Introduction

The world we live in is a strange and wonderful place. The vast majority of our interactions with objects around us can be described and predicted by relatively simple equations and relations, which have been fully understood for centuries now. The concept of this tactile world is however, a result of averaging a vast ensemble of smaller effects on an unimaginably small scale, all conspiring together to produce what we observe at the macroscopic level. In order to investigate this realm in which particles can fluctuate in and out of existence in the blink of an eye, we must turn to more sophisticated descriptions of the players on this stage.

This research has the goal of furthering our understanding of the interior of what are loosely called 'neutron stars', though as shall be shown herein, the contents are not necessarily only neutrons. With this in mind, further definitions include '*hyperon stars*', '*quark stars*', and '*hybrid stars*', to describe compact stellar objects containing hyperons[1], quarks, and a mixture of each, respectively. In order to do this, we require physics beyond that which describes the interactions of our daily lives; we require physics that describes the individual interactions between particles, and physics that describes the interactions between enormous quantities of particles.

Only by uniting the physics describing the realm of the large and that of the small can one contemplate so many orders of magnitude in scale; from individual particles with a diameter of less than 10^{-22} m, up to neutron stars with a diameter of tens of kilometers. Yet the physics at each end of this massive scale are unified in this field of nuclear matter in which interactions of the smallest theorized entities conspire in such a way that densities equivalent to the mass of humanity compressed to the size of a mere sugar cube become commonplace, energetically favourable, and stable.

The sophistication of the physics used to describe the world of the tiny and that of the enormous has seen much development over time. The current knowledge of particles has reached a point where we are able to make incredibly precise predictions about the properties of single particles and have them confirmed with equally astonishing accuracy from experiments. The physics describing neutron stars has progressed from relatively simple (yet sufficiently consistent with experiment) descriptions of neutron (and nucleon) matter to many more sophisticated descriptions involving various species of baryons, mesons, leptons and even quarks.

The outcome of work such as this is hopefully a better understanding of matter at both the microscopic and macroscopic scales, as well as the theory and formalism that unites these two extremes. The primary methods which we have used to construct models in this thesis are Quantum Hadrodynamics (QHD)—which shall be described in Sec. **??**—and the Quark-Meson Coupling (QMC) model, which shall be described in Sec. **??**.

In this thesis, we will outline the research undertaken in which we produce a model for neutron star structure which complies with current theories for dense matter at and above nuclear density and is consistent with current data for both finite nuclei and observed neutron stars. Although only experimental evidence can successfully validate any theory, we hope

[1] Baryons for which one or more of the three valence quarks is a strange quark. For example, the Σ^+ hyperon contains two up quarks and a strange quark. For a table of particle properties, including quark content, refer to Table **??**

to convey a framework and model that possesses a minimum content of inconsistencies or unjustified assumptions such that any predictions that are later shown to be fallacious can only be attributed to incorrect initial conditions.

As a final defense of any inconsistencies that may arise between this research and experiment, we refer the reader to one of the author's favourite quotes:

> “ There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable.
>
> There is another theory which states that this has already happened. ”
>
> – Douglas Adams, The Hitchhiker's Guide To The Galaxy.

Our calculations will begin at the particle interaction level, which will be described in more detail in Section 2, from which we are able to reproduce the bulk properties of matter at high densities, and which we shall discuss in Sections **??**–**??**. The methods for producing our simulations of the interactions and bulk properties will be detailed in Section **??**, along with a discussion on how this is applied to the study of compact stellar objects. The results of the simulations and calculations will be discussed in detail in Section **??**, followed by discussions on the interpretation of these results in Section **??**. For the convenience of the reader, and for the sake of completeness, derivations for the majority of the equations used herein are provided in Appendix **??**, and useful information regarding particles is provided in Appendix **??**. For now however, we will provide a brief introduction to this field of study.

1.1 The Four Forces

Theoretical particle physics has seen much success and found many useful applications; from calculating the individual properties of particles to precisions that rival even the best experimental setups, to determining the properties of ensembles of particles of greater and greater scale, and eventually to the properties of macroscopic objects as described by their constituents.

In order to do this, we need to understand each of the four fundamental forces in Nature. The weakest of these forces—gravity—attracts any two masses, and will become most important in the following section. Slightly stronger is electromagnetism; the force responsible for electric charge and magnetism. This force provides an attraction between opposite electric charges (and of course, repulsion between like charges), and thus helps to bind electrons to nuclei. The mathematical description of this effect is Quantum Electrodynamics (QED).

The ‘weak nuclear force’—often abbreviated to ‘the weak force’—is responsible for the decay of particles and thus radioactivity in general. At high enough energies, this force is unified with electromagnetism into the ‘electro-weak force’. The strongest of all the forces is the ‘strong nuclear force’, abbreviated to ‘the strong force’. This force is responsible for attraction between certain individual particles over a very short scale, and is responsible for the binding of protons within nuclei which would otherwise be thrown apart by the repulsive electromagnetic force between the positively charged protons. Each of these forces plays a part in the work contained herein, but the focus of our study will be the strong force.

A remarkably successful description of the strongest force at the microscopic level is 'Quantum Chromodynamics' (QCD) which is widely believed to be the true description of strong interactions, relying on quark and gluon degrees of freedom. The major challenge of this theory is that at low energies it is non-perturbative[2], in that the coupling constant —which one would normally perform a series-expansion in powers of—is large, and thus is not suitable for such an expansion. Regularisation techniques have been produced to create perturbative descriptions of QCD, but perturbative techniques fail to describe both dynamical chiral symmetry breaking and confinement; two properties observed in Nature.

Rather than working directly with the quarks and gluons of QCD, another option is to construct a model which reproduces the effects of QCD using an effective field theory. This is a popular method within the field of nuclear physics, and the route that has be taken for this work. More precisely, we utilise a balance between attractive and repulsive meson fields to reproduce the binding between fermions that the strong interaction is responsible for.

1.2 Neutron Stars

Although gravity may be the weakest of the four fundamental forces over comparable distance scales, it is the most prevalent over (extremely) large distances. It is this force that must be overcome for a star to remain stable against collapse. Although a description of this force that unifies it with the other three forces has not been (satisfactorily) found, General Relativity has proved its worth for making predictions that involve large masses.

At the time when neutron stars were first proposed by Baade and Zwicky [**?**], neutrons had only been very recently proven to exist by Chadwick [**?**]. Nonetheless, ever increasingly more sophisticated and applicable theories have continually been produced to model the interactions that may lead to these incredible structures; likely the most dense configuration of particles that can withstand collapse.

The current lack of experimental data for neutron stars permits a wide variety of models [**?**, **?**, **?**, **?**, **?**, **?**], each of which is able to successfully reproduce the observed properties of neutron stars, and most of which are able to reproduce current theoretical and experimental data for finite nuclei and heavy-ion collisions [**?**, **?**]. The limits placed on models from neutron star observations [**?**, **?**, **?**] do not sufficiently constrain the models, so we have the opportunity to enhance the models based on more sophisticated physics, while still retaining the constraints above.

The story of the creation of a neutron star begins with a reasonably massive star, with a mass greater than eight solar masses ($M > 8\ M_{\odot}$). After millions to billions of years or so (depending on the exact properties of the star), this star will have depleted its fuel by fusion of hydrogen into ^{3}He, ^{4}He, and larger elements up to iron (the most stable element since has the highest binding energy per nucleon).

At this point, the core of the star will consist of solid iron, as the heaviest elements are gravitationally attracted to the core of the star, with successively lighter elements layered on top in accordance with the traditional onion analogy. The core is unable to become any more stable via fusion reactions and is only held up against gravitational collapse by the

[2] At high energies, QCD becomes asymptotically free [**?**] and can be treated perturbatively. The physics of our world however is largely concerned with low energies.

degeneracy pressure of the electrons[3]. The contents of the upper layers however continue to undergo fusion to heavier elements which also sink towards the core, adding to the mass of the lower layers and thus increasing the gravitational pressure below.

This causes the temperature and pressure of the star to increase, which encourages further reactions in the upper levels. Iron continues to pile on top of the core until it reaches the Chandrasekhar limit of $M = 1.4\ M_{\odot}$, at which point the electron degeneracy pressure is overcome. The next step is not fully understood[4], but the result is a Type II supernova.

At the temperatures and pressures involved here, it is energetically favourable for the neutrons to undergo β-decay into protons, electrons (or muons), and antineutrinos according to

$$n \to p^+ + e^- + \bar{\nu}_{e^-}. \tag{1.1}$$

These antineutrinos have a mean-free path of roughly 10 cm [?] at these energies, and are therefore trapped inside the star, causing a neutrino pressure bubble with kinetic energy of order 10^{51} erg $= 6.2 \times 10^{56}$ MeV [?]. With the core collapsing (and producing even more antineutrinos) even the rising pressure of the bubble cannot support the mass of the material above and the upper layers begin falling towards the core.

The sudden collapse causes a shock-wave which is believed to '*bounce*' at the core and expel the outer layers of the star in a mere fraction of a second, resulting in what we know as a supernova, and leaving behind the expelled material which, when excited by radiation from another star, can be visible from across the galaxy as a supernova remnant (SNR).

At the very centre of the SNR, the remaining core of the star (naïvely a sphere of neutrons, with some fraction of protons, neutrons and electrons) retains the angular momentum of the original star, now with a radius on the order of 10 km rather than 10^9 km and thus neutron stars are thought to spin very fast, with rotational frequencies of up to 0.716 MHz [?]. Via a mechanism involving the magnetic field of the star, these spinning neutron stars may produce a beam of radiation along their magnetic axis, and if that beam happens to point towards Earth to the extent that we can detect it, we call the star a pulsar. For the purposes of this research, we shall assume the simple case that the objects we are investigating are static and non-rotating. Further calculations can be used to extrapolate the results to rotating solutions, but we shall not focus on this aspect here.

A further option exists; if the pressure and temperature (hence energy) of the system become great enough, other particles can be formed via weak reactions; for example, hyperons. The methods employed in this thesis have the goal of constructing models of matter at super-nuclear densities, and from these, models of neutron stars. The outcome of these calculations is a set of parameters which describe a neutron star (or an ensemble of them). Of these, the mass of a neutron star is an observable quantity. Other parameters, such as radius, energy, composition and so forth are unknown, and only detectable via higher-order (or proxy) observations.

The ultimate goal would be finding a physically realistic model based on the interactions of particles, such that we are able to deduce the structure and global properties of a neutron star based only on an observed mass. This however—as we shall endeavor to show—is easier said than done.

[3] In accordance with the Pauli Exclusion Principle, no two fermions can share the same quantum state. This limits how close two fermions—in this case, electrons—can be squeezed, leading to the degeneracy pressure.

[4] At present, models of supernova production have been unable to completely predict observations.

Particle Physics & Quantum Field Theory

In our considerations of the models that follow we wish to explore ensembles of particles and their interactions. In order to describe these particles we rely on Quantum Field Theory (QFT), which mathematically describes the 'rules' these particles obey. The particular set of rules that are believed to describe particles obeying the strong force at a fundamental level is Quantum Chromodynamics (QCD), but as mentioned in the introduction, this construction is analytically insolvable, so we rely on a model which simulates the properties that QCD predicts.

In the following sections, we will outline the methods of calculating the properties of matter from a field theoretic perspective.

2.1 Lagrangian Density

The first step to calculating any quantity in a Quantum Field Theory is to construct a Lagrangian density, which summarizes the dynamics of the system, and from which the equations of motion can be calculated. In order to do this, we must define precisely what it is that we wish to calculate the properties of.

The classification schemes of particle physics provide several definitions into which particles are identified, however each of these provides an additional piece of information about those particles. We wish to describe nucleons N (consisting of protons p, and neutrons n) which are hadrons[1], and are also fermions[2].

We will extend our description to include the hyperons Y (baryons with one or more valence strange quarks) consisting of Λ, Σ^-, Σ^0, Σ^+, Ξ^-, and Ξ^0 baryons. The hyperons, together with the nucleons, form the octet of baryons (see Fig. **??**).

