

# AA103

# Rocket Performance Metrics

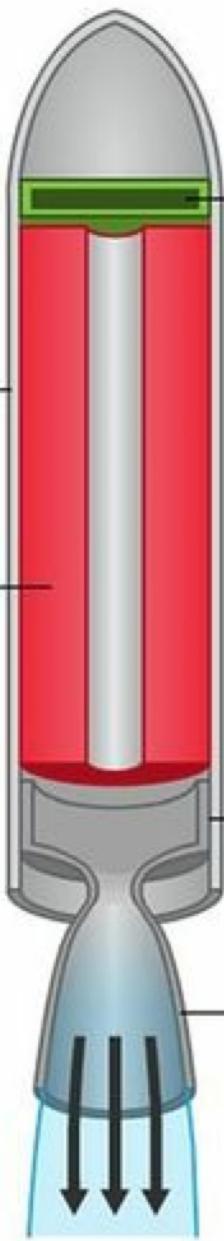
---

April 18, 2019

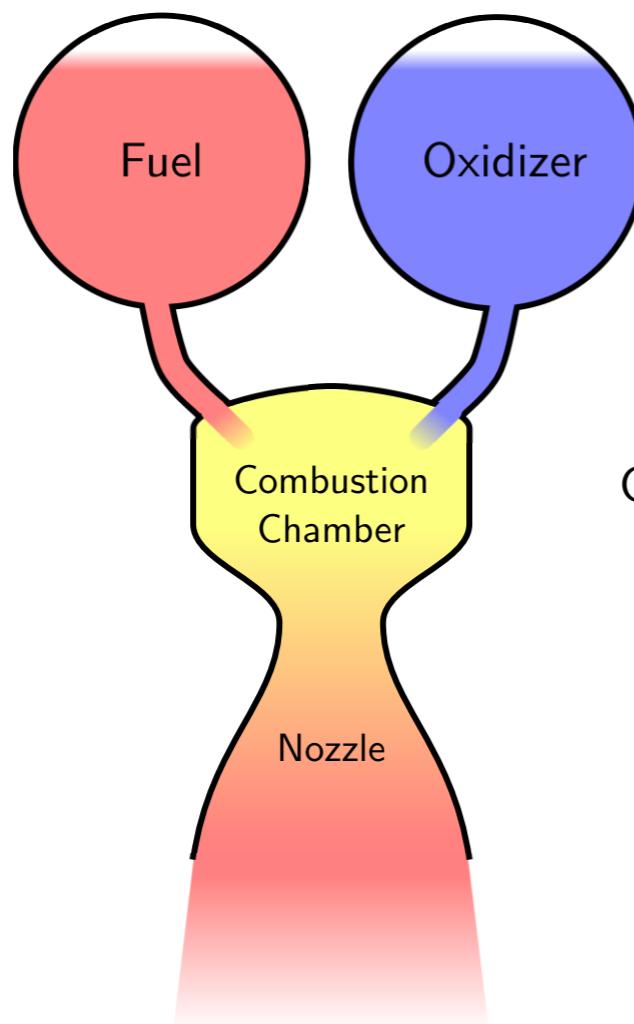
Jonny Dyer

# What is a rocket?

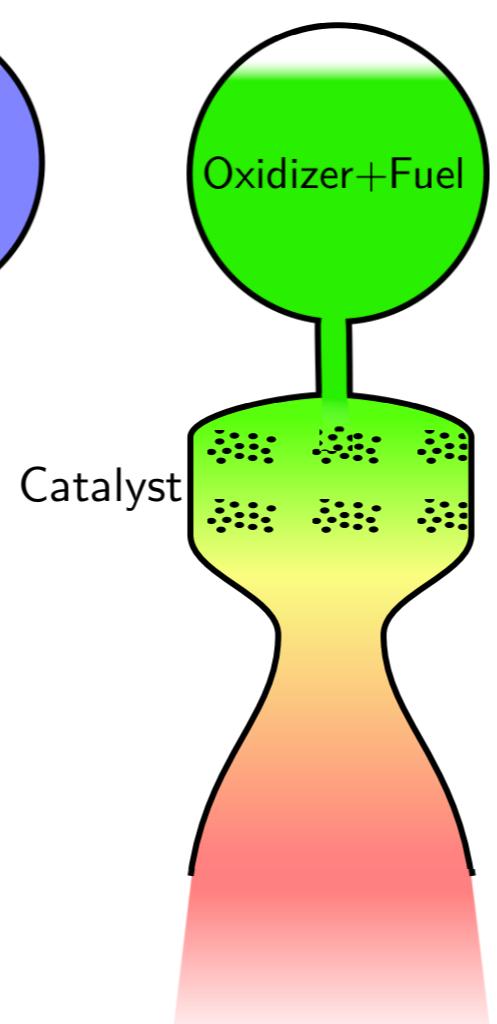
Solid



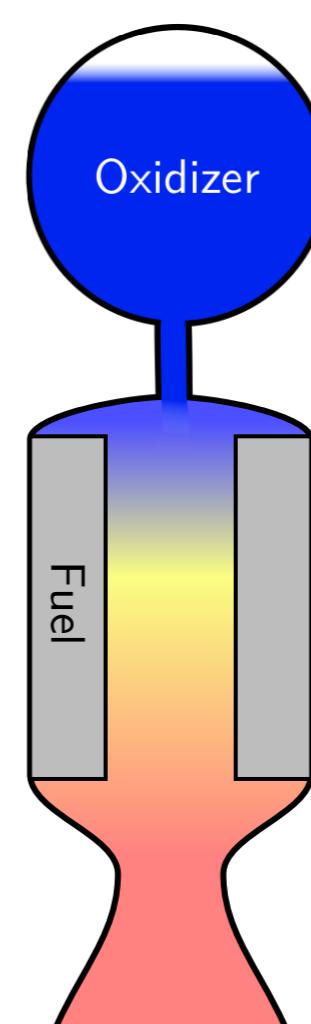
Bipropellant



Monopropellant

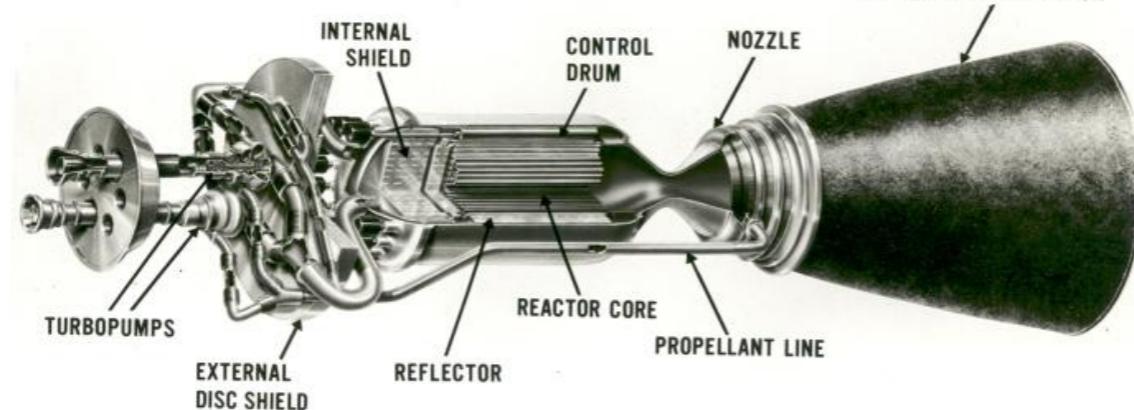
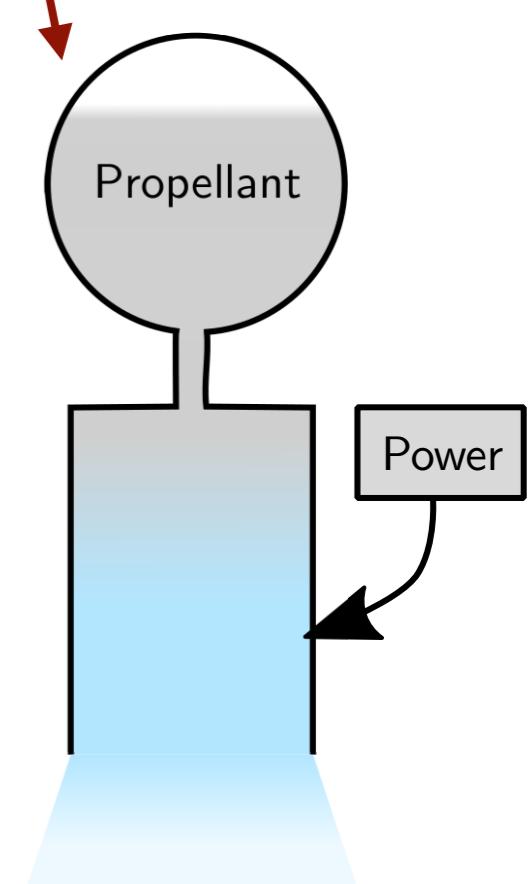


