Chapter 8
Heteroscedastic Data

In this chapter we discuss the generalization of the ANOM procedure to the heteroscedastic situation in which the process variances are not necessarily equal. In addition, this heteroscedastic analysis of means (HANOM) procedure allows an experimenter to set a goal of detecting differences among the I treatment means when two of the treatment means differ by a specified amount δ that does not depend on the process variances. See Dudewicz and Nelson (2003) for the mathematical details.

8.1 The One-Way Layout

As before, let I be the number of treatments being compared, and let y_{ij} be the jth observation from the ith population, where

$$\boxed{Y_{ij} = \mu_i + \epsilon_{ij}}$$

and all the observations are independent. Now, however, we assume that $\epsilon_{ij} \sim N(0, \sigma_i^2)$. Collecting the data for and performing a HANOM consists of the following steps:

1. Specify an initial sample size n_0 (≥ 2), take samples of size n_0 from each of the I populations, and calculate the sample means $\overline{y}_{0i}$ and the sample variances s_i^2.

2. Specify the level of significance α, a power γ, and the amount δ such that any two treatment means differing by δ will lead to rejection of the null hypothesis that all the treatment means are equal with power γ. From Figures A.17–A.24 (for $\alpha = 0.05$; for other α levels see Dudewicz and Nelson (2003)), find the figure with the appropriate α and I combination, and for the specified γ and df $= n_0 - 1$ find the corresponding value of w. Alternatively, one could choose a value of w using (8.1) based on a desired sample size n_i.

3. Compute
$$n_i = \max\{n_0 + 1, \lfloor (w/\delta)^2 s_i^2 \rfloor + 1\} \tag{8.1}$$
for each i, where $\lfloor y \rfloor$ denotes the greatest integer in y, and take $n_i - n_0$ additional observations $y_{i,n_0+1}, \ldots, y_{i,n_i}$ from population i.

4. For each i calculate the sample mean
$$\overline{y}_i = \frac{y_{i,n_0+1} + \cdots + y_{i,n_i}}{n_i - n_0} \tag{8.2}$$
of the second set of observations from population i.

5. For each i compute
$$b_i = \frac{n_i - n_0}{n_i}\left[1 + \sqrt{\left(\frac{n_0}{n_i - n_0}\right)\left(\left[\frac{\delta}{w}\right]^2 \frac{n_i}{s_i^2} - 1\right)}\right] \tag{8.3}$$
and
$$\tilde{\tilde{y}}_i = (1 - b_i)\overline{y}_{0i} + b_i \overline{y}_i, \tag{8.4}$$
and then compute
$$\tilde{\tilde{y}}_\bullet = (\tilde{\tilde{y}}_1 + \cdots + \tilde{\tilde{y}}_I)/I. \tag{8.5}$$

6. Compute decision lines
$$\boxed{\tilde{\tilde{y}}_\bullet \pm \mathcal{H}(\alpha; I, n_0 - 1)\frac{\delta}{w}}\,, \tag{8.6}$$
where $\mathcal{H}(\alpha; I, n_0 - 1)$ is found in Table B.7, and reject the hypothesis $H_0 : \mu_1 = \cdots = \mu_I$ if any of the $\tilde{\tilde{y}}_i$ fall outside these decision lines.

Example 8.1 (Adapted From Bishop and Dudewicz (1978)). An experiment was conducted to test the effects of different solvents on the ability of fungicide methyl-2-benzimidazole-carbamate to destroy the fungus *Penicillium expansum*. The fungicide was diluted in four different solvents and sprayed on the fungus, and the percentage of fungus destroyed was measured. The experimenter was interested in testing the hypothesis
$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$
that the four solvents resulted in the same average percentages of fungus being destroyed. Initial samples of size $n_0 = 10$ were taken with each of the four solvents. The data and summary statistics are given in Table 8.1. Using ANOMV to test for equal variances, one obtains the ANOMV chart in Figure 8.1, from which one would conclude that at the 0.05 level, the sample variance for solvent 3 is significantly large and the sample variance for solvent 4 is significantly small. Since the assumption of equal variances is not reasonable in

Table 8.1. *Data and Summary Statistics for the First-Stage Samples (Example* 8.1*).*

	Solvent 1		Solvent 2		Solvent 3		Solvent 4	
	95.39	98.07	92.15	94.25	95.15	94.28	95.99	96.68
	96.07	95.32	92.16	91.13	90.75	91.14	97.12	95.67
	96.58	97.41	94.75	92.09	91.97	91.45	97.07	96.49
	95.67	95.81	94.69	96.07	93.39	92.35	96.23	97.52
	96.73	97.79	93.53	96.15	86.97	94.92	97.19	95.30
$\overline{y}_{0i}$	96.48		93.70		92.24		96.53	
s_i^2	0.999		3.122		5.894		0.524	