We can describe fermions as four-component spinors ψ of plane-wave solutions to the Dirac Equation (see later), such that

$$\psi = u(\vec{p})e^{-ip_\mu x^\mu}, \tag{2.1}$$

where $u(\vec{p})$ are four-component Dirac spinors related to plane-waves with wave-vector $\vec{p}$ that carry the spin information for a particle, and which shall be discussed further in Appendix **??**. For convenience, we can group the baryon spinors by isospin group, since this is a degree of freedom that will become important. For example, we can collectively describe nucleons as a (bi-)spinor containing protons and neutrons, as

$$\psi_N = \begin{pmatrix} \psi_p(s) \\ \psi_n(s') \end{pmatrix}. \tag{2.2}$$

Here we have used the labels for protons and neutrons rather than explicitly using a label for isospin. We will further simplify this by dropping the label for spin, and it can be assumed

[1] Bound states of quarks. In particular, bound states of three 'valence' quarks plus any number of quark-antiquark pairs (the 'sea' quarks, which are the result of particle anti-particle production via gluons) are called baryons.

[2] Particles which obey Fermi–Dirac statistics, in which the particle wavefunction is anti-symmetric under exchange of particles; the property which leads to the Pauli Exclusion Principle.

that this label is implied. We will also require the Dirac Adjoint to describe the antibaryons, and this is written as

$$\bar{\psi} = \psi^\dagger \gamma^0. \tag{2.3}$$

Similarly, we can construct spinors for all the baryons. With these spinors we can construct a Lagrangian density to describe the dynamics of these particles. Since we are describing spin-$\frac{1}{2}$ particles we expect the spinors to be solutions of the Dirac equation which in natural units (for which $\hbar = c = 1$) is written as

$$(i\gamma^\mu \partial_\mu - M)\,\psi = (i\partial\!\!\!/ - M)\,\psi = 0, \tag{2.4}$$

and similarly for the antiparticle $\bar{\psi}$. Feynman slash notation is often used to contract and simplify expressions, and is simply defined as $A\!\!\!/ = \gamma_\mu A^\mu$. Here, ∂_μ is the four-derivative, M is the mass of the particle, and γ^μ are the (contravariant) Dirac Matrices, which due to the anti-commutation relation of

$$\left\{\gamma^\alpha, \gamma^\beta\right\} = \gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}\mathbb{I}, \tag{2.5}$$

(where $\eta = \mathrm{diag}(+1,-1,-1,-1)$ is the Minkoswki metric) generate a matrix representation of the Clifford Algebra $Cl(1,3)$. They can be represented in terms of the 2×2 identity matrix $\mathbb{I}$, and the Pauli Matrices $\vec{\sigma}$, as

$$\gamma^0 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.6}$$

Eq. (2.4) describes free baryons, so we can use this as the starting point for our Lagrangian density, and thus if we include each of the isospin groups, we have

$$\mathcal{L} = \sum_k \bar{\psi}_k \left(i\partial\!\!\!/ - M_k\right)\psi_k \ ; \ k \in \{N, \Lambda, \Sigma, \Xi\}, \tag{2.7}$$

where the baryon spinors are separated into isospin groups, as

$$\psi_N = \begin{pmatrix} \psi_p \\ \psi_n \end{pmatrix}, \quad \psi_\Lambda = \begin{pmatrix} \psi_\Lambda \end{pmatrix}, \quad \psi_\Sigma = \begin{pmatrix} \psi_{\Sigma^+} \\ \psi_{\Sigma^0} \\ \psi_{\Sigma^-} \end{pmatrix}, \quad \psi_\Xi = \begin{pmatrix} \psi_{\Xi^0} \\ \psi_{\Xi^-} \end{pmatrix}. \tag{2.8}$$

This implies that the mass term is also a diagonal matrix. In many texts this term is simply a scalar mass term multiplied by a suitable identity matrix, but that would imply the existence of a charge symmetry; that the mass of the proton and of the neutron were degenerate, and exchange of charges would have no effect on the Lagrangian density. We shall not make this assumption, and will rather work with the physical masses as found in Ref. [?], so M_k will contain distinct values along the diagonal.

To this point, we have constructed a Lagrangian density for the dynamics of free baryons. In order to simulate QCD, we require interactions between baryons and mesons to produce the correct phenomenology. Historically, the scalar-isoscalar meson[3] σ and vector-isoscalar meson ω have been used to this end. Additionally, the vector-isovector ρ meson has been included (for asymmetric matter) to provide a coupling to the isospin channel [?].

[3] Despite it's dubious status as a distinct particle state, rather than a resonance of $\pi\pi$.

In order to describe interactions of the baryons with mesons, we can include terms in the Lagrangian density for various classes of mesons by considering the appropriate bilinears that each meson couples to. For example, if we wish to include the ω meson, we first observe that as a vector meson it will couple to a vector bilinear (to preserve Lorentz invariance) as

$$-\, i g_\omega \bar{\psi}\gamma_\mu \omega^\mu \psi \tag{2.9}$$

with coupling strength g_ω, which as we shall see, may be dependent on the baryon that the meson is coupled to. The particular coefficients arise from the Feynman rules for meson-baryon vertices (refer to Appendix **??**). This particular vertex is written in Feynman diagram notation as shown in Fig. 2.1(b).

This is not the only way we can couple a meson to a baryon. We should also consider the Yukawa couplings of mesons to baryons with all possible Lorentz characteristics; for example, the ω meson can couple to a baryon ψ, with several different vertices:

$$\bar{\psi}\gamma_\mu\omega^\mu\psi, \quad \bar{\psi}\sigma_{\mu\nu}q^\nu\omega^\mu\psi, \text{ and } \bar{\psi}q_\mu\omega^\mu\psi, \tag{2.10}$$

where q_μ represents the baryon four-momentum transfer $(q_f - q_i)_\mu$. The latter two of these provides a vanishing contribution when considering the mean-field approximation (which shall be defined in Section 2.2), since $\sigma_{00} = 0$ and $q_\mu = 0$, as the system is on average, static.

If we include the appropriate scalar and vector terms—including an isospin-coupling of the ρ meson—in our basic Lagrangian density we have

$$\mathcal{L} = \sum_k \bar{\psi}_k \left(\gamma_\mu \left[i\partial^\mu - g_{k\omega}\omega^\mu - g_\rho(\vec{\tau}_{(k)}\cdot\vec{\rho}^{\,\mu}) \right] - M_k + g_{k\sigma}\sigma\right)\psi_k \ ; \ k \in \{N, \Lambda, \Sigma, \Xi\}. \tag{2.11}$$

The isospin matrices $\vec{\tau}_{(k)}$ are scaled Pauli matrices of appropriate order for each of the isospin groups, the third components of which are given explicitly here as

$$\tau_{(N)3} = \tau_{(\Xi)3} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \tau_{(\Lambda)3} = 0, \quad \tau_{(\Sigma)3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{2.12}$$

for which the diagonal elements of $\tau_{(k)3}$ are the isospin projections of the corresponding baryons within an isospin group defined by Eq. (2.8), i.e. $\tau_{(p)3} = I_{3p} = +\frac{1}{2}$.

At this point it is important that we remind the reader that the conventions in this field do not distinguish the use of explicit Einstein summation, and that within a single equation, indices may represent summation over several different spaces. To make this clearer, we will show an example of a term where all the indices are made explicit; the interaction term for the ρ meson in Eq. (2.11) for which we explicitly state all of the indices

$$\begin{aligned} \mathcal{L}_\rho &= \sum_k g_\rho\, \bar{\psi}_k \gamma^\mu \vec{\tau}_{(k)} \cdot \vec{\rho}_{\,\mu} \psi_k \\ &= \sum_k \sum_{i,j=1}^{f_k} \sum_{\alpha,\beta=1}^{4} \sum_{\mu=0}^{3} \sum_{a=1}^{3} g_\rho \, (\bar{\psi}_k^i)_\alpha \, (\gamma^\mu)_{\alpha\beta} \left(\tau_{(k)}^a\right)^{ij} \rho_\mu^a \left(\psi_k^j\right)_\beta, \end{aligned} \tag{2.13}$$

Fig. 2.1: Interaction vertex for the (a) scalar and (b) vector mesons, where the solid lines represent baryons ψ, the dashed line represents a scalar meson (e.g. σ), and the wavy line represents a vector meson (e.g. ω_μ).

where here k is summed over isospin groups N, Λ, Σ, and Ξ; i and j are summed over flavor space (within an isospin group of size f_k, e.g. $f_N = 2$, $f_\Sigma = 3$); α and β are summed over Dirac space; μ is summed over Lorentz space; and a is summed over iso-vector space. The Pauli matrices, of which $(\tau_k^a)^{ij}$ are the elements, are defined in Eq. (2.12). This level of disambiguity is overwhelmingly cluttering, so we shall return to the conventions of this field and leave the indices as implicit.

In addition to the interaction terms, we must also include the free terms and field tensors for each of the mesons, which are chosen with the intent that applying the Euler–Lagrange equations to these terms will produce the correct phenomenology, leading to

$$\begin{aligned}\mathcal{L} = &\sum_k \bar{\psi}_k \left(\gamma_\mu \left[i\partial^\mu - g_{k\omega}\omega^\mu - g_\rho(\vec{\tau}_{(k)} \cdot \vec{\rho}^{\,\mu}) \right] - M_k + g_{k\sigma}\sigma\right) \psi_k \\ &+ \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma - m_\sigma^2\sigma^2) + \frac{1}{2}m_\omega^2\omega_\mu\omega^\mu + \frac{1}{2}m_\rho^2\rho_\mu\rho^\mu - \frac{1}{4}\Omega_{\mu\nu}\Omega^{\mu\nu} - \frac{1}{4}R^a_{\mu\nu}R_a^{\mu\nu},\end{aligned} \tag{2.14}$$

where the field tensors for the ω and ρ mesons are, respectively,

$$\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu, \quad R^a_{\mu\nu} = \partial_\mu\rho^a_\nu - \partial_\nu\rho^a_\mu - g_\rho\epsilon^{abc}\rho^b_\mu\rho^c_\nu. \tag{2.15}$$

This is the Lagrangian density that we will begin with for the models we shall explore herein. Many texts (for example, Refs. [?, ?, ?, ?]) include higher-order terms ($\mathcal{O}(\sigma^3)$, $\mathcal{O}(\sigma^4)$, ...) and have shown that these do indeed have an effect on the state variables, but in the context of this work, we shall continue to work at this order for simplicity. It should be noted that the higher order terms for the scalar meson can be included in such a way as to trivially reproduce a framework consistent with the Quark-Meson Coupling model that shall be described later, and thus we are not entirely excluding this contribution.

2.2 Mean-Field Approximation

To calculate properties of matter, we will use an approximation to simplify the quantities we need to evaluate. This approximation, known as a Mean-Field Approximation (MFA) is made on the basis that we can separate the expression for a meson field α into two parts: a constant classical component, and a component due to quantum fluctuations;

$$\alpha = \alpha_{\text{classical}} + \alpha_{\text{quantum}}. \tag{2.16}$$

If we then take the vacuum expectation value (the average value in the vacuum) of these components, the quantum fluctuation term vanishes, and we are left with the classical component

$$\langle\alpha\rangle \equiv \langle\alpha_{\text{classical}}\rangle. \tag{2.17}$$

This component is what we shall use as the meson contribution, and we will assume that this contribution (at any given density) is constant. This can be thought of as a background 'field' on top of which we place the baryon components. For this reason, we consider the case of *infinite matter*, in which there are no boundaries to the system. The core of lead nuclei (composed of over 200 nucleons) can be thought of in this fashion, since the effects of the outermost nucleons are minimal compared to the short-range strong nuclear force.

Furthermore, given that the ground-state of matter will contain some proportion of proton and neutron densities, any flavor-changing meson interactions will provide no contribution in the MFA, since the overlap operator between the ground-state $|\Psi\rangle$ and any other state $|\xi\rangle$ is orthogonal, and thus

$$\langle\, \Psi \mid \xi \,\rangle = \delta_{\Psi\xi}. \tag{2.18}$$

For this reason, any meson interactions which, say, interact with a proton to form a neutron will produce a state which is not the ground state, and thus provides no contribution to the MFA. We will show in the next section that this is consistent with maintaining isospin symmetry.

2.3 Symmetries

In the calculations that will follow, there are several terms that we will exclude from our considerations *ab initio* (including for example, some that appear in Eq. (2.14)) because they merely provide a vanishing contribution, such as the quantum fluctuations mentioned above. These quantities shall be noted here, along with a brief argument supporting their absence in further calculations.

2.3.1 Rotational Symmetry and Isospin

The first example is simple enough; we assume rotational invariance of the fields to conserve Lorentz invariance. In order to maintain rotational invariance in all frames, we require that the spatial components of vector quantities vanish, leaving only temporal components. For example, in the MFA the vector-isoscalar meson four-vector ω_μ can be reduced to the temporal component ω_0, and for notational simplicity, we will often drop the subscript and use $\langle\alpha\rangle$ for the α meson mean-field contribution.