Hybrid



Not really a rocket

Electric



Nuclear  
Thermal

## Thrust Equation

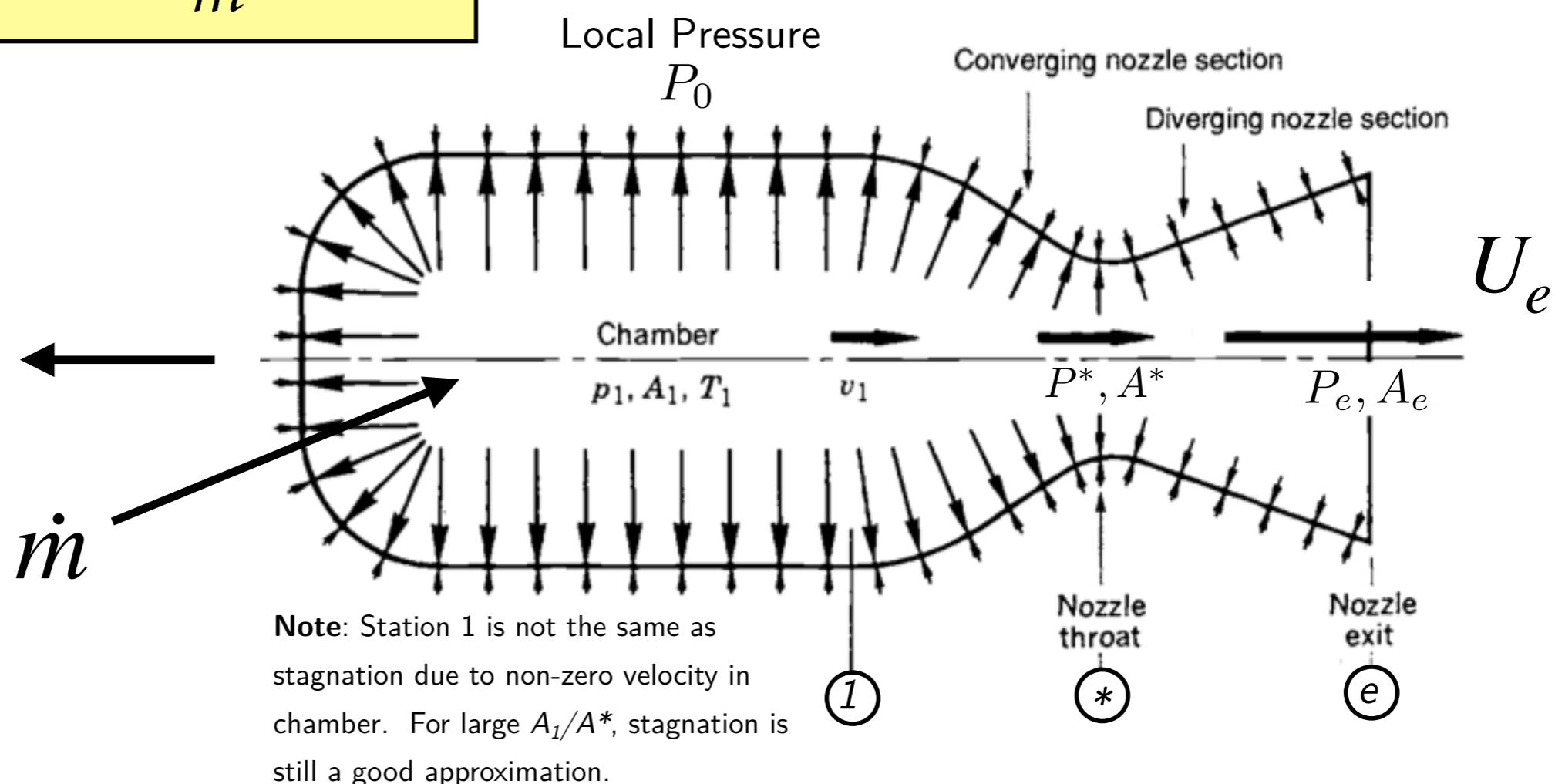
$$T = (P_e - P_0)A_e + \dot{m}_p U_e$$

## Effective Exhaust Velocity

$$\frac{T}{\dot{m}} = (P_e - P_0) \frac{A_e}{\dot{m}} + U_e$$

$$C = \frac{T}{\dot{m}}$$

Thrust  $\leftarrow$



## Rocket Equation

$$\Delta V = C \ln \frac{M_i}{M_f}$$

## Specific Impulse

$$I_{sp} = \frac{C}{g_0} = \frac{T}{g_0 \dot{m}}$$

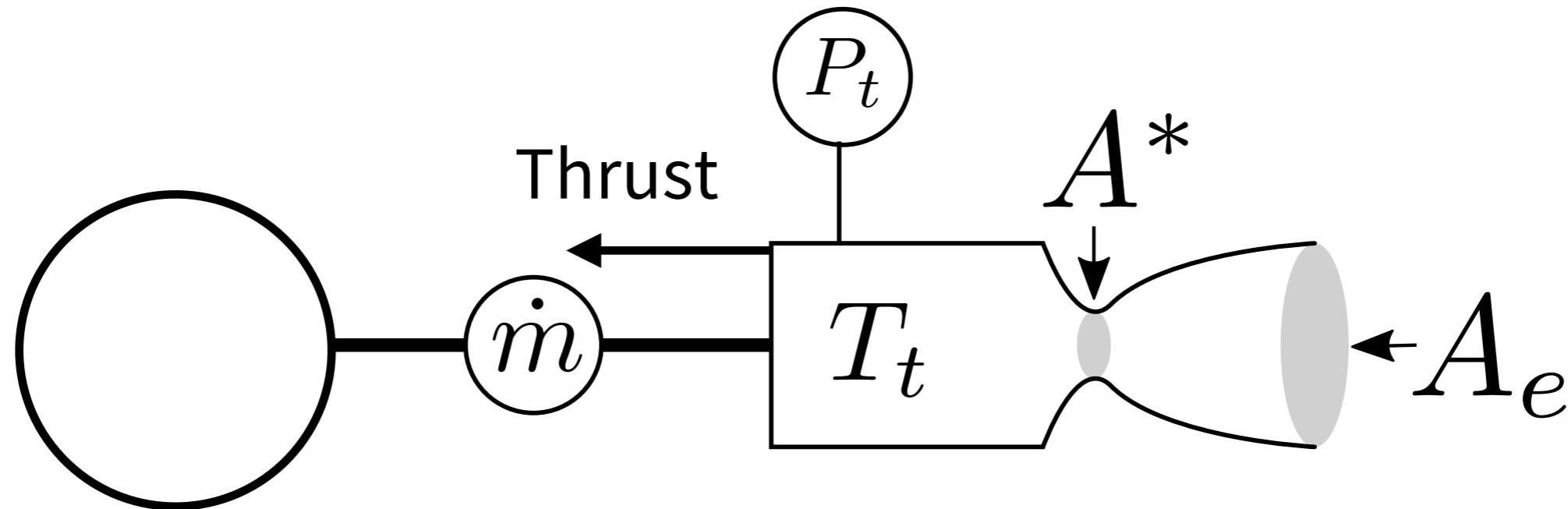
Any guesses as to what  $C = \frac{T}{\dot{m}}$  depends on?

Any guesses as to what  $C = \frac{T}{\dot{m}}$  depends on?

- Propellants
- Nozzle geometry
- Combustion mixing, completeness
- Fluid dynamics - friction, shocks, etc
- Heat transfer
- External environment - pressure
- ...

Need some practical tools to measure and evaluate performance

# A rocket test



Propellants

What parameters are practically observable  
to us?

# $c^*$ (“c star”)

Choked mass flow rate

$$\dot{m} = \rho^* a^* A^* = \rho_t \left[ \frac{\rho^*}{\rho_t} \right] a_t \left[ \frac{a^*}{a_t} \right] A^* = \frac{P_t A^*}{\sqrt{\frac{RT_t}{\gamma}}} \left[ \frac{\gamma + 1}{2} \right]^{\frac{2(\gamma - 1)}{\gamma + 1}}$$

Measure  
in the Lab

Compute from  
properties & chemistry

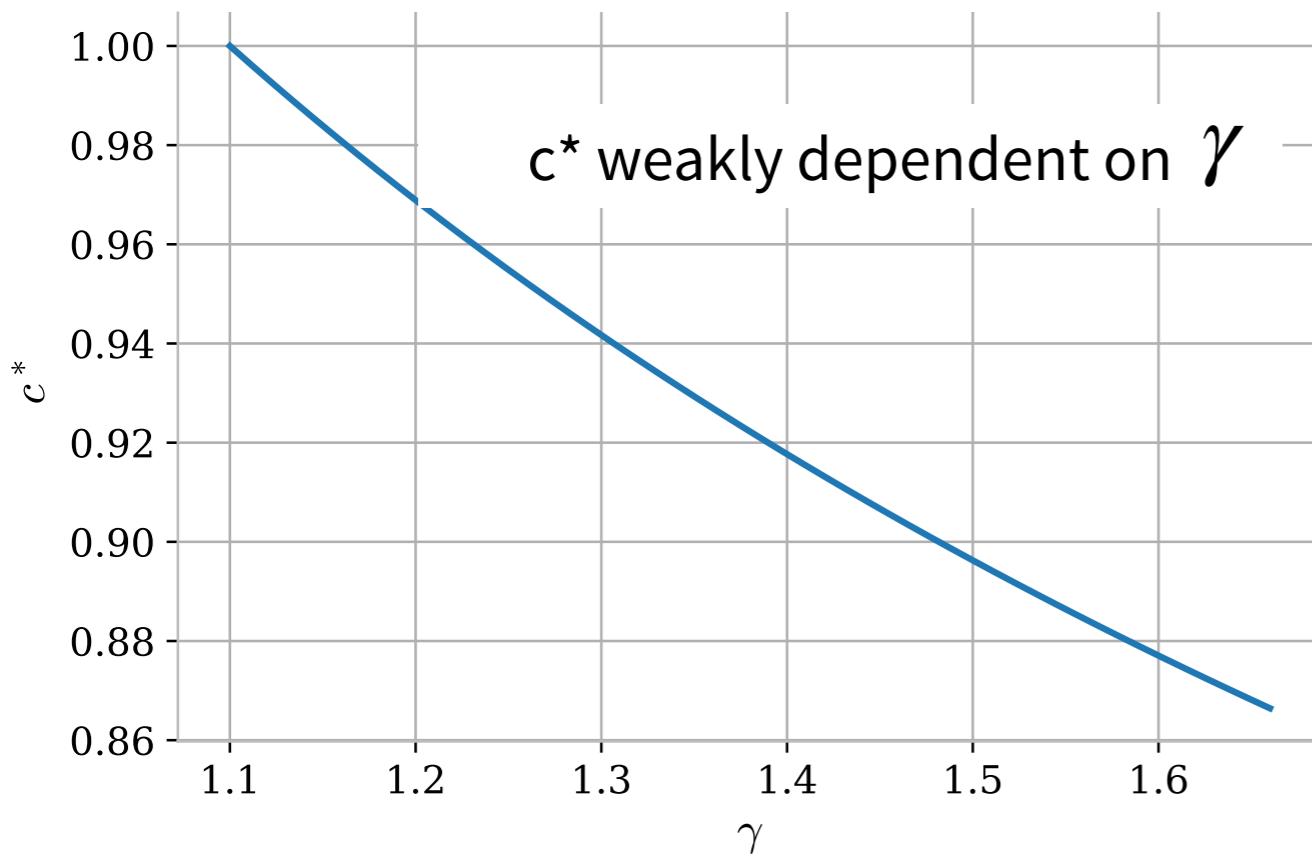
$$c^* = \frac{P_t A^*}{\dot{m}} = \sqrt{\frac{RT_t}{\gamma}} \left[ \frac{\gamma + 1}{2} \right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

$c^*$  depends on...

$$c^* = f(T_t, R, \gamma)$$

$$R = \frac{R_u}{M_w}$$

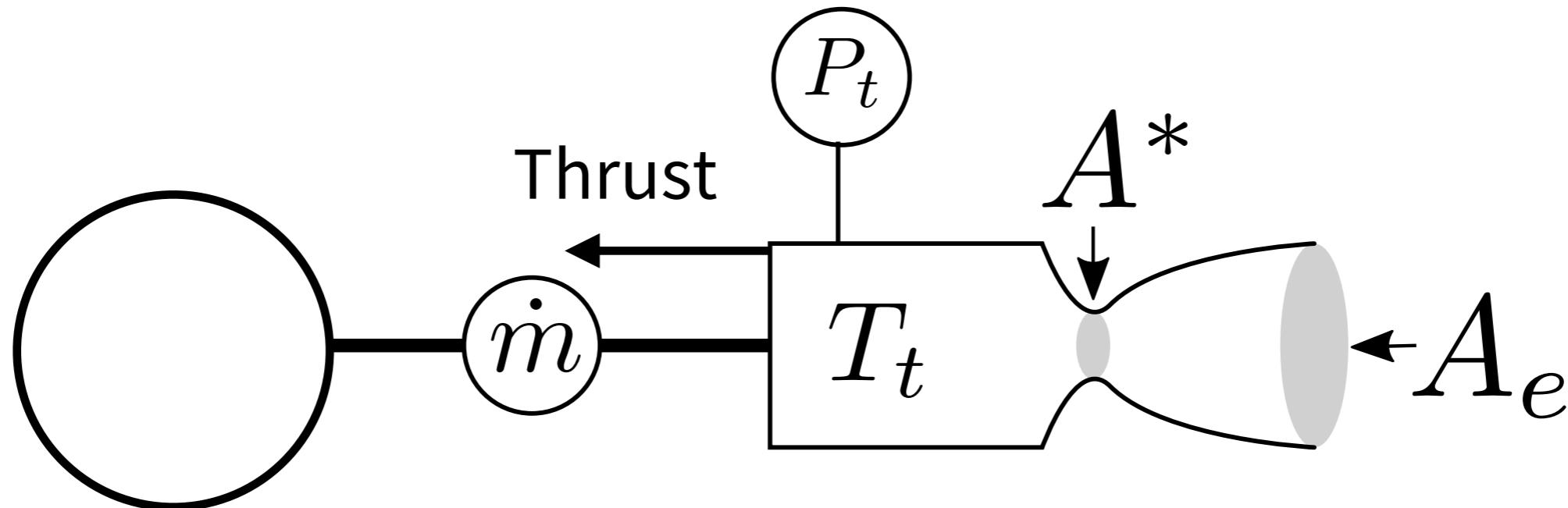
$$c^* = \sqrt{\frac{R_u T_t}{M_w}} f(\gamma)$$



$$\Delta h_t = \int_{T_0}^{T_t} C_p dT$$

$$c^* \propto \sqrt{\frac{T_t}{M_w}} \sim \sqrt{\frac{\Delta h_t}{C_p M_w}}$$

# A rocket test



Propellants

$$\eta_{c^*} = \frac{c_{measured}^*}{c_{ideal}^*} = \frac{P_t A^*}{\dot{m}_p c_{ideal}^*}$$

