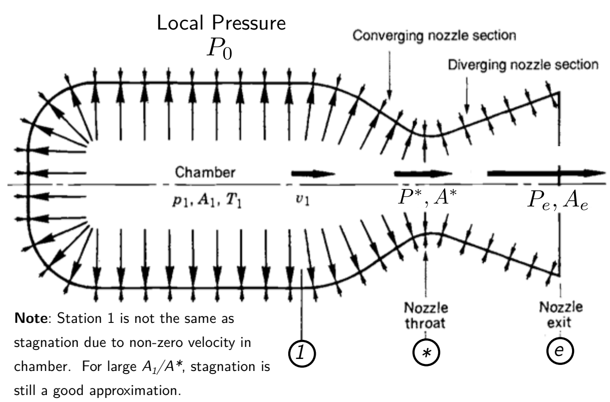
Rocket parameters and Efficiency

Assumptions for topics already covered

- Equation for thrust -
  - $T = \dot{m}_{in}(U_e - U_0) + (P_e - P_0)A_e + \dot{m}_p U_e$
  - Reduces to $T = (P_e - P_0)A_e + \dot{m}_p U_e$ for rocket case
- Rocket equation - $\Delta V = C \ln \frac{m_0}{m_i}$
- Isentropic flow through nozzles, critical flow, supersonic flow in de laval nozzle

0.1 Effective Exhaust Velocity and Specific Impulse

Now that we have covered the basic mechanics of thrust and the isentropic nozzle relations we will begin to show how these apply to rockets by defining a set of convenient parameters.



Let's start by remembering that, for a rocket

$$T = (P_e - P_0)A_e + \dot{m}_p U_e \tag{1}$$

Our intuition tells us that a good parameter for the effectiveness of a rocket is the quotient of thrust (what we want) with propellant mass flowrate (what we have to pay):

$$\frac{T}{\dot{m}} = (P_e - P_0)\frac{Ae}{\dot{m}} + U_e \tag{2}$$

This parameter is called effective exhaust velocity,

$$C = \frac{T}{\dot{m}} \tag{3}$$

and is one of the most important relations in rocketry. It figures prominently in the Rocket Equation

$$\Delta V = C \ln \frac{M_0}{M_f} \tag{4}$$

which we will be derived in a couple lectures.

If we assume T and $\dot{m}$ are completely independent (which we will see is not always true), we can also interpret C as

$$C = \frac{\int T dt}{\int \dot{m} dt} = \frac{I}{M_p} \tag{5}$$

which is how much total impulse we get from a unit mass of propellant. This interpretation points to a very closely related parameter called Specific Impulse, I_{sp}:

$$I_{sp} = \frac{C}{g_0} = \frac{T}{g_0 \dot{m}} \tag{6}$$

Notice that I_{sp} is just C normalized by Earth's gravitational acceleration with overall units of seconds. While the unit is at first a bit non-sensical, I_{sp} makes sense when you consider your propellant consumption as a weight flowrate rather than a mass-flowrate and so you are dividing thrust (a force) by weight flowrate (force per time) to arrive at a time. The etymology of I_{sp} is rooted in the fact that for imperial units we typically work in lbm. rather than slugs.

And while weird at first, the units of seconds do have some intuitive utility. For a rocket with $I_{sp} = 100$s a unit mass, m of propellant can generate enough thrust to support its weight in Earth's gravity for 100 seconds or 100 times its weight for one second.

Specific impulse is popularly spoken of as the "gas mileage" for a rocket cycle and this is fairly reasonable - it fundamentally indicates how much bang for the buck you get. I'll jump the gun just a bit for the sake of intuition and give some typical I_{sp} values for different types of propulsion in table 0.1.