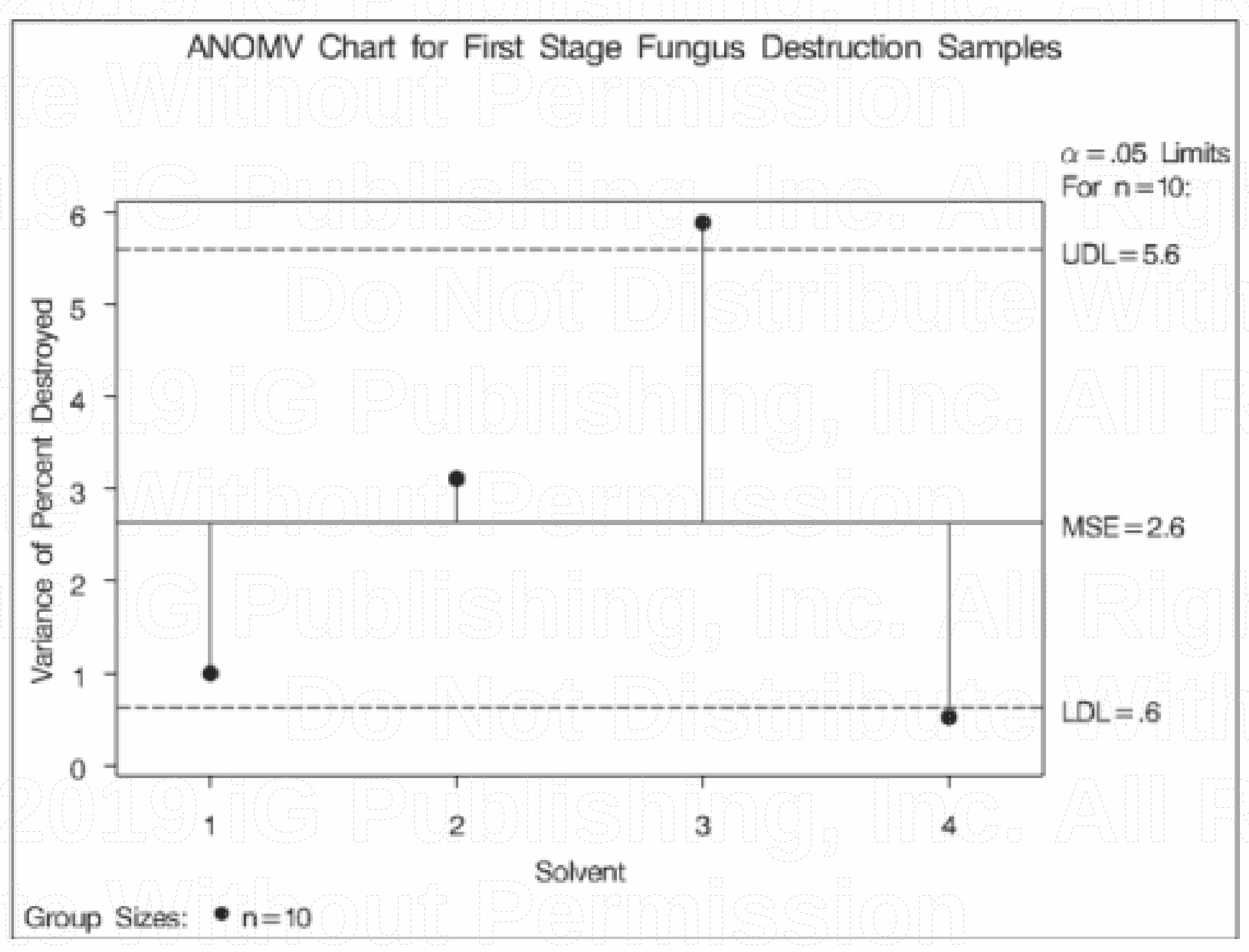

Figure 8.1. *ANOMV Chart for Example* 8.1.

this case, the HANOM procedure would be applicable. The experimenter wanted to conduct the test at level of significance $\alpha = 0.05$ and to have a power of at least $\gamma = 0.85$ if any two means differed by at least $\delta = 2.5$. From Figure A.18 ($\alpha = 0.05$ and $I = 4$), one finds that for $\gamma = 0.85$ and df $= 9$ the corresponding value of w is approximately 6. Using the sample variances from the first stage of the experiment, together with $\delta = 2.5$ and $w = 6$, one can compute the necessary values of n_i (equation (8.1)). For example,

$$\begin{aligned} n_1 &= \max\{11, \lfloor (6/2.5)^2 0.999 \rfloor + 1\} \\ &= \max\{11, 6\} \\ &= 11. \end{aligned}$$