$c^*$  and  $c^*$  efficiency are very useful parameters for static characterization

# Thrust Coefficient, $C_f$

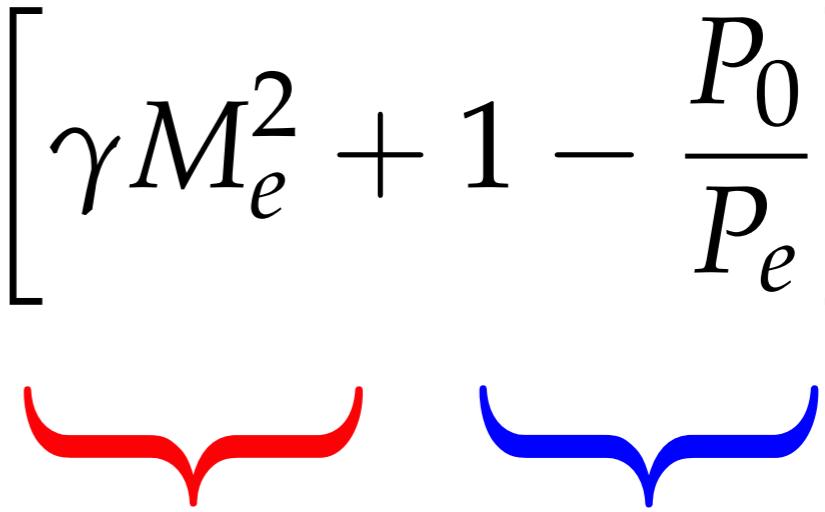
$$\begin{aligned} C_f = \frac{C}{c^*} &= \frac{(P_e - P_0) \frac{A_e}{\dot{m}} + U_e}{\frac{P_t A^*}{\dot{m}}} = \frac{P_e - P_0}{P_t} \frac{A_e}{A^*} + \frac{U_e}{c^*} \\ &\Rightarrow \left(1 - \frac{P_0}{P_e}\right) \frac{P_e A_e}{P_t A^*} + \frac{U_e}{c^*} \end{aligned}$$


Exhaust Pressure

Exhaust Momentum

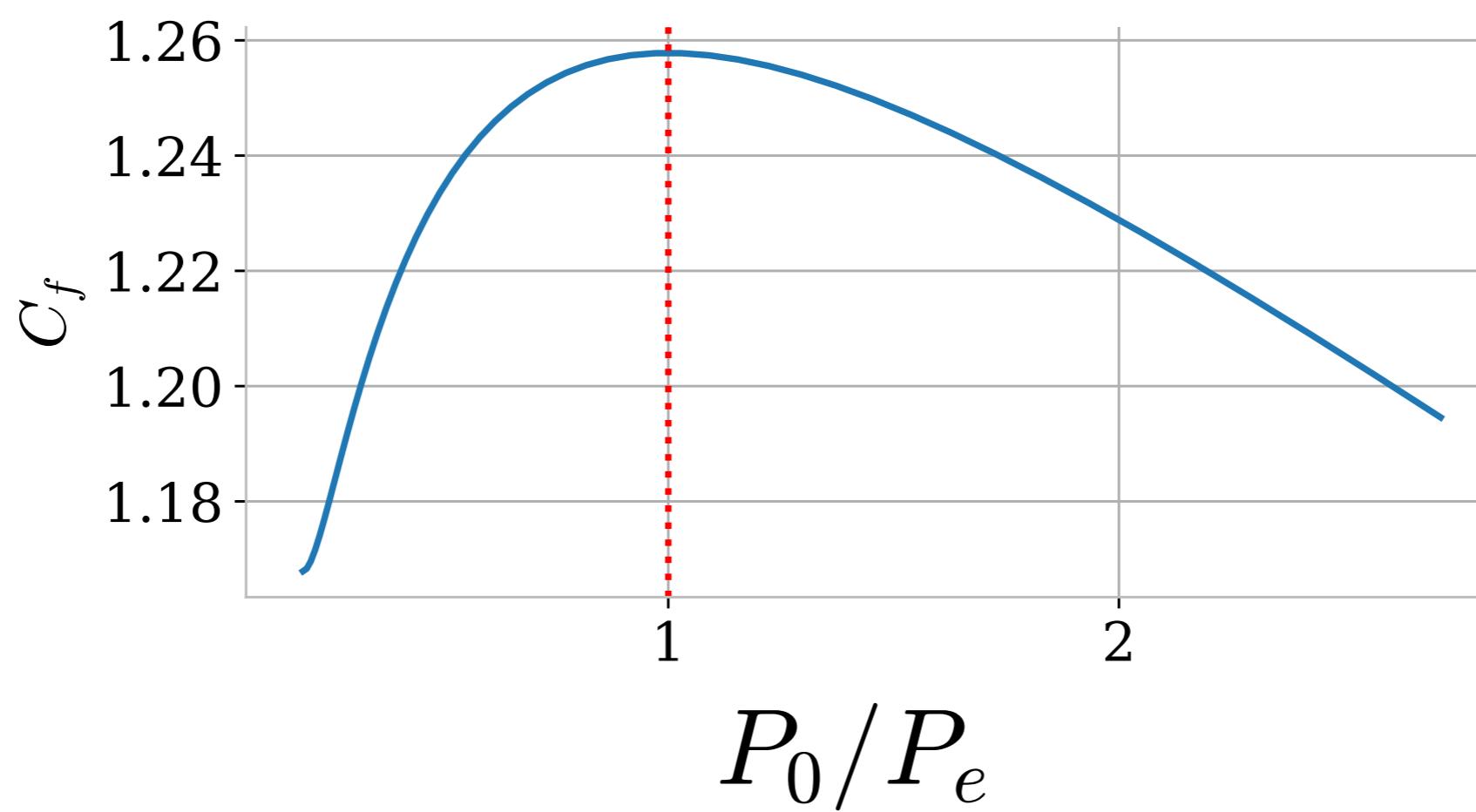
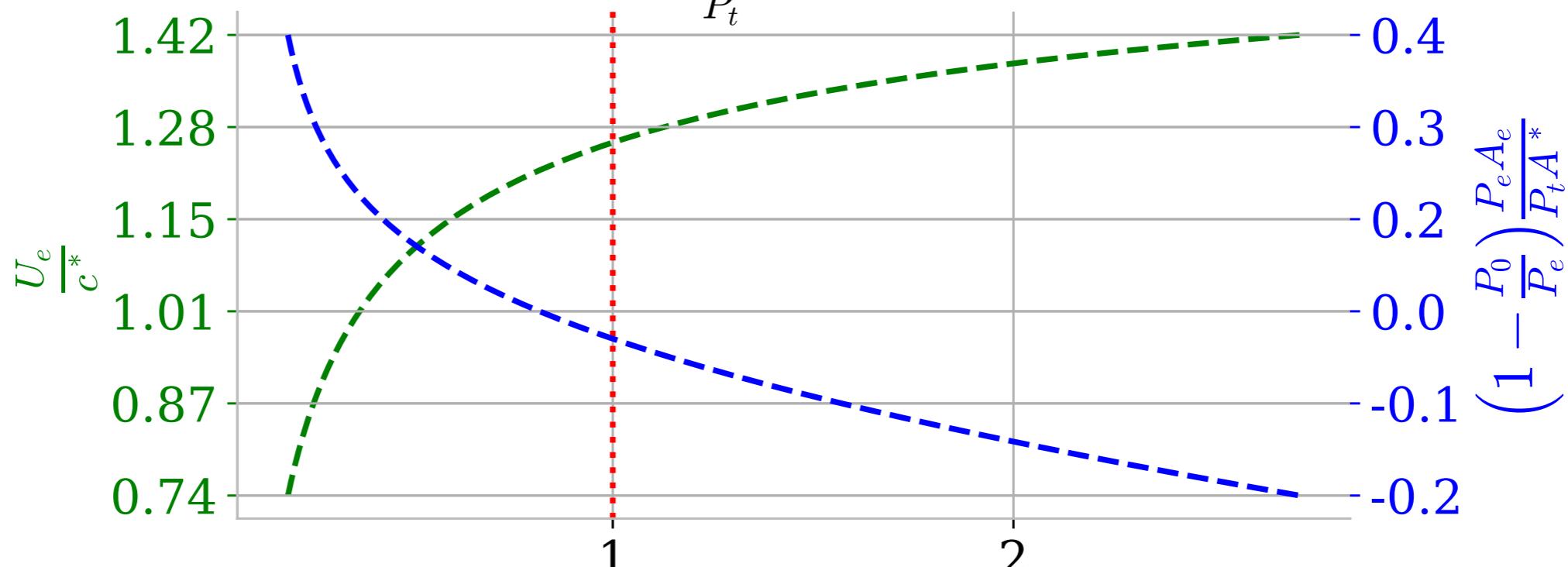
# Thrust Coefficient, $C_f$

$$C_f = \frac{\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{-2(\gamma-1)}}}{M_e \sqrt{1 + \frac{\gamma-1}{2} M_e^2}} \left[ \gamma M_e^2 + 1 - \frac{P_0}{P_e} \right]$$

