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HETEROSCEDASTIC ANALYSIS OF MEANS (HANOM)

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SYNOPTIC ABSTRACT

The analysis of means (ANOM) is a method that can compare a group of treatment means to see if any of those means are significantly different from the overall mean. It can be thought of as an alternative to the analysis of variance for analyzing fixed main effects in a designed experiment. The ANOM has advantages: it identifies any treatments that are different from the overall mean and has a graphical display that helps one to assess practical significance. Sample size tables and power curves have previously been developed for using ANOM to detect differences among I treatments when two of them differ by at least a specified amount $\Delta\sigma$, where σ is the common standard deviation of the treatments (or processes).

More recently results for the heteroscedastic situation where the different processes do not necessarily have equal standard deviations were presented. These new results allow an experimenter to set a goal of detecting differences among I treatment means when two of them differ by at least a specified amount δ, which does not depend on the (possibly different) standard deviations of the processes. In this paper we present power curves for $\alpha = 0.1, 0.05, 0.01$ and $I = 2(1)10, 12, 15, 20$; compare the power with that of heteroscedastic analysis of variance; and give two examples.

Key Words and Phrases: comparison of means; power; unequal variances.

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1 Introduction

The analysis of means (ANOM) is a technique for comparing a group of treatment means to see if any of them is significantly different from the overall mean. As such it can be thought of as an alternative to the analysis of variance (ANOVA) for analyzing fixed main effects in a designed experiment. The traditional assumptions are exactly the same as are traditional in the ANOVA, namely, that the observations are normally distributed and independent with means that depend on the particular treatment and common variance σ^2. With only two treatments the ANOM and the ANOVA are equivalent. While the ANOM is not an optimal test in any currently known mathematical sense, it has the advantages that it identifies any treatments that are different from the overall mean and provides a graphical display that aids in assessing practical significance. Further, when using exact critical values, no price is paid for these advantages in terms of decreased power (see P. R. Nelson (1983)).

The heteroscedastic analysis of means (HANOM, Nelson and Dudewicz (2002)) generalizes the ANOM procedure to the heteroscedastic situation where the process variances are not necessarily equal. The HANOM procedure allows an experimenter to set a goal of detecting differences among the I treatment means when two of the treatment means differ by a specified amount δ that does not depend on the process variances. In this paper we present power curves for $\alpha = 0.1, 0.05, 0.01$ and $I = 2(1)10, 12, 15, 20$; compare the power with that of heteroscedastic analysis of variance; and give two examples.

2 The One-Way Layout

In order to simplify the notation, we will start by considering the HANOM in the one-way layout situation. Let I be the number of treatments being compared, and let X_{ij} be the j^{th} observation from the i^{th} population, where we assume

$$X_{ij} = \mu_i + \epsilon_{ij}$$

with $\epsilon_{ij} \sim N(0, \sigma_i^2)$, and all the observations are independent.

Collecting the data for and performing HANOM consists of the following steps.

1. Specify an initial sample size $n_0 (\geq 2)$, take a sample of size n_0 from each of the I populations, and calculate the sample means $\overline{X}_{0i}$ and the sample variances s_i^2.

2. Specify the level of significance α, a power γ, and the amount δ such that any two treatment means differing by δ will lead to rejection of the null hypothesis that all the treatment means are equal with power γ. From Figures A1–A36 find the figure with the appropriate α and I combination, and for the specified γ and $\text{df} = n_0 - 1$ find the corresponding value of w.

3. Compute
$$n_i = \max\{n_0 + 1, \lfloor (w/\delta)^2 s_i^2 \rfloor + 1\} \tag{1}$$
for each i, where $\lfloor y \rfloor$ denotes the greatest integer in y, and take $n_i - n_0$ additional observations $X_{i,n_0+1}, \ldots, X_{i,n_i}$ from population i.

4. For each i calculate the sample mean
$$\overline{X}_i = \frac{X_{i,n_0+1} + \cdots + X_{i,n_i}}{n_i - n_0} \tag{2}$$
of the second set of observations from population i.

5. For each i compute
$$b_i = \frac{n_i - n_0}{n_i}\left[1 + \sqrt{\left(\frac{n_0}{n_i - n_0}\right)\left(\left\lfloor \frac{\delta}{w} \right\rfloor^2 \frac{n_i}{s_i^2} - 1\right)}\ \right] \tag{3}$$
and
$$\tilde{\tilde{X}}_i = (1 - b_i)\overline{X}_{0i} + b_i \overline{X}_i \tag{4}$$
and then compute
$$\tilde{\tilde{X}}_\bullet = (\tilde{\tilde{X}}_1 + \cdots + \tilde{\tilde{X}}_I)/I. \tag{5}$$

6. Compute decision lines
$$\begin{aligned} \text{UDL} &= \tilde{\tilde{X}}_\bullet + \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w} \\ \text{LDL} &= \tilde{\tilde{X}}_\bullet - \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w} \end{aligned} \tag{6}$$
where $\mathcal{H}(\alpha; I, n_0 - 1)$ is found in Table A.1 of Nelson and Dudewicz (2002), and reject the hypothesis $H_0 : \mu_1 = \cdots = \mu_I$ if any of the $\tilde{\tilde{X}}_i$'s falls outside these decision lines.

Example 2.1 Bishop and Dudewicz (1978) described an experiment to test the effects of different solvents on the ability of fungicide methyl-2-benzimidazole-carbamate to destroy the fungus Penicilium expansum. The fungicide was diluted in four different solvents and sprayed on the fungus, and the percentage of fungus destroyed was measured. The experimenter was interested in testing the hypothesis

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

that the four solvents resulted in the same average percentages of fungus being destroyed. Initial samples of size $n_0 = 15$ were taken with each of the four solvents. The data are given in Table A1, and the summary statistics are given in Table 1.

Table 1 – Summary Statistics from the First-Stage Samples for Example 2.1

i	$\overline{X}_{0i}$	s_i^2
1	96.842	2.1099
2	94.686	3.1709
3	94.383	5.8843
4	97.333	0.7797

One can test for equality of variances using the analysis of means for variances (ANOMV), which is described in Wludyka and Nelson (1997). The ANOMV chart is given in Figure 1, from which one would conclude that at the 0.05 level the sample variance for Solvent 3 is significantly large and the sample variance for Solvent 4 is significantly small.

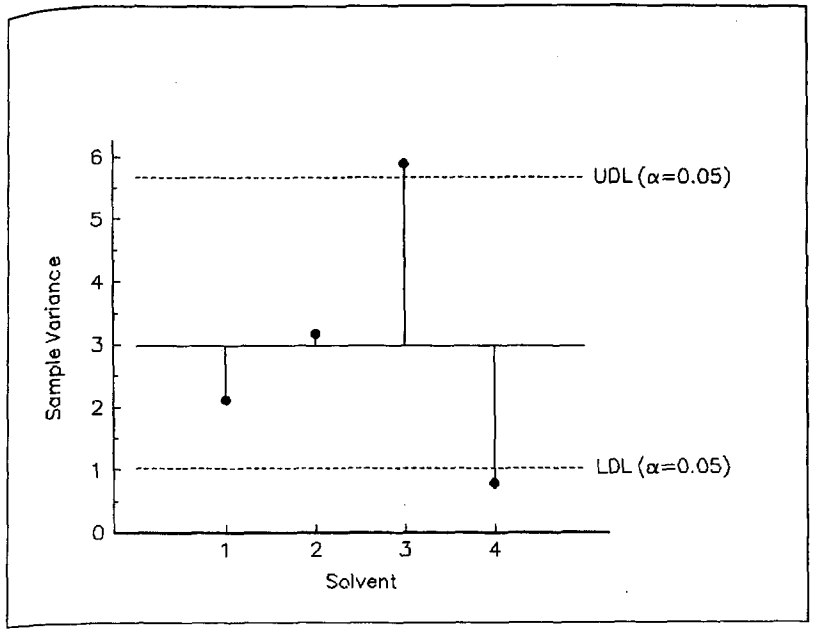

Figure 1 – The ANOMV Chart for Example 2.1.

Since the assumption of equal variances is not reasonable in this case, the HANOM procedure would be applicable (while ANOM would not). The experimenter wanted to conduct the test at level of significance $\alpha = 0.1$ and to have a power of at least $\gamma = 0.85$ if any two means differed by at least $\delta = \sqrt{2}$. From Figure A2 ($\alpha = 0.1$ and $I = 4$) one finds that for $\gamma = 0.85$ and df $= 14$ the corresponding value of w is approximately 5. Using the sample variances from the first stage of the experiment, together with $\delta = \sqrt{2}$ and $w = 5$, one can compute the necessary values of n_i (equation(1)). For example,

$$\begin{aligned} n_1 &= \max\{16, \lfloor (5/\sqrt{2})^2 2.1099 \rfloor + 1\} \\ &= \max\{16, 27\} \\ &= 27. \end{aligned}$$

Similarly, $n_2 = 40$, $n_3 = 74$, and $n_4 = 16$.

The second-stage data are given in Table A2, and using equations (2) – (4), one can compute the values in Table 2.