Table 8.2. *Second-Stage Data and Computed Statistics (Example* 8.1*).*

	Solvent 1	Solvent 2		Solvent 3			Solvent 4
	94.98	96.61	95.68	94.21	92.01	94.43	96.50
	95.12	96.95	92.25	90.86	93.22	95.09	96.85
	96.37	94.39	94.98	93.59	95.01	93.57	97.54
	99.03	94.17	97.71	93.62	94.46	95.05	95.56
	95.16	95.58	93.44	97.01	96.27	99.39	98.04
	97.21	92.34	96.99	91.12	94.77	90.73	97.73
	94.45	92.75	93.01	94.72	95.53	94.68	98.00
	95.68	93.64		90.82	94.07	92.25	97.35
	96.67	98.64		91.56	95.21	92.24	97.14
	94.27	94.07		93.47	95.39	92.25	96.77
	95.78	94.23		93.35	91.68	95.55	97.30
					93.05		
$\overline{y}_i$	95.884	94.857		93.830			97.162
b_i	0.3654	0.4766		0.7235			0.5584
$\tilde{\tilde{y}}_i$	96.264	94.250		93.390			96.881

Similarly, $n_2 = 18$, $n_3 = 34$, and $n_4 = 11$. Note that the stage-two sample size is proportional to the stage-one sample variance. Table 8.2 gives the second-stage data and the summary values calculated using (8.2)–(8.4). Using the values of $\tilde{\tilde{y}}_i$ from Table 8.2 and (8.5), one obtains $\tilde{\tilde{y}}_\bullet = 95.196$, and the decision lines (8.6) are

$$\begin{aligned} 95.196 &\pm \mathcal{H}(0.05; 4, 9)\frac{\delta}{w} \\ &\pm 2.55\frac{2.5}{6} \\ &\pm 1.0625 \\ &(94.134, 96.259). \end{aligned}$$

The HANOM chart in Figure 8.2 shows that all the solvents are significantly different from the grand mean at level $\alpha = 0.05$. (Note that this chart lists $n = 10$, which was the initial sample size, not the overall sample size.) Solvents 1 and 4 destroyed significantly more fungus, and solvents 2 and 3 destroyed significantly less fungus.

8.2 Higher-Order Layouts

In designs with more than one factor, one must first test for possible interactions among the factors. As in the homogeneous variance case, this can be done using ANOVA. We will consider here only a two-way layout. Details on heteroscedastic ANOVA for a two-way layout can be found in Bishop and Dudewicz (1978), and still higher-order layouts are discussed in Bishop and Dudewicz (1981).

The usual model for the two-way layout is

$$Y_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \alpha_{ij}^{AB} + \epsilon_{ijk},$$