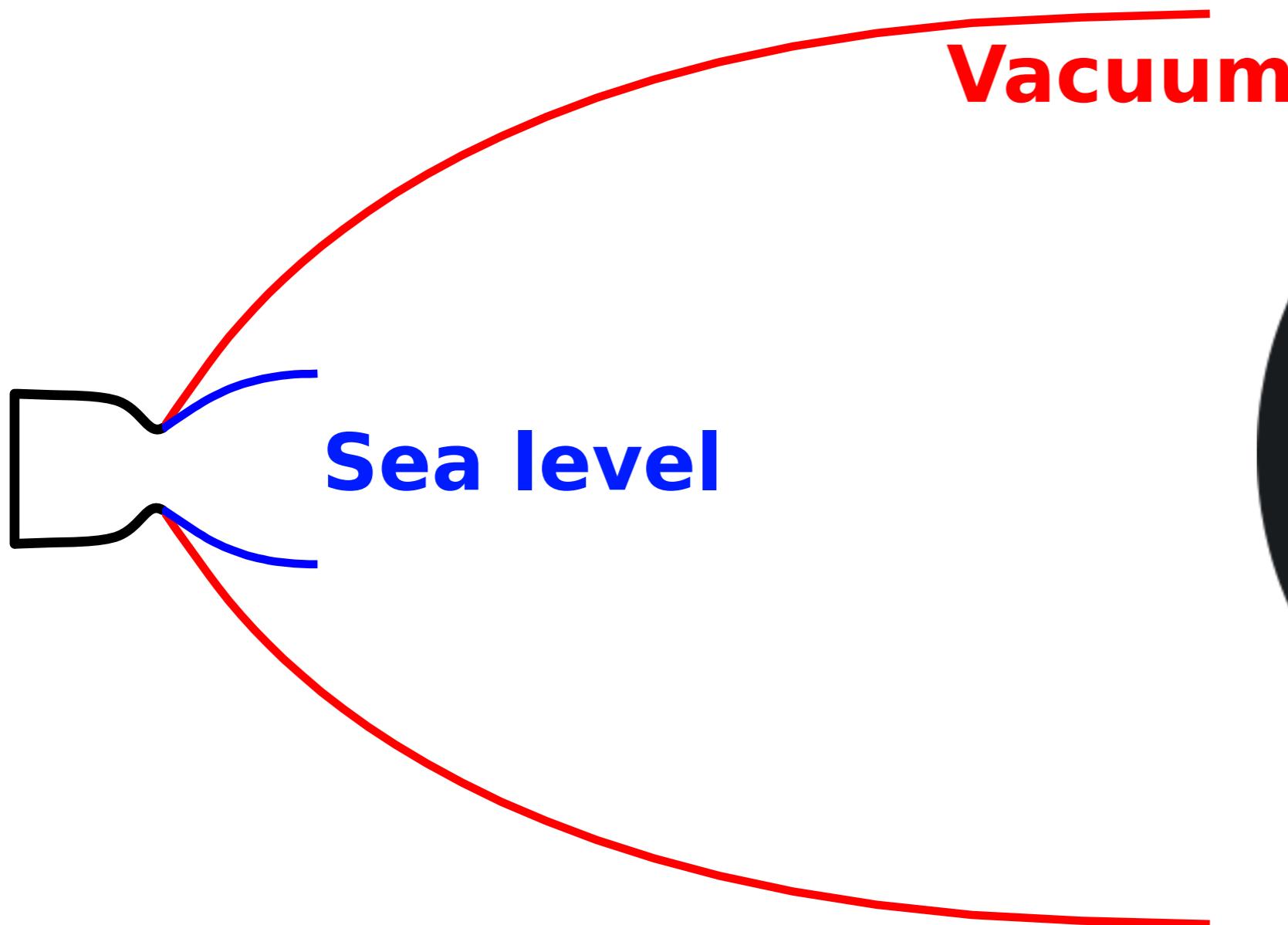
  
Exhaust Momentum      Exhaust Pressure

# Optimum Expansion

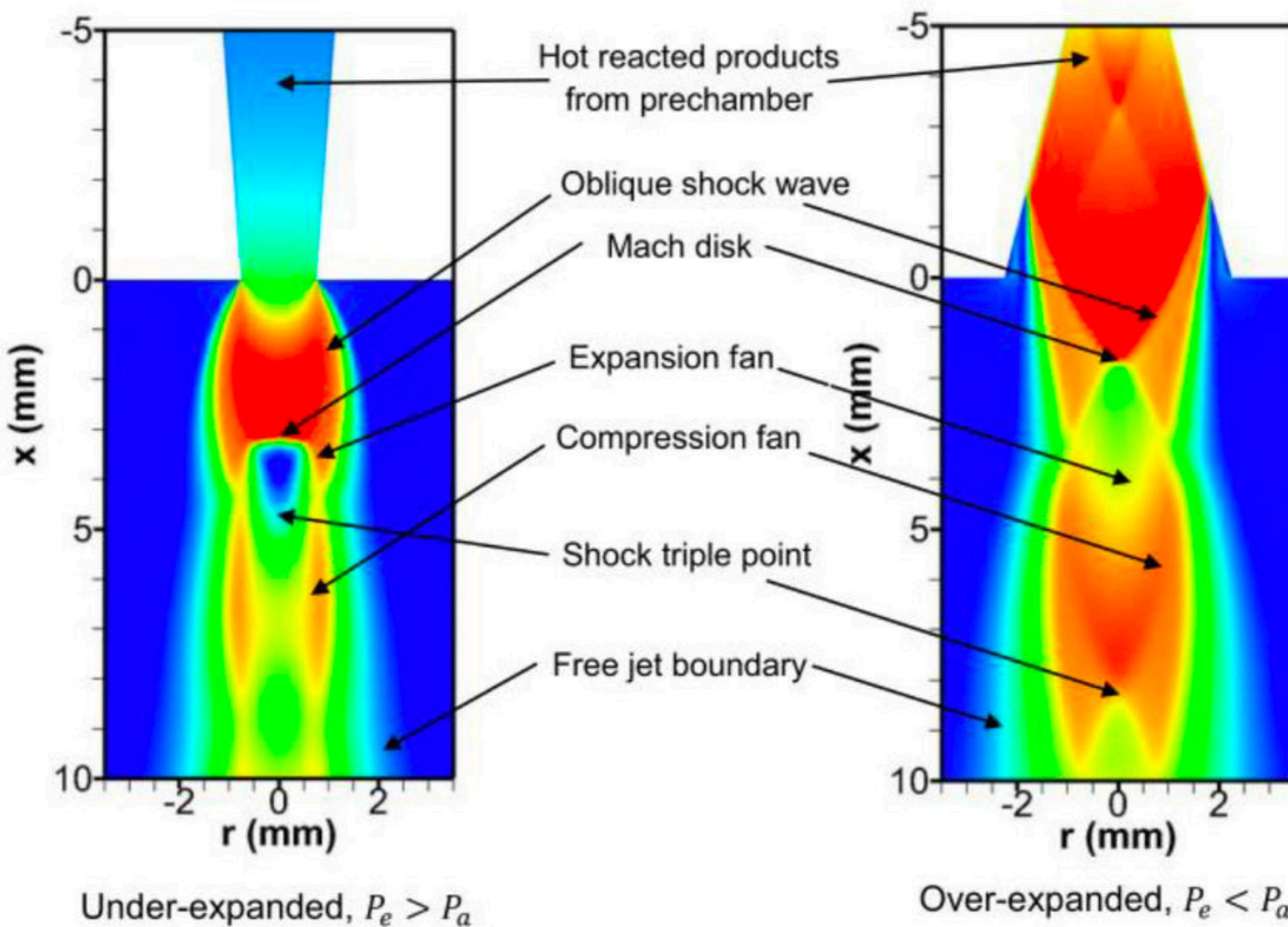
$$\frac{P_0}{P_t} = 0.1$$



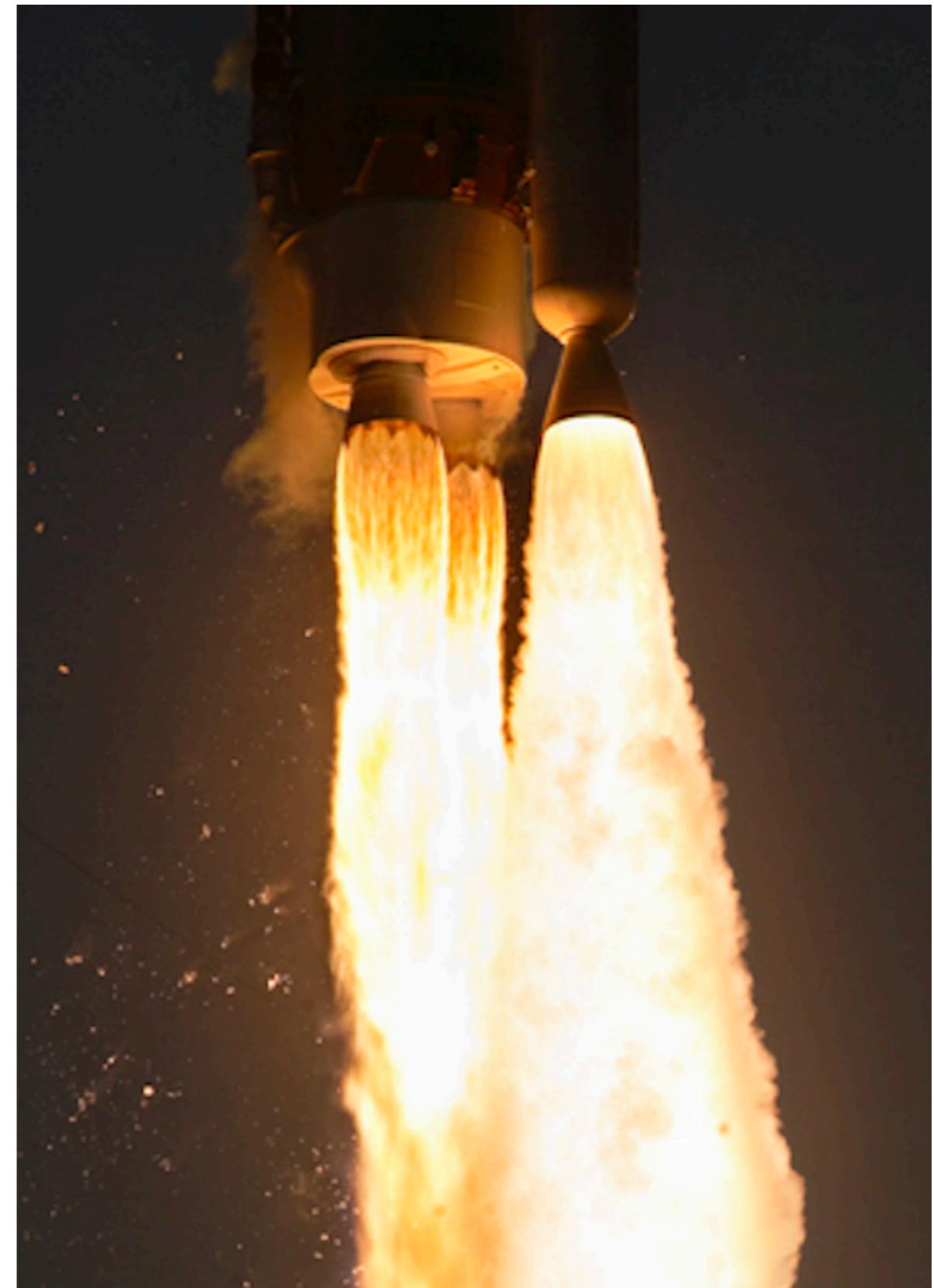
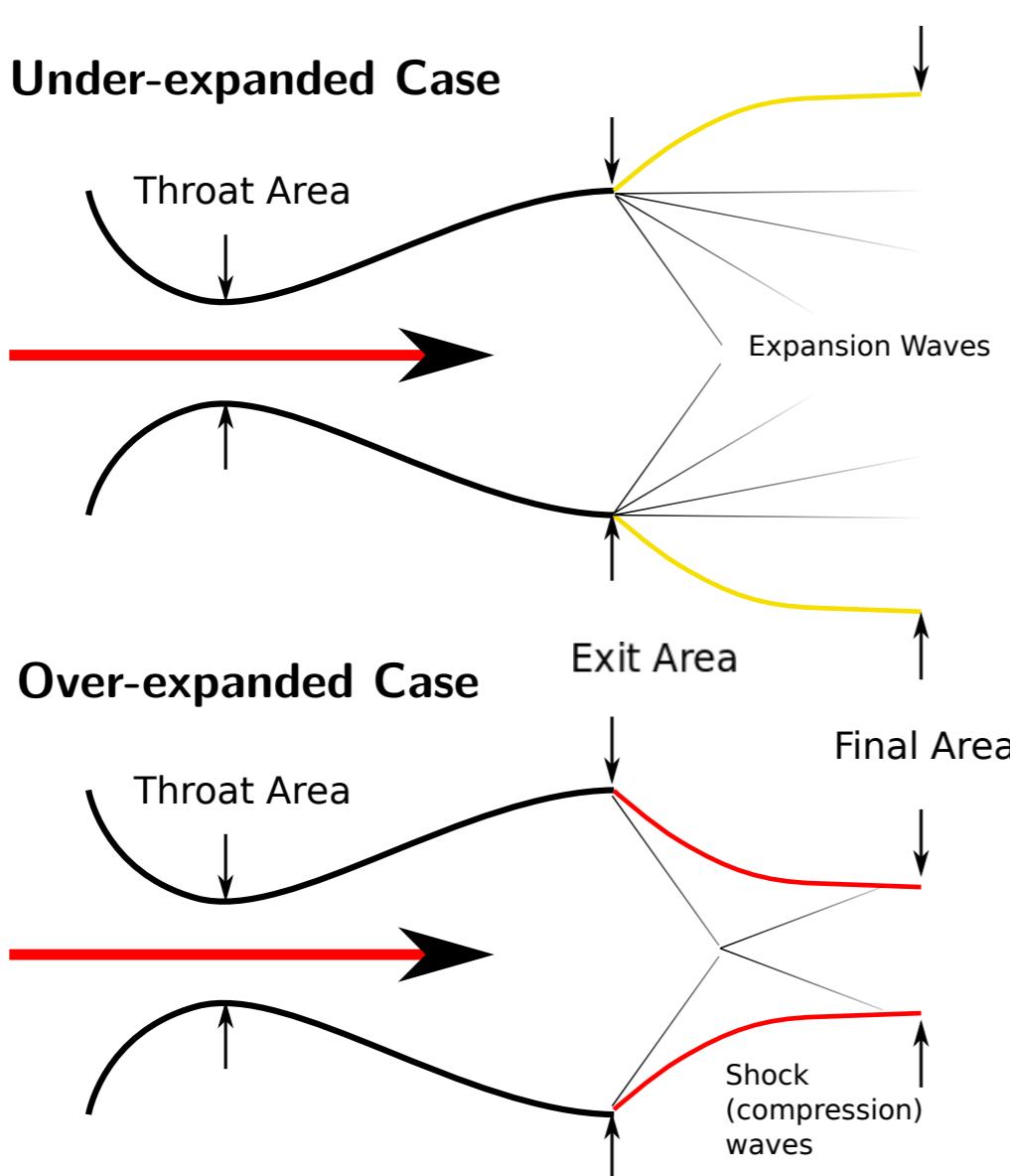
# Pressure Effect on Nozzles