Table 2 – Statistics Computed Using the Second-Stage Samples

i	$\overline{X}_i$	b_i	$\tilde{\overline{X}}_i$
1	98.235	0.5210	97.568
2	95.530	0.6714	95.253
3	94.834	0.8286	94.757
4	97.970	0.2564	97.496

Using the values of $\tilde{\overline{X}}_i$ from Table 2 and equation (5), one obtains $\tilde{\overline{X}}_\bullet = 96.268$, and the decision lines (6) are

$$\begin{aligned} \text{UDL} &= 96.268 + \mathcal{H}(0.1; 4, 14)\frac{\delta}{w} \\ &= 96.268 + 2.08\frac{\sqrt{2}}{5} \\ &= 96.268 + 0.588 \\ &= 96.856 \\ \text{LDL} &= 96.268 - 0.588 \\ &= 95.680. \end{aligned}$$

The HANOM chart given in Figure 2 shows that all the solvents are significantly different from the grand mean at level $\alpha = 0.1$. Solvents 1 and 4 destroyed significantly more fungus, and Solvents 2 and 3 destroyed significantly less fungus.

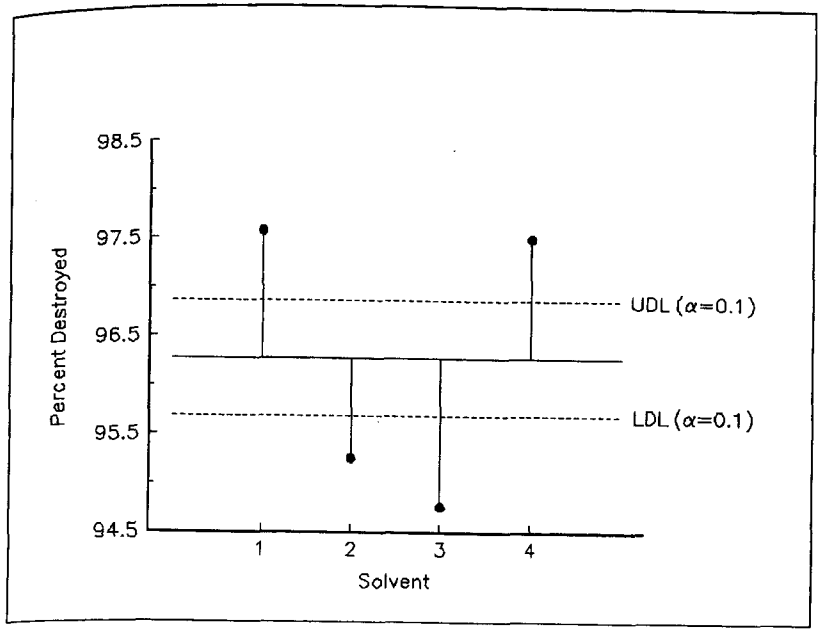

Figure 2 – The HANOM Chart for Example 2.1.

3 Higher-Order Layouts

In designs with two or more factors one first tests for possible interaction(s) among the factors. As in the homogeneous variance case this is done using ANOVA. We consider here only a two-way layout. Details on heteroscedastic ANOVA for a two-way layout are given in Bishop and Dudewicz (1978), and details for higher-order layouts are given in Bishop and Dudewicz (1981).

The two-way layout's usual model is

$$X_{ijk} = \mu + \alpha_i^A + \alpha_j^B + \alpha_{ij}^{AB} + \epsilon_{ijk}$$

where there are I levels of factor A and J levels of factor B. In the heteroscedastic case one assumes $\epsilon_{ijk} \sim N(0, \sigma_{ij}^2)$. Initially one takes a sample

of size $n_0 \geq 2$ from each of the IJ treatment combinations and computes the unbiased estimates s_{ij}^2 of the σ_{ij}^2 and the averages of the observations for each treatment combination, $\overline{X}_{0ij}$. If it is known in advance that the σ_{ij}^2 are not all equal, or if a test (on variances) such as the ANOMV indicates that the variances are not all equal, then heteroscedastic ANOVA and HANOM are each appropriate. Sample sizes at the second stage are determined using equation (1) with s_i replaced with s_{ij} and n_i replaced with n_{ij}. That is,

$$n_{ij} = \max\{n_0 + 1, \lfloor (w/\delta)^2 s_{ij}^2 \rfloor + 1\}. \tag{7}$$

An appropriate value of w depends on whether or not one detects significant interaction, and if there is how one will proceed with the analysis. For fixed α, γ, and δ the value w increases as the number of means to be compared increases. Thus, a choice of w based on comparing IJ means, the situation that would occur if one found significant interaction and was then interested in comparing the IJ treatment combinations, is conservative. Alternatively, if one is interested in the main effects of a factor, then it would make sense to choose w based on $\max(I, J)$.

The heteroscedastic ANOVA test of $H_{AB} : \alpha_{ij}^{AB} = 0$ for all (i, j) is conducted using the usual interaction sum of squares with $\overline{X}_{ij}$ replaced by $\tilde{\overline{X}}_{ij}$. That is, for each (i, j) one would compute

$$b_{ij} = \frac{n_{ij} - n_0}{n_{ij}} \left[1 + \sqrt{\left(\frac{n_0}{n_{ij} - n_0} \right) \left(\left[\frac{\delta}{w} \right]^2 \frac{n_{ij}}{s_{ij}^2} - 1 \right)} \right] \tag{8}$$

and

$$\tilde{\overline{X}}_{ij} = (1 - b_{ij})\overline{X}_{0ij} + b_{ij}\overline{X}_{ij} \tag{9}$$

$$\tilde{\overline{X}}_{i\bullet} = \frac{1}{J} \sum_{j=1}^{J} \tilde{\overline{X}}_{ij} \tag{10}$$

$$\tilde{\overline{X}}_{\bullet j} = \frac{1}{I} \sum_{i=1}^{I} \tilde{\overline{X}}_{ij} \tag{11}$$

$$\tilde{\overline{X}}_{\bullet\bullet} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \tilde{\overline{X}}_{ij} . \tag{12}$$

To test hypothesis H_{AB} one uses

$$\tilde{F}_{AB} = \left(\frac{w}{\delta} \right)^2 \sum_{i=1}^{I} \sum_{j=1}^{J} (\tilde{\overline{X}}_{ij} - \tilde{\overline{X}}_{i\bullet} - \tilde{\overline{X}}_{\bullet j} + \tilde{\overline{X}}_{\bullet\bullet})^2, \tag{13}$$

which is compared with the appropriate quantile of the $((n_0-1)/(n_0-3))\chi^2_{\mathtt{df}}$ distribution, `df = (I-1)(J-1)`.

If significant interaction is not found, then one may test the main effects of factor A by comparing the $\tilde{\overline{X}}_{i\bullet}$ with decision lines

$$\begin{aligned} \text{UDL} &= \tilde{\overline{X}}_{\bullet\bullet} + \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w} \\ \text{LDL} &= \tilde{\overline{X}}_{\bullet\bullet} - \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w}. \end{aligned} \tag{14}$$

Similarly, one tests the main effects of factor B by comparing the $\tilde{\overline{X}}_{\bullet j}$ with the decision lines

$$\begin{aligned} \text{UDL} &= \tilde{\overline{X}}_{\bullet\bullet} + \mathcal{H}(\alpha; J, n_0 - 1)\tfrac{\delta}{w} \\ \text{LDL} &= \tilde{\overline{X}}_{\bullet\bullet} - \mathcal{H}(\alpha; J, n_0 - 1)\tfrac{\delta}{w}. \end{aligned} \tag{15}$$

If one finds significant interaction, then one either compares the IJ treatment combinations using the $\tilde{\overline{X}}_{ij}$ and the decision lines

$$\begin{aligned} \text{UDL} &= \tilde{\overline{X}}_{\bullet\bullet} + \mathcal{H}(\alpha; IJ, n_0 - 1)\tfrac{\delta}{w} \\ \text{LDL} &= \tilde{\overline{X}}_{\bullet\bullet} - \mathcal{H}(\alpha; IJ, n_0 - 1)\tfrac{\delta}{w} \end{aligned} \tag{16}$$

or (assuming $I > J$) compares the I levels of factor A separately for each level j' of factor B using the $\tilde{\overline{X}}_{ij'}$ and decision lines

$$\begin{aligned} \text{UDL} &= \tilde{\overline{X}}_{\bullet j'} + \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w} \\ \text{LDL} &= \tilde{\overline{X}}_{\bullet j'} - \mathcal{H}(\alpha; I, n_0 - 1)\tfrac{\delta}{w}. \end{aligned} \tag{17}$$

Example 3.1 A 3×4 factorial experiment was conducted to study the effects of pH and catalyst on the adhesion of a lens coating. The values in Table A3 are the resulting adhesions, and the sample means and variances are given in Table 3. The ANOMV chart given in Figure 3 indicates that at the 0.05 level the σ_{ij}^2 are not all equal. Thus, heteroscedastic ANOVA and HANOM are each appropriate ways to continue the analysis.

