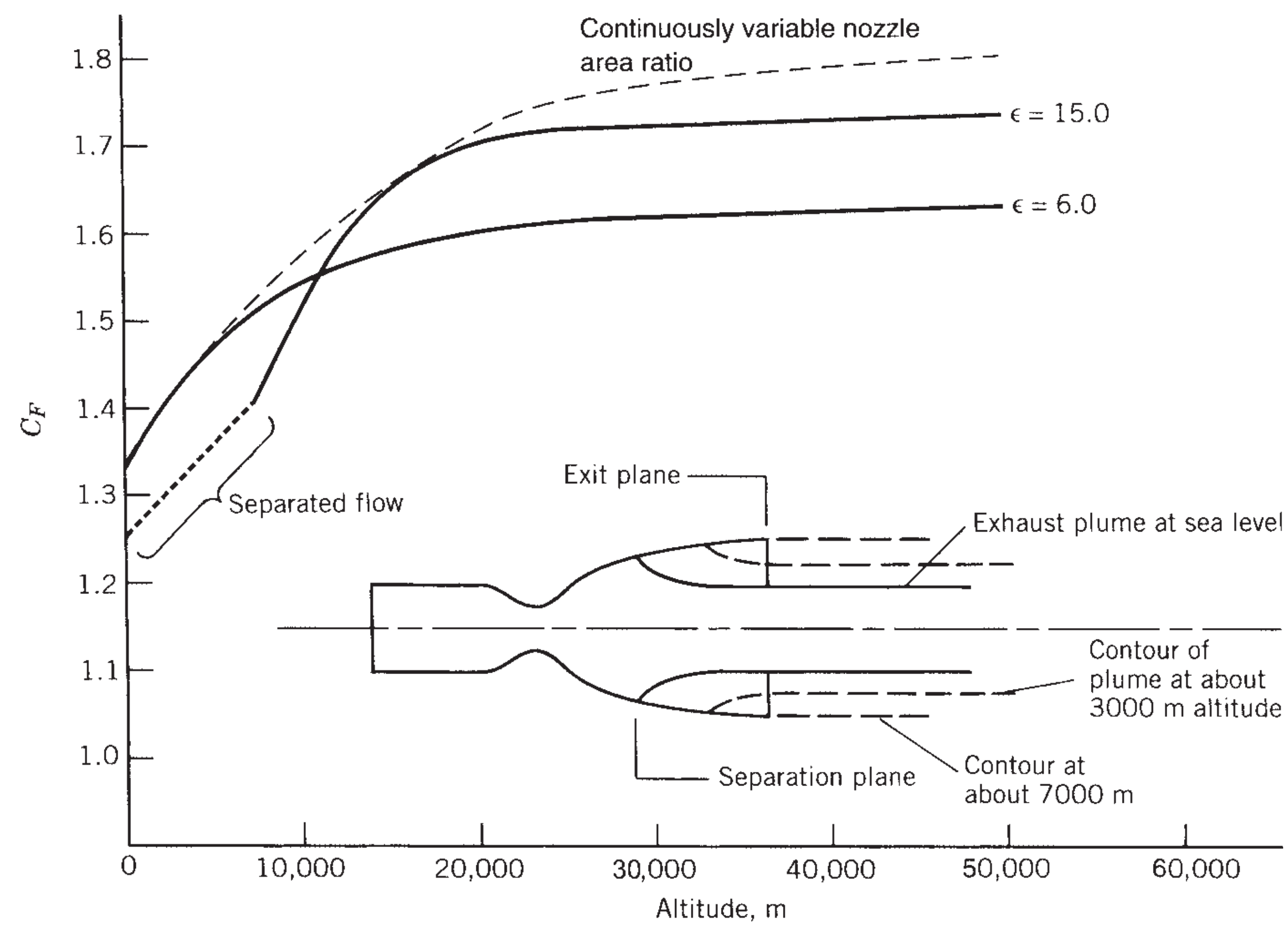
A real example



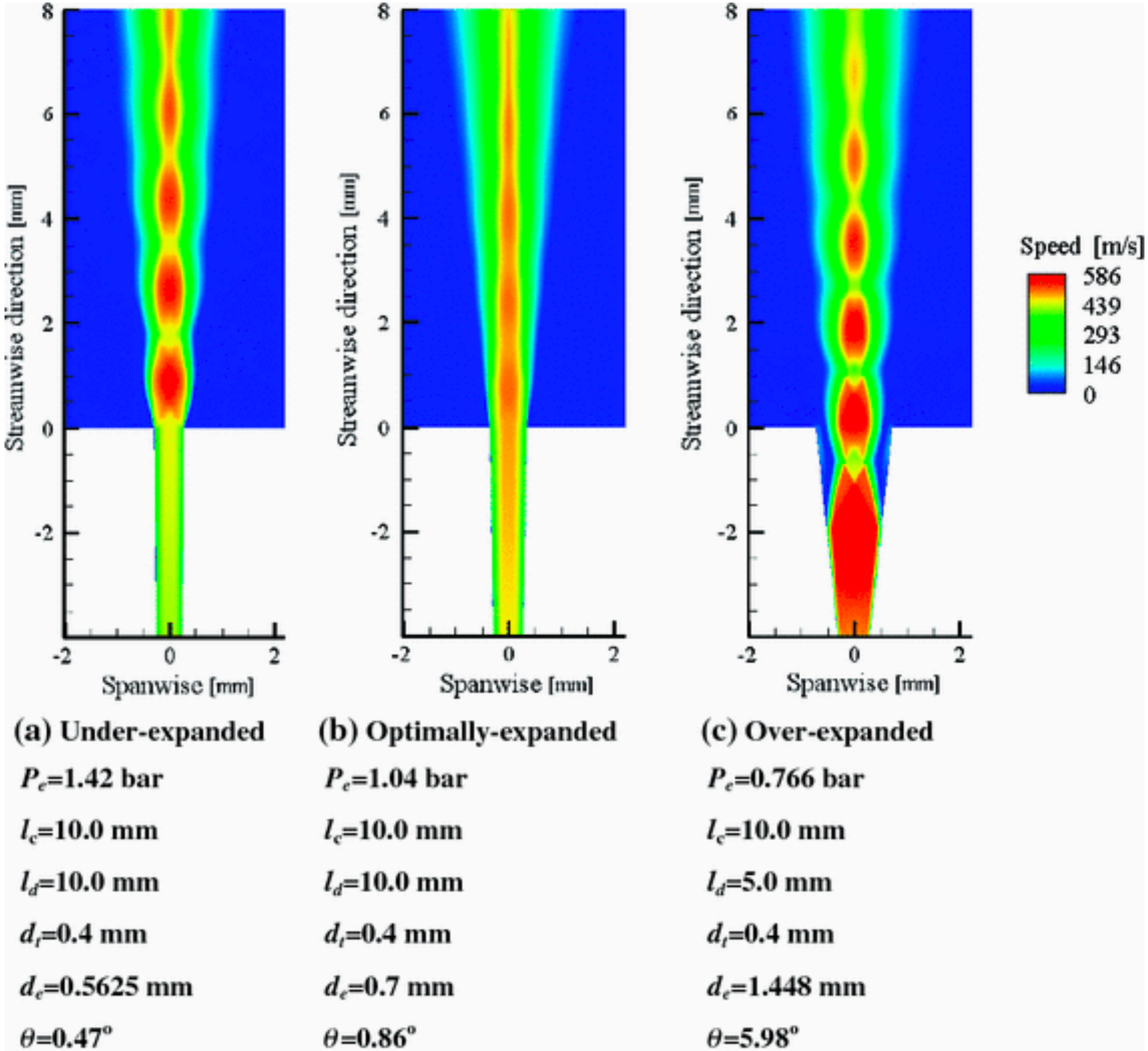

ENGINE TESTING
Over 1200 seconds of firing across 42 main engine tests
Longest test 100 seconds; 40 seconds typical for Mars landing
Test engine operates at up to 200 atmospheres