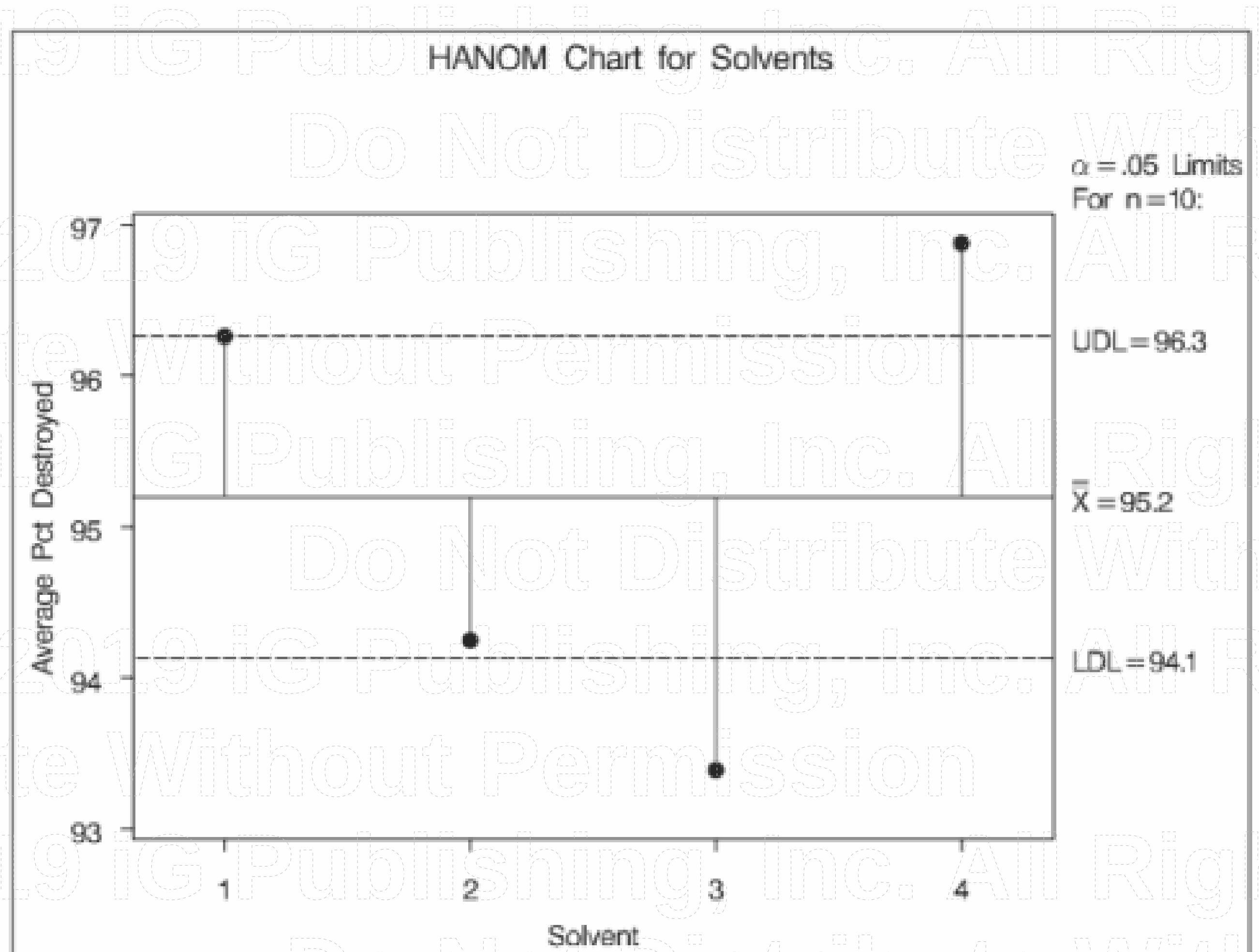

Figure 8.2. *HANOM Chart for Solvents (Example* 8.1*).*

where there are I levels of factor A and J levels of factor B. In the heteroscedastic case, one assumes that $\epsilon_{ijk} \sim N(0, \sigma_{ij}^2)$. Initially one would take a sample of size $n_0 \geq 2$ from each of the IJ treatment combinations and compute the usual unbiased estimates s_{ij}^2 of the σ_{ij}^2 and the averages of the observations for each treatment combination, $\overline{y}_{0ij}$. If it is known a priori that the σ_{ij}^2 are not all equal, or if a test on the variances such as the ANOMV indicates that the variances are not all equal, then heteroscedastic ANOVA and HANOM are appropriate. Sample sizes at the second stage are computed using (8.1), where s_i is replaced with s_{ij} and n_i is replaced with n_{ij}. That is,

$$n_{ij} = \max\{n_0 + 1, \lfloor (w/\delta)^2 s_{ij}^2 \rfloor + 1\}. \tag{8.7}$$

An appropriate value for w depends on whether there is significant interaction and, if there is, how one will proceed with the analysis. For fixed α, γ, and δ the value of w increases as the number of means being compared increases. Thus, a conservative choice for w would be based on comparing IJ means, which is the situation that would occur if one found significant interaction and were then interested in comparing the IJ treatment combinations. Alternatively, if one were interested in the main effects of a factor, then it would make sense to choose w based on $\max(I, J)$.

The heteroscedastic ANOVA test of $H_{AB} : \alpha_{ij}^{AB} = 0$ for all (i, j) is conducted using essentially the usual interaction sum of squares with $\overline{y}_{ij}$ replaced by $\tilde{\overline{y}}_{ij}$. Specifically, for each (i, j) one would compute

$$b_{ij} = \frac{n_{ij} - n_0}{n_{ij}} \left[1 + \sqrt{\left(\frac{n_0}{n_{ij} - n_0}\right)\left(\left[\frac{\delta}{w}\right]^2 \frac{n_{ij}}{s_{ij}^2} - 1\right)} \right] \tag{8.8}$$

and

$$\tilde{\tilde{y}}_{ij} = (1 - b_{ij})\overline{y}_{0ij} + b_{ij}\overline{y}_{ij}, \tag{8.9}$$

$$\tilde{\tilde{y}}_{i\bullet} = \frac{1}{J}\sum_{j=1}^{J}\tilde{\tilde{y}}_{ij}, \tag{8.10}$$

$$\tilde{\tilde{y}}_{\bullet j} = \frac{1}{I}\sum_{i=1}^{I}\tilde{\tilde{y}}_{ij}, \tag{8.11}$$

$$\tilde{\tilde{y}}_{\bullet\bullet} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J}\tilde{\tilde{y}}_{ij}\ . \tag{8.12}$$