Table 3 – Summary Statistics from the First-Stage Samples (Example 3.1)

i	j	$\overline{X}_{0ij}$	s^2_{ij}	n_{ij}
1	1	13.60	0.140	5
1	2	14.375	0.029	5
1	3	13.975	1.63	13
1	4	15.10	0.060	5
2	1	16.05	0.0033	5
2	2	14.075	0.096	5
2	3	12.125	0.149	5
2	4	13.625	0.069	5
3	1	13.975	0.069	5
3	2	13.00	0.087	5
3	3	12.775	0.043	5
3	4	16.575	0.029	5

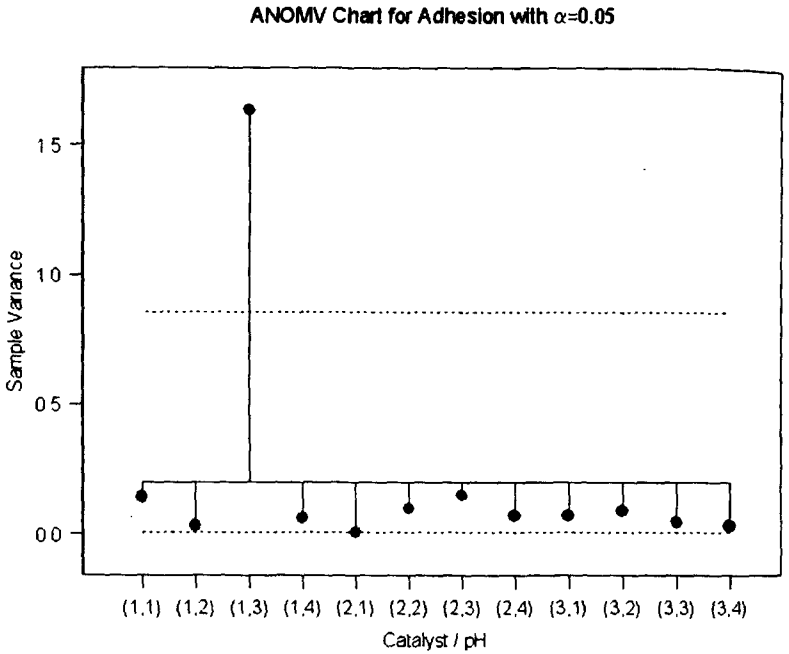

Figure 3 – The ANOMV Chart for Example 3.1.

The experimenters were interested in conducting tests using $\alpha = 0.05$ and being able to detect any effect of $\delta = 5$ or more with a power of 0.7. From Figure A22 ($\alpha = 0.05$ and $I = 12$) one finds that for $\delta = 5$ and df= 3 the value of w is approximately 14. Using equation (7) the sample sizes for the second stage were computed and are given in Table 3. The second-stage data are given in Tables A4, and the statistics computed using equations (8) – (12) are given in Table 4.

Table 4 – Statistics Computed Using the Second-Stage Samples (Example 3.1)

i	j	$\overline{X}_{ij}$	b_{ij}	$\tilde{\overline{X}}_{ij}$
1	1	13.7	0.9542	13.695
1	2	14.2	2.027	14.020
1	3	14.23	0.7539	14.167
1	4	15.2	1.441	15.244
2	1	16.1	5.718	16.336
2	2	14.2	1.151	14.219
2	3	12.0	0.9239	12.010
2	4	13.6	1.347	13.591
3	1	14.1	1.347	14.143
3	2	13.1	1.209	13.121
3	3	12.7	1.697	12.648
3	4	16.6	2.027	16.626

The row means, column means, and the grand mean (of the $\tilde{\overline{X}}_{ij}$) are given in Table 5.

Table 5 – Average Adhesions for Catalyst/pH Combinations Together with Row Means, Column Means, and the Grand Mean

		pH				
		1	2	3	4	
	1	13.695	14.020	14.167	15.244	14.282
Catalyst	2	16.336	14.219	12.010	13.591	14.039
	3	14.143	13.121	12.648	16.626	14.135
		14.725	13.787	12.941	15.154	14.152

Using equation (13), one then would compute $\tilde{F}_{AB} = (14/5)^2(11.63) = 91.18$, which when compared with $(3/1)\chi^2(0.01; 6) = (3)(16.812) = 50.44$ is found to be significant.

Since the main purpose of this experiment was to determine the best Catalyst/pH combination, the decision lines (16) were used to compare the Catalyst/pH combinations.

$$\begin{aligned} \text{UDL} &= 14.152 + \mathcal{H}(0.05; 12, 3)(5/14) \\ &= 14.152 + (7.28)(5/14) = 16.75 \\ \text{LDL} &= 14.152 - (7.28)(5/14) = 11.55 \end{aligned}$$

and from the HANOM chart in Figure 4 one sees that there is no effect due to the catalyst/pH combinations at the 0.05 level.

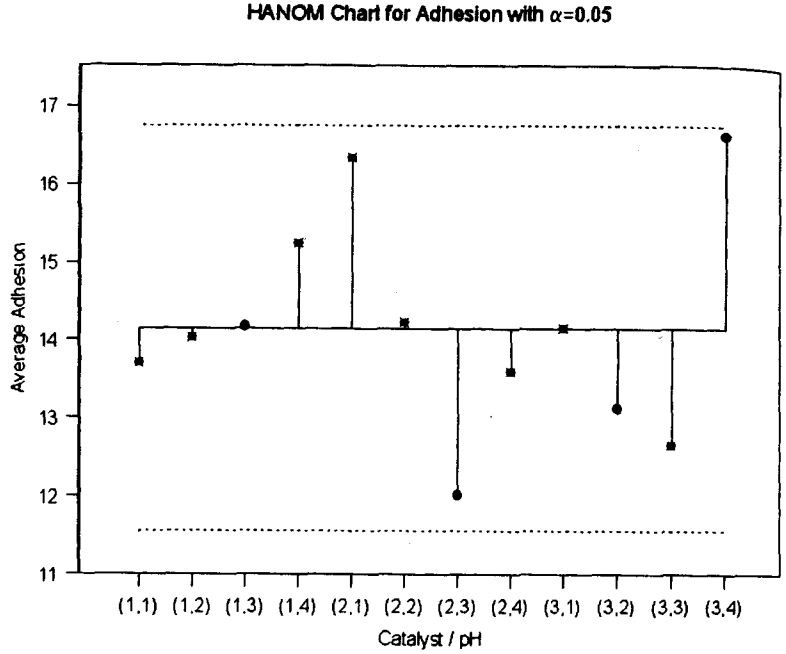

Figure 4 – The HANOM Chart for Catalyst/pH Combinations.

4 Comparison with Heteroscedastic ANOVA

The power function of HANOM depends on the configuration of the μ_i's, and in order to provide power curves it is necessary, as it is for the ANOM and

the ANOVA, to consider subspaces of the μ_i's. For HANOM if one chooses subspaces

$$M_\delta = \{\mu = \{\mu_1, \ldots, \mu_I\} : \max_{i,j} |\mu_i - \mu_j| \geq \delta\}$$

then the power is determined using the least favorable configuration (LFC) of the means on these subspaces. It is shown in Nelson and Dudewicz (2002) that on M_δ the LFC for HANOM is a configuration of the form

$$\mu = (\delta/2, -\delta/2, 0, \ldots, 0). \tag{18}$$

The power values used to construct the curves in Figures A1– A36 and in the comparisons given below were computed as described in Nelson and Dudewicz (2002) using the LFC (18).

The power values for heteroscedastic ANOVA were computed as described in Bishop and Dudewicz (1978). For the comparisons given below we used 10^6 trials for both the simulated critical points and power values for each procedure. As in the homoscedastic case, heteroscedastic ANOVA and HANOM are in fact equivalent when $I = 2$ means are being compared. When $I = 2$ critical values for heteroscedastic ANOVA are $2[\mathcal{H}(\alpha; 2, \nu)]^2$. For more than two means the powers for the two procedures are compared in Tables 6 and 7.

Table 6a – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $I = 3$

	$n_0 = 5$			$n_0 = 15$			$n_0 = 25$		
		α			α			α	
w/δ	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
2	0.188(-3)	0.091(-1)	0.014(0)	0.288(-2)	0.180(0)	0.051(0)	0.304(-4)	0.196(-2)	0.064(-1)
3	0.327(-7)	0.165(-5)	0.022(-1)	0.516(-6)	0.374(-3)	0.141(-5)	0.543(-7)	0.407(-5)	0.179(-6)
4	0.531(-14)	0.306(-14)	0.039(-4)	0.749(-7)	0.621(-8)	0.319(-16)	0.774(-7)	0.659(-8)	0.388(-15)
5	0.738(-15)	0.512(-25)	0.077(-13)	0.904(-4)	0.828(-7)	0.564(-26)	0.919(-4)	0.856(-8)	0.643(-21)
6	0.882(-9)	0.724(-25)	0.155(-34)	0.972(-2)	0.942(-4)	0.789(-23)	0.979(-2)	0.956(-4)	0.846(-17)
7	0.953(-3)	0.873(-15)	0.298(-70)	0.994(0)	0.985(-1)	0.925(-12)	0.996(0)	0.990(-1)	0.952(-8)
8	0.982(-1)	0.949(-6)	0.504(-94)	0.999(0)	0.997(0)	0.980(-4)	0.999(0)	0.998(0)	0.989(-2)
9	0.993(0)	0.980(-2)	0.718(-78)	1.000(0)	0.999(0)	0.996(-1)	1.000(0)	1.000(0)	0.998(0)

Table 6b – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $I = 4$

	$n_0 = 5$			$n_0 = 15$			$n_0 = 25$		
		α			α			α	
w/δ	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
2	0.152(-9)	0.071(-5)	0.012(0)	0.242(-9)	0.142(-7)	0.036(-3)	0.260(-9)	0.160(-6)	0.046(-4)
3	0.242(-21)	0.110(-15)	0.015(-2)	0.437(-18)	0.294(-18)	0.095(-13)	0.471(-15)	0.334(-15)	0.129(-13)
4	0.398(-34)	0.191(-32)	0.022(-5)	0.671(-19)	0.522(-26)	0.227(-33)	0.706(-16)	0.574(-20)	0.297(-31)
5	0.604(-38)	0.336(-56)	0.035(-14)	0.856(-14)	0.750(-24)	0.441(-53)	0.880(-10)	0.794(-18)	0.538(-43)
6	0.792(-29)	0.540(-69)	0.062(-33)	0.954(-5)	0.903(-14)	0.683(-54)	0.965(-4)	0.927(-10)	0.769(-38)
7	0.911(-14)	0.744(-56)	0.119(-75)	0.989(-1)	0.972(-4)	0.866(-33)	0.993(-1)	0.982(-3)	0.916(-20)
8	0.967(-5)	0.885(-29)	0.227(-139)	0.998(0)	0.994(-1)	0.958(-13)	0.999(0)	0.997(0)	0.978(-6)
9	0.988(-1)	0.955(-11)	0.402(-189)	1.000(0)	0.999(0)	0.990(-3)	1.000(0)	1.000(0)	0.996(-1)

