SECTIONS 5.2/5.4   BASIC PROPERTIES OF EIGENVALUES AND EIGENVECTORS / SIMILARITY TRANSFORMATIONS

Eigenvalues of an $n \times n$ matrix A are $\{\lambda : \text{there exists a vector } x \neq 0 \text{ for which } Ax = \lambda x\}$. Such a vector x is called an eigenvector, and (λ, x) is called an eigenpair of A.

Some applications -- see Section 5.1.

Clearly any nonzero scalar multiple of an eigenvector is also an eigenvector:

$$Ax = \lambda x \quad \Rightarrow \quad A(\alpha x) = \lambda(\alpha x).$$

More generally, if $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ are eigenvectors associated with the same eigenvalue λ, then any nonzero linear combination of them is also an eigenvector:

$$A\left(\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \cdots + \alpha_k x^{(k)}\right) = \alpha_1(\lambda x^{(1)}) + \alpha_2(\lambda x^{(2)}) + \cdots + \alpha_k(\lambda x^{(k)})$$
$$= \lambda\left(\alpha_1 x^{(1)} + \alpha_2 x^{(2)} + \cdots + \alpha_k x^{(k)}\right)$$

The eigenvalues are the n roots of the characteristic equation

$$\det(A - \lambda I) = 0.$$

This is a polynomial of degree n in the variable λ.

EXAMPLE

$$A = \begin{bmatrix} 8 & -1 & -5 \\ -4 & 4 & -2 \\ 18 & -5 & -7 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 8-\lambda & -1 & -5 \\ -4 & 4-\lambda & -2 \\ 18 & -5 & -7-\lambda \end{bmatrix} = -\lambda^3 + 5\lambda^2 - 24\lambda + 20$$

The eigenvalues are the zeros of this cubic polynomial: $1,\ 2+4i,\ 2-4i$. For each λ, an eigenvector can be obtained from the homogeneous system of linear equations $(A-\lambda I)x=0$. For example, for $\lambda=1$,

$$\begin{bmatrix} 7 & -1 & -5 \\ -4 & 3 & -2 \\ 18 & -5 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that the coefficient matrix of this system is necessarily <u>singular</u>. It can be solved by choosing an arbitrary value for one of the variables, and then solving for the other 2 variables. For example, letting $x_3=1$, gives the system

$$7x_1-x_2=5$$
$$-4x_1+3x_2=2$$
$$18x_1-5x_2=8$$

which gives

$$x=\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

NOTE: Real matrices can have complex eigenvalues. Any complex eigenvalues occur in complex conjugate pairs. Eigenvectors associated with complex eigenvalues are in general complex valued.

Matrices A and B are said to be <u>similar</u> if there exists a nonsingular matrix T such that

$$A=T^{-1}BT .$$

Similar matrices have the <u>same eigenvalues</u>, as their characteristic polynomials are identical:

$$\det(A-\lambda I)=\det(T^{-1}BT-\lambda I)$$
$$=\det\left(T^{-1}(B-\lambda I)T\right)$$
$$=\left(\det T^{-1}\right)\det(B-\lambda I)\left(\det T\right)$$
$$=\det(B-\lambda I),\quad \text{since}\quad \det T^{-1}=\frac{1}{\det T}.$$

The eigenvectors of A are easily obtained from those of B by multiplying by T^{-1}:

$$Bx = \lambda x \quad \Rightarrow \quad T^{-1}BTT^{-1}x = T^{-1}(\lambda x) \quad \Rightarrow \quad A(T^{-1}x) = \lambda(T^{-1}x).$$

OTHER SIMPLE EIGENVALUE/EIGENVECTOR RELATIONSHIPS

(i) eigenvalues and eigenvectors of a shifted matrix $A - \alpha I$:

$$Ax = \lambda x \quad \Rightarrow \quad (A - \alpha I)x = (\lambda - \alpha)x$$

(ii) A is nonsingular if and only if 0 is not an eigenvalue of A.

-- follows since 0 is an eigenvalue if and only if there exists a nonzero vector such that $Ax = 0$

(iii) eigenvalues and eigenvectors of A^{-1} :

$$Ax = \lambda x \quad \Rightarrow \quad A^{-1}x = \frac{1}{\lambda}x$$

(iv) The eigenvalues of A and A^T are the same (as their characteristic polynomials are the same), but there is no simple relationship between their eigenvectors.

(v) eigenvalues and eigenvectors of powers of A:

$$Ax = \lambda x \quad \Rightarrow \quad A^2x = A(\lambda x) = \lambda(Ax) = \lambda^2 x.$$

In general, the eigenvalues of A^k are λ^k, and the eigenvectors are the same as those of A.

(vi) The eigenvalues of a triangular matrix are clearly its diagonal entries. The eigenvalues of a block triangular matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ & A_{22} & \cdots & A_{2k} \\ & & \ddots & \vdots \\ & & & A_{kk} \end{bmatrix}$$

are the union of the eigenvalues of the diagonal submatrices $A_{11}, A_{22}, \ldots, A_{kk}$.

How many <u>linearly independent eigenvectors</u> can an $n \times n$ matrix have?

Answer: anywhere from 1 to n. The following examples illustrate the possible cases when $n = 3$.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ has 3 linearly independent eigenvectors.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ has 2 linearly independent eigenvectors.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$ has 1 linearly independent eigenvector.

A matrix A is called <u>defective</u> (or <u>not semisimple</u> in your textbook) if it does <u>not</u> have a set of n linearly independent eigenvectors.

A matrix A with n linearly independent eigenvectors is <u>nondefective</u> or (in our text) <u>semisimple</u>.