The hypothesis H_{AB} is tested using

$$\tilde{F}_{AB} = \left(\frac{w}{\delta}\right)^2 \sum_{i=1}^{I}\sum_{j=1}^{J}(\tilde{\tilde{y}}_{ij} - \tilde{\tilde{y}}_{i\bullet} - \tilde{\tilde{y}}_{\bullet j} + \tilde{\tilde{y}}_{\bullet\bullet})^2, \tag{8.13}$$

which is compared with the appropriate quantile of an $((n_0 - 1)/(n_0 - 3))\chi^2_{(I-1)(J-1)}$ distribution.

If no significant interaction is found, then the main effects of factor A can be tested by comparing the $\tilde{\tilde{y}}_{i\bullet}$ with the decision lines

$$\boxed{\tilde{\tilde{y}}_{\bullet\bullet} \pm \mathcal{H}(\alpha; I, n_0 - 1)\frac{\delta}{w}}\ . \tag{8.14}$$

Similarly, the main effects of factor B can be tested by comparing the $\tilde{\tilde{y}}_{\bullet j}$ with the decision lines

$$\boxed{\tilde{\tilde{y}}_{\bullet\bullet} \pm \mathcal{H}(\alpha; J, n_0 - 1)\frac{\delta}{w}}\ . \tag{8.15}$$

If significant interaction is found, then one can either compare the IJ treatment combinations using the $\tilde{\tilde{y}}_{ij}$ and decision lines

$$\boxed{\tilde{\tilde{y}}_{\bullet\bullet} \pm \mathcal{H}(\alpha; IJ, n_0 - 1)\frac{\delta}{w}} \tag{8.16}$$

or (assuming $I > J$) compare the I levels of factor A separately for each level j' of factor B using the $\tilde{\tilde{y}}_{ij'}$ and decision lines

$$\boxed{\tilde{\tilde{y}}_{\bullet j'} \pm \mathcal{H}(\alpha; I, n_0 - 1)\frac{\delta}{w}}\ . \tag{8.17}$$

Table 8.3. *Summary Statistics from the First-Stage and Second-Stage Samples (Example* 8.2*).*

i	j	n_0	$\overline{y}_{0ij}$	s_{ij}^2	n_{ij}	$\overline{y}_{ij}$	b_{ij}	$\tilde{\overline{y}}_{ij}$
1	1	6	53.467	1.8827	7	49.8	0.6184	51.199
1	2	6	55.083	2.2496	7	52.3	0.5543	53.540
1	3	6	55.250	5.827	8	57.1	0.3479	55.894
1	4	6	58.683	2.3496	7	59.0	0.5389	58.854
2	1	6	52.250	1.259	7	51.7	0.7744	51.824
2	2	6	54.250	2.187	7	54.1	0.5643	54.165
2	3	6	56.850	5.619	8	52.35	0.3799	55.140
2	4	6	59.367	3.2546	7	60.9	0.4243	60.017
3	1	6	51.333	2.6906	7	53.2	0.4914	52.250
3	2	6	53.217	3.1576	7	57.5	0.4351	55.080
3	3	6	57.433	14.4386	19	55.338	0.7245	55.915
3	4	6	59.067	1.8067	7	58.4	0.6336	58.644

Example 8.2 (Insulation Data). An experiment was conducted to study the effects of three different types of insulation (factor A) at four different temperatures (factor B) on the insulation's ability to maintain a fixed temperature. Initial samples of size six were taken for each of the 12 treatment combinations. The data are a coded version of the rise in temperature after a fixed amount of time. The sample means and variances are given in Table 8.3. The ANOMV chart in Figure 8.3 indicates that at the 0.05 level, the σ_{ij}^2 are not all equal. Thus, heteroscedastic ANOVA and HANOM are appropriate ways to continue the analysis.