A corollary of the MFA is that the field tensor for the rho meson vanishes;

$$R^a_{\mu\nu} = \partial_\mu \rho^a_\nu - \partial_\nu \rho^a_\mu - g_\rho \epsilon^{abc} \rho^b_\mu \rho^c_\nu \underset{\text{MFA}}{\longrightarrow} R^a_{00} = 0, \tag{2.19}$$

since the derivatives of the constant terms vanish and $(\vec{\rho}_0 \times \vec{\rho}_0) = 0$. The same occurs for the omega meson field tensor

$$\Omega_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \underset{\text{MFA}}{\longrightarrow} \Omega_{00} = 0. \tag{2.20}$$

We also require rotational invariance in isospin space along a quantization direction of $\hat{z} = \hat{3}$ (isospin invariance) as this is a symmetry of the strong interaction, thus only the neutral components of an isovector have a non-zero contribution. This can be seen if we examine the general 2×2 unitary isospin transformation, and the Taylor expansion of this term

$$\psi(x) \to \psi'(x) = e^{i\vec{\tau}\cdot\vec{\theta}/2}\psi(x) \underset{|\theta|\ll 1}{\longrightarrow} \left(1 + i\vec{\tau}\cdot\vec{\theta}/2\right)\psi(x), \tag{2.21}$$

where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ is a triplet of real constants representing the (small) angles to be rotated through, and $\vec{\tau}$ are the usual Pauli matrices as defined in Eq. (2.12). As for the ρ mesons, we can express the triplet as linear combinations of the charged states, as

$$\vec{\rho} = (\rho_1, \rho_2, \rho_3) = \left(\frac{1}{\sqrt{2}}(\rho_+ + \rho_-),\ \frac{i}{\sqrt{2}}(\rho_- - \rho_+),\ \rho_0\right). \tag{2.22}$$

The transformation of this triplet is then

$$\vec{\rho}\,(x) \to \vec{\rho}\,'(x) = e^{i\vec{T}\cdot\vec{\theta}}\vec{\rho}\,(x). \tag{2.23}$$

where $(T^i)_{jk} = -i\epsilon_{ijk}$ is the adjoint representation of the SU(2) generators; the spin-1 Pauli matrices in the isospin basis, a.k.a. SO(3). We can perform a Taylor expansion about $\vec{\theta} = \vec{0}$, and we obtain

$$\rho_j(x) \underset{|\theta|\ll 1}{\longrightarrow} \left[\delta_{jk} + i(T^i)_{jk}\theta_i\right]\rho_k(x). \tag{2.24}$$

We can therefore write the transformation as

$$\vec{\rho}\,(x) \underset{|\theta|\ll 1}{\longrightarrow} \vec{\rho}\,(x) - \vec{\theta} \times \vec{\rho}\,(x). \tag{2.25}$$

Writing this out explicitly for the three isospin states, we obtain the individual transformation relations

$$\vec{\rho}\,(x) \to \vec{\rho}\,(x) - \vec{\theta} \times \vec{\rho}\,(x) = (\rho_1 - \theta_2\rho_3 + \theta_3\rho_2,\ \rho_2 - \theta_1\rho_3 + \theta_3\rho_1,\ \rho_3 - \theta_1\rho_2 + \theta_2\rho_1). \tag{2.26}$$

If we now consider the rotation in only the $\hat{z} = \hat{3}$ direction, we see that the only invariant component is ρ_3

$$\vec{\rho}\,(x) \underset{\substack{\theta_1=0\\ \theta_2=0}}{\longrightarrow} (\rho_1 + \theta_3\rho_2,\ \rho_2 + \theta_3\rho_1,\ \rho_3)\,. \tag{2.27}$$

If we performed this rotation along another direction—i.e. $\hat{1}$, $\hat{2}$, or a linear combination of directions—we would find that the invariant component is still a linear combination of charged states. By enforcing isospin invariance, we can see that the only surviving ρ meson state will be the charge-neutral state $\rho_3 \equiv \rho_0$.

2.3.2 Parity Symmetry

We can further exclude entire isospin classes of mesons from contributing since the ground-state of nuclear matter (containing equal numbers of up and down spins) is a parity eigenstate, and thus the parity operator $\mathcal{P}$ acting on the ground-state produces

$$\mathcal{P}|\mathcal{O}\rangle = \pm|\mathcal{O}\rangle. \tag{2.28}$$

Noting that the parity operator is idempotent ($\mathcal{P}^2 = \mathbb{I}$), inserting the unity operator into the ground-state overlap should produce no effect;

$$\langle\mathcal{O}|\mathcal{O}\rangle = \langle\mathcal{O}|\mathbb{I}|\mathcal{O}\rangle = \langle\mathcal{O}|\mathcal{P}\mathcal{P}|\mathcal{O}\rangle = \langle\mathcal{O}|(\pm)^2|\mathcal{O}\rangle = \langle\mathcal{O}|\mathcal{O}\rangle. \tag{2.29}$$

We now turn our attention to the parity transformations for various bilinear combinations that will accompany meson interactions. For Dirac spinors $\psi(x)$ and $\bar{\psi}(x)$ the parity transformation produces

$$\begin{aligned} \mathcal{P}\psi(t,\vec{x}\,)\mathcal{P} &= \gamma^0\psi(t,-\vec{x}\,), \\ \mathcal{P}\bar{\psi}(t,\vec{x}\,)\mathcal{P} &= \bar{\psi}(t,-\vec{x}\,)\gamma^0, \end{aligned} \tag{2.30}$$

where we have removed the overall phase factor $\exp(i\phi)$ since this is unobservable and can be set to unity without loss of generalisation. We can also observe the effect of the parity transformation on the various Dirac field bilinears that may appear in the Lagrangian density. The five possible Dirac bilinears are:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi, \tag{2.31}$$

for scalar, vector, tensor, pseudo-vector and pseudo-scalar meson interactions respectively, where γ_5 is defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{2.32}$$

in the commonly used Dirac basis. By acting the above transformation on these bilinears we obtain a result proportional to the spatially reversed wavefunction $\psi(t, -\vec{x}\,)$,

$$\mathcal{P}\bar{\psi}\psi\mathcal{P} \quad = \quad +\bar{\psi}\psi(t, -\vec{x}\,), \tag{2.33}$$

$$\mathcal{P}\bar{\psi}\gamma^\mu\psi\mathcal{P} \quad = \quad \begin{cases} +\bar{\psi}\gamma^\mu\psi(t, -\vec{x}\,) & \text{for } \mu = 0, \\ -\bar{\psi}\gamma^\mu\psi(t, -\vec{x}\,) & \text{for } \mu = 1, 2, 3, \end{cases} \tag{2.34}$$

$$\mathcal{P}\bar{\psi}\gamma^\mu\gamma^5\psi\mathcal{P} \quad = \quad \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi(t, -\vec{x}\,) & \text{for } \mu = 0, \\ +\bar{\psi}\gamma^\mu\gamma^5\psi(t, -\vec{x}\,) & \text{for } \mu = 1, 2, 3, \end{cases} \tag{2.35}$$

$$\mathcal{P}i\bar{\psi}\gamma^5\psi\mathcal{P} \quad = \quad -i\bar{\psi}\gamma^5\psi(t, -\vec{x}\,). \tag{2.36}$$

By inserting the above pseudo-scalar and pseudo-vector bilinears into the ground-state overlap as above, and performing the parity operation, we obtain a result equal to its negative, and so the overall expression *must* vanish. For example

$$\langle\mathcal{O}|i\bar{\psi}\gamma^5\psi|\mathcal{O}\rangle = \langle\mathcal{O}|\mathcal{P}i\bar{\psi}\gamma^5\psi\mathcal{P}|\mathcal{O}\rangle = \langle\mathcal{O}| - i\bar{\psi}\gamma^5\psi|\mathcal{O}\rangle = 0. \tag{2.37}$$

Thus all pseudo-scalar and pseudo-vector meson contributions—such as those corresponding to π and K—provide no contribution to the ground-state in the lowest order. We will show later in Chapter **??** that mesons can provide higher order contributions, and the pseudo-scalar π mesons are able to provide a non-zero contribution via Fock terms, though we will not calculate these contributions here.

2.4 Fermi Momentum

Since we are dealing with fermions that obey the Pauli Exclusion Principle[4], and thus Fermi–Dirac statistics[5], there will be restrictions on the quantum numbers that these fermions may possess. When considering large numbers of a single type of fermion, they will each require a unique three-dimensional momentum $\vec{k}$ since no two fermions may share the same quantum numbers.

For an ensemble of fermions we produce a 'Fermi sea' of particles; a tower of momentum states from zero up to some value 'at the top of the Fermi sea'. This value—the Fermi momentum—will be of considerable use to us, thus it is denoted k_F.

[4] That no two fermions can share a single quantum state.

[5] The statistics of indistinguishable particles with half-integer spin. Refer to Appendix **??**.

Although the total baryon density is a useful control parameter, many of the parameters of the models we wish to calculate are dependent on the density via k_F. The relation between the Fermi momentum and the total density is found by counting the number of momentum states in a spherical volume up to momentum k_F (here, this counting is performed in momentum space). The total baryon density—a number density in units of baryons/fm^3, usually denoted as just fm^{-3}—is simply the sum of contributions from individual baryons, as

$$\rho_{\rm total} = \sum_i \rho_i = \sum_i \frac{(2J_i+1)}{(2\pi)^3}\int \theta(k_{F_i} - |\vec{k}|)\, d^3k = \sum_i \frac{k_{F_i}^3}{3\pi^2}, \tag{2.38}$$

where here, i is the set of baryons in the model, J_i is the spin of baryon i (where for the leptons and the octet of baryons, $J_i = \frac{1}{2}$), and θ is the Heaviside step function defined as

$$\theta(x) = \begin{cases} 1, & \text{if} \quad x > 0 \\ 0, & \text{if} \quad x < 0 \end{cases}, \tag{2.39}$$

which restricts the counting of momentum states to those between 0 and k_F.

We define the species fraction for a baryon B, lepton ℓ, or quark q as the density fraction of that particle, denoted by Y_i, such that

$$Y_i = \frac{\rho_i}{\rho_{\rm total}} \ ; \quad i \in \{B, \ell, q\}\,. \tag{2.40}$$

Using this quantity we can investigate the relative proportions of particles at a given total density.

2.5 Chemical Potential

In order to make use of statistical mechanics we must define the some important quantities. One of these will be the chemical potential μ, also known as the Fermi energy ϵ_F; the energy of a particle at the top of the Fermi sea, as described in Appendix ??. This energy is the relativistic energy of such a particle, and is the energy associated with a Dirac equation for that particle. For the simple case of a non-interacting particle, this is

$$\mu_B = \epsilon_{F_B} = \sqrt{k_{F_B}^2 + M_B^2}. \tag{2.41}$$

In the case that the baryons are involved in interactions with mesons, we need to introduce scalar and temporal self-energy terms, which (for example) for Hartree-level QHD using a mean-field approximation are given by

$$\Sigma_B^s = -g_{B\sigma}\langle\sigma\rangle, \quad \Sigma_B^0 = g_{B\omega}\langle\omega\rangle + g_\rho I_{3B}\langle\rho\rangle, \tag{2.42}$$

where I_{3B} is the isospin projection of baryon B, defined by the diagonal elements of Eq. (2.12), and where the scalar self-energy is used to define the baryon effective mass as

$$M_B^* = M_B + \Sigma_B^s = M_B - g_{B\sigma}\langle\sigma\rangle, \tag{2.43}$$

These self-energy terms affect the energy of a Dirac equation, and thus alter the chemical potential, according to

$$\mu_B = \sqrt{k_{F_B}^2 + (M_B + \Sigma_B^s)^2} + \Sigma_B^0\ . \tag{2.44}$$

Eq (2.43) and Eq. (2.44) define the important in-medium quantities, and the definition of each will become dependent on which model we are using.