Table 6c – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $I = 5$

	$n_0 = 5$			$n_0 = 15$			$n_0 = 25$		
		α			α			α	
w/δ	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
2	0.134(-12)	0.063(-6)	0.011(0)	0.214(-14)	0.120(-12)	0.028(-4)	0.230(-14)	0.137(-11)	0.037(-5)
3	0.193(-30)	0.086(-17)	0.013(-2)	0.384(-24)	0.246(-25)	0.071(-16)	0.418(-22)	0.286(-21)	0.100(-16)
4	0.306(-52)	0.135(-38)	0.017(-5)	0.610(-27)	0.452(-36)	0.173(-39)	0.650(-23)	0.511(-27)	0.240(-35)
5	0.483(-67)	0.230(-71)	0.023(-11)	0.812(-19)	0.686(-35)	0.357(-65)	0.843(-16)	0.742(-25)	0.462(-52)
6	0.689(-59)	0.390(-103)	0.036(-26)	0.934(-8)	0.866(-21)	0.597(-73)	0.949(-7)	0.900(-14)	0.703(-50)
7	0.851(-35)	0.598(-105)	0.061(-58)	0.982(-2)	0.958(-8)	0.808(-51)	0.988(-1)	0.972(-5)	0.879(-30)
8	0.941(-14)	0.790(-70)	0.111(-121)	0.996(0)	0.990(-2)	0.933(-23)	0.998(0)	0.994(-1)	0.965(-11)
9	0.979(-4)	0.911(-32)	0.207(-211)	0.999(0)	0.998(0)	0.982(-7)	1.000(0)	0.999(0)	0.992(-2)

Table 6d – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $I = 10$

	$n_0 = 5$			$n_0 = 15$			$n_0 = 25$		
		α			α			α	
w/δ	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
2	0.109(-12)	0.054(-5)	0.010(0)	0.156(-22)	0.082(-15)	0.017(-5)	0.171(-20)	0.094(-15)	0.022(-5)
3	0.125(-31)	0.059(-14)	0.011(-1)	0.256(-40)	0.146(-34)	0.034(-15)	0.293(-33)	0.181(-29)	0.052(-16)
4	0.156(-61)	0.070(-30)	0.012(-2)	0.429(-49)	0.277(-52)	0.079(-36)	0.490(-32)	0.346(-37)	0.128(-34)
5	0.218(-101)	0.089(-57)	0.013(-5)	0.647(-38)	0.480(-57)	0.179(-70)	0.710(-21)	0.571(-33)	0.280(-54)
6	0.328(-140)	0.128(-102)	0.015(-9)	0.835(-19)	0.706(-41)	0.356(-100)	0.878(-8)	0.785(-20)	0.505(-61)
7	0.498(-150)	0.200(-163)	0.018(-18)	0.943(-5)	0.876(-19)	0.588(-98)	0.963(-1)	0.921(-7)	0.735(-48)
8	0.696(-115)	0.328(-217)	0.023(-34)	0.986(0)	0.962(-5)	0.799(-64)	0.992(0)	0.979(-1)	0.896(-24)
9	0.857(-59)	0.515(-218)	0.032(-68)	0.997(0)	0.991(0)	0.928(-28)	0.999(0)	0.996(0)	0.971(-8)

Table 6e – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $I = 20$

	$n_0 = 5$			$n_0 = 15$			$n_0 = 25$		
	α			α			α		
w/δ	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
2	0.103(-9)	0.051(-3)	0.010(0)	0.127(-22)	0.065(-14)	0.013(-4)	0.140(-18)	0.073(-14)	0.015(-5)
3	0.108(-22)	0.053(-9)	0.010(0)	0.182(-43)	0.097(-31)	0.020(-12)	0.217(-31)	0.122(-26)	0.030(-12)
4	0.117(-43)	0.056(-19)	0.011(-1)	0.295(-55)	0.172(-49)	0.039(-27)	0.363(-28)	0.232(-30)	0.071(-22)
5	0.133(-74)	0.060(-34)	0.011(-2)	0.477(-44)	0.315(-54)	0.088(-51)	0.570(-6)	0.418(-15)	0.166(-30)
6	0.162(-118)	0.069(-58)	0.011(-5)	0.690(-15)	0.523(-37)	0.190(-79)	0.776(14)	0.645(7)	0.339(-28)
7	0.217(-169)	0.083(-97)	0.012(-8)	0.862(5)	0.741(-10)	0.370(-93)	0.914(18)	0.837(20)	0.570(-13)
8	0.315(-211)	0.110(-154)	0.013(-13)	0.955(8)	0.896(5)	0.600(-76)	0.976(10)	0.946(15)	0.786(0)
9	0.471(-212)	0.159(-228)	0.014(-21)	0.989(4)	0.969(5)	0.806(-42)	0.995(3)	0.987(6)	0.922(5)

Table 7 – Power for HANOM (HANOM Power Minus Heteroscedastic ANOVA Power $\times 10^3$), $n_0 = 5$

	$I = 4$			$I = 5$			$I = 10$			$I = 20$	
	α			α			α			α	
w/δ	0.05	0.01		0.05	0.01		0.05	0.01		0.05	0.01
10	0.984(-3)	0.617(-172)		0.967(-11)	0.368(-273)		0.697(-81)	0.037(-67)		0.205(36)	0.007(-2)
11	0.994(-1)	0.805(-103)		0.988(-3)	0.579(-244)		0.859(-44)	0.066(-144)		0.358(42)	0.010(-6)
12	0.998(0)	0.917(-45)		0.996(-1)	0.777(-148)		0.945(-15)	0.123(-267)		0.565(40)	0.015(-16)
13	0.999(0)	0.969(-16)		0.999(0)	0.904(-65)		0.981(-4)	0.226(-387)		0.765(30)	0.023(-37)
14	1.000(0)	0.988(-5)		0.999(0)	0.964(-24)		0.994(0)	0.396(-407)		0.900(21)	0.038(-85)
15	1.000(0)	0.995(-1)		1.000(0)	0.987(-7)		0.998(0)	0.609(-306)		0.963(13)	0.067(-181)
16	1.000(0)	0.998(0)		1.000(0)	0.995(-2)		0.999(0)	0.798(-167)		0.988(7)	0.122(-324)
17	1.000(0)	0.999(0)		1.000(0)	0.998(0)		1.000(0)	0.917(-69)		0.996(3)	0.226(-444)
18	1.000(0)	1.000(0)		1.000(0)	0.999(0)		1.000(0)	0.970(-24)		0.998(1)	0.394(-447)
19	1.000(0)	1.000(0)		1.000(0)	1.000(0)		1.000(0)	0.990(-7)		0.999(0)	0.606(-326)
20	1.000(0)	1.000(0)		1.000(0)	1.000(0)		1.000(0)	0.996(-2)		1.000(0)	0.798(-174)

Neither procedure is uniformly better, and with one exception the differences are not of practical significance. When $I = 20$, the differences are the most extreme. For $I = 20$ and $n_0 = 25$, HANOM becomes more powerful when powers above about 0.7 are specified. However, for $I = 20$ and $n_0 = 5$, heteroscedastic ANOVA is much more powerful. The biggest difference occurs when $I = 20, n_0 = 5$, and $\alpha = 0.01$ (see Table 7). Tables 6 and 7 indicate that HANOM has lower power when power is moderate; and that HANOM has equal or higher power when power is high and as it approaches 1.

5 Conclusions

The usual procedures for testing to see if a set of treatments contains any with means that are different require equal variances in the populations being compared. When variances are not all equal, and it is either inconvenient or even impossible to transform the data to make variances equal, heteroscedastic procedures are required. Heteroscedastic ANOM allows the experimenter to set as a goal: detecting differences among I treatment means when two of them differ by at least a specified amount δ, which does not depend on the variances of the processes.

Comparison of HANOM with heteroscedastic ANOVA shows that neither procedure is uniformly better; the differences in power are small (not great enough to make a practical difference except when I is large and both n_0 and α are small; i.e., $I = 20, n_0 = 5$, and $\alpha = 0.01$; and there is enough variability that second samples of more than one observation are needed). Note, however, that if one anticipates large variability and wants to use $\alpha = 0.01$, initial samples larger than $n_0 = 5$ should generally be chosen. HANOM also has the same advantages as the ANOM; namely, it points out which means are different from the grand mean if any of them are, and makes it easy to judge the practical significance of any differences.

6 Acknowledgment

This research was begun while E. J. Dudewicz was Visiting Distinguished Professor at Clemson University, whose support is gratefully acknowledged.