The experimenters were interested in conducting tests using $\alpha = 0.1$ and being able to detect any effect of $\delta \geq 7$ with a power of 0.8. For $\alpha = 0.1$, $I = 12$, and df $= 5$, the value of w (from Dudewicz and Nelson (2002)) is approximately 8. Using (8.7) the sample sizes for the second stage were computed and are given in Table 8.3. The statistics computed from the second-stage data using (8.8)–(8.12) are also given in Table 8.3. Using (8.13), one then would compute $\tilde{F}_{AB} = (8/7)^2(2.4855) = 3.246$, which when compared with $(5/3)\chi^2(0.05; 6) = (5/3)(12.592) = 21.0$ is found not to be significant. Since there is no significant interaction, the main effect of the two factors can be studied using HANOM. For convenience, the row means, column means, and grand mean (of the $\tilde{\overline{y}}_{ij}$) are given in Table 8.4. For factor A one would compute decision lines (8.14)

$$\begin{aligned} 55.210 &\pm \mathcal{H}(0.1; 3, 5)(7/8) \\ &\pm (2.16)(7/8) \\ &\pm 1.89 \\ &(53.320, 57.100), \end{aligned}$$

and from the HANOM chart in Figure 8.4, one sees that there is no effect due to the

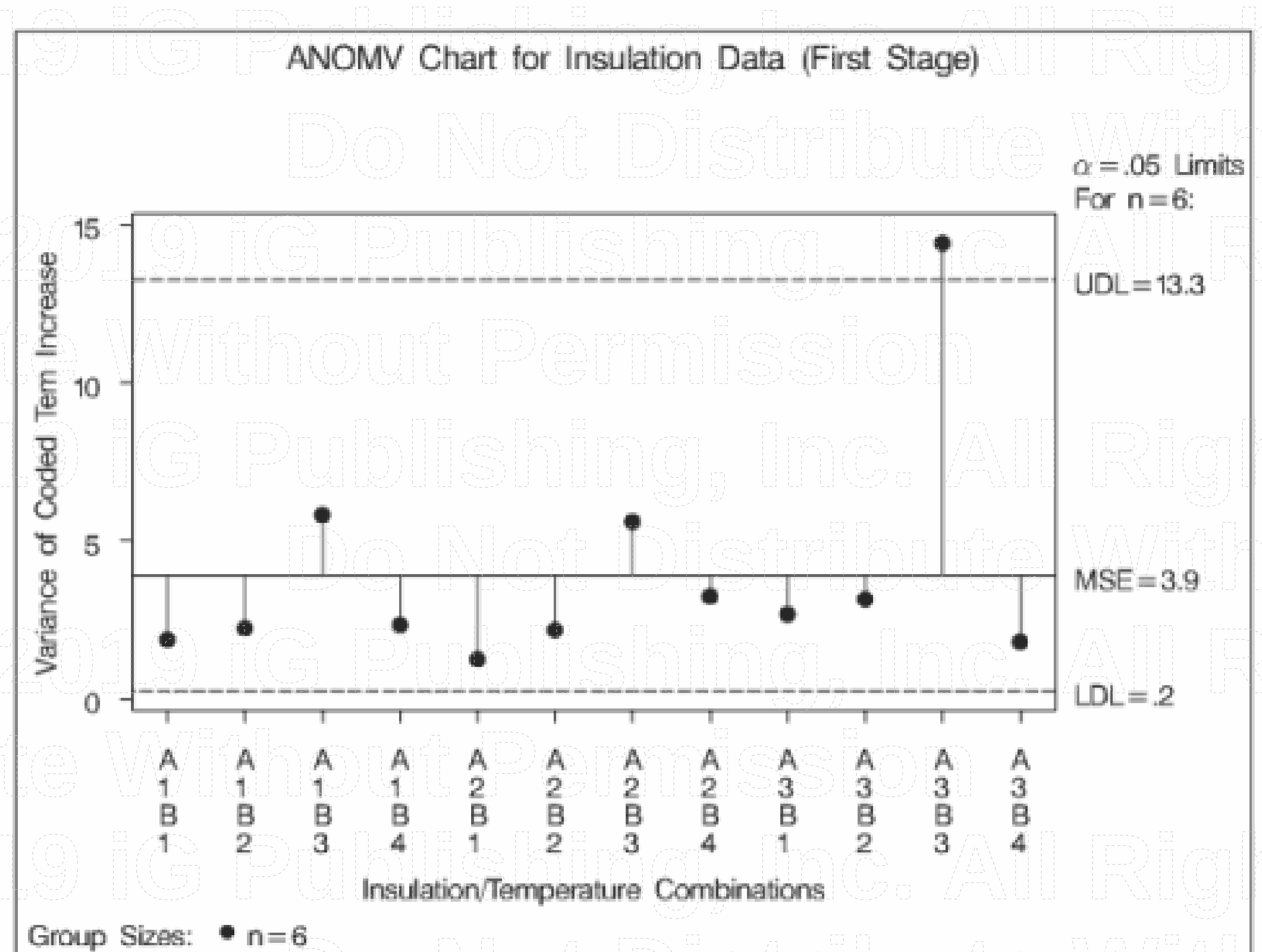

Figure 8.3. *The ANOMV Chart for Example* 8.2.