This is all well and good, but all we have really done at this point is some algebra. What we really are interested in

Technology	Isp (s)	Exhaust Velocity (m/s)
Electric Propulsion	1000 - 10,000	9,800 - 98,000
Nuclear Thermal / Beamed Energy Propulsion	600 - 1,000	5,900 - 9,800
Bipropellant Chemical Propulsion	200 - 500	2,000 - 4,900
Monopropellant Chemical Propulsion	100 - 250	980 - 2,450
Cold Gas Propulsion	10 - 120	100 - 1,150

Table 0.1: Table 1 - Representative effective exhaust velocities for different propulsion technologies

as engineers is how do I get the most gas mileage out of my rocket. And for that discussion, we'll first define another couple useful parameters.

0.2 $c*$

We will go back, for a moment, to choked compressible flow. With the isentropic flow equations and the $M = 1$ choked condition, we can derive (assuming constant C_p, C_v and R):

$$\dot{m} = \rho_c a^* A^* = \rho_t \left[\frac{\rho_c}{\rho_t}\right] a_0 \left[\frac{a^*}{a_t}\right] A^* = \frac{P_t A^*}{\sqrt{\frac{RT_t}{\gamma}}} \left[\frac{\gamma+1}{2}\right]^{\frac{2(\gamma-1)}{\gamma+1}} \tag{7}$$

remembering that the t subscript denotes the total or stagnation condition and the c subscript denotes choked ($M = 1$) condition. This is a very useful relationship as it allows us to compute the mass flowrate through a choked nozzle as a function of only nozzle throat area, A^*, ideal gas properties and the stangation condition temperature and pressure.

Let's define a new parameter, called characteristic velocity and denoted c^* using this result:

$$c^* = \frac{P_t A^*}{\dot{m}} = \sqrt{\frac{RT_t}{\gamma}} \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{8}$$

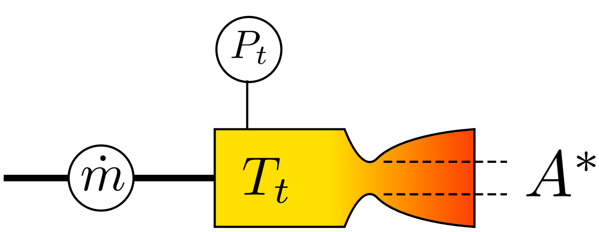


Figure 0.1: c^* parameters of interest. $\dot{m}$, P_t and A^* are all easily measureable. T_t is difficult due to the extremely high temperatures of chemical rockets.

c^* is useful far beyond simple choked flow considerations because it can be both **measured** and **computed**. And indeed the equation above shows this directly - the LHS is a function measure-able variables stagnation pressure P_t[1], area A^* and mass flowrate $\dot{m}$. The RHS is a function of intrinsic gas properties γ, R and stagnation temperature. c^* gives us the tools to calculate a parameter (RHS) that we can go and easily measure in the lab (LHS).

Given that c^* depends on γ, R and T_t it is essentially a function only of the working fluids thermodynamic properties and stagnation (chamber) state. This is not 100% true - when we get to combustion we will see how a (weak) dependency on pressure comes back into c^*, but for conceptual purposes we should think of c^* this way - as only a function of intrinsic thermodynamic properties of our propellants.

And so finally we come to a very interesting result regarding the functional dependence of c^*:

$$c^* = f(T_t, R, \gamma) \tag{9}$$

R is the specific gas constant $R = \frac{R_u}{M_w}$ which is inversely dependent on gas molecular weight, M_w. Thus

$$c^* \propto \sqrt{\frac{T_t}{M_w}} \tag{10}$$

Moreover T_t is related to the gas internal stagnation enthalpy by $\Delta h_t = \int_{T_0}^{T_t} C_p dT$ and so we should see T_t as a representation of the **energy content** of the rocket gasses.

The effect of γ on c^* is a bit more subtle. γ is fundamentally related to the number of vibratory degrees of freedom a molecule has. For monatomic systems (such as helium or atomic hydrogen) γ assymptotes to an upper limit of 1.66. For most simple diatoms (nitrogen, oxygen, hydrogen) it is around 1.4 and for larger molecules or those with more complex bonding it is lower. For the conditions we are interested in within a rocket, we wouldn't expect to find γ much lower than 1.1.