Nozzle flow separation





Source: Sutton, *Rocket Propulsion Elements*



Source: https://link.springer.com/article/10.1007/s11666-010-9542-8

Unsteady Separation Video

Thrust Coefficient from Sutton

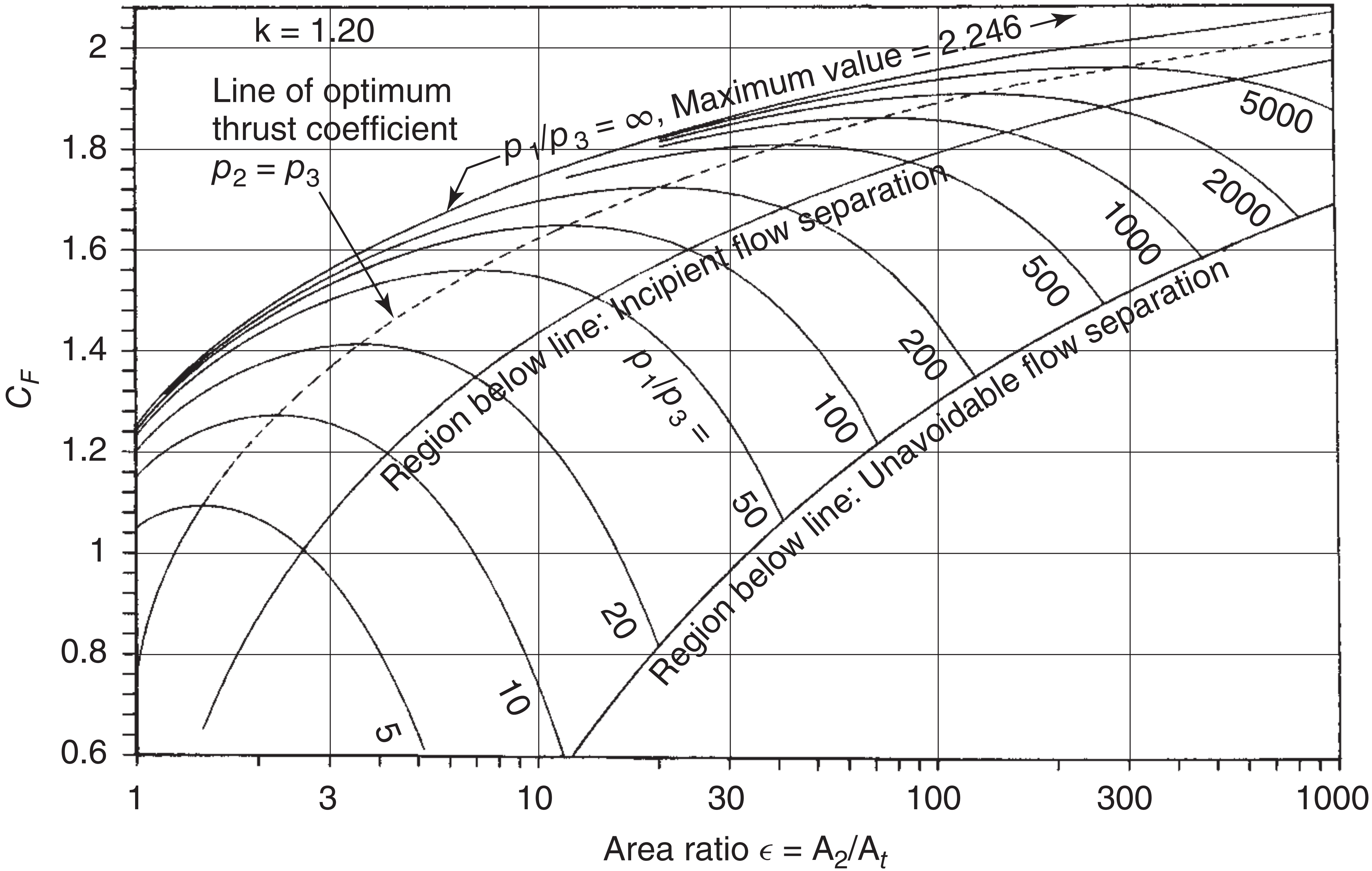


FIGURE 3–6. Thrust coefficient C_F versus nozzle area ratio for $k = 1.20$.

Source: Sutton, *Rocket Propulsion Elements*

Putting it all together

$$C = \frac{T}{\dot{m}} = \frac{T}{P_t A^*} \frac{P_t A^*}{\dot{m}}$$

$$\boxed{\Rightarrow C = \frac{T}{\dot{m}} = C_f c^*}$$

- c* captures the effect of propellants and combustion on rocket performance
- Cf captures the effect of nozzle geometry and pressures on performance

$$\boxed{C = \sqrt{\frac{2\gamma}{\gamma - 1} \frac{R_u T_t}{M_w} \left[1 - \left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right]}}$$