For a relativistic system such as that which will consider here, each conserved quantity is associated with a chemical potential, and we can use the combination of these associated chemical potentials to obtain relations between chemical potentials for individual species. In our case, we will consider two conserved quantities: total baryon number and total charge, and so we have a chemical potential related to each of these. We can construct the chemical potential for each particle species by multiplying each conserved charge by its associated chemical potential to obtain a general relation. Thus

$$\mu_i = B_i\mu_n - Q_i\mu_e, \tag{2.45}$$

where; i is the particle species (which can be any of the baryons) for which we are constructing the chemical potential; B_i and Q_i are the baryon number ('baryon charge', which is unitless) and electric charge (normalized to the proton charge) respectively; and μ_n and μ_e are the chemical potentials of neutrons and electrons, respectively. Leptons have $B_\ell = 0$, and all baryons have $B_B = +1$. The relations between the chemical potentials for the octet of baryons are therefore derived to be

$$\begin{array}{ccccccc} \mu_\Lambda & = & \mu_{\Sigma^0} & = & \mu_{\Xi^0} & = & \mu_n, \\ & & \mu_{\Sigma^-} & = & \mu_{\Xi^-} & = & \mu_n + \mu_e, \\ & & \mu_p & = & \mu_{\Sigma^+} & = & \mu_n - \mu_e, \\ & & & & \mu_\mu & = & \mu_e. \end{array} \tag{2.46}$$

A simple example of this is to construct the chemical potential for the proton (for which the associated charges are $B_p = +1$ and $Q_p = +1$);

$$\mu_p = \mu_n - \mu_e. \tag{2.47}$$

This can be rearranged to a form that resembles neutron β-decay

$$\mu_n = \mu_p + \mu_e. \tag{2.48}$$

If we were to consider further conserved charges, such as lepton number for example, we would require a further associated chemical potential. In that example, the additional chemical potential would be for (anti)neutrinos $\mu_{\bar{\nu}}$. The antineutrino would be required to preserve the lepton number on both sides of the equation; the goal of such an addition. Since we shall consider that neutrinos are able to leave the system considered, we can ignore this contribution *ab inito*. The removal of this assumption would alter Eq. (2.48) to include the antineutrino, as would normally be expected in β-decay equations

$$\mu_n = \mu_p + \mu_e + \mu_{\bar{\nu}}. \tag{2.49}$$

2.6 Explicit Chiral Symmetry (Breaking)

One of the most interesting symmetries of QCD is chiral symmetry. If we consider the QCD Lagrangian density to be the sum of quark and gluon contributions, then in the massless quark limit ($m_q = 0$);

$$
\begin{aligned}
\mathcal{L}_{\text{QCD}} &= \mathcal{L}_g + \mathcal{L}_q \\
&= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}_i i \gamma^\mu (D_\mu)_{ij} \psi_j \\
&= -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \bar{\psi}_i i \gamma^\mu \partial_\mu \psi_i - g A^a_\mu \bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j \ ,
\end{aligned} \tag{2.50}
$$

where here, $\psi_i(x)$ is a quark field of color $i \in \{r, g, b\}$, $A^a_\mu(x)$ is a gluon field with color index $a \in \{1, \ldots, 8\}$, T^a_{ij} is a generator[6] for SU(3), g is the QCD coupling constant, and $G^a_{\mu\nu}$ represents the gauge-invariant gluonic field strength tensor, given by

$$
G^a_{\mu\nu} = [\partial_\mu, A^a_\nu] - g f^{abc} A^b_\mu A^c_\nu \ , \tag{2.51}
$$

written with the structure constants f^{abc}. Left- and right-handed components of Dirac fields can be separated using the projection operators

$$
\psi_{\substack{L \\ R}} = \frac{1 \mp \gamma_5}{2} \psi, \tag{2.52}
$$

using the definition of γ_5 of Eq. (2.32), and so the quark terms in the QCD Lagrangian density (the gluon terms are not projected) can be written in terms of these components as

$$
\mathcal{L}^{(f)}_q = i \bar{\psi}^{(f)}_L D_\mu \gamma^\mu \psi^{(f)}_L + i \bar{\psi}^{(f)}_R D_\mu \gamma^\mu \psi^{(f)}_R . \tag{2.53}
$$

This Lagrangian density is invariant under rotations in U(1) of the left- and right-handed fields

$$
\mathrm{U}(1)_L : \quad \psi_L \to e^{i\alpha_L} \psi_L, \quad \psi_R \to \psi_R, \tag{2.54}
$$
$$
\mathrm{U}(1)_R : \quad \psi_R \to e^{i\alpha_R} \psi_R, \quad \psi_L \to \psi_L, \tag{2.55}
$$

where α_L and α_R are arbitrary phases. This invariance is the chiral $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ symmetry. The Noether currents associated with this invariance are then

$$
J^\mu_L = \bar{\psi}_L \gamma^\mu \psi_L, \qquad J^\mu_R = \bar{\psi}_R \gamma^\mu \psi_R, \tag{2.56}
$$

and as expected, these currents are conserved, such that $\partial_\mu J^\mu_L = \partial_\mu J^\mu_R = 0$ according to the Dirac Equation. These conserved currents can be alternatively written in terms of conserved vector and axial-vector currents, as

$$
J^\mu_L = \frac{V^\mu - A^\mu}{2}, \qquad J^\mu_R = \frac{V^\mu + A^\mu}{2}, \tag{2.57}
$$

where here, V^μ and A^μ denote the vector and axial-vector currents respectively—the distinction of A^μ here from the gluon fields in Eq. (2.50) is neccesary—and these are defined by

$$
V^\mu = \bar{\psi} \gamma^\mu \psi, \qquad A^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi, \tag{2.58}
$$

[6]For example, $T^a = \lambda^a/2$ using the Hermitian Gell-Mann matrices λ_a.

and which are also conserved, thus $\partial_\mu V^\mu = \partial_\mu A^\mu = 0$. The chiral symmetry of $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ is therefore equivalent to invariance under transformations under $\mathrm{U}(1)_V \otimes \mathrm{U}(1)_A$, where we use the transformations

$$\mathrm{U}(1)_V: \quad \psi \to e^{i\alpha_V}\psi, \qquad \bar{\psi} \to \psi^\dagger e^{-i\alpha_V}\gamma_0, \tag{2.59}$$
$$\mathrm{U}(1)_A: \quad \psi \to e^{i\alpha_A\gamma_5}\psi, \qquad \bar{\psi} \to \psi^\dagger e^{-i\alpha_A\gamma_5}\gamma_0. \tag{2.60}$$

Using the anticommutation relation

$$\{\gamma_5, \gamma_\mu\} = \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0 \tag{2.61}$$

we can evaluate the effect that the vector and axial-vector transformations have on the QCD Lagrangian density, and we find that both transformations are conserved. If we now consider a quark mass term $\mathcal{L}_m$ in the QCD Lagrangian density, the fermionic part becomes

$$\mathcal{L}^{\psi}_{\mathrm{QCD}} = \mathcal{L}_q + \mathcal{L}_m = \bar{\psi}_i\left(i\gamma^\mu(D_\mu)_{ij} - m\delta_{ij}\right)\psi_j. \tag{2.62}$$

For the purposes of these discussions, we can set the masses of the quarks to be equal without loss of generality. Although the massless Lagrangian density possesses both of the above symmetries, the axial vector symmetry—and hence chiral symmetry—is explicitly broken by this quark mass term;

$$\mathcal{L}_m = -\bar{\psi}m\psi \xrightarrow{\mathrm{U}(1)_A} -\bar{\psi}me^{2i\alpha_A}\psi \neq -\bar{\psi}m\psi. \tag{2.63}$$

The vector symmetry is nonetheless preserved when including this term.

2.7 Dynamical Chiral Symmetry (Breaking)

Even with a massless Lagrangian density, it is possible that chiral symmetry becomes dynamically broken, and we refer to this as Dynamically Broken Chiral Symmetry, or DCSB.

Following the description of Ref. [?], if we consider the basic Lagrangian density of QCD to be

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}_i\left(i\gamma^\mu(D_\mu)_{ij} - m\delta_{ij}\right)\psi_j - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a, \tag{2.64}$$

with the definitions as in the previous section, of

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^{abc}A^b_\mu A^c_\nu, \quad D_\mu = \partial_\nu + igA^a_\mu T^a, \tag{2.65}$$

with standard definitions of other terms, then we can write the sum of all QCD One-Particle Irreducible (1-PI) diagrams[7] with two external legs as shown in Fig. 2.2; illustrating the quark self-energy. The expression for the renormalized quark self-energy in d dimensions is

$$-i\Sigma(p) = \frac{4}{3}Z_r\, g^2 \int \frac{d^dq}{(2\pi)^d}(i\gamma_\mu)(iS(q))(iD^{\mu\nu}(p-q))(i\Gamma_\nu(q,p)), \tag{2.66}$$

where Z_r is a renormalization constant, g is the QCD coupling, and q is the loop momentum.

[7] Diagrams that cannot be made into two separate disconnected diagrams by cutting an internal line are called One-Particle Irreducible, or 1-PI.

In the absence of matter fields or background fields (the Lorentz-covariant case), we can write this self-energy as a sum of Dirac-vector and Dirac-scalar components, as

$$\Sigma(p) = \not{p}\, \Sigma_{\rm DV}(p^2) + \Sigma_{\rm DS}(p^2). \tag{2.67}$$

where $\Sigma_{\rm DV}(p^2)$ is the Dirac-vector component, and $\Sigma_{\rm DS}(p^2)$ is the Dirac-scalar component. These must both be functions of p^2, since there are no other Dirac-fields to contract with, and $\Sigma(p)$ is a Lorentz invariant quantity in this case.

For the purposes of our discussion in this section, we will approximate the Dirac-vector component of the self-energy to be $\Sigma_{\rm DV} \sim 1$, in which case the self-energy is dependent only on the Dirac-scalar component.

Even with a massless theory ($m = 0$) it is possible that the renormalized self-energy develops a non-zero Dirac-scalar component, thus $\Sigma_{\rm DS}(p^2) \neq 0$. This leads to a non-zero value for the quark condensate $\langle\bar{\psi}_q\psi_q\rangle$, and in the limit of exact chiral symmetry, leads to the pion becoming a massless Goldstone boson. Thus chiral symmetry can be dynamically broken. With the addition of a Dirac-scalar component of the self-energy, the Lagrangian density becomes

$$\mathcal{L}_{QCD} = \bar{\psi}_i \left(i\gamma^\mu (D_\mu)_{ij} - (m + \Sigma_{\rm DS})\delta_{ij} \right) \psi_j - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a, \tag{2.68}$$

Fig. 2.2: Feynman diagram for the QCD self-energy for a quark, as given by the Dyson–Schwinger Equation (DSE). The full expression for this is given in Eq. (2.66).

and we can define a dynamic quark mass via the gap equation;

$$m^* = m + \Sigma_{\rm DS}. \tag{2.69}$$

We will continue this discussion in Section ??, in which we will describe a particular model for $\Sigma_{\rm DS}$ in order to describe DCSB.

2.8 Equation of State

In order to investigate models of dense matter, we need to construct an Equation of State (EOS), which is simply a relation between two or more state variables—those which thermodynamically describe the current state of the system, such as temperature, pressure, volume, or internal energy—under a given set of physical conditions. With this, we will be able to investigate various aspects of a model and compare differences between models in a consistent fashion.

For our purposes, we use the total baryon density $\rho_{\rm total}$ as the control parameter of this system, and so we need to obtain the connection between, say, the energy density $\mathcal{E}$, the pressure P, and this total baryon density, i.e.

$$\mathcal{E} = \mathcal{E}(\rho_{\rm total}), \qquad P = P(\rho_{\rm total}). \tag{2.70}$$

State variables are important quantities to consider. Within any transition between states the total change in any state variable will remain constant regardless of the path taken, since the change is an exact differential, by definition. For the hadronic models described herein, the EOS are exact, in that they have an analytic form;

$$P(\rho_{\rm total}) = \rho_{\rm total}^2 \frac{\partial}{\partial \rho_{\rm total}} \left(\frac{\mathcal{E}(\rho_{\rm total})}{\rho_{\rm total}} \right). \tag{2.71}$$

As simple as this exact form may seem, the derivative complicates things, and we will find it easier to calculate the pressure independently. Nonetheless, this expression will hold true. More interestingly, this expression is equivalent to the first law of thermodynamics in the absence of heat transfer; i.e.

$$PdV = -dE, \tag{2.72}$$

(the proof of which can be found in Appendix ??) which assures us that the theory is thermodynamically consistent.

A notable feature of each symmetric matter EOS we calculate is the effect of saturation; whereby the energy per baryon for the system possesses a global minimum at a particular value of the Fermi momentum. This can be considered as a binding energy of the system. In symmetric matter (in which the densities of protons and neutrons are equal), the nucleon Fermi momenta are related via $k_F = k_{F_n} = k_{F_p}$, and the energy per baryon (binding energy) E is determined via

$$E = \left[\frac{1}{\rho_{\rm total}} \left(\mathcal{E} - \sum_B \rho_B M_B \right) \right]. \tag{2.73}$$

In order to reproduce (a chosen set of) experimental results, this value should be an extremum of the curve with a value of $E_0 = -15.86$ MeV at a density of $\rho_0 = 0.16\ {\rm fm}^{-3}$ (or the corresponding Fermi momentum k_{F_0}).