[1] In a finite sized rocket chamber we are not actually measuring stagnation pressure because there will be non-zero gas velocity. For reasonable *contraction ratios*, A_1/A^*, this is a small effect and it can be mostly corrected analytically using isentropic flow relations

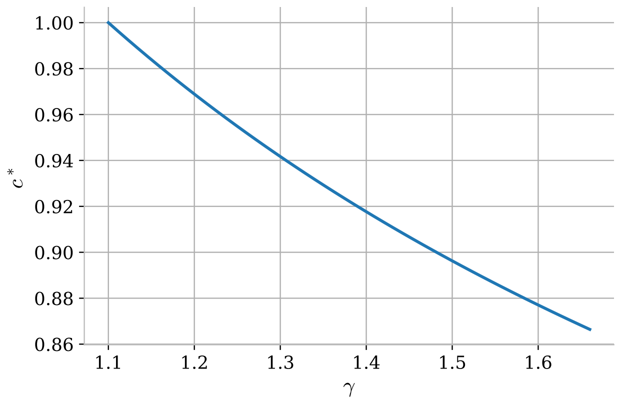


Figure 0.2: c^* shows only a weak dependence on γ

The plot below shows the effect of γ on c^* which is fairly limited compared with T_t and M_w. And so if you take one thing away from this discussion, remember that

$$c^* \propto \sqrt{\frac{\Delta h_t}{\overline{C_p} M_w}} \tag{11}$$

0.3 C_f

Ok, that's interesting, but for the purposes of rocket performance we want C not c^*. Take a look at the definition of c^* and it looks a lot like our definition of effective velocity and specific impulse above:

$$c^* = \frac{P_t A^*}{\dot{m}} \sim \frac{T}{\dot{m}} = C \tag{12}$$

and indeed c^* and C **are** closely related.

In order to see how, let's define another new parameter, C_f that related thrust to the nozzle throat area and rocket total pressure:

$$C_f = \frac{T}{P_t A^*} \tag{13}$$

In prose this says:

> *C_f represents the amount of thrust a rocket can produce given the stagnation pressure of its propellants and a useful characteristic fluid area - the choked area of its nozzle.*

Stated differently it is a measure of how effectively we take the stagnation pressure we generate in the chamber and turn it into thrust.

And since c^* is also defined by chamber pressure and nozzle throat area, C_f becomes the connection between C and c^*:

$$C = \frac{T}{\dot{m}} = \frac{C_f P_t A^*}{\dot{m}} = C_f c^* \tag{14}$$

But why split C into C_f and c^* this way?

Remember that c^* represents the potential of the rocket propellants themselves to create thrust and essentially depends exclusively on the thermodynamic characteristics of those propellants. C_f represents how well our nozzle can convert the propellant's latent utility into real thrust and thus depends almost exclusively on the physical nature of our nozzle. And so as we go about maximizing C, we can divide that into two separate problems - one of picking propellants (c^*) and the other of chosing system pressures and nozzle geometry (C_f).

Since the primary role of the nozzle is to convert gasses into thrust C_f can also be seen as a measure of the "goodness" of the nozzle. It is called **thrust coefficient**.

C_f can be expanded from its definition above:

$$\begin{aligned} C_f = \frac{C}{c^*} = \frac{(P_e - P_0)\frac{A_e}{\dot{m}} + U_e}{\frac{P_t A^*}{\dot{m}}} = \frac{P_e - P_0}{P_t}\frac{A_e}{A^*} + \frac{U_e}{c^*} \\ \Rightarrow \left(1 - \frac{P_0}{P_e}\right)\frac{P_e A_e}{P_t A^*} + \frac{U_e}{c^*} \end{aligned} \tag{15}$$

It is worth noting that, like the thrust equation, there are two pieces to C_f: $\left(1 - \frac{P_0}{P_e}\right)\frac{P_e A_e}{P_t A^*}$ is representative of thrust created through pressure force and $\frac{U_e}{c^*}$ is representative of the contribution of gas momentum to thrust. We will refer to these two components when we discuss optimal nozzle expansion in a minute.