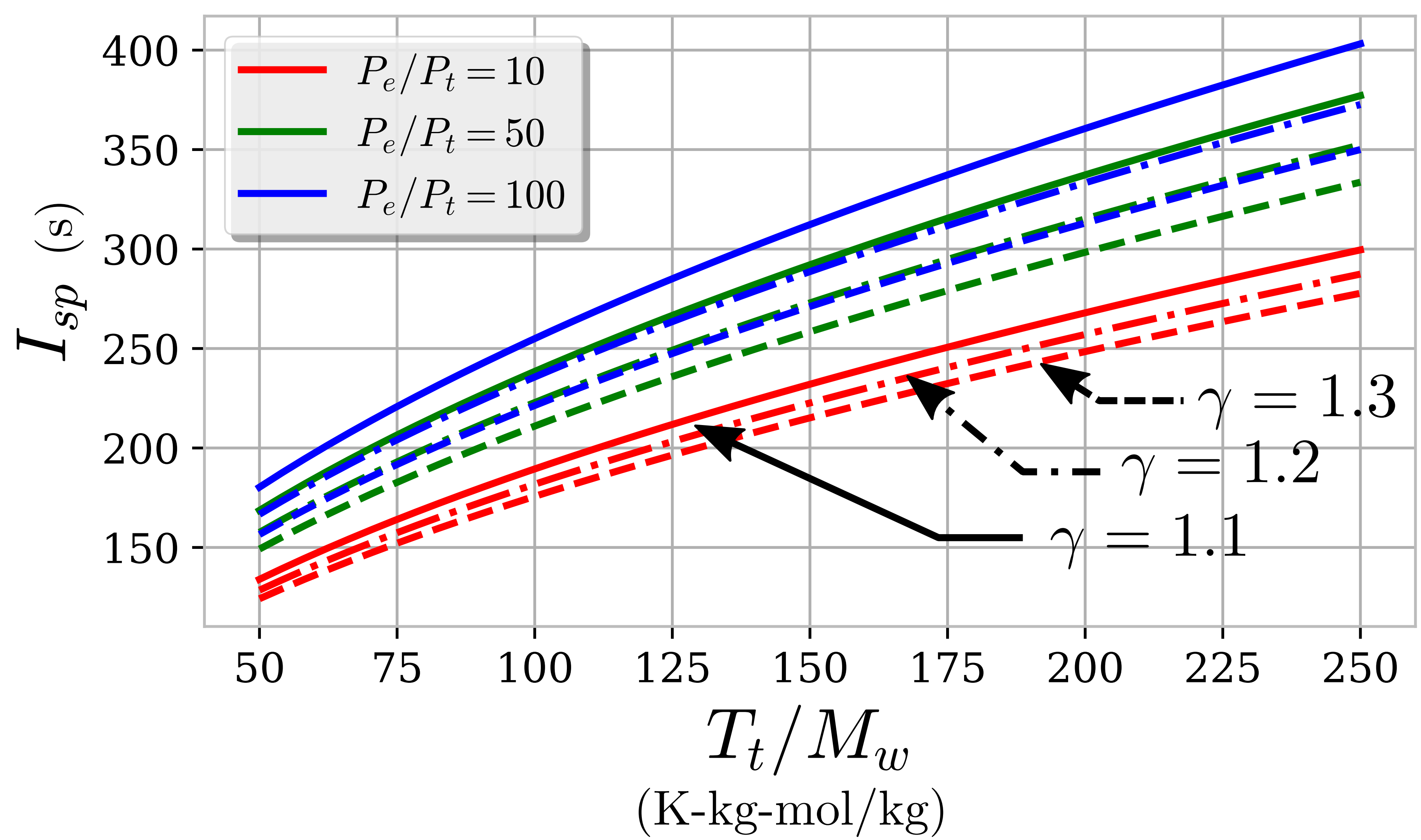
Cold Gas Performance

Name	γ	M_w	c^* (m/s)	C_f	C (m/s)
N_2	1.40	28	434.4	1.26	546.4
H_2	1.41	2	1621.5	1.26	2039.2
He	1.66	4	1085.2	1.25	1356.2

Table 0.2: c^*, C_f, C for several pure gasses at room temperature, $P_0/P_e = 1$ and $P_0/P_t = 0.1$. Note how C_f is essentially constant despite the differences in gas properties while c^* and C vary substantially.