The nucleon symmetry energy $a_{\rm sym}$ is approximately a measure of the energy difference between the energy per baryon (binding energy) of a neutron-only model and a symmetric nuclear model (essentially a measure of the breaking of isospin symmetry). A more formal expression (without assuming degeneracy between nucleon masses, as derived in Appendix **??**) is

$$a_{\rm sym} = \frac{g_\rho^2}{3\pi^2 m_\rho^2} k_F^3 + \frac{1}{12}\frac{k_F^2}{\sqrt{k_F^2 + (M_p^*)^2}} + \frac{1}{12}\frac{k_F^2}{\sqrt{k_F^2 + (M_n^*)^2}}\,. \tag{2.74}$$

At saturation, this should take the value of $(a_{\rm sym})_0 = 32.5$ MeV (for an analysis of values, see Ref. [**?**]).

Another important aspect of an EOS is the compression modulus K which represents the *stiffness* of the EOS; the ability to withstand compression. This ability is intimately linked to the Pauli Exclusion Principle in that all other things being equal, a system with more available states (say, distinguishable momentum states) will have a softer EOS, and thus a smaller compression modulus. The compression modulus itself is defined as the curvature of the binding energy at saturation, the expression for which is

$$K = \left[k_F^2 \frac{d^2}{dk_F^2}\left(\frac{\mathcal{E}}{\rho_{\rm total}}\right)\right]_{k_{F_{\rm sat}}} = 9\left[\rho_{\rm total}^2 \frac{d^2}{d\rho_{\rm total}^2}\left(\frac{\mathcal{E}}{\rho_{\rm total}}\right)\right]_{\rho=\rho_0}\,. \tag{2.75}$$

The motivation for this is that by compressing the system, the energy per baryon will rise. The curvature at saturation determines how fast that rise will occur, and thus how resistant to compression the system is. Experimentally, this is linked to the properties of finite nuclei, particularly those with a large number of nucleons, and the binding of these within a nucleus.

According to Ref. [**?**] this should have a value in the range 200–300 MeV, and we will calculate the value of K for each of the models to follow for comparison.

2.9 Phase Transitions

In order to consider transitions between different phases of matter we must use statistical mechanics. The simplest method of constructing a phase transition—known as a 'Maxwell transition'—is an isobaric (constant pressure) transition constructed over a finite density range. A transition of this form remains useful in understanding the liquid–gas style phase transition that occurs within QHD, which is a first-order transition (similar to that of ice melting in a fluid) with the phases being separated by a non-physical negative-pressure region. The inclusion of a Maxwell transition to this simple model for QHD removes this unphysical region and replaces it with a constant pressure phase.

The method for constructing a Maxwell transition will not be covered here, though in-depth details can be found in Ref. [**?**]. We can however extract the transition densities from Ref. [**?**] to reproduce the results, which are shown later in Fig. **??** for the various varieties of QHD. The more sophisticated method of constructing a phase transition—the 'Gibbs transition' [**?**] that we have used for the results produced herein—relies on a little more statistical mechanics. A comparison between the Maxwell and Gibbs methods for models similar to those used in this work can be found in Ref. [**?**]. For a full in-depth discussion of this topic, see Ref. [**?**].

If we consider a homogeneous (suitable for these mean-field calculations) system with energy E, volume V, N_m particles of type m, and entropy S which depends on these parameters such that

$$S = S(E, V, N_1, \ldots, N_m), \tag{2.76}$$

then we can consider the variation of the entropy in the system as a function of these parameters, resulting in

$$dS = \left(\frac{\partial S}{\partial E}\right)_{V,N_1,\ldots,N_m} dE + \left(\frac{\partial S}{\partial V}\right)_{E,N_1,\ldots,N_m} dV + \sum_{i=1}^{m} \left(\frac{\partial S}{\partial N_i}\right)_{V,N_{j\neq i}} dN_i, \tag{2.77}$$

using standard statistical mechanics notation whereby a subscript X on a partial derivative $(\partial A/\partial B)_X$ denotes that X is explicitly held constant. Eq. (2.77) should be equal to the fundamental thermodynamic relation when the number of particles is fixed, namely

$$dS = \frac{đQ}{T} = \frac{dE + PdV}{T}. \tag{2.78}$$

Here, the symbol $đ$ denotes the inexact differential, since the heat Q is not a state function —does not have initial and final values—and thus the integral of this expression is only true for infinitesimal values, and not for finite values. Continuing to keep the number of each type of particle N_i constant, a comparison of coefficients between Eqs. (2.77) and (2.78) results in the following relations:

$$\left(\frac{\partial S}{\partial E}\right)_{V,N_i,\ldots,N_m} = \frac{1}{T}, \quad \left(\frac{\partial S}{\partial V}\right)_{E,N_i,\ldots,N_m} = \frac{P}{T}. \tag{2.79}$$

To provide a relation similar to Eq. (2.79) for the case where $dN_i \neq 0$, one defines μ_j—the chemical potential per molecule—as

$$\mu_i = -T \left(\frac{\partial S}{\partial N_i}\right)_{E,V,N_{j\neq i}}. \tag{2.80}$$

We can now re-write Eq. (2.77) with the definitions in Eq. (2.79) for the case where the particle number can change, as

$$dS = \frac{1}{T}dE + \frac{P}{T}dV - \sum_{i=1}^{m} \frac{\mu_i}{T} dN_i, \tag{2.81}$$

which can be equivalently written in the form of the fundamental thermodynamic relation for non-constant particle number,

$$dE = TdS - PdV + \sum_{i=1}^{m} \mu_i dN_i. \tag{2.82}$$

If we now consider a system X of two phases A and B, then we can construct relations between their parameters by considering the following relations:

$$\begin{aligned} E_X &= E_A + E_B, \\ V_X &= V_A + V_B, \\ N_X &= N_A + N_B. \end{aligned} \tag{2.83}$$

If we consider that these quantities are conserved between phases, we find the following conservation conditions

$$\begin{aligned} dE_X = 0 \quad &\Rightarrow \quad dE_A + dE_B = 0 \quad \Rightarrow \quad dE_A = -dE_B, \\ dV_X = 0 \quad &\Rightarrow \quad dV_A + dV_B = 0 \quad \Rightarrow \quad dV_A = -dV_B, \\ dN_X = 0 \quad &\Rightarrow \quad dN_A + dN_B = 0 \quad \Rightarrow \quad dN_A = -dN_B. \end{aligned} \tag{2.84}$$

The condition for phase equilibrium for the most probable situation is that the entropy must be a maximum for $S = S_X(E_X, V_X, N_X) = S(E_A, V_A, N_A; E_B, V_B, N_B)$, which leads to

$$dS_X = dS_A + dS_B = 0. \tag{2.85}$$

Thus, inserting Eq. (2.81) we find

$$dS = \left(\frac{1}{T_A} dE_A + \frac{P_A}{T_A} dV_A - \frac{\mu_A}{T_A} dN_A\right) + \left(\frac{1}{T_B} dE_B + \frac{P_B}{T_B} dV_B - \frac{\mu_B}{T_B} dN_B\right). \tag{2.86}$$

If we now apply the result of Eq. (2.84) we can simplify this relation to

$$dS = 0 = \left(\frac{1}{T_A} - \frac{1}{T_B}\right) dE_A + \left(\frac{P_A}{T_A} - \frac{P_B}{T_B}\right) dV_A - \left(\frac{\mu_A}{T_A} - \frac{\mu_B}{T_B}\right) dN_A \tag{2.87}$$

and thus for arbitrary variations of E_A, V_A and N_A, each bracketed term must vanish separately, so that

$$\frac{1}{T_A} = \frac{1}{T_B}, \quad \frac{P_A}{T_A} = \frac{P_B}{T_B}, \quad \frac{\mu_A}{T_A} = \frac{\mu_B}{T_B}. \tag{2.88}$$

Eq. (2.88) implies that at the phase transition the system will be isentropic ($dS = 0$), isothermal ($dT = 0$), isobaric ($dP = 0$), and isochemical ($d\mu = 0$), where the terms S, T, V, and μ now refer to the mean values rather than for individual particles.

We only require two systems at any one time when considering a mixture of phases, for example, a neutron ('neutron phase') can transition to a proton and an electron ('proton and electron phase') provided that the condition $\mu_n = \mu_p + \mu_e$ is met.

For a phase transition between hadronic- and quark-matter phases then, the conditions for stability are therefore that chemical, thermal, and mechanical equilibrium between the hadronic H, and quark Q phases is achieved, and thus that the independent quantities in each phase are separately equal. Thus the two independent chemical potentials (as described in Sec. 2.5) μ_n and μ_e are each separately equal to their counterparts in the other phase, i.e. $[(\mu_n)_H = (\mu_n)_Q]$, and $[(\mu_e)_H = (\mu_e)_Q]$ for chemical equilibrium; $[T_H = T_Q]$ for thermal equilibrium; and $[P_H = P_Q]$ for mechanical equilibrium.

An illustrative example of these relations is shown in Fig. 2.3 in which the values of the independent chemical potentials μ_n and μ_e, as well as the pressure P for a hadronic phase and a quark phase are plotted for increasing values of total density $\rho_{\rm total}$. In this case, the quark matter data is calculated based on the hadronic matter data, using the chemical potentials in the hadronic phase as inputs for the quark phase calculations, and as such the chemical potentials are—by construction—equal between the phases. As this is an illustrative example of the relations between the phases, no constraints have been imposed to reproduce a phase transition yet.

In this figure, the low-density points correspond to small values of μ_n, and we see that for densities lower than some phase transition density $\rho_{\rm total} < \rho_{\rm PT}$ the hadronic pressure is

Fig. 2.3: (Color Online) Illustrative locus of values for the independent chemical potentials μ_e and μ_n, as well as the pressure P for phases of hadronic matter and deconfined quark matter. Note that pressure in each phase increases with density. and that a projection onto the $\mu_n\mu_e$ plane is a single line, as ensured by the chemical equilibrium condition.

greater than the quark pressure and thus the hadronic phase is dominant. At the transition the pressures are equal, and thus both phases can be present in a mixed phase, and beyond the transition the quark pressure is greater than the hadronic pressure indicating that the quark phase becomes dominant.

Note that for all values of the total density, the chemical potentials in each phase are equal, as shown by the projection onto the $\mu_n\mu_e$ plane.

In our calculations, we will only investigate these two phases independently up to the phase transition, at which point we will consider a mixed phase, as shall be described in the next section.

We consider both phases to be cold on the nuclear scale, and assume $T = 0$ so the temperatures are also equal, again by construction. We must therefore find the point—if it exists—at which, for a given pair of independent chemical potentials, the pressures in both the hadronic phase and the quark phase are equal.

To find the partial pressure of any baryon, quark, or lepton species i we use

$$P_i = \frac{(2J_B+1)\mathcal{N}_c}{3(2\pi)^3}\int \frac{\vec{k}^2\,\theta(k_{F_i}-|\vec{k}|)}{\sqrt{\vec{k}^2+(M_i^*)^2}}\,d^3k, \tag{2.89}$$

where the number of colors is $\mathcal{N}_c = 3$ for quarks, $\mathcal{N}_c = 1$ for baryons and leptons, and where θ is the Heaviside step function defined in Eq. (2.39). To find the total pressure in each phase, we sum the pressures contributions in that phase. The pressure in the hadronic phase H is given by

$$P_H = \sum_j P_j + \sum_\ell P_\ell + \sum_m P_m, \tag{2.90}$$

in which j represents the baryons, ℓ represents the leptons, m represents the mesons appearing in the particular model being considered, if they appear, and the pressure in the quark phase Q is given by

$$P_Q = \sum_q P_q + \sum_\ell P_\ell - B, \tag{2.91}$$

where q represents the quarks, and B denotes the bag energy density, which we shall discuss further in Section **??**.

In order to determine the EOS beyond the point at which the pressures are equal, we need to consider the properties of a mixed phase.

2.10 Mixed Phase

Once we have defined the requirements for a phase transition between two phases, we must consider the possibility of a mixed phase (MP) containing proportions of the two phases. This adds a further degree of sophistication to a model; we can not only find equations of state for hadronic matter and quark matter and simply stitch them together, but we can also allow the transition between these to occur gradually.