Beyond this things get a little messy and different people attack the derivation different ways. Rather than putting a whole bunch of alegbra in here, I'm going to provide you the tools needed to compute C_f practically.

$\frac{U_e}{c^*}$, $\frac{P_e}{P_t}$ and $\frac{A_e}{A^*}$ are all related to the isentropic expansion of gasses through the nozzle to its exit. The exit mach number, M_e thus becomes the common parameter and using classic isentropic relations we can derive the functional relationship of each with M_e as the independent variable:

$$\frac{U_e}{c^*} = \frac{\gamma M_e}{\sqrt{1 + \frac{\gamma-1}{2}M_e^2}}\left[\frac{\gamma+1}{2}\right]^{\frac{\gamma+1}{-2(\gamma-1)}} \tag{16}$$

$$\frac{P_e}{P_t} = \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\frac{-\gamma}{\gamma-1}} \tag{17}$$

$$\frac{A_e}{A^*} = \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma+1}{-2(\gamma-1)}} \frac{\left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\frac{\gamma+1}{2(\gamma-1)}}}{M_e} \tag{18}$$

This is a useful form because it shows very clearly that $\frac{U_e}{c^*}$, $\frac{P_e}{P_t}$ and $\frac{A_e}{A^*}$ are not all independent (they all depend on M_e). In fact the dimensionality of this set is one - picking a number for any one of these directlys sets the others. Furthermore note that these equations, unlike c^*, have **no**

dependency on stagnation temperature or molecular weight and thus no direct dependency on propellant properties. They do depend on γ which is a property of the working fluid but as with c^*, the dependence is not terribly strong and is really set for us by the propellant choice we made in optimizing c^*.

I'd like to look at how the parameter we can control directly, A_e/A^*, affects the others and C_f. Using the equations above, we will sweep through M_e and compute the other parameters directly.

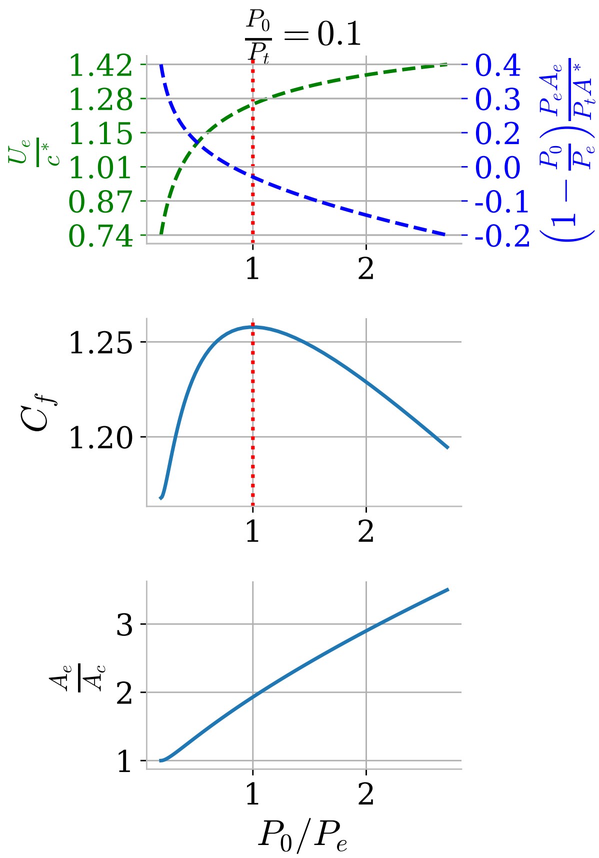


Figure 0.3: Relationship of the components of C_f to each other and to C_f itself.

Note that the momentum component grows as exit pressure decreases because the gasses are accelerated further before release. Conversely, the pressure component decreases as exit pressure decreases for obvious reasons. In fact the pressure component of C_f takes on negative values for exit pressures below ambient - this is called over-expansion.

A_e/A^* increases monotonically with decreasing exit pressure - this simply represents the fact that to achieve lower exit pressures we must use a larger nozzle expansion ratio.