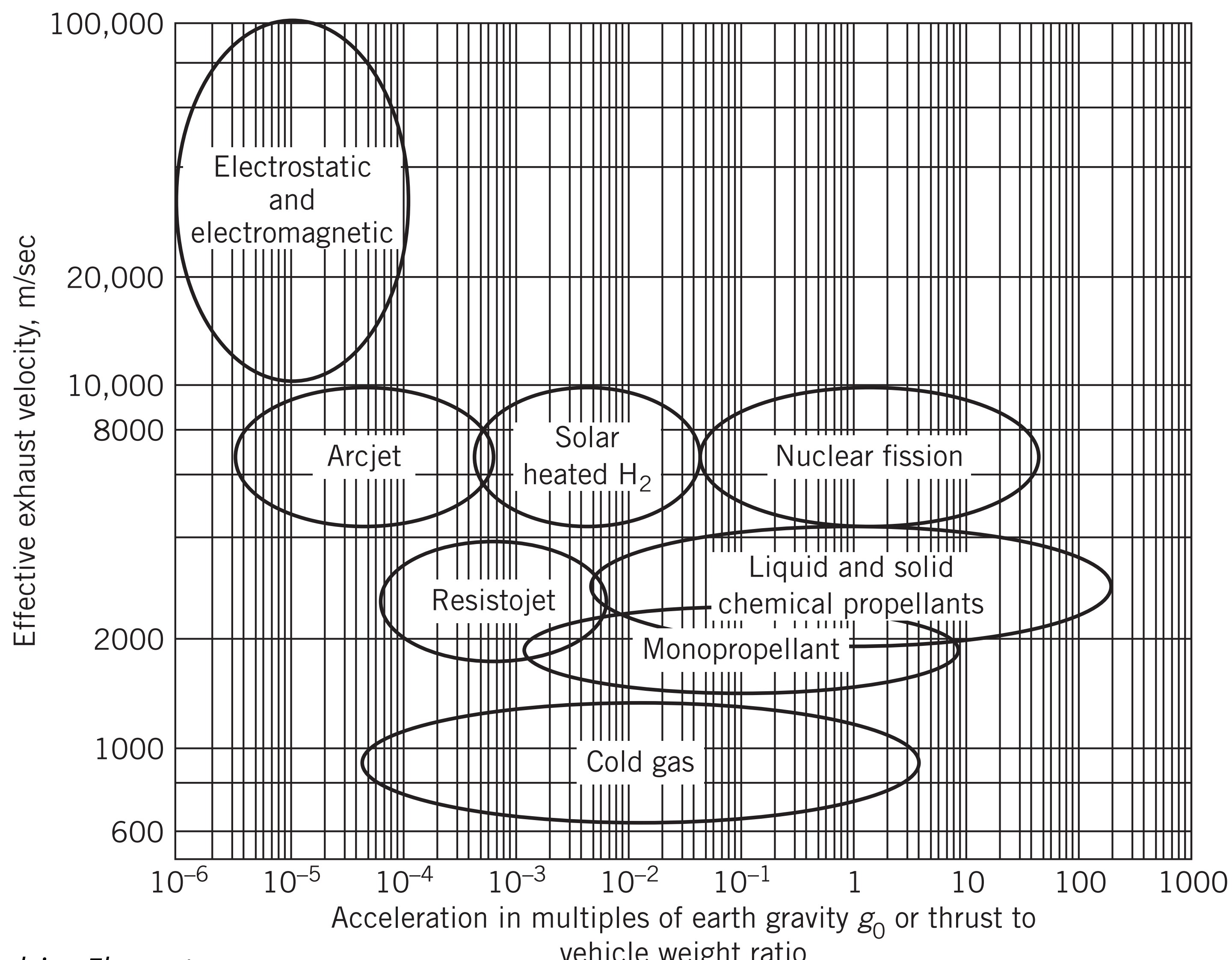
Effective Exhaust Velocity & Isp

$$C = (P_e - P_0)\frac{A_e}{\dot{m}} + \sqrt{\frac{2\gamma}{\gamma - 1}\frac{R_u T_t}{M_w}\left[1 - \left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right]}$$