7 Appendix

Table A1 – First-Stage Data for Example 2.1

Solvent 1	Solvent 2	Solvent 3	Solvent 4
96.44	93.63	93.58	97.18
96.87	93.99	93.02	97.42
97.24	94.61	93.86	97.65
95.41	91.69	92.90	95.90
95.29	93.00	91.43	96.35
95.61	94.17	92.68	97.13
95.28	92.62	91.57	96.06
94.63	93.41	92.87	96.33
95.58	94.67	92.65	96.71
98.20	95.28	95.31	98.11
98.29	95.13	95.33	98.38
98.30	95.68	95.17	98.35
98.65	97.52	98.59	98.05
98.43	97.52	98.00	98.25
98.41	97.37	98.79	98.12

Table A2 – Second-Stage Data for Example 2.1

Solvent 1	Solvent 2		Solvent 3				Solvent 4
98.59	96.97	92.15	96.36	93.43	98.15	92.42	97.97
98.20	97.21	92.09	96.69	92.72	96.73	92.38	
98.37	97.44	94.03	96.89	93.56	97.55	92.06	
98.57	96.86	92.43	96.13	94.13	94.44	92.50	
98.42	97.26	92.62	97.65	93.57	93.61	92.54	
98.29	98.27	94.47	97.81	96.27	93.61	92.52	
98.51	97.57	94.14	97.71	98.05	94.20	91.80	
98.89	97.81	93.09	97.48	97.67	94.20	93.00	
98.66	98.20	98.47	97.96	98.93	93.34	91.69	
97.39	93.92	98.06	94.30	97.23	93.33	95.42	
97.41	93.86	98.35	93.29	95.95	93.51	92.29	
97.52	92.57	97.09	94.21	97.79	93.91	92.57	
	93.32		92.90	97.41	94.05	94.96	
			93.02	96.94	93.76	94.34	
			93.43	97.08	93.76		

Table A3 – First-Stage Data for Example 3.1

	pH1	pH2	pH3	pH4
Catalyst 1	13.2	14.6	12.3	15.1
	14.1	14.2	14.2	14.8
	13.5	14.3	15.4	15.4
	13.6	14.4	14.0	15.1
Catalyst 2	16.0	13.8	11.6	13.5
	16.1	13.9	12.3	13.4
	16.1	14.5	12.5	14.0
	16.0	14.1	12.1	13.6
Catalyst 3	13.6	12.6	12.8	16.5
	14.1	13.1	13.0	16.4
	14.2	13.3	12.5	16.8
	14.0	13.0	12.8	16.6

Table A4a – Second-Stage Data for Example 3.1

	pH1	pH2	pH3	pH4
Catalyst 1	13.7	14.2	Table A4b	15.2
Catalyst 2	16.1	14.2	12.0	13.6
Catalyst 3	14.1	13.1	12.7	16.6

Table A4b – Second-Stage Data for the (Catalyst 1, pH3) Treatment Combination in Example 3.1

14.2	13.5	15.1
15.3	12.2	13.3
14.5	15.2	14.8

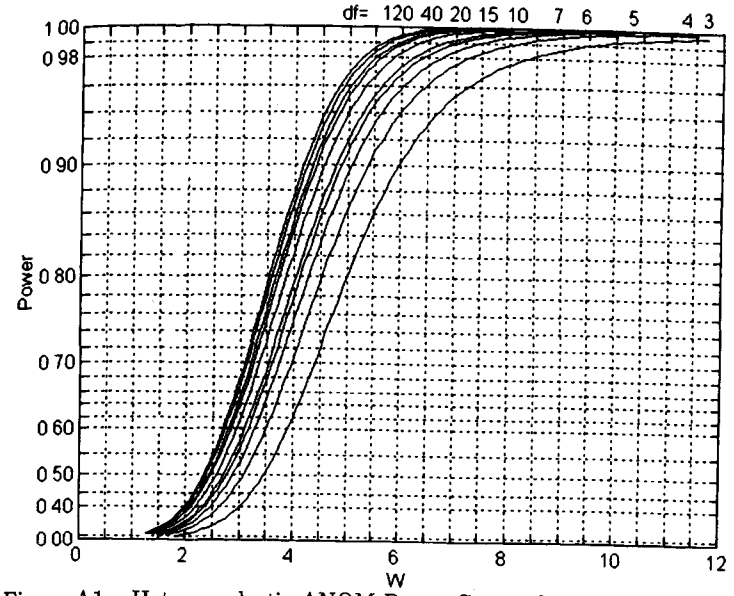

Figure A1 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 2$.

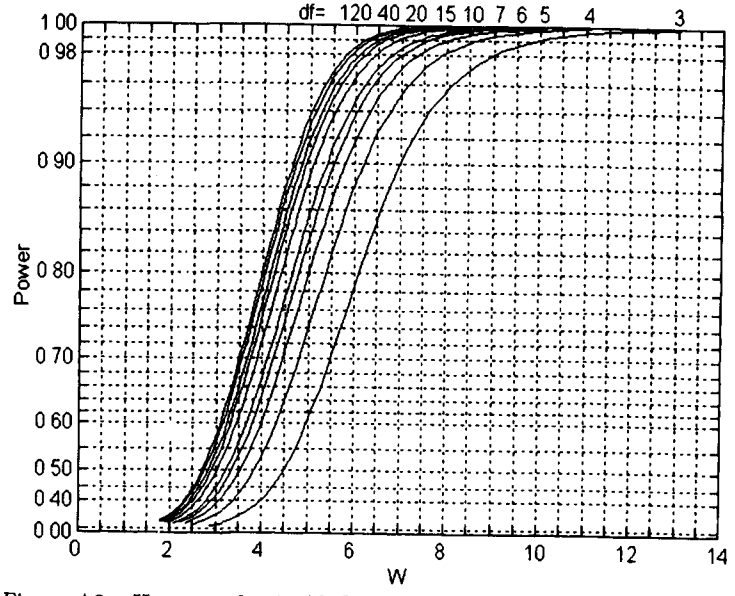

Figure A2 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 3$.

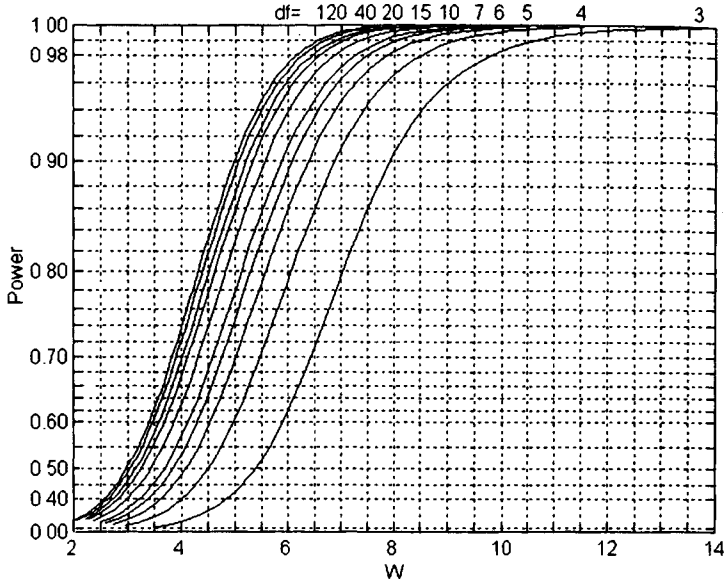

Figure A3 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 4$.

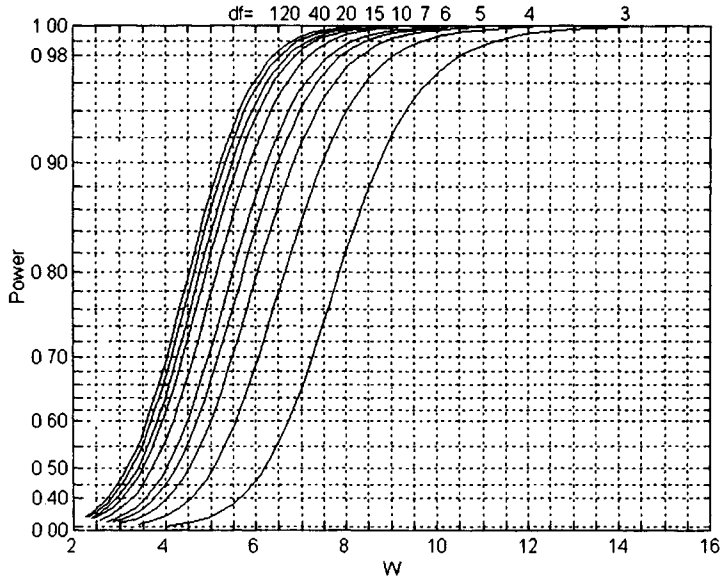

Figure A4 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 5$.

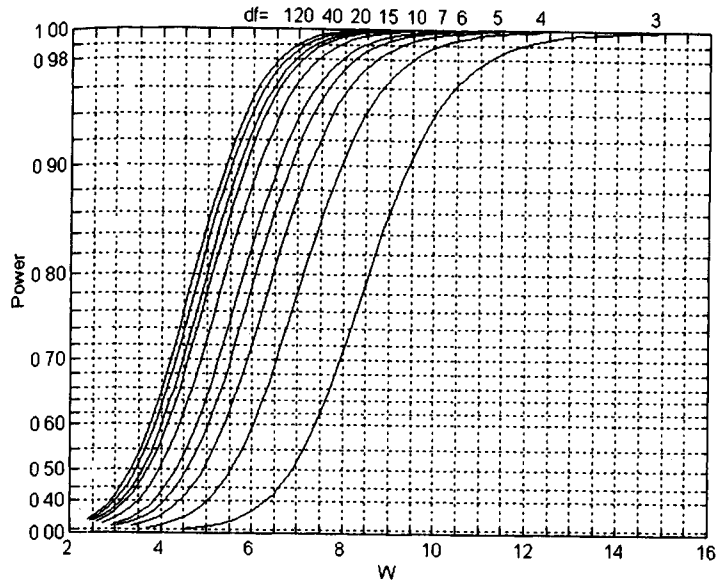

Figure A5 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 6$.