# Under- and Over- Expansion

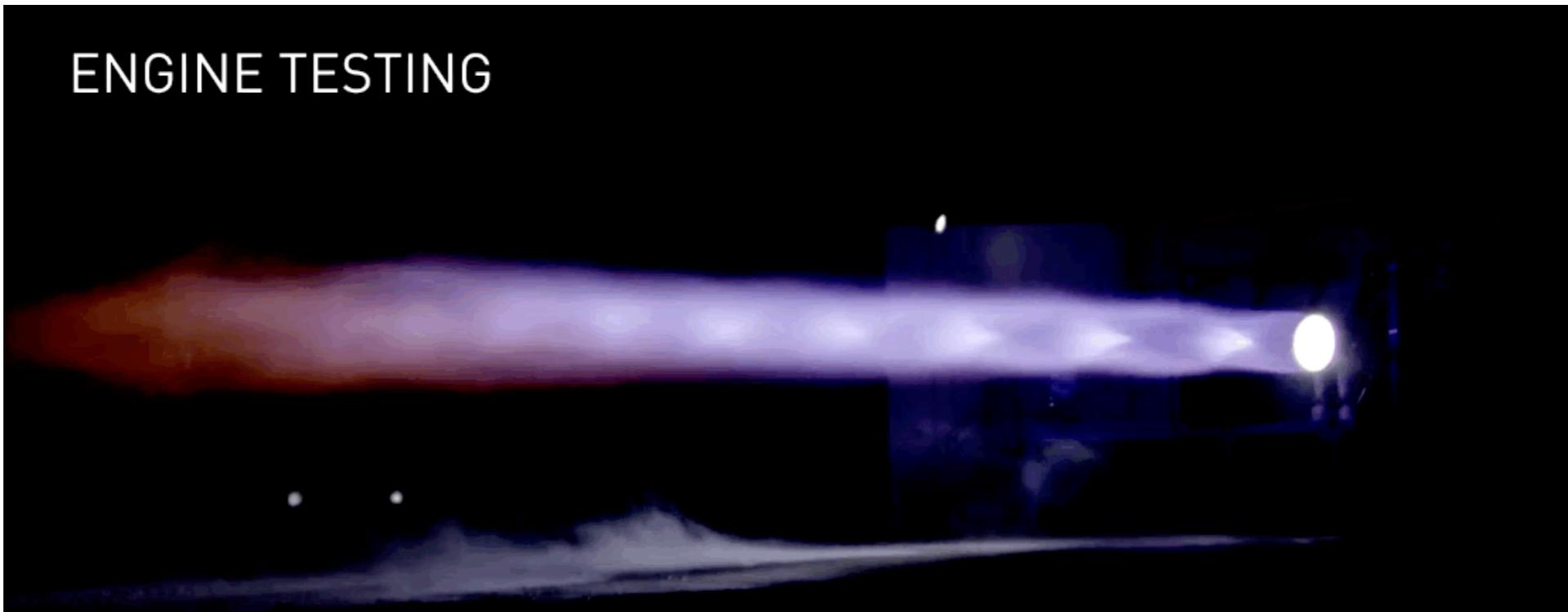


## A real example





## ENGINE TESTING

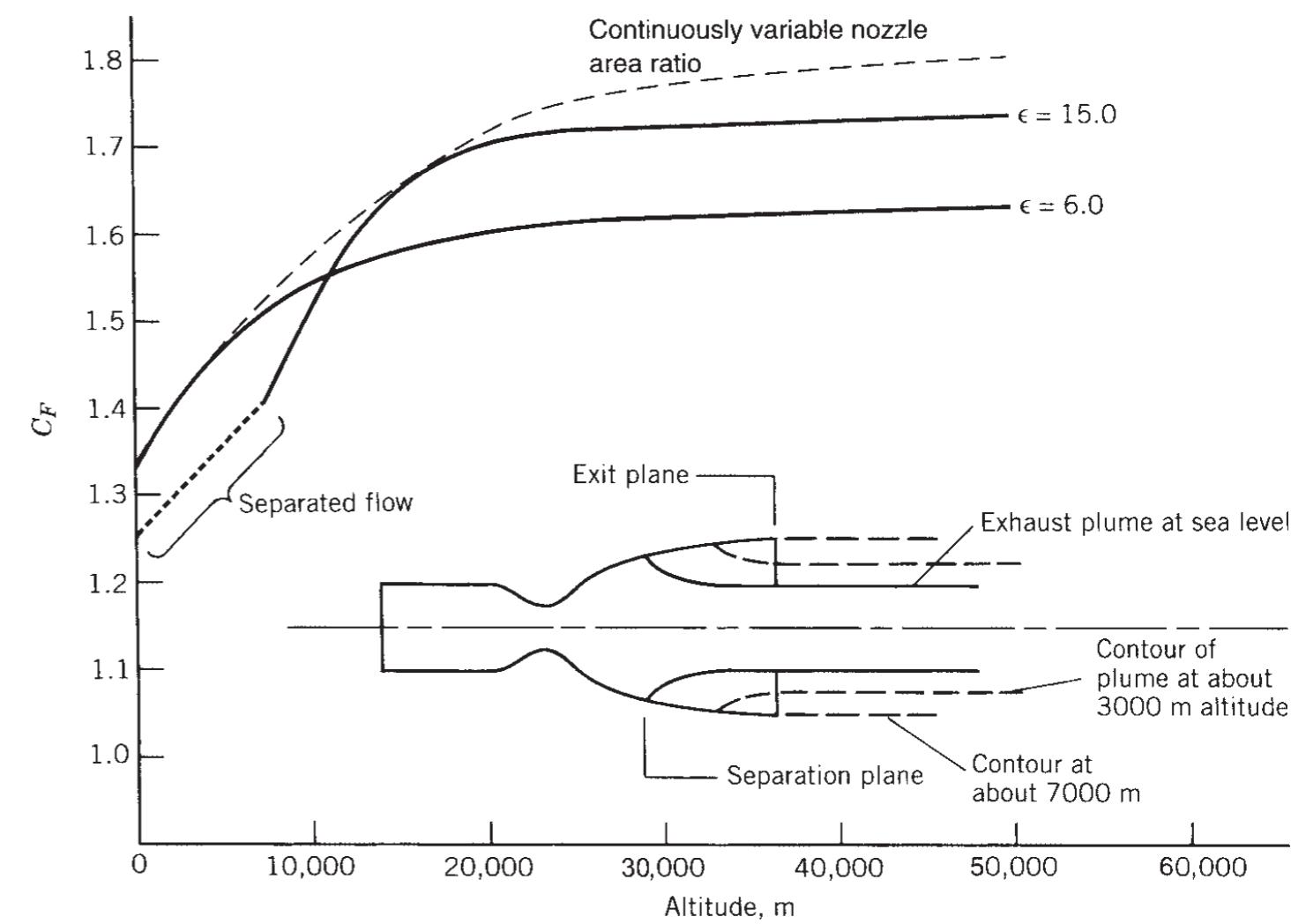
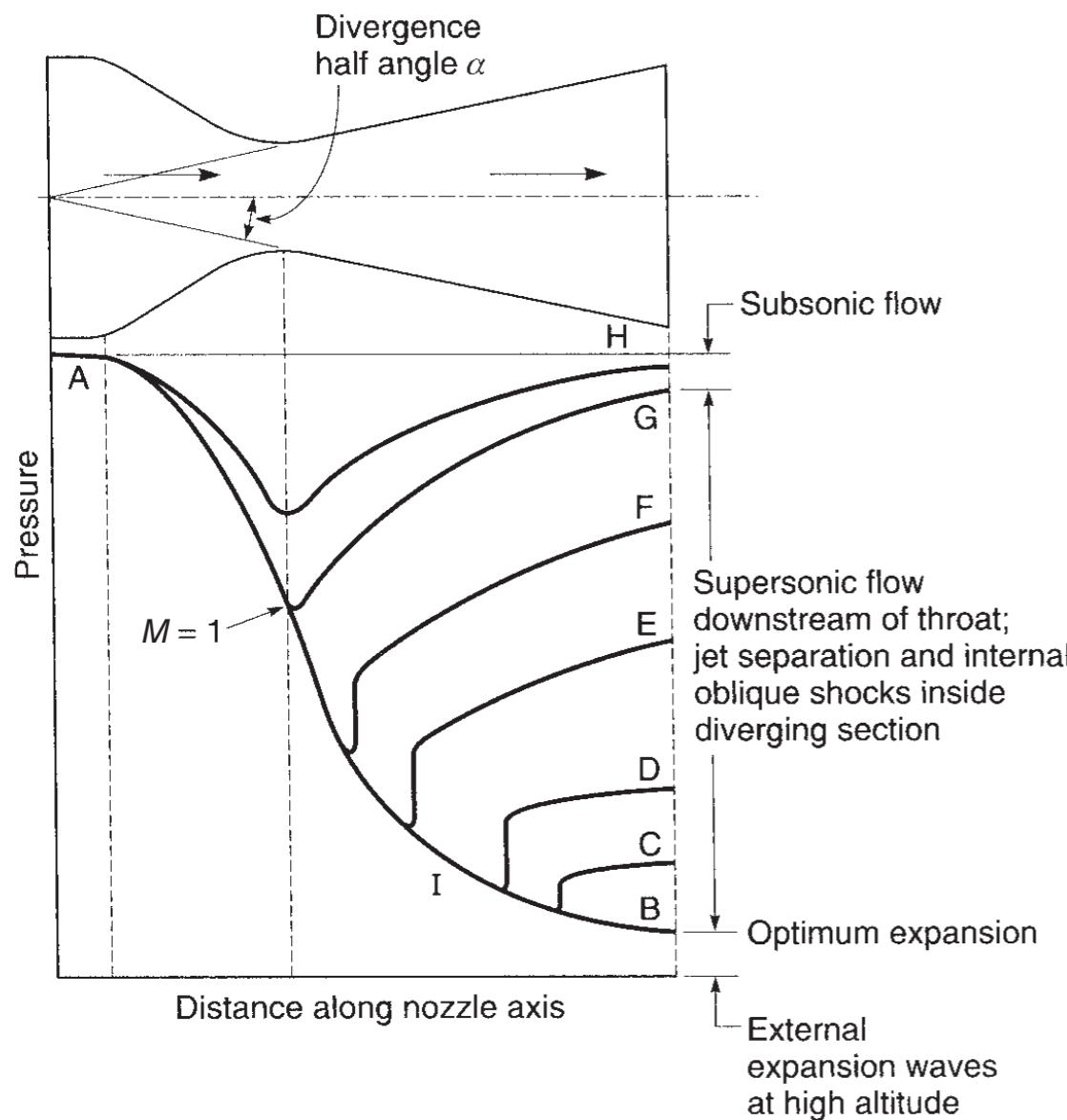