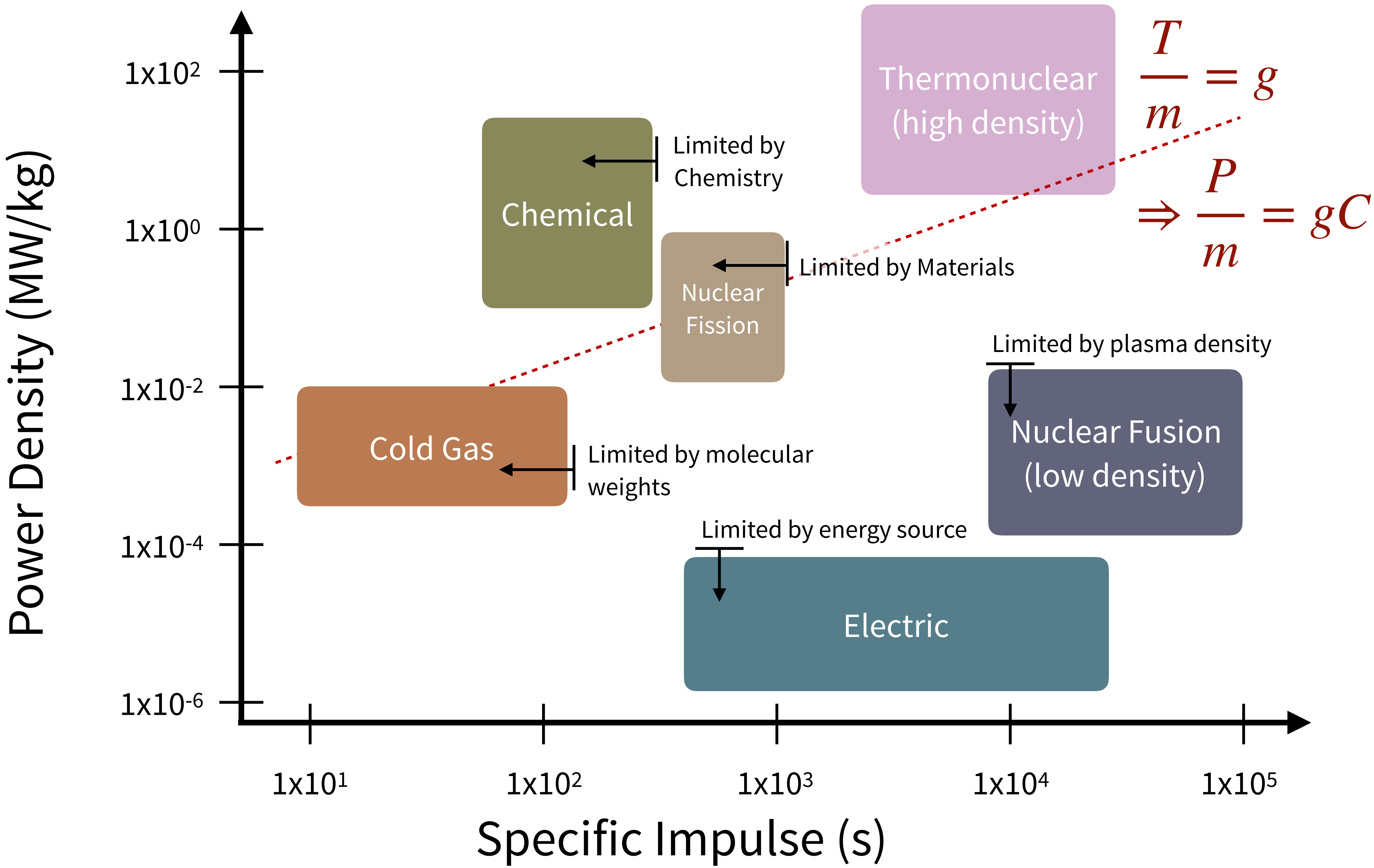
Performance by Technology

Technology	Isp (s)	Exhaust Velocity (m/s)
Nuclear Fusion	10,000 - 50,000+	98,000 - 490,000+
Electric Propulsion	1,000 - 10,000	9,800 - 98,000
Nuclear Thermal (fission)	600 - 1,000	5,900 - 9,800
Beamed Thermal (microwave / laser)	600 - 1,000	5,900 - 9,800
Bipropellant Chemical Propulsion	200 - 500	2,000 - 4,900
Monopropellant Chemical Propulsion	100 - 250	980 - 2,450
Cold Gas Propulsion	10 - 120	100 - 1,150



Source: Sutton, *Rocket Propulsion Elements*

Propulsion Energy & Power Density



Rocket Efficiency

$$\eta_{thermal} = \frac{\dot{W}_{exhaust}}{P_{in}} = \frac{\dot{m}C^2}{2P_{in}}$$

$$\eta_{thermal} = \frac{\gamma}{\gamma - 1}\frac{R}{\overline{\overline{C_p}}}\left[1 - \left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right]$$

$$\eta_{thermal} \approx 1 - \frac{T_e}{T_t}$$

Carnot Efficiency!