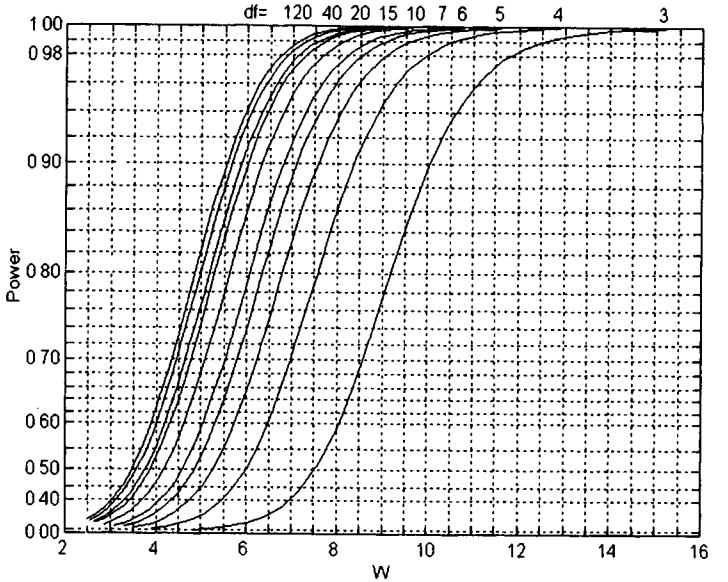

Figure A6 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 7$.

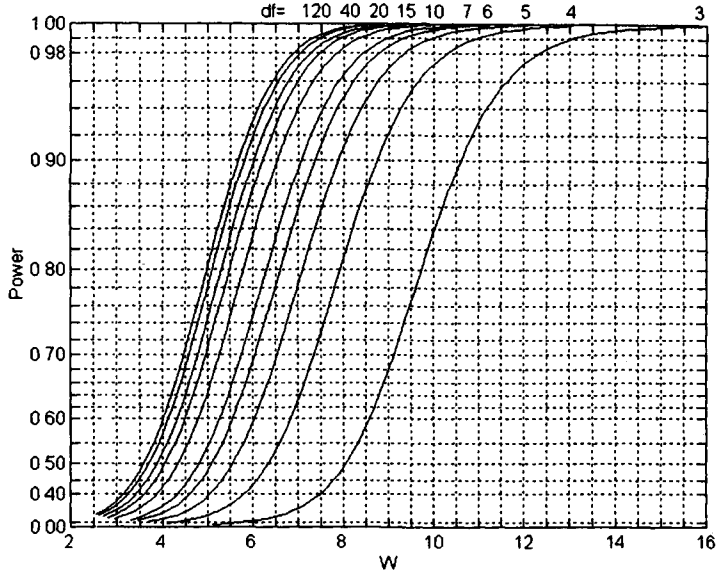

Figure A7 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 8$.

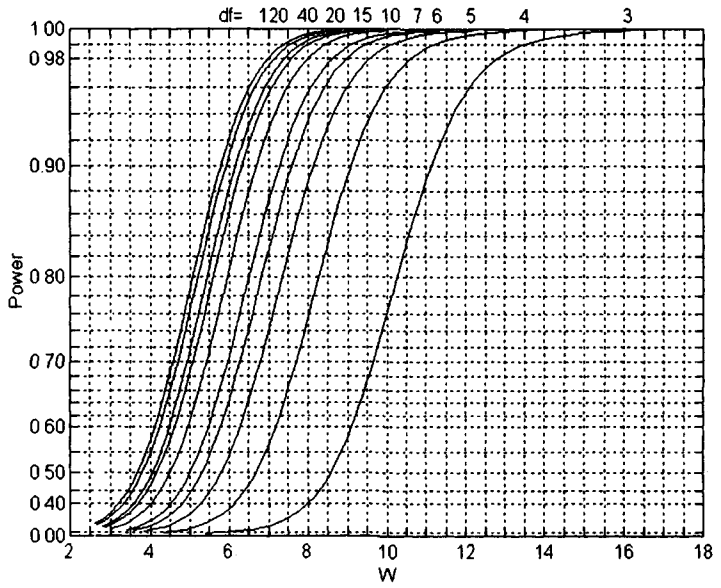

Figure A8 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 9$.

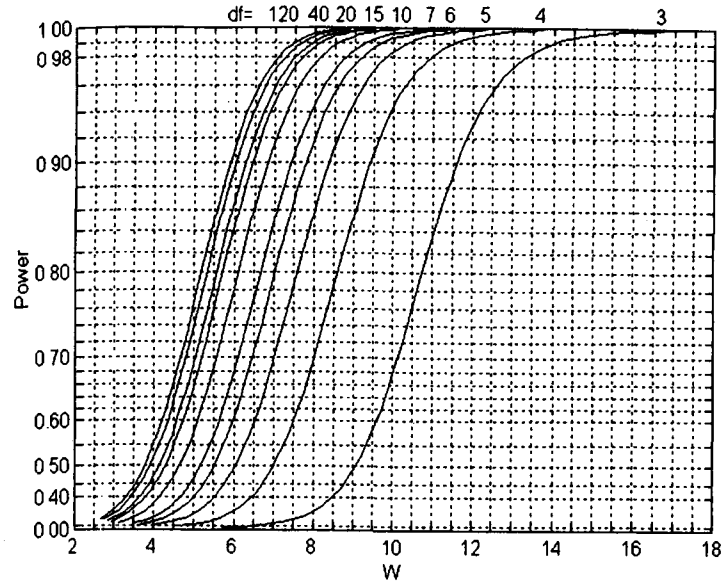

Figure A9 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 10$.

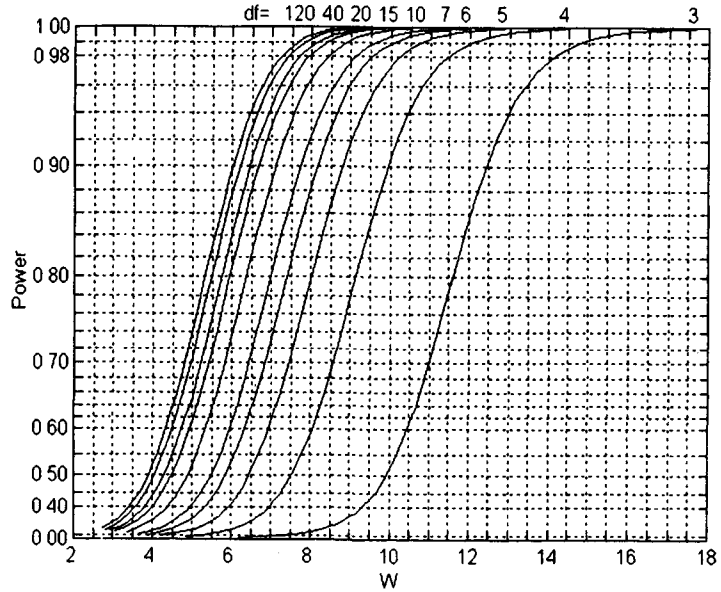

Figure A10 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 12$.

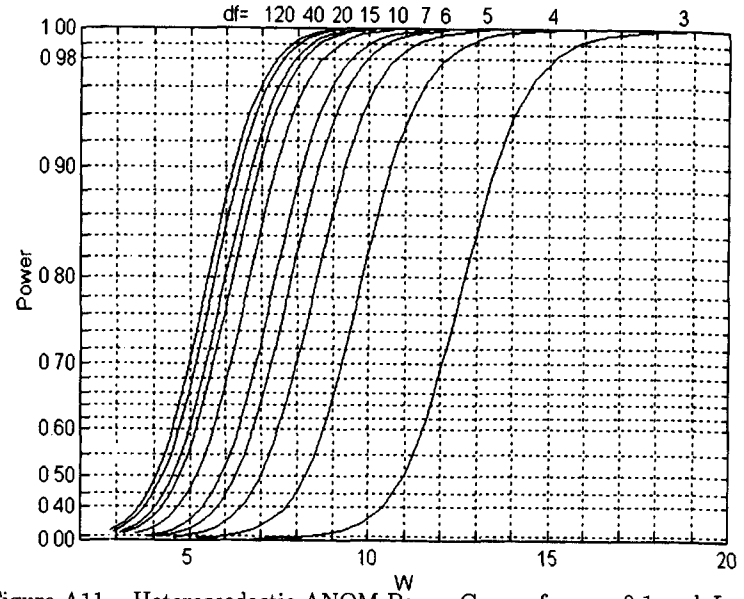

Figure A11 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 15$.