Table 8.4. *Average Outputs for Insulation-Temperature Combinations Together with Row Means, Column Means, and Grand Mean (Example* 8.2*).*

		Insulation 1	Insulation 2	Insulation 3	
Temperature	1	51.199	51.824	52.250	51.758
	2	53.540	54.165	55.080	54.262
	3	55.894	55.140	55.915	55.650
	4	58.854	60.017	58.644	59.172
		54.872	55.287	55.473	55.210

insulations at the 0.1 level. Similarly, for factor B one would compute decision lines (8.14)

$$\begin{aligned} 55.210 &\pm \mathcal{H}(0.1; 4, 5)(7/8) \\ &\pm (2.53)(7/8) \\ &\pm 2.21 \\ &\quad (53.0, 57.4), \end{aligned}$$

and from the HANOM chart in Figure 8.5, one sees that the temperatures have an effect at the 0.1 level.

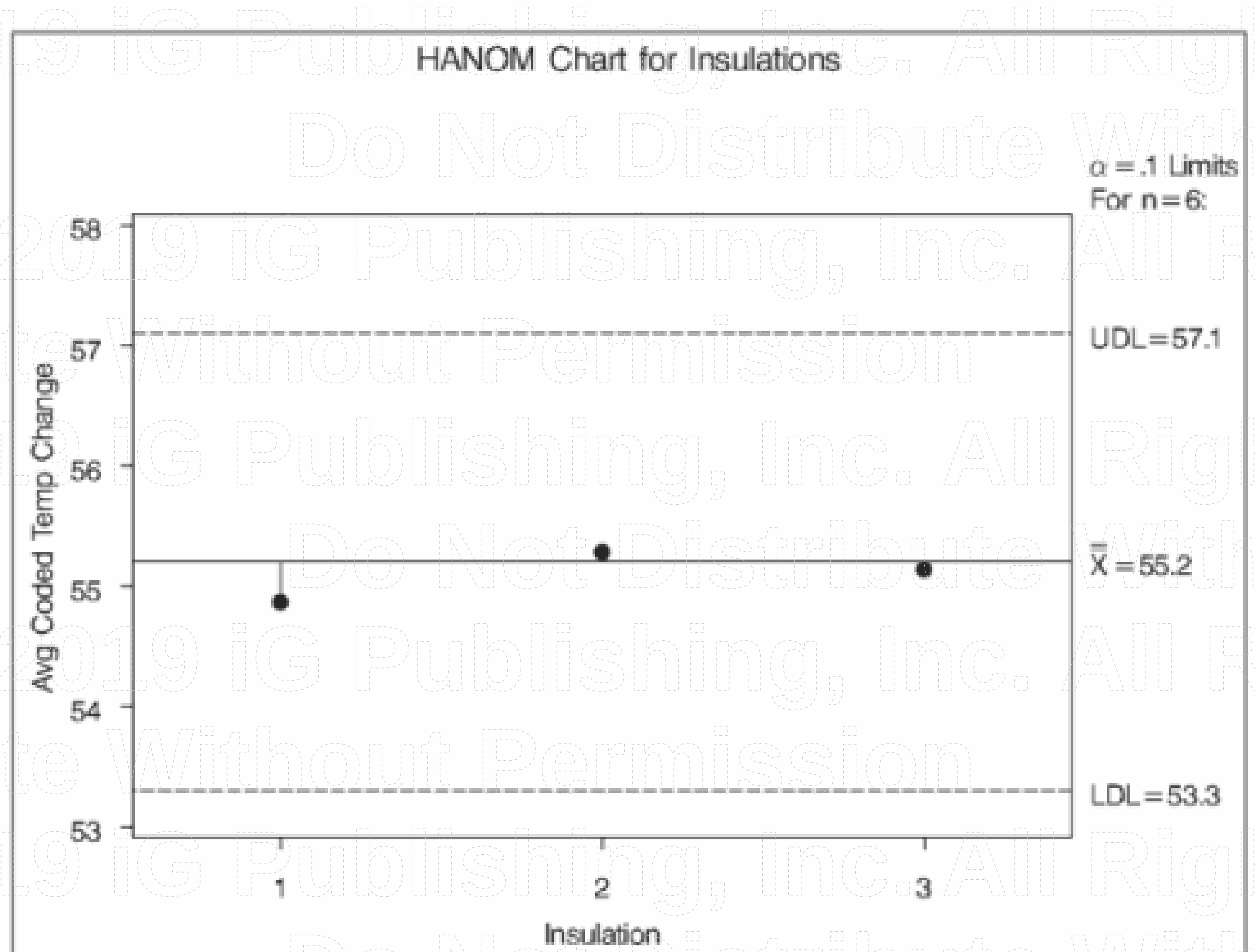

Figure 8.4. *The HANOM Chart for Insulation (Example* 8.2*).*

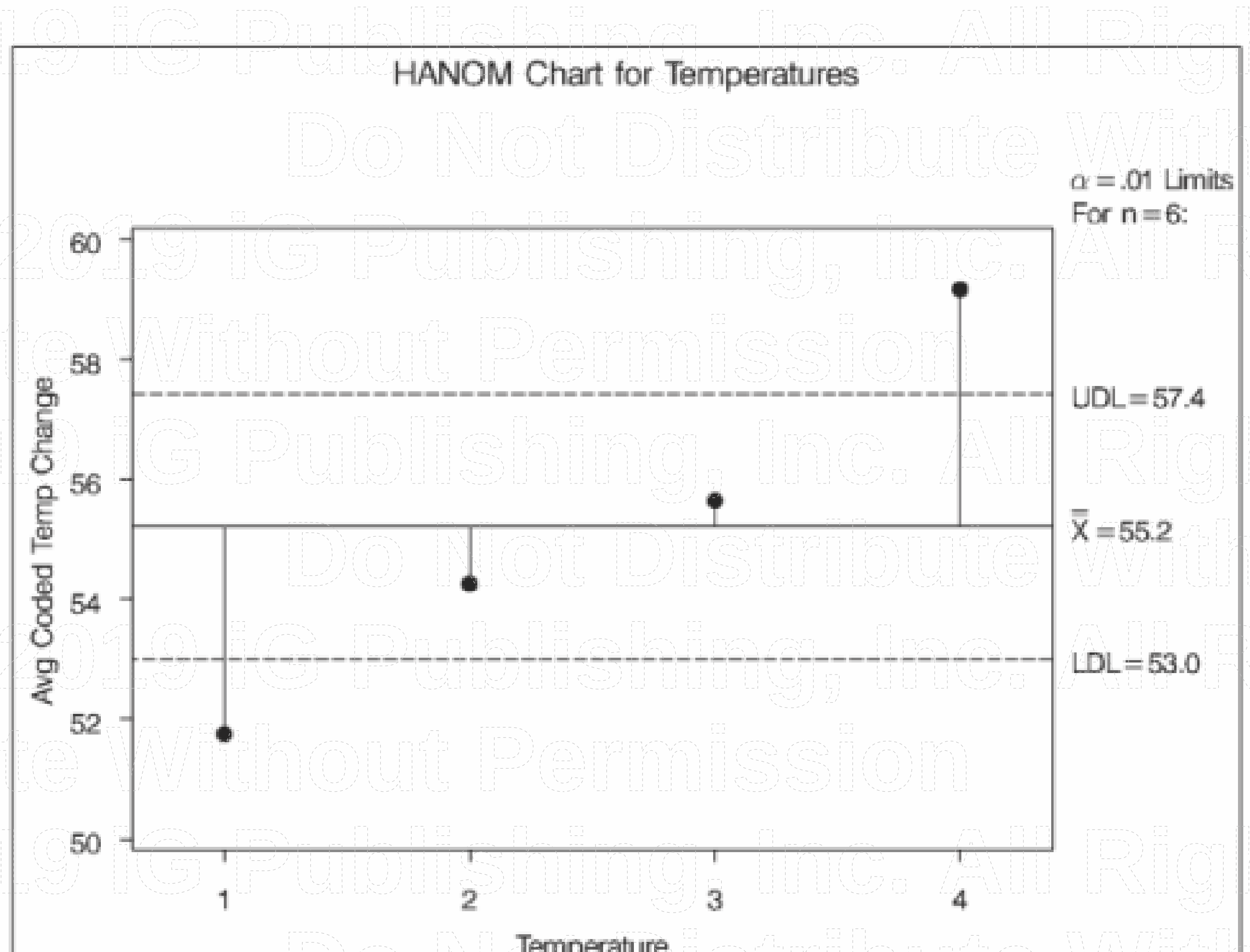

Figure 8.5. *The HANOM Chart for Temperature (Example* 8.2*).*