To calculate the mixed phase EOS, we calculate the hadronic EOS with control parameter ρ_{total}, and use the independent chemical potentials μ_n and μ_e as inputs to determine the quark matter EOS, since we can determine all other Fermi momenta given these two quantities. We increase ρ_{total} until we find a density—if it exists—at which the pressure in the quark phase is equal the pressure in the hadronic phase (if such a density cannot be found, then the transition is not possible for the given models).

Assuming that such a transition is possible, once we have the density and pressure at which the phase transition occurs, we change the control parameter to the quark fraction χ (which is an order parameter parameterizing the transition to the quark matter phase) which determines the proportions of hadronic matter and quark matter. If we consider the mixed phase to be composed of some fraction of hadronic matter and some fraction of quark matter, then the mixed phase of matter will have the following properties: the total density will be

$$\rho_{\text{MP}} = (1-\chi)\,\rho_{\text{HP}} + \chi\,\rho_{\text{QP}}, \tag{2.92}$$

where ρ_{HP} and ρ_{QP} are the densities in the hadronic and quark phases, respectively. A factor of three in the equivalent baryon density in the quark phase,

$$\rho_{\text{QP}} = \frac{1}{3}\sum_q \rho_q = (\rho_u + \rho_d + \rho_s)/3, \tag{2.93}$$

arises because of the restriction that a baryon contains three quarks.

According to the condition of mechanical equilibrium detailed earlier, the pressure in the mixed phase will be

$$P_{\text{MP}} = P_{\text{HP}} = P_{\text{QP}}. \tag{2.94}$$

We can step through values $0 < \chi < 1$ and determine the properties of the mixed phase, keeping the mechanical stability conditions as they were above. In the mixed phase we need to alter our definition of charge neutrality; while previously we have used the condition that two phases were independently charge-neutral, such as $n \to p^+ + e^-$, it now becomes possible that one phase is (locally) charged, while the other phase carries the opposite charge, making the system globally charge-neutral. This is achieved by enforcing

$$0 = (1-\chi)\,\rho^c_{\text{HP}} + \chi\,\rho^c_{\text{QP}} + \rho^c_\ell\,, \tag{2.95}$$

where this time we are considering charge-densities, which are simply the sum of densities multiplying their respective charges

$$\rho^c_i = \sum_j Q_j \rho_j\ ; \quad i \in \{\text{HP}, \text{QP}, \ell\}, \tag{2.96}$$

where j are the all individual particles modelled within the grouping i. For example, the quark charge-density in a non-interacting quark phase is given by

$$\rho^c_{\text{QP}} = \sum_q Q_q \rho_q = \frac{2}{3}\rho_u - \frac{1}{3}\rho_d - \frac{1}{3}\rho_s. \tag{2.97}$$

We continue to calculate the properties of the mixed phase for increasing values of χ until we reach $\chi = 1$, at which point the mixed phase is now entirely charge-neutral quark matter. This corresponds to the density at which the mixed phase ends, and a pure quark phase begins. We can therefore continue to calculate the EOS for pure charge-neutral quark matter, once again using $\rho_{\rm total}$ as the control parameter, but now where the total density is the equivalent density as defined in Eq. (2.93).

2.11 Stellar Matter

The equations of state described above are derived for homogeneous infinite matter. If we wish to apply this to a finite system we must investigate the manner in which large ensembles of particles are held together. The focus of this work is 'neutron stars', and we must find a way to utilise our knowledge of infinite matter to provide insight to macroscopic objects. For this reason, we turn to the theory of large masses; General Relativity.

The Tolman–Oppenheimer–Volkoff (TOV) equation [**?**] describes the conditions of stability against gravitational collapse for an EOS, i.e. in which the pressure gradient is sufficient to prevent gravitational collapse of the matter. The equations therefore relate the change in pressure with radius to various state variables from the EOS. To preserve continuity, the equations are solved under the condition that the pressure at the surface of the star *must* be zero.

The TOV equation is given by

$$\frac{dP}{dr} = -\frac{G\left(P/c^2+\mathcal{E}\right)\left(M(r)+4r^3\pi P/c^2\right)}{r(r-2GM(r)/c^2)}, \tag{2.98}$$

or, in Planck units[8]

$$\frac{dP}{dr} = -\frac{(P+\mathcal{E})\left(M(r)+4\pi r^3 P\right)}{r(r-2M(r))}, \tag{2.99}$$

where the mass within a radius R is given by integrating the energy density, as

$$M(R) = \int_0^R 4\pi r^2 \mathcal{E}(r)\, dr. \tag{2.100}$$

For a full derivation of these equations, refer to Appendix **??**.

Supplied with these equations and a derived EOS, we can calculate values for the total mass and total radius of a star[9] for a given central density. We refer to these values as the 'stellar solutions.'

This is particularly interesting, since the mass of a neutron star is observable (either via observing a pair of objects rotating about a barycenter[10], or some other indirect/proxy measurement), yet the radius is not directly observable, as stars are sufficiently distant that they all appear as 'point-sources'. With these calculations, we produce a relationship between two quantities: the stellar mass and the stellar radius, of which only the mass is currently observable, and even this is not always so. This provides useful data for further theoretical

[8] In which certain fundamental physical constants are normalized to unity, viz $\hbar = c = G = 1$.

[9] Stellar objects in these calculations are assumed to be static, spherically symmetric, and non-rotating, as per the derivation of this equation. For studies of the effect of rapid rotation in General Relativity see Refs. [**?**, **?**].

[10] A common centre of mass for the system, the point about which both objects will orbit, which is the balance point of the gravitational force. In this case, the mass measurements are simplified.

work requiring both quantities, as well as an opportunity to place theoretical bounds on future experimental observations.

In addition to this data, since we are able to solve our equations for the radial distance from the centre of the star, we can provide data that current experiments can not; we can investigate the interior of a neutron star, by calculating the proportions of various particles at successive values of internal radius and/or density. This allows us to construct a cross-section of a neutron star, investigate the possible contents, and examine the effects that various changes to the models have on both the internal and external properties.

2.12 SU(6) Spin-Flavor Baryon-Meson Couplings

We have noted earlier that the coupling of baryons to mesons is dependent on isospin group. The physics leading to this result is highly non-trivial, but is often neglected in the literature. We will therefore outline the process involved in determining the relations between baryon-meson couplings.

In order to determine the normalized relations between the point vertex couplings of various mesons to the full baryon octet g_{Bm} it is common to express the octet as a 3×3 matrix in flavor space as

$$B = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ -\Xi^- & \Xi^0 & -\frac{2\Lambda}{\sqrt{6}} \end{pmatrix}. \tag{2.101}$$

This has been constructed as an array where rows and columns are distinguished by rotations in flavor space, which can be seen if we observe the quark content of these baryons;

$$B = \begin{pmatrix} uds & \mathbf{d} \to \mathbf{u} & uus & \mathbf{s} \to \mathbf{d} & uud \\ \mathbf{u} \to \mathbf{d} & & & & \\ dds & & dus & & ddu \\ \mathbf{d} \to \mathbf{s} & & & & \\ ssd & & uss & & sud \end{pmatrix}. \tag{2.102}$$

The vector meson octet ($J^P = 1^-$) can be written in a similar fashion as

$$P^{\text{vec}}_{\text{oct}} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega_8}{\sqrt{6}} & K^{*0} \\ K^{*-} & \overline{K^{*0}} & -2\left(\frac{\omega_8}{\sqrt{6}}\right) \end{pmatrix}, \tag{2.103}$$

which, along with the singlet state, $P^{\text{vec}}_{\text{sing}} = \frac{1}{\sqrt{3}}\text{diag}(\omega_0, \omega_0, \omega_0)$ defines the vector meson nonet

$$P^{\text{vec}} = P^{\text{vec}}_{\text{oct}} + P^{\text{vec}}_{\text{sing}}. \tag{2.104}$$

Furthermore, the scalar meson octet ($J^P = 0^+$) can be written as

$$P^{\text{sca}}_{\text{oct}} = \begin{pmatrix} \frac{a_0^0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{6}} & a_0^+ & \kappa^+ \\ a_0^- & -\frac{a_0^0}{\sqrt{2}} + \frac{\sigma_8}{\sqrt{6}} & \kappa^0 \\ \kappa^- & \overline{\kappa^0} & -2\frac{\sigma_8}{\sqrt{6}} \end{pmatrix}, \tag{2.105}$$

and along with singlet state $P^{\rm sca}_{\rm sing} = \frac{1}{\sqrt{3}}{\rm diag}(\sigma_0, \sigma_0, \sigma_0)$, these define the scalar meson nonet. The meson octet matrices are constructed in a similar fashion to the baryon octet matrix;

$$P_{\rm oct} = \begin{pmatrix} u\bar{u} & \bar{\mathbf{u}} \to \bar{\mathbf{d}} & u\bar{d} & \bar{\mathbf{d}} \to \bar{\mathbf{s}} & u\bar{s} \\ \mathbf{u} \to \mathbf{d} & & & & \\ d\bar{u} & & d\bar{d} & & d\bar{s} \\ \mathbf{d} \to \mathbf{s} & & & & \\ s\bar{u} & & s\bar{d} & & s\bar{s} \end{pmatrix}. \tag{2.106}$$

The singlet and octet representations of both ω and σ ($\omega_0, \omega_8, \sigma_0, \sigma_8$, appearing in Eqs. (2.103)–(2.105)) are not however the physical particles which we wish to include in the model; these are linear combinations of the physical particles. Due to explicit SU(3) flavor-symmetry breaking ($m_s > m_u,\ m_d$), a mixture of the unphysical ω_8 and ω_0 states produces the physical ω and ϕ mesons, while a mixture of the unphysical σ_8 and σ_0 states produces the physical σ and f_0 mesons, the properties of which we list in Appendix **??**.

The octet and singlet states are represented by linear combinations of quark-antiquark pairs. The state vectors for these are

$$|\omega_8\rangle = |\sigma_8\rangle = \frac{1}{\sqrt{6}}\left(|\bar{u}u\rangle + |\bar{d}d\rangle - 2|\bar{s}s\rangle\right), \quad |\omega_0\rangle = |\sigma_0\rangle = \frac{1}{\sqrt{3}}\left(|\bar{u}u\rangle + |\bar{d}d\rangle| + |\bar{s}s\rangle\right), \tag{2.107}$$

where the normalizations arise by ensuring that

$$\langle\xi|\xi\rangle = 1; \quad \xi \in \{\omega_8, \omega_0, \sigma_8, \sigma_0\}. \tag{2.108}$$

Since the quark contents of the physical states are predominantly

$$\omega = \sigma = \frac{1}{\sqrt{2}}\,(u\bar{u} + d\bar{d}), \quad \text{and} \quad \phi = f_0 = -s\bar{s}, \tag{2.109}$$

we can replace the octet and singlet combinations with the physical states via the replacements of

$$\omega_8 = \tfrac{1}{\sqrt{3}}\,\omega + \tfrac{2}{\sqrt{6}}\,\phi, \qquad \omega_0 = \sqrt{\frac{2}{3}}\,\omega - \frac{1}{\sqrt{3}}\,\phi, \tag{2.110}$$

$$\sigma_8 = \tfrac{1}{\sqrt{3}}\,\sigma + \tfrac{2}{\sqrt{6}}\,f_0, \qquad \sigma_0 = \sqrt{\frac{2}{3}}\,\sigma - \frac{1}{\sqrt{3}}\,f_0. \tag{2.111}$$

Using these definitions, we can express the octet and singlet states in terms of the physical states

$$P^{\rm vec}_{\rm oct} = \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{18}} + \frac{\phi}{\sqrt{9}} & \rho^+ & K^{*+} \\ \rho^- & -\frac{\rho^0}{\sqrt{2}} + \frac{\omega}{\sqrt{18}} + \frac{\phi}{\sqrt{9}} & K^{*0} \\ K^{*-} & \overline{K^{*0}} & -2\left(\frac{\omega}{\sqrt{18}} + \frac{\phi}{\sqrt{9}}\right) \end{pmatrix}, \tag{2.112}$$

$$P^{\rm sca}_{\rm oct} = \begin{pmatrix} \frac{a_0^0}{\sqrt{2}} + \frac{\sigma}{\sqrt{18}} + \frac{f_0}{\sqrt{9}} & a_0^+ & \kappa^+ \\ a_0^- & -\frac{a_0^0}{\sqrt{2}} + \frac{\sigma}{\sqrt{18}} + \frac{f_0}{\sqrt{9}} & \kappa^0 \\ \kappa^- & \overline{\kappa^0} & -2\left(\frac{\sigma}{\sqrt{18}} + \frac{f_0}{\sqrt{9}}\right) \end{pmatrix}. \tag{2.113}$$

Each of the above mesons can interact with a pair of baryons in three possible SU(3) invariant ways, which we shall identify as F-style (anti-symmetric), D-style (symmetric) and S-style (singlet). The singlet mesons ω_0 and σ_0 are associated with an S-style coupling, while the octet particles are associated with F- and D-style couplings.