Finally note that C_f reaches a maximum where $P_e/P_0 = 1$. This is called optimal expansion. It is a fundamental result in rocket theory and can be stated:

Rocket effective exhaust velocity (and therefore specific impulse) is maximized when the nozzle expands the exhaust gasses such that they match the local pressure at the nozzle exit.

In general, we want to design a rocket nozzle to match the exit pressure to ambient pressure. This is difficult to do for a rocket ascending through the atmosphere where the pressure is continuosly changing. For traditional nozzles, a compromise that looks at the average performance over the ascent pressure profile is often chosen.

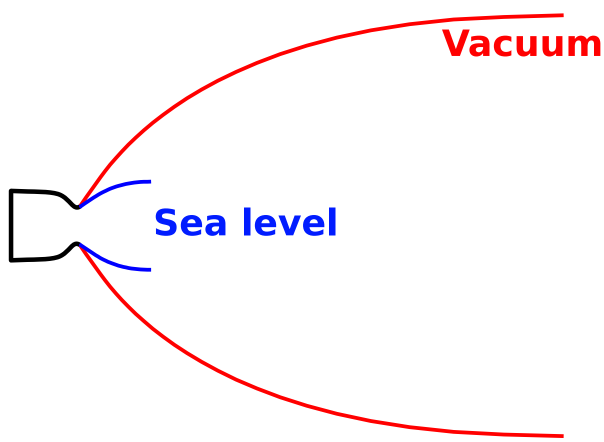


Figure 0.4: Note the size difference between a sea-level and vacuum- optimized nozzle. The rocket itself is identical except for the nozzle expansion.

And since the way we lower exit pressure is with a bigger expansion ratio, $\epsilon = \frac{A_e}{A^*}$, this result explains why upper stage "vacuum nozzles" are so much bigger than first stage "sea-level" nozzles as can be seen in graphic above and the side-by-side of the SpaceX Merlin 1D vacuum (left) and Merlin 1D sea-level (right) engines:

Figure 0.5: SpaceX' Merlin engine has very different size nozzles for its first and second stage even though the engine itself is nearly identical.

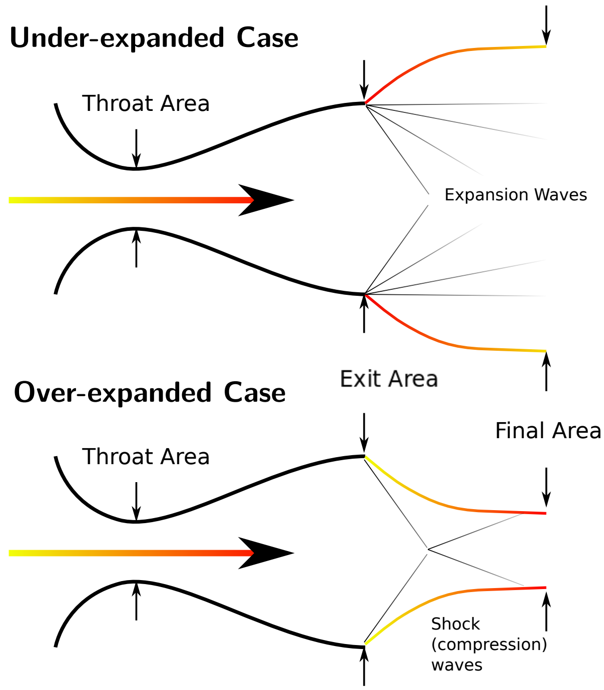


Figure 0.6: In a super-sonic exhaust stream, expansion or compression waves form to equilibrate the jet pressure to the surrounding pressures.

Figure 0.7: An example of mach diamonds in the plume of one of Stanford's research hybrid rockets

So what actually happens when a nozzle is under- or over-expanded? In a subsconic (incompressivle) jet, the fluid dynamics allow for communication of information upstream and the pressure distribution in the nozzle would relax to a state where $P_e = P_0$. In supersonic (compressible flow), pressure information travels more slowly than the fluid velocity so there is no information transfer upstream. In order to equilibrate pressure with the ambient, a set of expansion or compression waves form from the exit of the nozzle.