Over 1200 seconds of firing across 42 main engine tests

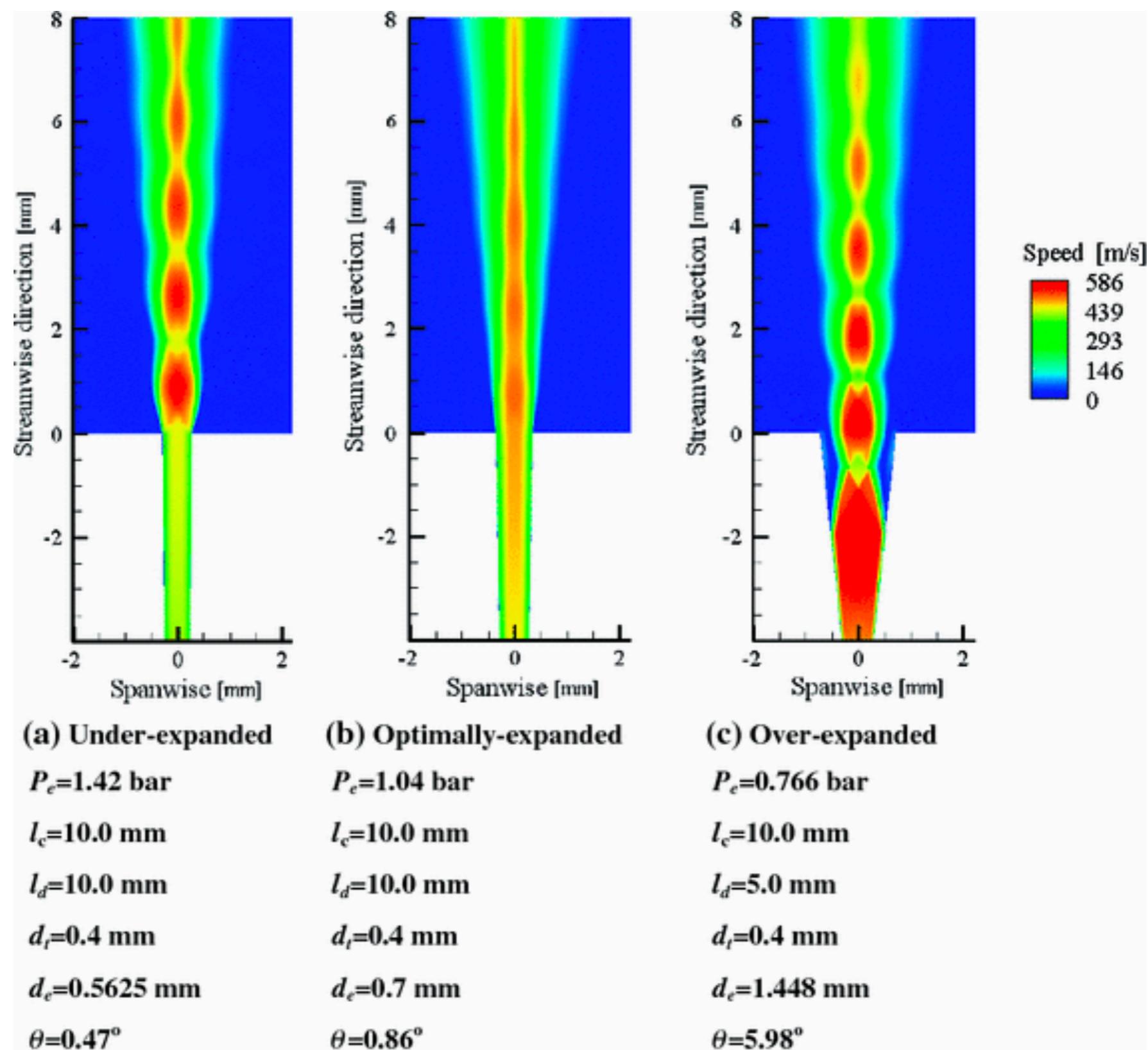
Longest test 100 seconds; 40 seconds typical for Mars landing

Test engine operates at up to 200 atmospheres

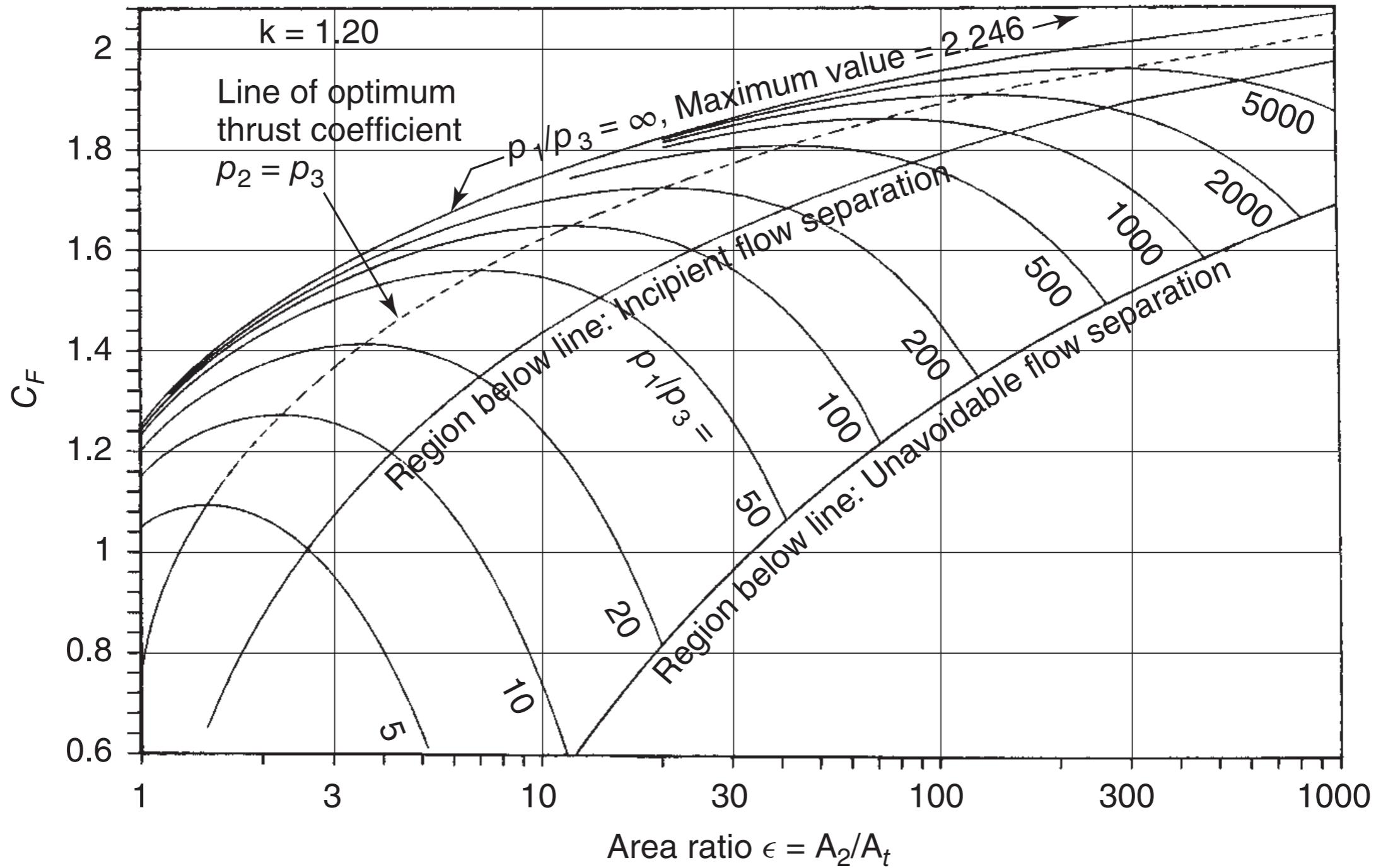
# Nozzle flow separation



Source: [Sutton, Rocket Propulsion Elements](#)

Unsteady Separation Video

# Thrust Coefficient from Sutton



**FIGURE 3–6.** Thrust coefficient  $C_F$  versus nozzle area ratio for  $k = 1.20$ .

# Putting it all together

---

$$C = \frac{T}{\dot{m}} = \frac{T}{P_t A^*} \frac{P_t A^*}{\dot{m}}$$

$$\Rightarrow C = \frac{T}{\dot{m}} = C_f c^*$$

- $c^*$  captures the effect of propellants and combustion on rocket performance
- $C_f$  captures the effect of nozzle geometry and pressures on performance

$$C = \frac{P_e - P_0}{P_t} \frac{A_e}{A^*} \sqrt{\frac{RT_t}{\gamma}} \left[ \frac{\gamma + 1}{2} \right]^{\frac{-2(\gamma-1)}{\gamma+1}} + \sqrt{\frac{2\gamma}{\gamma-1} \frac{R_u T_t}{M_w} \left[ 1 - \left( \frac{P_e}{P_t} \right)^{(\gamma-1)/\gamma} \right]}$$