NOTE: If a matrix has an eigenvalue of multiplicity greater than 1, it may be defective or nondefective: see the above 3 examples (where each matrix has one eigenvalue of multiplicity 3).

THEOREM 5.2.11 (page 308)
        Eigenvectors corresponding to distinct eigenvalues are linearly independent.

COROLLARY 5.2.12
        If A has n distinct eigenvalues, then A is nondefective.

THEOREM 5.4.6 (page 335)
        A matrix A is nondefective if and only if there exists a nonsingular matrix V such that $V^{-1}AV = D$, where

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

and $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ are the eigenvalues of A.

<u>Proof</u>. Let $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ be a set of n linearly independent eigenvectors, and suppose that

$$Ax^{(i)} = \lambda_i x^{(i)}, \qquad 1 \le i \le n. \qquad (*)$$

Let $V = \begin{bmatrix} x^{(1)} & x^{(2)} & \cdots & x^{(n)} \end{bmatrix}$. Then the n equations in (*) are equivalent to the matrix equation

$$AV = VD, \quad \text{which implies that } V^{-1}AV = D.$$

For the converse, if there exists a matrix V such that $V^{-1}AV = D$, then $AV = VD$. This matrix equation is equivalent to the n equations

$$Ax^{(i)} = \lambda_i x^{(i)}, \qquad 1 \le i \le n,$$

where $x^{(i)}$ denotes the i-th column vector of V. Since V is nonsingular, the n eigenvectors $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ are linearly independent, implying that A is nondefective.

NOTE: if matrix V diagonalizes A (that is, $V^{-1}AV = D$ is diagonal), then the above proof shows that the column vectors of V are a set of n linearly independent eigenvectors of A.

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<u>Definitions</u>
A is a real symmetric matrix if $A = A^T$.
A complex matrix A is Hermitian if $A = A^*$.
U is a (real) orthogonal matrix if $U^TU = UU^T = I$ or $U^T = U^{-1}$.
U is a (complex) unitary matrix if $U^*U = UU^* = I$ or $U^* = U^{-1}$.
A and B are similar if $A = T^{-1}BT$.
A and B are orthogonally similar if $A = U^TBU$, where U is orthogonal.
A and B are unitarily similar if $A = U^*BU$, where U is unitary.

THEOREM 5.4.11 (Schur's Theorem, page 337)
Given any square matrix A, there exists a unitary matrix U such that $T = U^{-1}AU = U^*AU$ is upper triangular. The diagonal entries of T are the eigenvalues of A.

THEOREM 5.4.12 (Spectral Theorem, page 339)
  If A is a Hermitian matrix, then there exists a unitary matrix U such that
$D = U^{-1}AU = U^*AU$ is diagonal. The diagonal entries of D are the eigenvalues of A.

Proof. By Schur's Theorem, U^*AU is upper triangular. But
$(U^*AU)^* = U^*A^*U = U^*AU$ implies that U^*AU is also Hermitian, and thus is diagonal.

NOTE: If A is real symmetric, then the matrix U of Theorem 5.4.12 is a (real) orthogonal matrix such that U^TAU is diagonal (see Theorem 5.4.19, page 340).

It follows from the Spectral Theorem that the eigenvalues of a Hermitian matrix (including a real symmetric matrix) are real: since U^*AU is Hermitian and diagonal, its diagonal entries (which are the eigenvalues of A) are real. See Corollary 5.4.13, page 339.

The most important result for arbitrary real matrices A using (real) orthogonal similarity transformations is

THEOREM 5.4.22 (the Real Schur Theorem or the Wintner-Murnaghan Theorem, page 341)
  If A is a real matrix, then there exists a (real) orthogonal matrix U such that
$T = U^TAU$ is a quasi-triangular matrix

$$
T = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1k} \\ & T_{22} & \cdots & T_{2k} \\ & & \ddots & \vdots \\ & & & T_{kk} \end{bmatrix}.
$$

(That is, T is block upper triangular and each main diagonal submatrix T_{ii} is either 1×1 or 2×2).

NOTE: the eigenvalues of A are the union of the eigenvalues of the main diagonal submatrices T_{ii}. Any real eigenvalues of A will occur as 1×1 submatrices and any pairs of complex conjugate eigenvalues of A will occur as the 2 eigenvalues of some 2×2 submatrix.

SUMMARY OF RESULTS USING SIMILARITY TRANSFORMATIONS

GENERAL SIMILARITY TRANSFORMATIONS

A is nondefective (semisimple)	A is defective (not semisimple)
-- there exists V such that $V^{-1}AV$ is diagonal	-- there exists X such that $X^{-1}AX$ is bidiagonal (the Jordan normal form)

UNITARY SIMILARITY TRANSFORMATIONS

arbitrary matrix A	matrix A is Hermitian
-- there exists a unitary matrix U such that U^*AU is upper triangular	-- there exists a unitary matrix U such that U^*AU is diagonal

ORTHOGONAL SIMILARITY TRANSFORMATIONS

arbitrary real matrix A	A is real symmetric
-- there exists an orthogonal matrix P such that P^TAP is quasi-triangular (with all diagonal blocks of order 1 or 2)	-- there exists an orthogonal matrix P such that P^TAP is diagonal

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The equivalence of computing <u>eigenvalues</u> and computing the <u>zeros of polynomials</u>

We have already seen that the eigenvalues are the zeros of the characteristic polynomnial.

On the other hand, given any (monic) polynomial

$$p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

the companion matrix of $p(\lambda)$ is

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix},$$

where all unspecified entries of A are 0. Then

$$\det(\lambda I - A) = p(\lambda)$$

and thus the zeros of $p(\lambda)$ are the eigenvalues of A.