Due to flow radial inhomogeneity in the flow in real nozzles, there will always be some equilibration wave structure at the exit. The pattern of compression and expansion waves and associated changes in gas temperature and luminosity are what are generate *Mach Diamonds* in the exhaust stream.

0.4 c^* Efficiency

Due to imperfect mixing, combustion, chamber heat-loss and other effects the realized rocket performance is typically less than that computed theoretically. This will be reflected in the rocket thrust, effective velocity and c^* and potentially C_f.

Becuase many of the larger losses are associated with the process of combustion in the chamber, the non-ideal performance is most easily and directly measured in c^*. It is with this in mind that c^* *efficiency* is defined:

$$\eta_{c^*} = \frac{c^*_{measured}}{c^*_{ideal}} = \frac{P_t A^*}{\dot{m}_p c^*_{ideal}} \tag{19}$$

where c^*_{ideal} is computed using eq. (8) for simple substances or, more commonly, using a complex chemical equillibrium code as will be discussed in the next lecture.

Optimizing η_{c^*} becomes a primary activity in the development of a rocket engine, often consuming a singificant fraction of the development budget and time. In liquid rockets this might manifest as injector tuning, while in a solid rocket it is attacked by adjusting propellant composition, ratios and particle sizes.

0.5 Putting it all together

The separation of C into c^* and C_f separates concerns between propellant thermodyanmics in c^* and nozzle physical parameters in C_f.

Let's quickly compute C, c^* and C_f to get a sense of how we would begin to use them practically. Assume we start with a plenum full of room-temperature, high pressure gas and then vent it into vacuum through a nozzle.

Now let's compute c^*, C_f and then C. For now we will assume optimal expansion ($P_0/P_e = 1$) and calculate the associated area ratio. A useful relation we will use to determine M_e is:

$$M_e^2 = 2\left(\frac{\frac{Pe}{Pt}^{\frac{1-\gamma}{\gamma}} - 1}{\gamma - 1}\right) = 2\left(\frac{\left(\frac{P_e}{P_0}\frac{P_0}{P_t}\right)^{\frac{1-\gamma}{\gamma}} - 1}{\gamma - 1}\right) \tag{20}$$

Name	γ	M_w	c^* (m/s)	C_f	C (m/s)
N_2	1.40	28	434.4	1.26	546.4
H_2	1.41	2	1621.5	1.26	2039.2
He	1.66	4	1085.2	1.25	1356.2

Table 0.2: c^*, C_f, C for several pure gasses at room temperature, $P_0/P_e = 1$ and $P_0/P_t = 0.1$. Note how C_f is essentially constant despite the differences in gas properties while c^* and C vary substantially.

This simple exercise confirms what we said before - c^* depends on the nature of working fluid while C_f is largely fixed by the nozzle parameters. Another interesting conclusion from this is that even before we get to chemistry, we can see how wonderful of a working fluid Hydrogen is due to it's low molecular weight. Remember this because hydrogen will emerge again and again as we talk about rockets.

If we expand eq. (14) and simplify we arrive at a useful form for C:

$$C = \sqrt{\frac{2\gamma}{\gamma-1}\frac{R_u T_t}{M_w}\left[1 - \left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right]} \tag{21}$$

In eq. (21) we can see all the impacts of propellant choice and nozzles on effective exhaust velocity. The term involving T_t and M_w clearly shows the effect of propellant choice on C, as we noted with c^* earlier. The term involving $\frac{P_e}{P_t}$ demonstrates the impact of the system pressures and nozzle geometry we choose. fig. 0.8 summarizes all of this.

0.6 Energy and efficiency

So far we have concentrated on propulsion primarily from a conservation of momentum perspective. But there are a lot interesting observations to be made when we look at the thermodynamics of propulsion as well.