}  $(P_e - P_0) \frac{A_e}{\dot{m}}$   
}  $U_e$

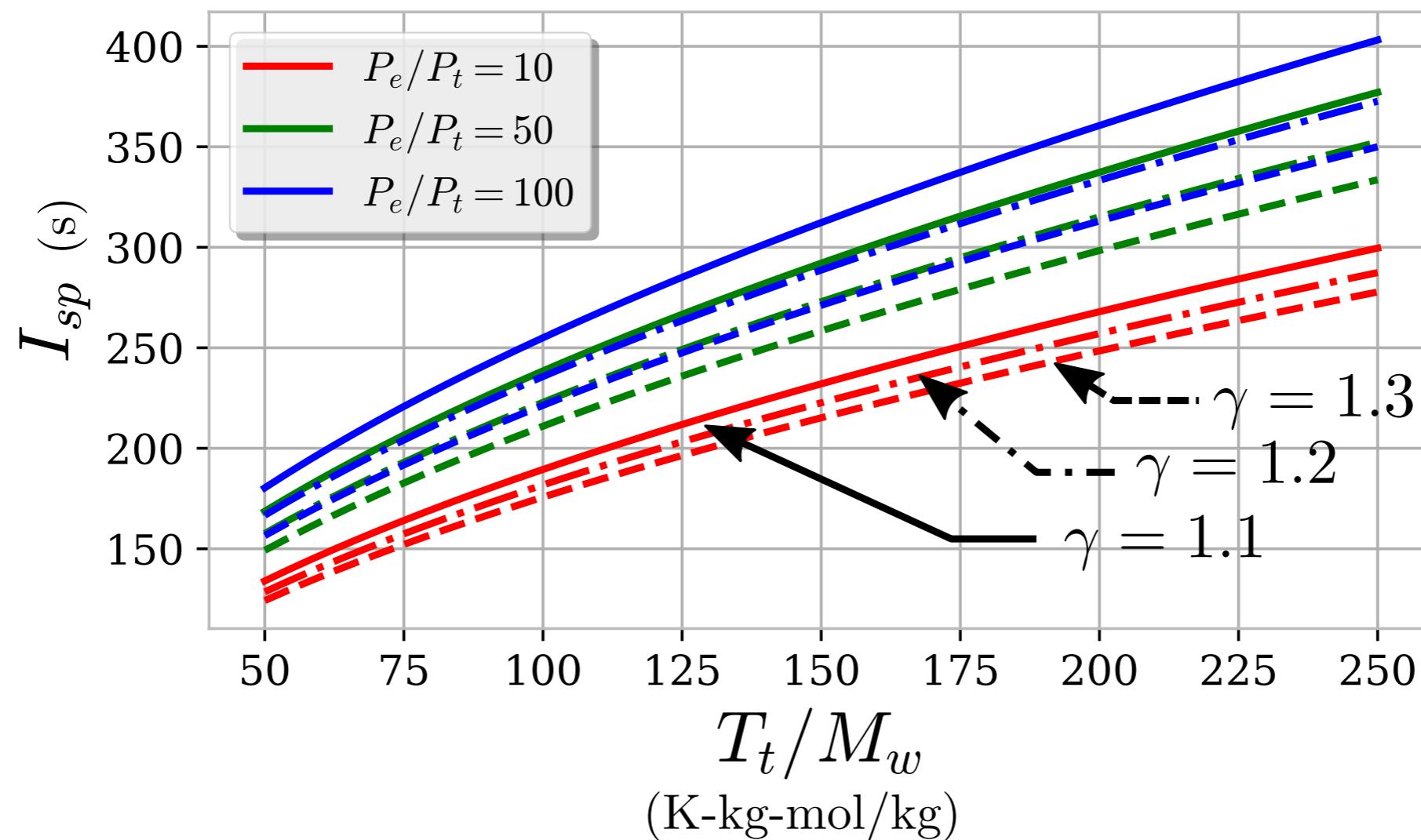
## Cold Gas Performance

Name	$\gamma$	$M_w$	$c^*$ (m/s)	$C_f$	C (m/s)
$N_2$	1.40	28	434.4	1.26	546.4
$H_2$	1.41	2	1621.5	1.26	2039.2
He	1.66	4	1085.2	1.25	1356.2

Table 0.2:  $c^*$ ,  $C_f$ , C for several pure gasses at room temperature,  $P_0/P_e = 1$  and  $P_0/P_t = 0.1$ . Note how  $C_f$  is essentially constant despite the differences in gas properties while  $c^*$  and C vary substantially.

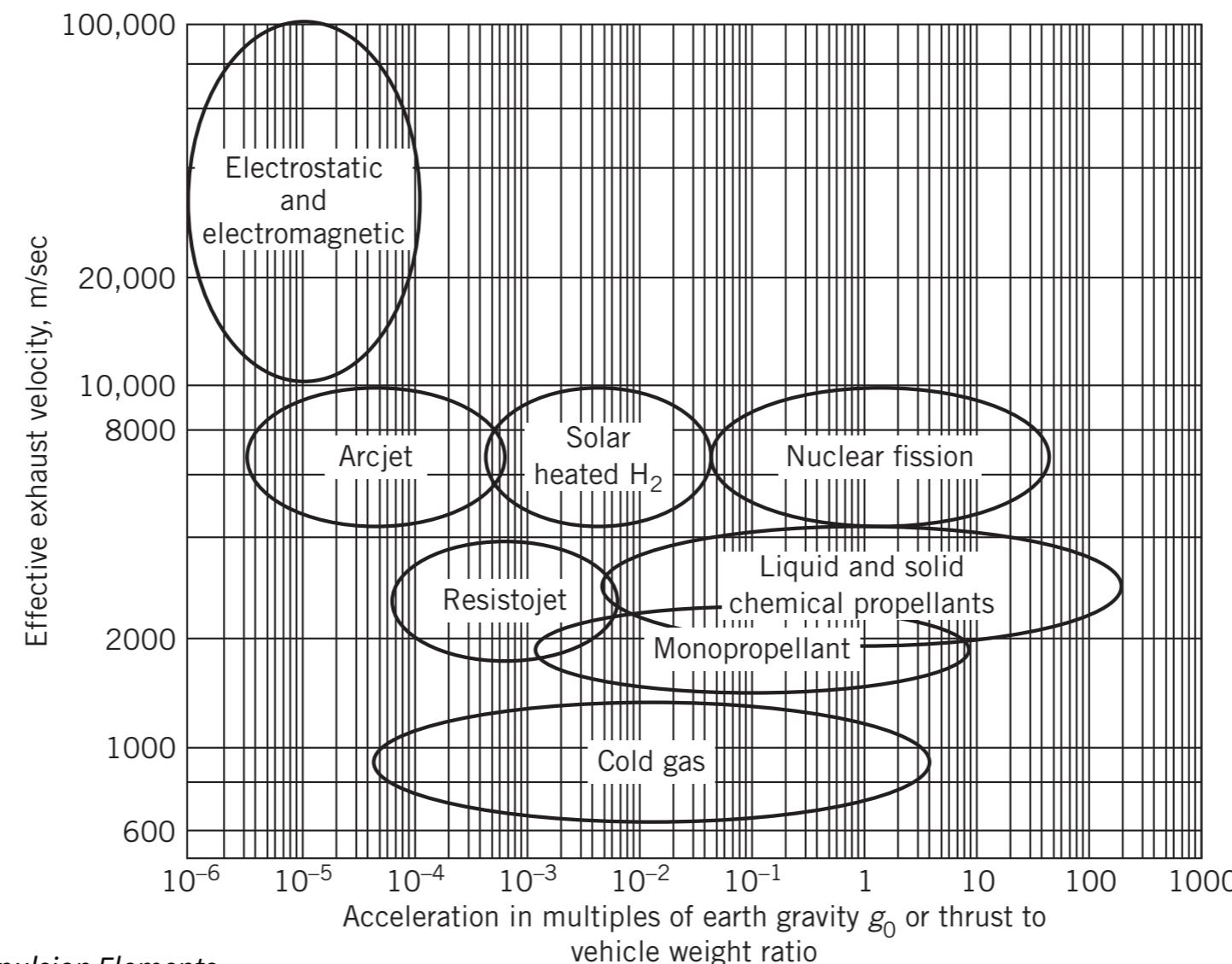
## Effective Exhaust Velocity & Isp

$$C = (P_e - P_0) \frac{A_e}{\dot{m}} + \sqrt{\frac{2\gamma}{\gamma-1} \frac{R_u T_t}{M_w} \left[ 1 - \left( \frac{P_e}{P_t} \right)^{(\gamma-1)/\gamma} \right]}$$

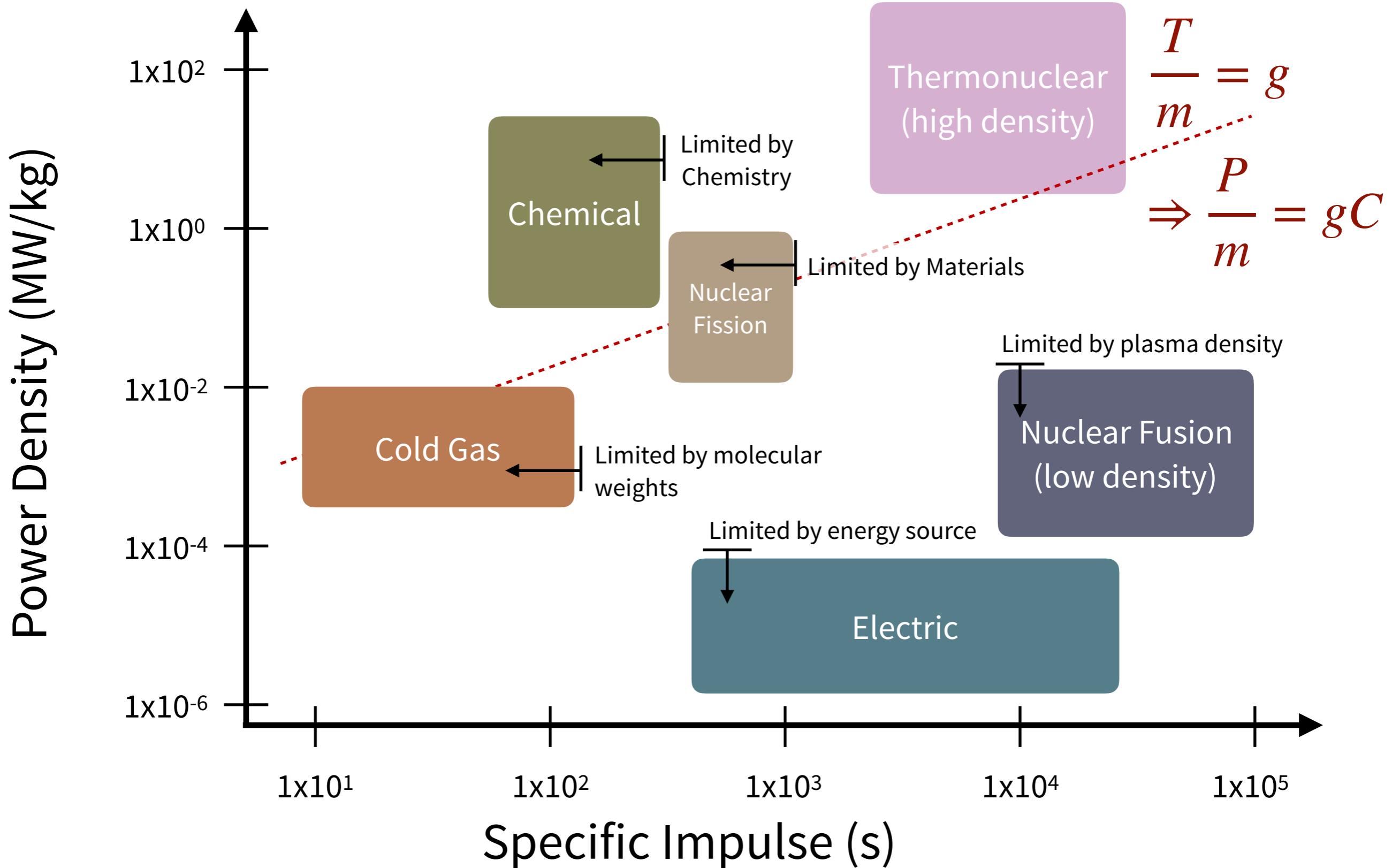


# Performance by Technology

Technology	Isp (s)	Exhaust Velocity (m/s)
Nuclear Fusion	10,000 - 50,000+	98,000 - 490,000+
Electric Propulsion	1,000 - 10,000	9,800 - 98,000
Nuclear Thermal (fission)	600 - 1,000	5,900 - 9,800
Beamed Thermal (microwave / laser)	600 - 1,000	5,900 - 9,800
Bipropellant Chemical Propulsion	200 - 500	2,000 - 4,900
Monopropellant Chemical Propulsion	100 - 250	980 - 2,450
Cold Gas Propulsion	10 - 120	100 - 1,150



# Propulsion Energy & Power Density



# Rocket Efficiency

---

$$\eta_{thermal} = \frac{\dot{W}_{exhaust}}{P_{in}} = \frac{\dot{m}C^2}{2P_{in}}$$

$$\eta_{thermal} = \frac{\gamma}{\gamma - 1} \frac{R}{\overline{C}_p} \left[ 1 - \left( \frac{P_e}{P_t} \right)^{(\gamma-1)/\gamma} \right]$$

$$\eta_{thermal} \approx 1 - \frac{T_e}{T_t}$$

Carnot Efficiency!