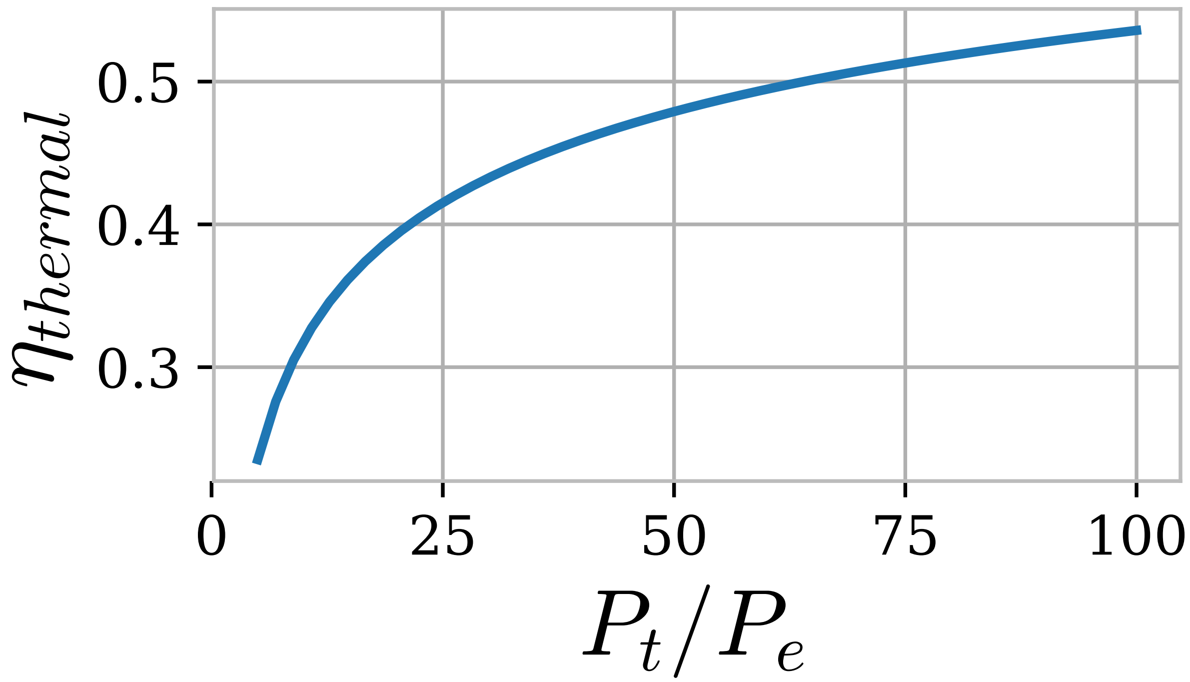
Propulsive Efficiency

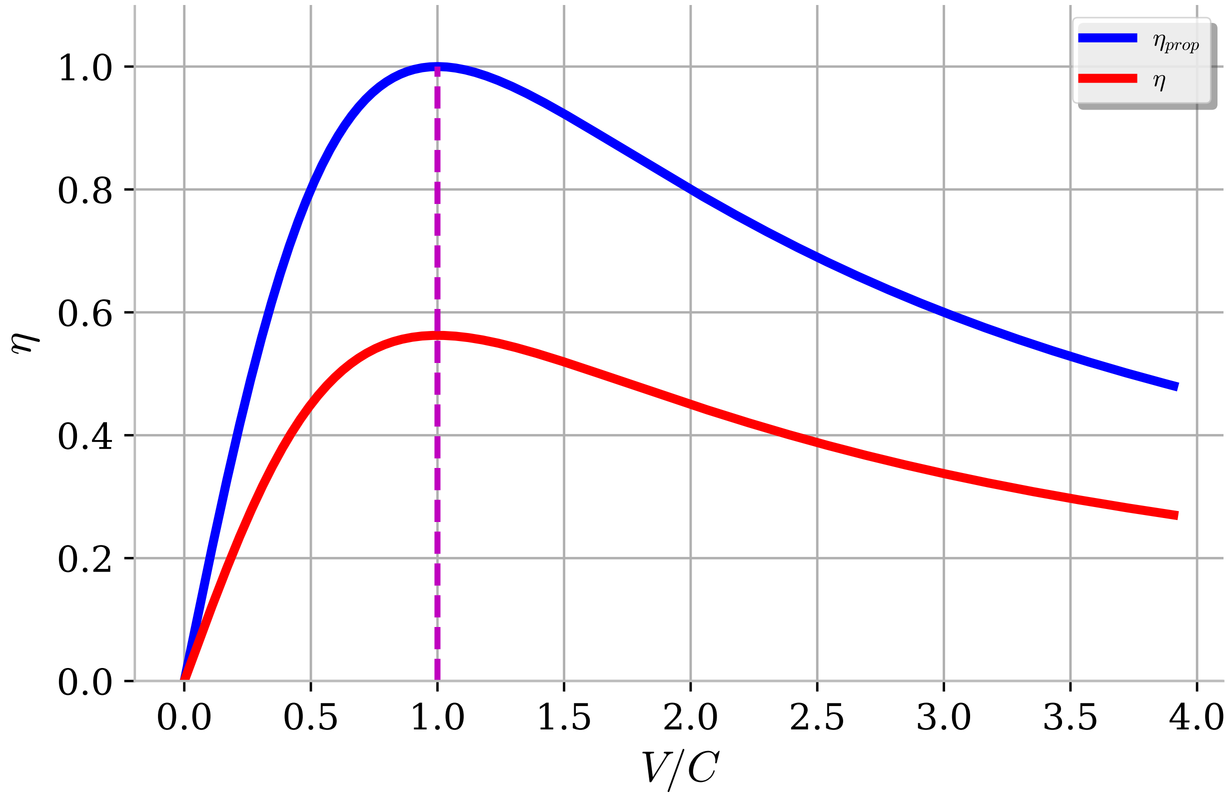
$$\eta_{prop} = \frac{\dot{W}_r}{\dot{W}_r + \dot{W}_{exhaust}} = \frac{Tv}{Tv + \frac{1}{2}\dot{m}(C - v)^2}$$