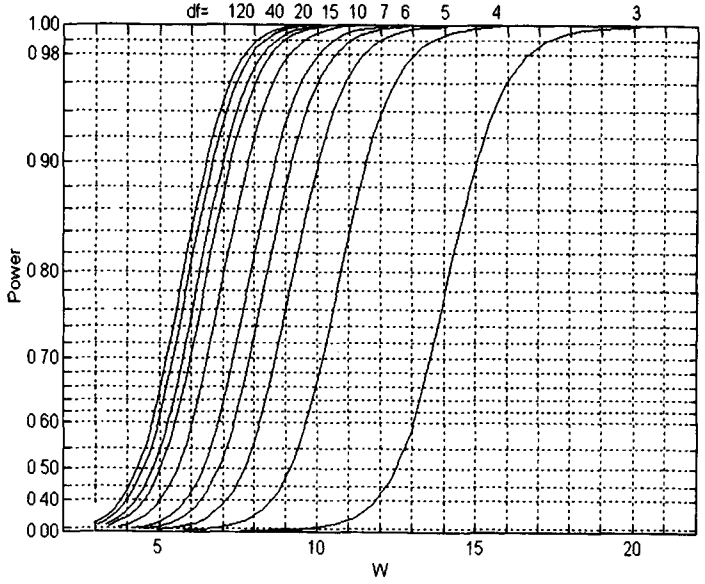

Figure A12 – Heteroscedastic ANOM Power Curves for $\alpha = 0.1$ and $I = 20$.

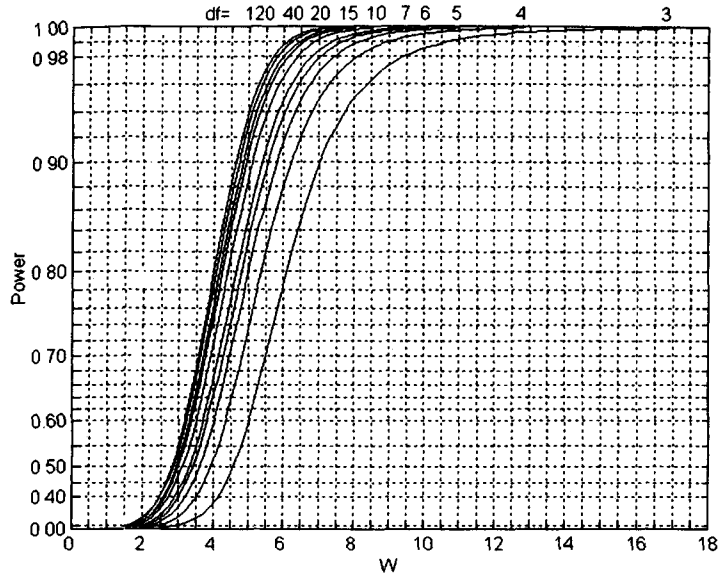

Figure A13 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 2$.

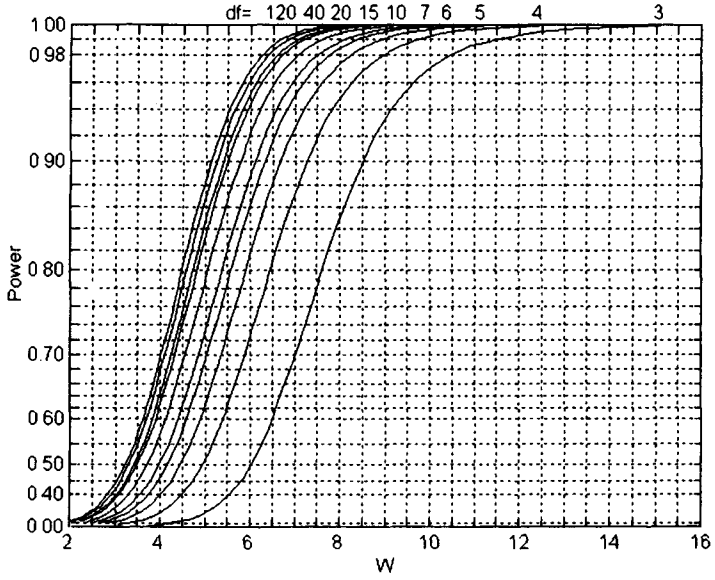

Figure A14 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 3$.

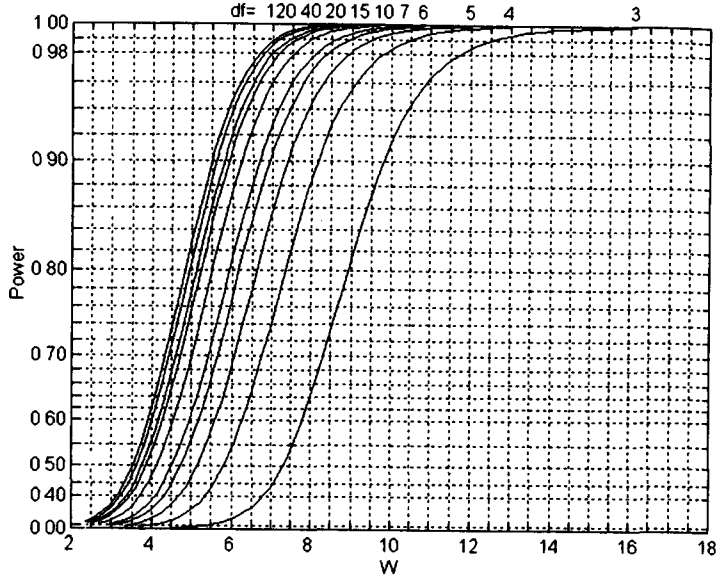

Figure A15 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 4$.

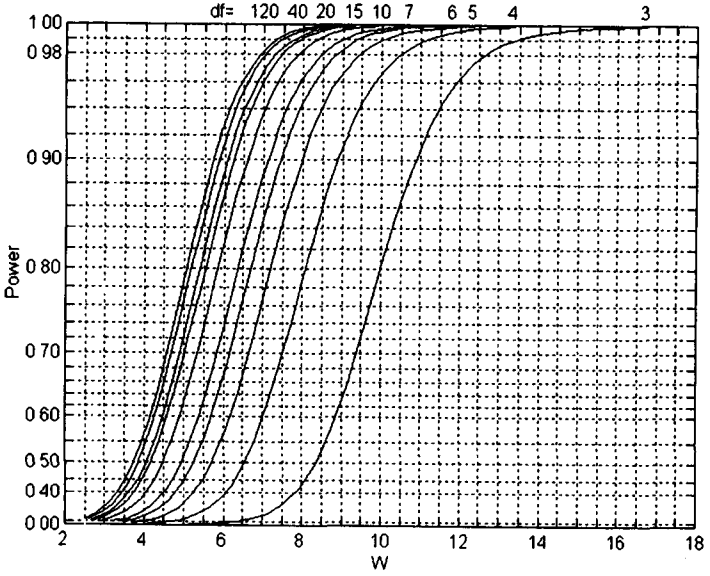

Figure A16 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 5$.

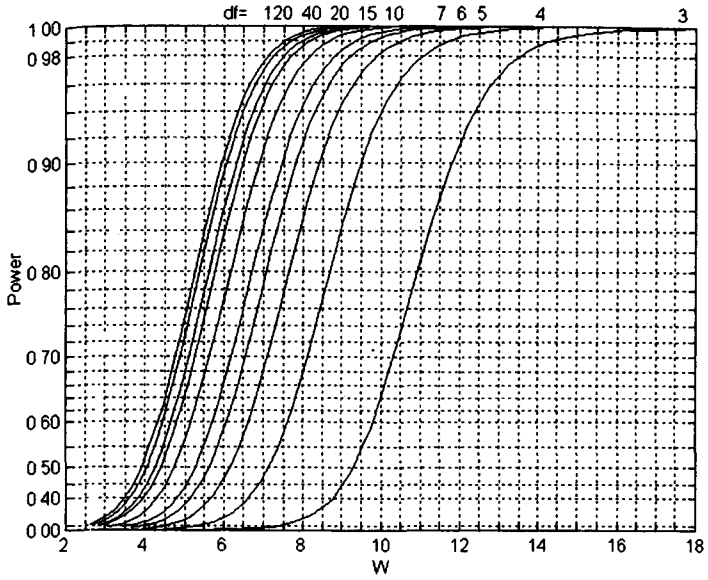

Figure A17 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 6$.

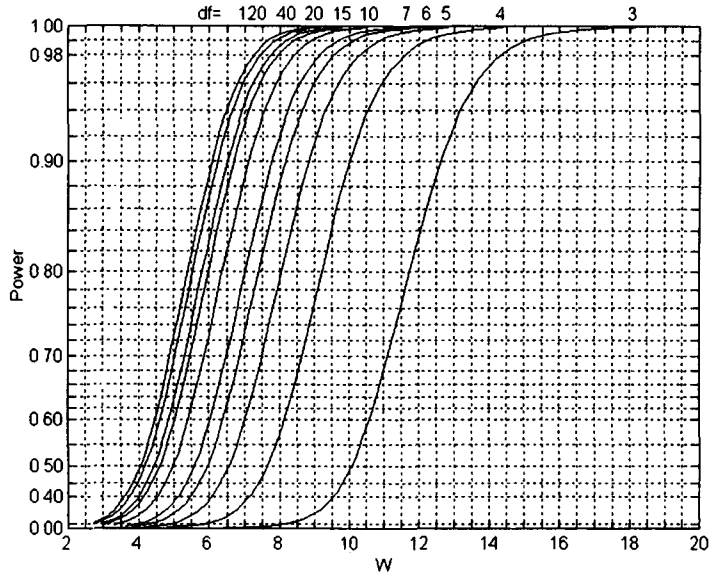

Figure A18 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 7$.