# Propulsive Efficiency

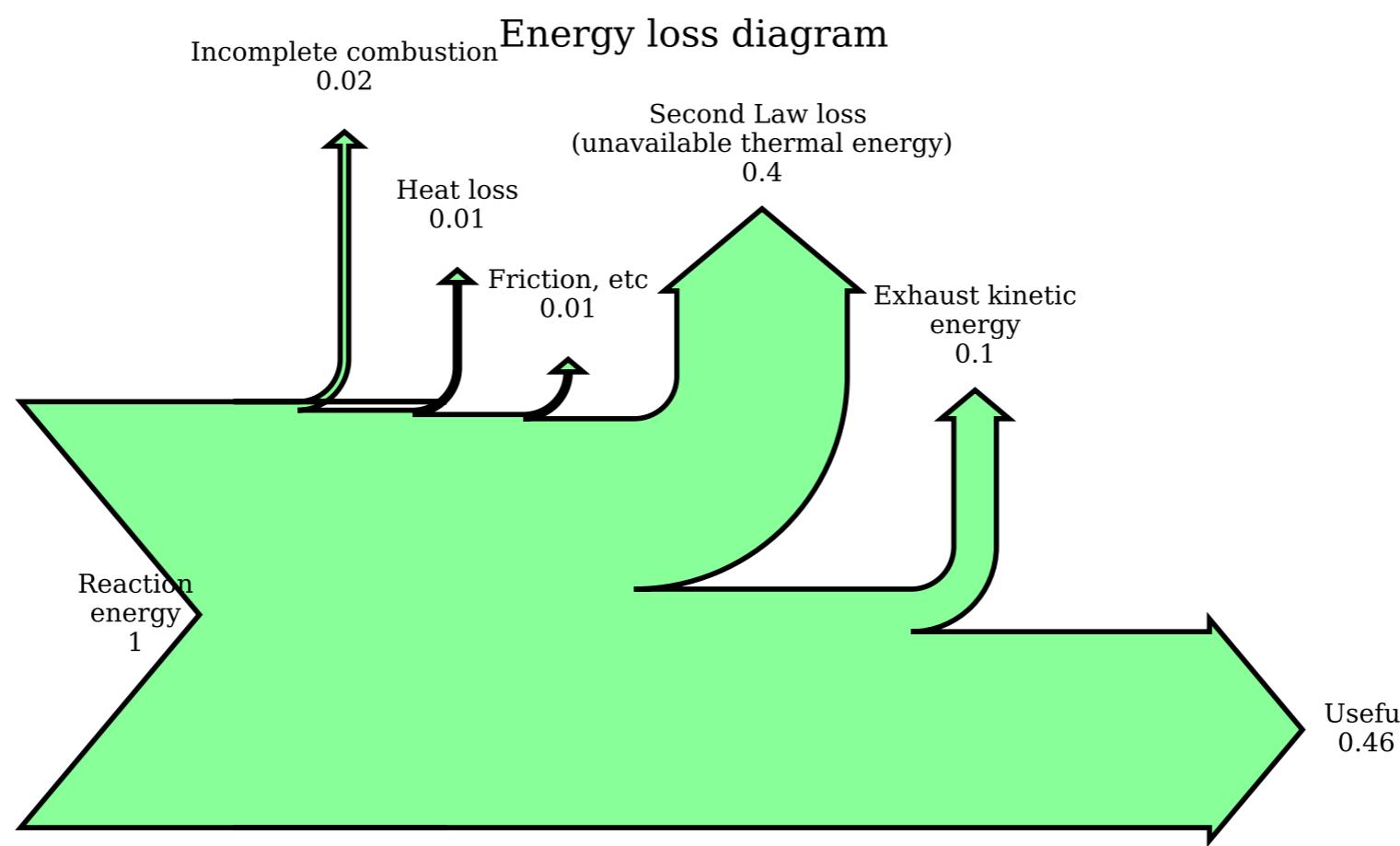
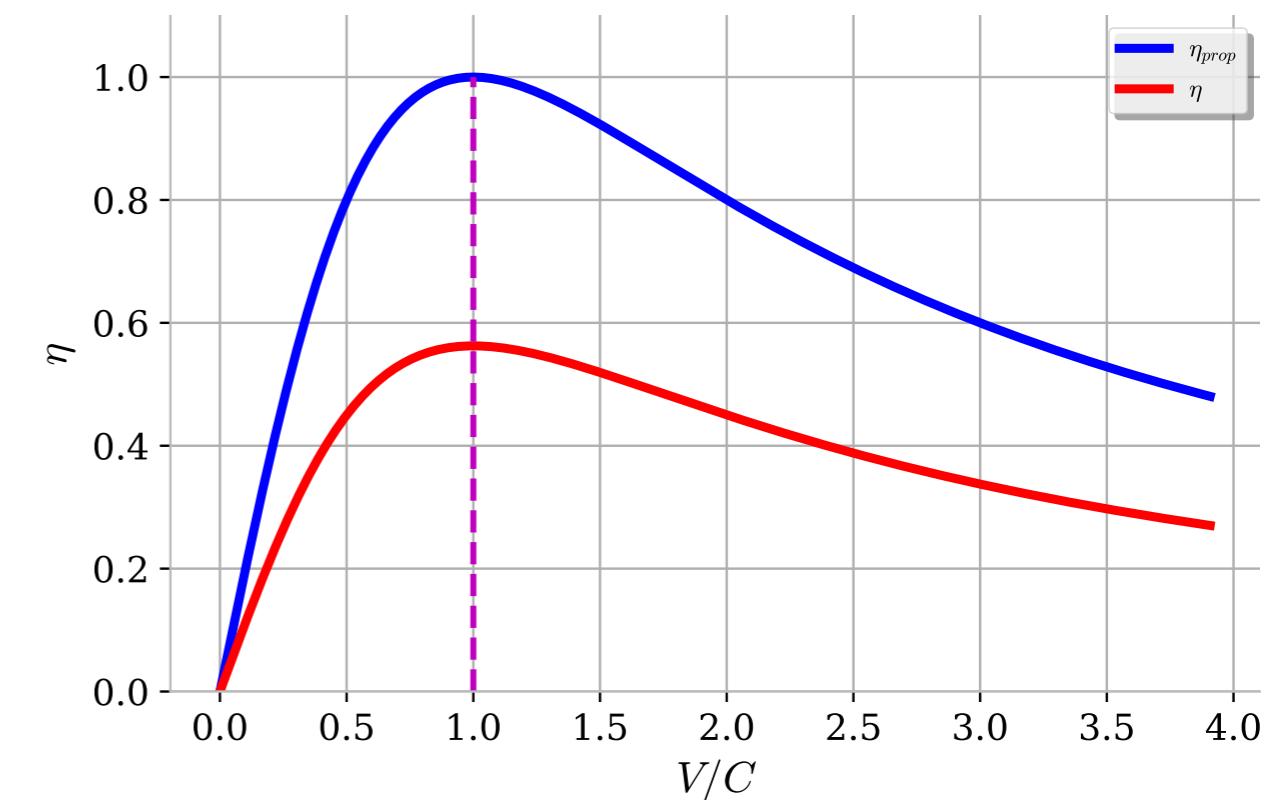
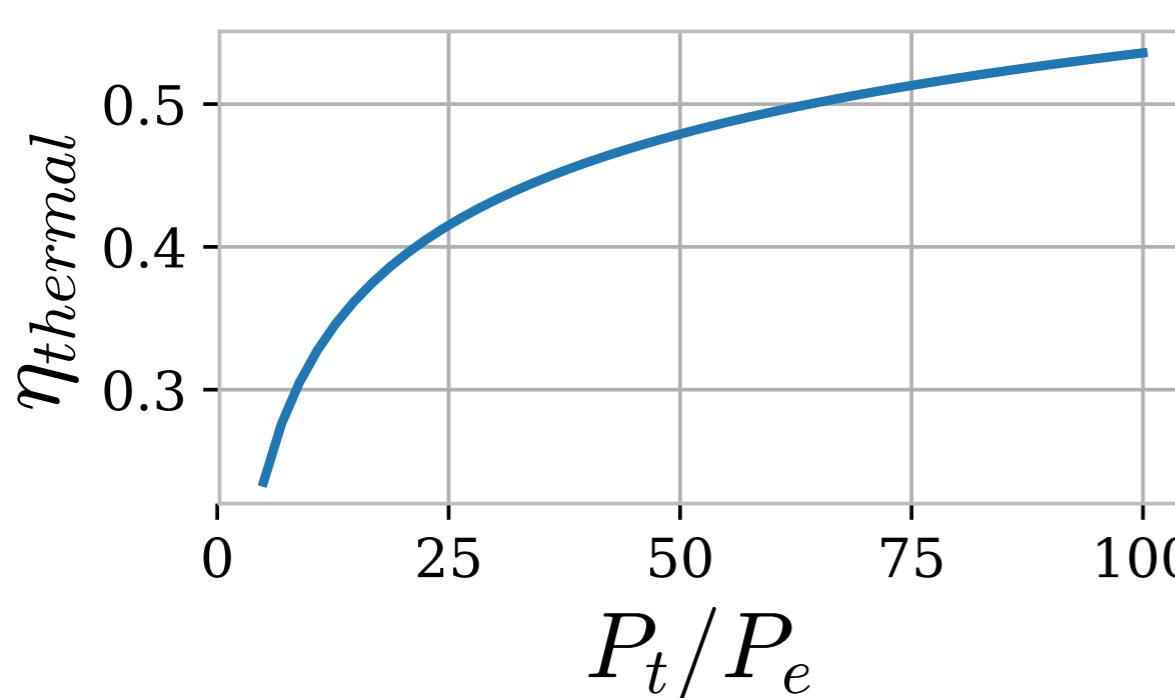
---

$$\eta_{prop} = \frac{\dot{W}_r}{\dot{W}_r + \dot{W}_{exhaust}} = \frac{Tv}{Tv + \frac{1}{2}\dot{m}(C - v)^2}$$

$$\frac{Tv}{Tv + \frac{1}{2}\dot{m}(C - v)^2} = \frac{\dot{m}Cv}{\dot{m}[Cv + (C - v)^2]}$$

$$\rightarrow \eta_{prop} = 2 \frac{\frac{v}{C}}{1 + (\frac{v}{C})^2}$$

# Efficiency



# Three Balloons



What is  $\Delta V$ ?

1. Your air-filled balloon  
→  $T_t = 298 \text{ K}$   $M_w \approx 28 \text{ g/mol}$
2. A  $2\text{H}_2 + \text{O}_2$  balloon  
→  $M_{wi} \approx 12 \text{ g/mol}$   $M_{wf} \approx 18 \text{ g/mol}$   
$$\frac{Cp_{H_2O}}{R_u} \approx 5 \quad \Delta_f H_{H_2O}(g) = 241.8 \text{ kJ/mol}$$
3. A deuterium+tritium balloon  
→  $M_{wi} \approx 2.5 \text{ g/mol}$   $M_{wf} \approx 4 \text{ g/mol}$   
$$\frac{Cp_{^4He}}{R_u} \approx 2.5 \quad \Delta H_{D+T} = 17.5 \text{ MeV}$$
  
$$\Delta H_{D+T} = 1,717 \text{ GJ/mol}$$