To determine these F-, D-, and S-style couplings for the vector and scalar mesons we need to calculate the SU(3) invariant Lagrangian density coefficients symbolically for each isospin group, with each SU(3) invariant combination[11] given by:

$$\begin{aligned}
\left[\bar{B}BP\right]_F &= \mathrm{Tr}(\bar{B}PB) - \mathrm{Tr}(\bar{B}BP) \\
&= \mathrm{Tr}(\bar{B}P_{\mathrm{oct}}B) - \mathrm{Tr}(\bar{B}BP_{\mathrm{oct}}), \\
\left[\bar{B}BP\right]_D &= \mathrm{Tr}(\bar{B}PB) + \mathrm{Tr}(\bar{B}BP) - \frac{2}{3}\mathrm{Tr}(\bar{B}BP)\mathrm{Tr}(P) \\
&= \mathrm{Tr}(\bar{B}P_{\mathrm{oct}}B) + \mathrm{Tr}(\bar{B}BP_{\mathrm{oct}}), \\
\left[\bar{B}BP\right]_S &= \mathrm{Tr}(\bar{B}B)\mathrm{Tr}(P) \\
&= \mathrm{Tr}(\bar{B}B)\mathrm{Tr}(P_{\mathrm{sing}}).
\end{aligned} \tag{2.114}$$

where we have expanded the meson matrices P according to Eq. (2.104), and we note that the octet matrices $\bar{B}$, and B are traceless.

Together, these terms can be combined to form an interaction Lagrangian density for all possible SU(3) invariant interactions involving each meson isospin group, with octet and singlet coupling coefficients F, D, and S (each defined separately for each isospin group) as

$$\mathcal{L}^{\mathrm{int}} = -\sqrt{2}\left\{F\left[\bar{B}BP\right]_F + D\left[\bar{B}BP\right]_D\right\} - S\frac{1}{\sqrt{3}}\left[\bar{B}BP\right]_S, \tag{2.115}$$

where the remaining numerical factors are introduced for convenience.

If we evaluate this Lagrangian density by matrix multiplication of B, the octet and singlet matrices of P_{vec}, and $\bar{B} = B^\dagger\gamma^0$ in the combinations stated in Eq. (2.115), we can extract the coefficients of each baryon-meson vertex in terms of F, D and S factors. These are summarized in Table 2.1 for vertices involving the physical vector mesons ω and ρ^0 and ϕ, for pairs of like baryons[12]. The summary for the scalar mesons is the same under replacements of $\omega \to \sigma$, $\vec{\rho} \to \vec{a}_0$, and $\phi \to f_0$.

The physical ϕ and f_0 states are purely strange quark components. These do not couple to nucleons significantly (since nucleons contain only up and down valence quarks) the only way to produce these mesons is via gluons. Thus we set the (normalized or not) couplings of these mesons to zero; $g_{B\phi} = g_{Bf_0} = 0$. We are then left with the physical σ and ω mesons as the effective meson degrees of freedom.

If we denote the *total* (but not normalized) coupling (now including all prefactors of Eq. (2.115)) of a (like) baryon pair $\bar{B}B$ to a meson α by $f_{B\alpha}$, and we calculate the SU(3)

[11] Following the notation of Ref. [?].

[12] As discussed in Section 2.2, any flavor-changing meson-baryon interactions would produce a null overlap of ground-state operators, and as such we only focus on the like-baryon interactions of the form $g_{B\alpha}\bar{\psi}_B\alpha\psi_{B'}\delta_{BB'}$ for a meson α in this discussion.

Table 2.1: F-, D-, and S-style couplings of like baryon-baryon pairs to vector mesons used in these models, according to vertices of type $B + P \to \bar{B}$. The summary for the scalar mesons is the same under the replacements of $\omega \to \sigma$, $\vec{\rho} \to \vec{a}_0$, and $\phi \to f_0$.

$\overline{\Sigma^-}\Sigma^-\rho^0$	$\propto$	$2F$	$\overline{\Lambda}\Lambda\omega$	$\propto$	$\frac{1}{9}\left(6D - \sqrt{6}S\right)$
$\overline{\Sigma^-}\Sigma^-\omega$	$\propto$	$\frac{1}{9}\left(-6D - \sqrt{6}S\right)$	$\overline{\Lambda}\Lambda\phi$	$\propto$	$\frac{1}{9}\left(6\sqrt{2}D + \sqrt{3}S\right)$
$\overline{\Sigma^-}\Sigma^-\phi$	$\propto$	$\frac{1}{9}\left(-6\sqrt{2}D + \sqrt{3}S\right)$	$\overline{p}p\rho^0$	$\propto$	$-D - F$
$\overline{\Sigma^0}\Sigma^0\omega$	$\propto$	$\frac{1}{9}\left(-6D - \sqrt{6}S\right)$	$\overline{p}p\omega$	$\propto$	$\frac{1}{9}\left(3D - 9F - \sqrt{6}S\right)$
$\overline{\Sigma^0}\Sigma^0\phi$	$\propto$	$\frac{1}{9}\left(-6\sqrt{2}D + \sqrt{3}S\right)$	$\overline{p}p\phi$	$\propto$	$\frac{1}{9}\left(3\sqrt{2}D - 9\sqrt{2}F + \sqrt{3}S\right)$
$\overline{\Sigma^+}\Sigma^+\rho^0$	$\propto$	$-2F$	$\overline{n}n\rho^0$	$\propto$	$D + F$
$\overline{\Sigma^+}\Sigma^+\omega$	$\propto$	$\frac{1}{9}\left(-6D - \sqrt{6}S\right)$	$\overline{n}n\omega$	$\propto$	$\frac{1}{9}\left(3D - 9F - \sqrt{6}S\right)$
$\overline{\Sigma^+}\Sigma^+\phi$	$\propto$	$\frac{1}{9}\left(-6\sqrt{2}D + \sqrt{3}S\right)$	$\overline{n}n\phi$	$\propto$	$\frac{1}{9}\left(3\sqrt{2}D - 9\sqrt{2}F + \sqrt{3}S\right)$
$\overline{\Xi^-}\Xi^-\rho^0$	$\propto$	$-D + F$			
$\overline{\Xi^-}\Xi^-\omega$	$\propto$	$\frac{1}{9}\left(3D + 9F - \sqrt{6}S\right)$			
$\overline{\Xi^-}\Xi^-\phi$	$\propto$	$\frac{1}{9}\left(3\sqrt{2}D + 9\sqrt{2}F + \sqrt{3}S\right)$			
$\overline{\Xi^0}\Xi^0\rho^0$	$\propto$	$D - F$			
$\overline{\Xi^0}\Xi^0\omega$	$\propto$	$\frac{1}{9}\left(3D + 9F - \sqrt{6}S\right)$			
$\overline{\Xi^0}\Xi^0\phi$	$\propto$	$\frac{1}{9}\left(3\sqrt{2}D + 9\sqrt{2}F + \sqrt{3}S\right)$			

invariant combinations for the singlet ω_0 and mixed state ω_8, we can use the relation between these normalizations from Eq. (2.107) to relate the S-style couplings to the remaining couplings via

$$-\frac{S}{3} = f_{N\omega_0} = \sqrt{2} f_{N\omega_8} = \frac{\sqrt{2}}{\sqrt{3}}(D - 3F), \tag{2.116}$$

and so we can reduce the relation of the couplings to

$$S = \sqrt{6}(3F - D), \tag{2.117}$$

and we can therefore find the couplings of the singlet ω_0 meson in terms of just F and D factors. After removing the strange quark components and substituting the result of Eq. (2.117) we obtain a summary of couplings as shown in Table 2.2.

We note however that the couplings in Tables 2.1 and 2.2 do not display isospin symmetry manifestly, though our original Lagrangian density (refer to Section 2.1) was constructed in terms of isospin groups only with common coefficients. This can be remedied by considering a *general* Lagrangian density constructed from isospin groups, which we shall restrict to terms involving like baryons and the mesons we are interested in, to give

$$\begin{aligned}\mathcal{L}_{\rm int}^{\rm oct} &= -f_{N\rho}(\overline{N}\vec{\tau}^{\rm T}N)\cdot\vec{\rho} + i f_{\Sigma\rho}(\vec{\overline{\Sigma}}\times\vec{\Sigma})\cdot\vec{\rho} - f_{\Xi\rho}(\overline{\Xi}\vec{\tau}^{\rm T}\Xi)\cdot\vec{\rho} \\ &\quad -f_{N\omega}(\overline{N}N)\omega - f_{\Lambda\omega}(\overline{\Lambda}\Lambda)\omega - f_{\Sigma\omega}(\vec{\overline{\Sigma}}\cdot\vec{\Sigma})\omega - f_{\Xi\omega}(\overline{\Xi}\Xi)\omega, \end{aligned} \tag{2.118}$$

where the N, Λ, and Ξ isospin groups are defined as before as

$$N = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \Lambda \end{pmatrix}, \quad \Xi = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix}. \tag{2.119}$$

The ρ mesons terms are defined in isospin space as linear combinations of the physical charged states (as we did in Section 2.3.1) as

$$\rho^- = \frac{1}{\sqrt{2}}(\rho_1 - i\rho_2), \quad \rho^+ = \frac{1}{\sqrt{2}}(\rho_1 + i\rho_2), \quad \rho^0 = \rho_3,$$

or, equivalently as

$$\rho_1 = \frac{1}{\sqrt{2}}(\rho^+ + \rho^-), \quad \rho_2 = \frac{i}{\sqrt{2}}(\rho^- - \rho^+), \quad \rho_3 = \rho^0, \tag{2.120}$$

with the same convention for the replacement of $\vec{\rho}\to\vec{\Sigma}$. This gives the expansion

$$\vec{\Sigma}\cdot\vec{\rho} = \Sigma^+\rho^- + \Sigma^0\rho^0 + \Sigma^-\rho^+. \tag{2.121}$$

We can expand the Lagrangian density term by term to find the individual interactions

$$\begin{aligned}(\overline{N}\vec{\tau}^{\rm T}N)\cdot\vec{\rho} &= (\overline{p}\ \ \overline{n})\ \tau_i^{\rm T}\rho^i \begin{pmatrix} p \\ n \end{pmatrix} \\ &= (\overline{p}n + \overline{n}p)\rho_1 + i(\overline{p}n - \overline{n}p)\rho_2 + (\overline{p}p - \overline{n}n)\rho_3 \\ &= \frac{1}{\sqrt{2}}(\overline{p}n + \overline{n}p)(\rho_+ + \rho_-) - \frac{1}{\sqrt{2}}(\overline{p}n - \overline{n}p)(\rho_- - \rho_+) + (\overline{p}p - \overline{n}n)\rho_0 \\ &= \overline{p}p\rho_0 - \overline{n}n\rho_0 + \sqrt{2}\overline{p}n\rho_+ + \sqrt{2}\overline{n}p\rho_- \ , \end{aligned} \tag{2.122}$$

Table 2.2: Couplings of like baryon-baryon pairs to vector mesons used in these models, according to vertices of type $B + P \to \bar{B}$ using the relation of Eq. (2.117)