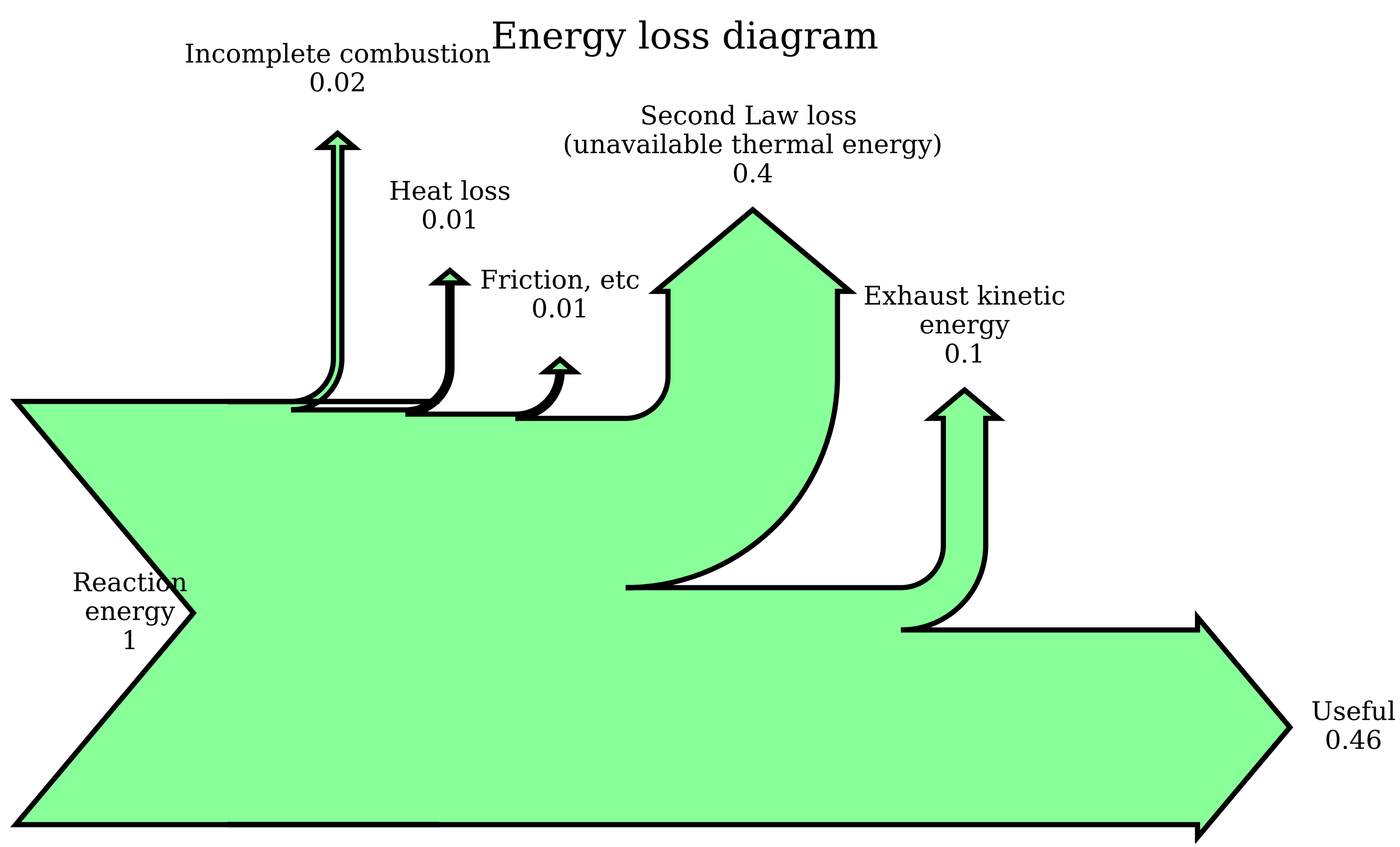
$$\frac{Tv}{Tv + \frac{1}{2}\dot{m}(C - v)^2} = \frac{\dot{m}Cv}{\dot{m}\left[Cv + (C - v)^2\right]}$$

$$\rightarrow \eta_{prop} = 2\frac{\frac{v}{C}}{1 + \left(\frac{v}{C}\right)^2}$$

Efficiency


0.5
0.4
0.3
$\eta_{thermal}$
0
25
50
75
100
P_t/P_e


η_{prop}
η
1.0
0.8
0.6
0.4
0.2
0.0
η
0.0 0.5 1.0 1.5 2.0 2.5 3.0 3.5 4.0
V/C


Energy loss diagram
Incomplete combustion
0.02
Heat loss
0.01
Friction, etc
0.01
Second Law loss
(unavailable thermal energy)
0.4
Exhaust kinetic
energy
0.1
Reaction
energy
1
Useful
0.46

Three Balloons

What is ΔV?

1. Your air-filled balloon
   - ➡ $T_t = 298\text{ K} \quad M_w \approx 28\text{ g/mol}$
2. A $2H_2 + O_2$ balloon
   - ➡ $M_{wi} \approx 12\text{ g/mol} \qquad M_{wf} \approx 18\text{ g/mol}$

   $$\frac{Cp_{H2O}}{R_u} \approx 5 \qquad \Delta_f H_{H2O}(g) = 241.8\text{ kJ/mol}$$
3. A deuterium+tritium balloon
   - ➡ $M_{wi} \approx 2.5\text{ g/mol} \qquad M_{wf} \approx 4\text{ g/mol}$

   $$\frac{Cp_{^4He}}{R_u} \approx 2.5 \qquad \Delta H_{D+T} = 17.5\text{ MeV}$$
   $$\Delta H_{D+T} = 1{,}717\text{ GJ/mol}$$