With rocket propulsion and things in space we are talking about **a lot of energy** and to put that in to context, let's

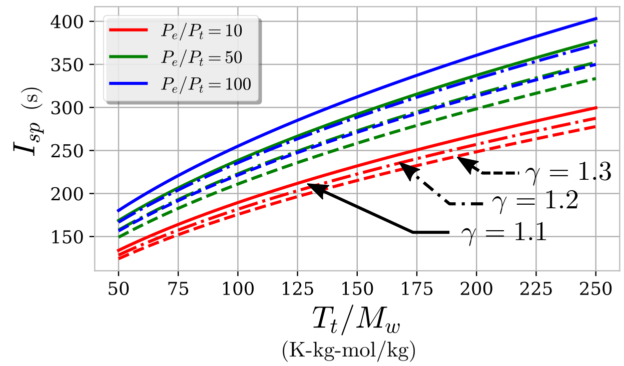


Figure 0.8: How specific impulse varies

see how much energy our Low Earth Orbit satellite ends up with.

An object moving in a conservative potential field (like Earth's gravity) has total energy:

$$E = \text{KE} + \text{PE} = \frac{mv^2}{2} + \frac{GMm}{r}$$

or in **mass-specific energy**

$$\epsilon = \frac{v^2}{2} + \frac{GM}{r}$$

In the case of our satellite in a circular Low Earth Orbit this comes out to:

$$\epsilon = \frac{v^2}{2} + \frac{\mu}{r_0} - \frac{\mu}{r} = 29.1\text{MJ/kg} + 4.5\text{MJ/kg} = \mathbf{33.6MJ/kg}$$

Again note how much of this is due to the orbital velocity of the rocket. And to put in context just how much energy this is, it is equivalent to:

- 1000x the specific energy of a Jet airliner at 650MPH
- 5x the specific chemical energy content of nitroglycerin
- 3x the required specific energy to melt aluminum

It is no wonder that not much is left when object with this much energy slams into the atmosphere!

0.7 Rocket efficiency

Clearly there is a lot of energy being liberated in rocket engines in order to put things into space. But how efficient are they?

There are different ways to define energy efficiency, but the the first we'll look at is the **thermal efficiency** or how effectively a rocket takes input energy and converts it to gas kinetic energy:

$$\eta_{thermal} = \frac{\dot{W}_{exhaust}}{P_{in}} = \frac{\dot{m}C^2}{2P_{in}}$$

where P_{in} is the amount of energy being liberated in time to power the rocket whether it be latent, chemical, nuclear or electrical energy. In the case of chemical rockets we can define it as the energy released by the propellants when they combust or:

$$P_{in} = \dot{m}\Delta h$$

where Δh is the amount of thermal energy going into the rocket chamber.

Substituting eq. (21) gives

$$\eta_{thermal} = \frac{\frac{2\gamma}{\gamma-1}RT_t\left[1-\left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right]}{2\Delta h} \tag{22}$$

Because

$$T_t \sim \frac{\Delta h}{\overline{\overline{C_p}}}$$

we can subsitute such that

$$\eta_{thermal} = \frac{\gamma}{\gamma-1}\frac{R}{\overline{\overline{C_p}}}\left[1-\left(\frac{P_e}{P_t}\right)^{(\gamma-1)/\gamma}\right] \tag{23}$$

And finally recognizing that for an ideal gas

$$\frac{R}{\overline{\overline{C_p}}} = \frac{\gamma-1}{\gamma}$$

and for a perfectly isentropic nozzle expansion process

$$\frac{T_e}{T_t} = \frac{P_e}{P_t}^{(\gamma-1)/\gamma}$$

we see that eq. (22) reduces to the Carnot efficiency.

fig. 0.9 shows $\eta_{thermal}$ as a function of P_t/P_e:

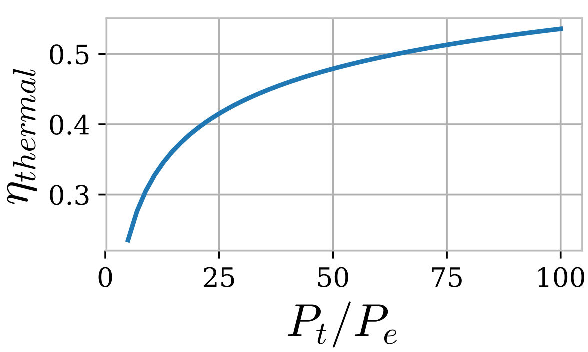


Figure 0.9: How specific impulse varies

This is pretty good for a heat engine and a lot of early rocket work focused on optimizing thermal efficiency. And thermal efficiency will matter when we get to electric propulsion, but it is less useful a construct for chemical rockets where it is largely defined by the propellant properties rather than the machine they burn in. Furthermore the efficiency of accelerating gasses is not really what we care about - we care about the efficiency of accelerating the vehicle.

To understand that, we will define another, more useful effiency metric, called **propulsive efficiency** as such

$$\eta_{prop} = \frac{\dot{W}_r}{\dot{W}_r + \dot{W}_{exhaust}} = \frac{Tv}{Tv + \frac{1}{2}\dot{m}(C-v)^2}$$

or the fraction of total work that actually goes to accelerating the vehicle.

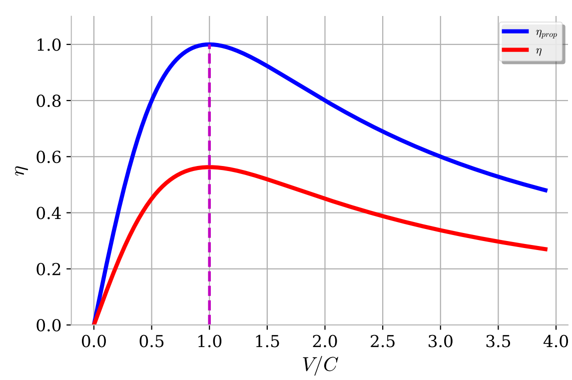
$$\frac{Tv}{Tv + \frac{1}{2}\dot{m}(C-v)^2} = \frac{\dot{m}Cv}{\dot{m}\left[Cv + (C-v)^2\right]}$$

$$\rightarrow \eta_{prop} = 2\frac{\frac{v}{C}}{1+(\frac{v}{C})^2}$$

The overall efficiency is

$$\eta = \eta_{thermal}\eta_{prop}$$

A small example demonstrating how η varies over ratio of vehicle speed to exhaust speed, $\frac{v}{C}$ is shown below.

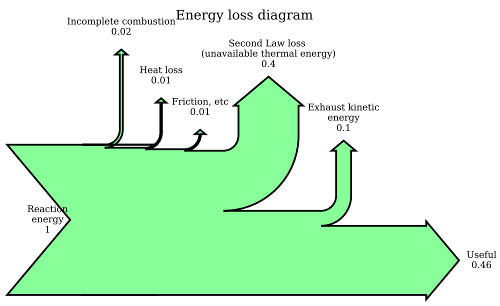


And here we see another interesting result much like optimal nozzle expansion. The rocket is most efficient when the exhaust velocity equals the vehicle velocity. This is actually quite intuitive - in this condition the exhaust gasses are left with zero velocity in the static frame. There is no residual kinetic energy in the exhaust so all of it's kinetic energy have been transferred to the vehicle. Anywhere else the exhaust is left with residual kinetic energy which is a loss.

Propulsive efficiency will come back in a big way when discuss electrical propulsion as it is very important to system optimization.

0.8 Energy losses

Finally let's also look at where the inefficiencies are for a typical launch vehicle powered by a chemical engine with $I_{sp} = 340$ s and moving at 5,000 m/s for intuition.



Notice that almost half of the chemical energy is unavailable for conversion to work due to second law of thermodynamics. This is expected given the Carnot efficiency we developed earlier. The biggest contributor after second law loss is residual exhaust kinetic energy which we saw is something is minimized when $V/C = 1$. The other losses are quite small and we can say

> *When a rocket operates with exhaust velocity near vehicle velocity, it is a surprisingly efficient heat engine*