$\overline{\Sigma^-}\Sigma^-\rho^0$	$\propto$	$2F$	$\overline{p}p\rho^0$	$\propto$	$-D-F$	$\overline{\Xi^-}\Xi^-\rho^0$	$\propto$	$-D+F$
$\overline{\Sigma^-}\Sigma^-\omega$	$\propto$	$-\frac{2D}{3}$	$\overline{p}p\omega$	$\propto$	$\frac{D}{3}-F$	$\overline{\Xi^-}\Xi^-\omega$	$\propto$	$\frac{D}{3}+F$
$\overline{\Sigma^0}\Sigma^0\omega$	$\propto$	$-\frac{2D}{3}$	$\overline{n}n\rho^0$	$\propto$	$D+F$	$\overline{\Xi^0}\Xi^0\rho^0$	$\propto$	$D-F$
$\overline{\Sigma^+}\Sigma^+\rho^0$	$\propto$	$-2F$	$\overline{n}n\omega$	$\propto$	$\frac{D}{3}-F$	$\overline{\Xi^0}\Xi^0\omega$	$\propto$	$\frac{D}{3}+F$
$\overline{\Sigma^+}\Sigma^+\omega$	$\propto$	$-\frac{2D}{3}$	$\overline{\Lambda}\Lambda\omega$	$\propto$	$\frac{2D}{3}$			

where we note that a term $\bar{B}BP$ indicates the annihilation of a baryon B with a meson P, and the creation of a baryon $\bar{B}$ according to the reaction $B+P \to \bar{B}$. Continuing to expand terms, for the Σ baryons we have

$$(\vec{\bar{\Sigma}} \times \vec{\Sigma}) \cdot \vec{\rho} \;=\; -i\rho^+ \left(\Sigma^-\overline{\Sigma^0} - \Sigma^0\overline{\Sigma^+}\right) - i\rho^- \left(\Sigma^0\overline{\Sigma^-} - \Sigma^+\overline{\Sigma^0}\right) - i\rho^0 \left(\Sigma^+\overline{\Sigma^+} - \Sigma^-\overline{\Sigma^-}\right) \;, \tag{2.123}$$

and for the Ξ baryons,

$$\begin{aligned}
(\bar{\Xi}\vec{\tau}^{\mathrm{T}}\Xi) \cdot \vec{\rho} &= \left(\overline{\Xi^0} \;\; \overline{\Xi^-}\right) \tau_i^{\mathrm{T}} \rho^i \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix} \\
&= (\overline{\Xi^0}\Xi^- + \overline{\Xi^-}p)\rho_1 + i(\overline{\Xi^0}\Xi^- - \overline{\Xi^-}\Xi^0)\rho_2 + (\overline{\Xi^0}\Xi^0 - \overline{\Xi^-}\Xi^-)\rho_3 \\
&= \overline{\Xi^0}\Xi^0\rho_0 - \overline{\Xi^-}\Xi^-\rho_0 + \sqrt{2}\overline{\Xi^0}\Xi^-\rho_+ + \sqrt{2}\overline{\Xi^-}\Xi^0\rho_- .
\end{aligned} \tag{2.124}$$

The iso-scalar terms are more straightforward;

$$(\overline{N}N)\omega = \overline{p}p\omega + \overline{n}n\omega, \tag{2.125}$$

$$\overline{\Lambda}\Lambda\omega \quad \text{(requires no expansion)}, \tag{2.126}$$

$$(\vec{\bar{\Sigma}} \cdot \vec{\Sigma})\omega = \overline{\Sigma^+}\Sigma^+\omega + \overline{\Sigma^0}\Sigma^0\omega + \overline{\Sigma^-}\Sigma^-\omega, \tag{2.127}$$

$$(\bar{\Xi}\Xi)\omega = \overline{\Xi^0}\Xi^0\omega + \overline{\Xi^-}\Xi^-\omega. \tag{2.128}$$

Once we have calculated the full interaction Lagrangian density, and the F and D coefficients of each interaction, we have factors of the following form:

$$\mathcal{L}_{\mathrm{int}} = \sum_B \sum_m A_{Bm} f_{Bm} X_{Bm}; \quad X_{Bm} = C_{Bm}\bar{B}Bm, \tag{2.129}$$

where $A_{\Sigma\rho} = i$, and for all other interactions $A_{Bm} = -1$. The term X_{Bm} is the expanded interaction term (after expanding cross products, etc.) arising from the *general* Lagrangian density, Eq. (2.118), and contains factors of $C_{Bm} = \pm 1, \pm\sqrt{2}$. We also require a term calculated from the SU(3) invariant combinations M_{Bm}; the coefficient of the interaction $\bar{B}Bm$ in terms of F and D factors as found in Table 2.2.

To calculate the values of f_{Bm} we apply the following formula:

$$f_{Bm} = \frac{C_{Bm}}{A_{Bm}} M_{Bm}. \tag{2.130}$$

For example, consider the interaction vertex $\omega + \Sigma^0 \to \overline{\Sigma^0}$:

$$A_{\Sigma\omega} = -1, \quad C_{\Sigma\omega} = +1, \quad M_{\Sigma\omega} = -\frac{2D}{3}, \quad \Rightarrow \quad f_{\Sigma\omega} = \frac{+1}{-1}(-\frac{2D}{3}) = \frac{2D}{3}. \tag{2.131}$$

Performing these calculations for every possible interactions provides (consistently) the following couplings of the octet of baryons to the octet of mesons:

$$\begin{gathered}
f_{N\rho} = D + F, \quad f_{\Lambda\rho} = 0, \quad f_{\Sigma\rho} = 2F, \quad f_{\Xi\rho} = F - D, \\
f_{N\omega} = 3F - D, \quad f_{\Lambda\omega} = -\tfrac{4}{3}D + 2F, \quad f_{\Sigma\omega} = 2F, \quad f_{\Xi\omega} = F - D.
\end{gathered} \tag{2.132}$$

Table 2.3: F-, D- and S-style Baryon currents for all types of meson vertices of the form $\bar{B}BX$ where X is a meson with either scalar (S), vector (V), tensor (T), axial(pseudo-) vector (A) or pseudo-scalar (P) spin form. Adapted from Ref. [?].

	F	D	S
S	$\frac{1}{3}H\bar{\psi}\psi$	0	$\frac{1}{3}H\bar{\psi}\psi$
V$_1$	$\frac{1}{3}\left(H-\frac{1}{6}\frac{q^2}{M^2}\right)\bar{\psi}\gamma_\mu\psi$	$\frac{q^2}{6M^2}\bar{\psi}\gamma_\mu\psi$	$\frac{1}{3}\left(H-\frac{1}{3}\frac{q^2}{M^2}\right)\bar{\psi}\gamma_\mu\psi$
V$_2$	$-\frac{1}{9M}i\bar{\psi}\sigma_{\mu\nu}\psi$	$+\frac{1}{3M}i\bar{\psi}\sigma_{\mu\nu}\psi$	$-\frac{2}{9M}i\bar{\psi}\sigma_{\mu\nu}\psi$
V$_3$	0	0	0
A	$\frac{2}{9}H\bar{\psi}\gamma_5\gamma_\mu\psi$	$\frac{1}{3}H\bar{\psi}\gamma_5\gamma_\mu\psi$	$\frac{1}{9}H\bar{\psi}\gamma_5\gamma_\mu\psi$
P	$\frac{2}{9}H\bar{\psi}\gamma_5\psi$	$\frac{1}{3}H\bar{\psi}\gamma_5\psi$	$\frac{1}{9}H\bar{\psi}\gamma_5\psi$

We can further simplify our calculations by examining all the different currents that one can form using a baryon, an antibaryon and a meson. As discussed in Ref. [?], all possible couplings of baryons to vector mesons (denoted by $\bar{B}BV$) should be considered when writing out the most general Lagrangian density. By calculating the currents (prior to making any approximations or assumptions that appear in earlier sections here) we are able to find the F-, D-, and S-style couplings of the form $\bar{B}BX$ where X is a meson with either scalar (S), vector (V), tensor (T), axial-vector (A) or pseudo-scalar (P) spin form.

Under an expanded SU(6) spin-flavor symmetry, the currents are shown in Table 2.3, where the various vector couplings are of the forms

$$V_1 = \bar{\psi}\gamma_\mu\psi,\ V_2 = \bar{\psi}\sigma_{\mu\nu}q^\nu\psi,\ V_3 = \bar{\psi}q_\mu\psi, \tag{2.133}$$

and we use the convenience definitions of

$$\sigma_{\mu\nu} = \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right], \quad H = \frac{4M^2+q^2}{2M^2}. \tag{2.134}$$

If we now consider this as a low energy effective field theory, we can consider the case of $q^2 = 0$. We can also enforce rotational symmetry due to lack of a preferred frame (or direction) and thus remove the spatial components of both the mesons and the momenta, so that $V^\mu = (V^0, \vec{0})$ and $q^\mu = (q^0, \vec{0})$, as per Section 2.3.1. Along with $\sigma_{00} = 0$, all terms proportional to q^2 vanish, and $H = 2$. Using these assumptions, the currents are reduced to those found in Table 2.4.

We can now observe the relations between the F- and D-style couplings (with the S-style coupling now contributing to F and D); First, as a check, we observe that the ratio D/F for the pseudo-scalars (and the axial-vectors for that matter) is indeed $\frac{3}{2}$ as commonly noted in the literature [?, ?] under SU(6) symmetry [?]. Less commonly found in the literature is that the γ_μ-type vector coupling is purely F-style, thus the vector analogy of the above relation is $D/F = 0$, implying $D = 0$.

Using the couplings of Table 2.2, we can evaluate the couplings of the vector mesons to the entire baryon octet. This provides us with a unified description of the couplings in terms

Table 2.4: F-, D- and S-style Baryon currents with mean-field assumptions $V^\mu = (V^0, \vec{0})$, $q^\mu = (q^0, \vec{0})$, and $q^2 = 0$.

	F	D	S
S	$\frac{2}{3}\bar{\psi}\psi$	0	$\frac{2}{3}\bar{\psi}\psi$
V_1	$\frac{2}{3}\bar{\psi}\gamma_\mu\psi$	0	$\frac{2}{3}\bar{\psi}\gamma_\mu\psi$
V_2	0	0	0
V_3	0	0	0
A	$\frac{4}{9}\bar{\psi}\gamma_5\gamma_\mu\psi$	$\frac{2}{3}\bar{\psi}\gamma_5\gamma_\mu\psi$	$\frac{2}{9}\bar{\psi}\gamma_5\gamma_\mu\psi$
P	$\frac{4}{9}\bar{\psi}\gamma_5\psi$	$\frac{2}{3}\bar{\psi}\gamma_5\psi$	$\frac{2}{9}\bar{\psi}\gamma_5\psi$

of an arbitrary parameter F. These couplings are thus

$$\begin{gathered} f_{N\rho} = F, \quad f_{\Lambda\rho} = 0, \quad f_{\Sigma\rho} = 2F, \quad f_{\Xi\rho} = F, \\ f_{N\omega} = 3F, \quad f_{\Lambda\omega} = 2F, \quad f_{\Sigma\omega} = 2F, \quad f_{\Xi\omega} = F. \end{gathered} \tag{2.135}$$

We can normalize these results to the nucleon-ω coupling, since we will fit this parameter to saturation properties (refer to Section 2.8). Thus the normalized couplings are

$$g_{Bm} = g_{N\omega}\frac{f_{Bm}}{f_{N\omega}}. \tag{2.136}$$

We can then separate the meson couplings, since the the normalization above results in the following relations, using isospin I_B, and strangeness S_B of baryon B;

$$g_{B\omega} = \frac{(3 - S_B)}{3}\, g_{N\omega}, \quad g_{B\rho} = \frac{2I_B}{3}\, g_{N\omega}. \tag{2.137}$$

These results are consistent with a commonly used naïve assumption that the ω meson couples to the number of light quarks, and that the ρ meson couples to isospin. To emphasize the isospin symmetry in our models, we will include the isospin as a factor in our Lagrangian densities in the form of the $\vec{\tau}$ matrices. In doing so, rather than having an independent coupling for each isospin group, we will have a global coupling for the ρ meson, g_ρ.

Similarly to the above relations for the vector mesons, we have the same relation for the scalar mesons; that the coupling is purely F-style ($D = 0$). Therefore the couplings for the scalar mesons are the same as for the vector mesons, under the replacements $\omega \to \sigma$, $\vec{\rho} \to \vec{a}_0$. In the calculations that follow, we shall further neglect the contributions from the scalar iso-vector $\vec{a}_0$ due to their relatively large mass (refer to Table **??**).

As an alternative to the SU(6) relations for the ρ meson coupling g_ρ, we can use an experimental constraint. As we have shown above, the ρ meson couples to isospin, and as we will show in Section **??** the isospin density is proportional to the asymmetry between members of an isospin group; for example the asymmetry between protons and neutrons. This asymmetry is measured by the symmetry energy $a_4 \equiv a_{\mathrm{sym}}$ (derived in Appendix **??**) which appears in the semi-empirical mass formula (the connection is derived in Appendix **??**) which in the absence of charge symmetry is defined by Eq. (2.74). The coupling of ρ to the nucleons is found such that the experimental value of the asymmetry energy of $a_{\mathrm{sym}} = 32.5$ MeV is reproduced at saturation. The coupling of ρ to the remaining baryons follows the relations above.