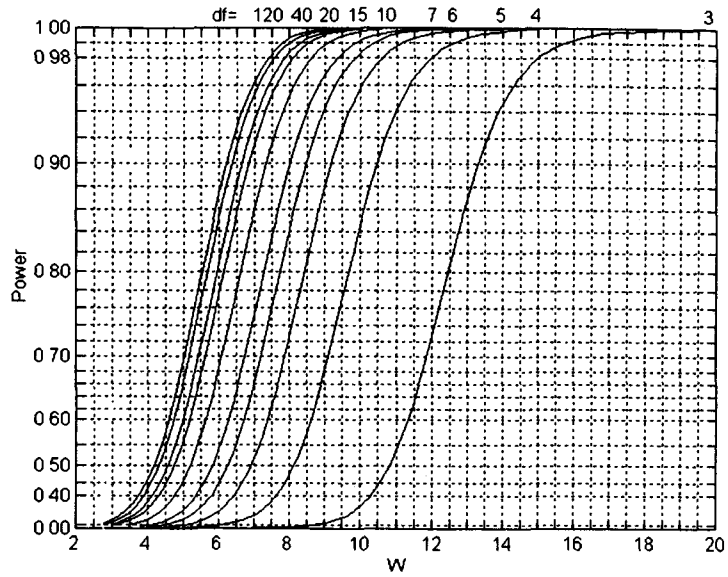

Figure A19 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 8$.

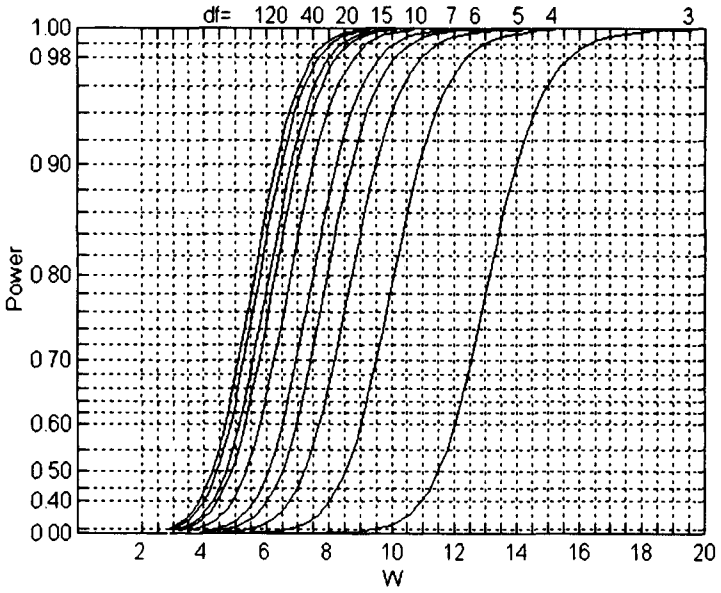

Figure A20 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 9$.

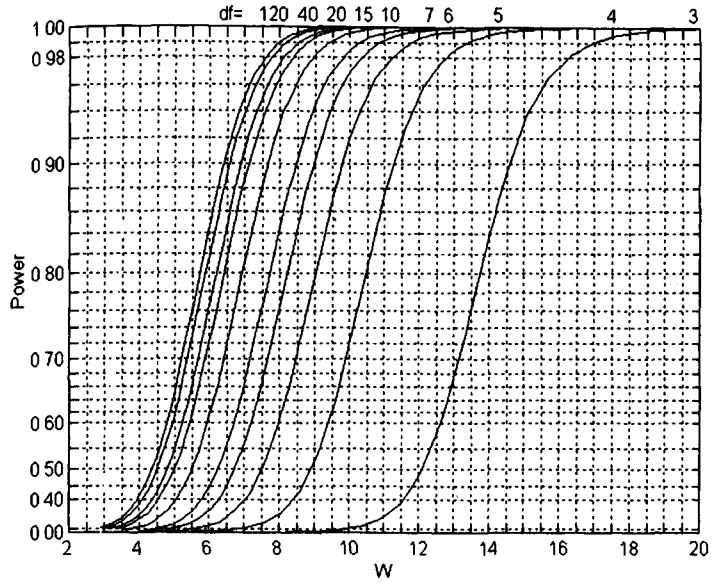

Figure A21 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 10$.

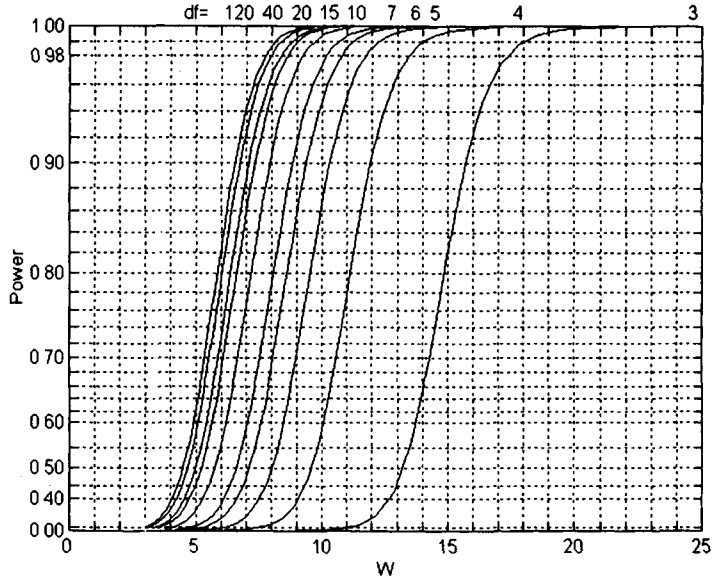

Figure A22 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 12$.

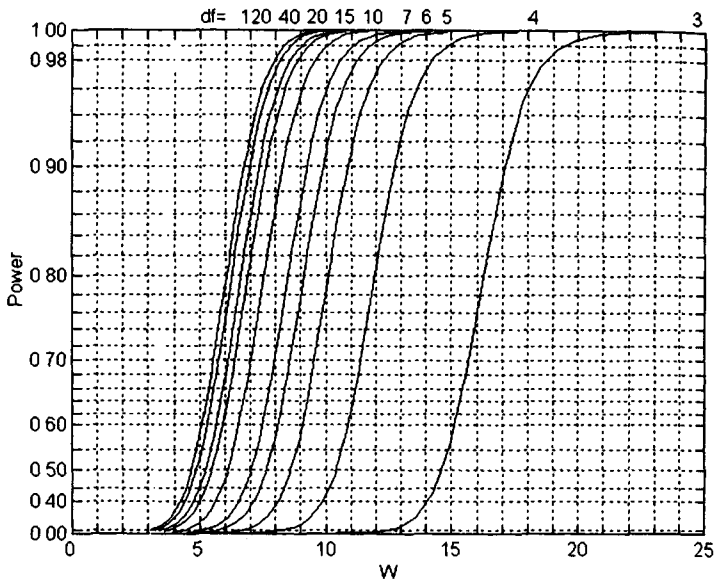

Figure A23 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 15$.

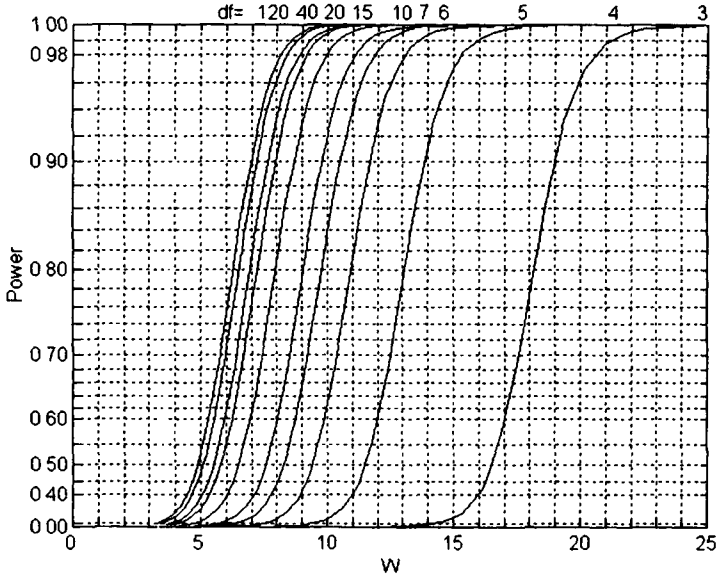

Figure A24 – Heteroscedastic ANOM Power Curves for $\alpha = 0.05$ and $I = 20$.

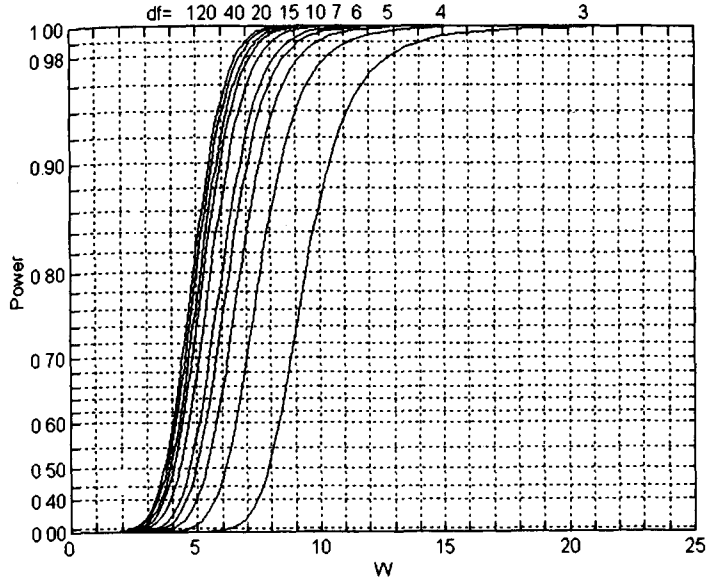

Figure A25 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 2$.

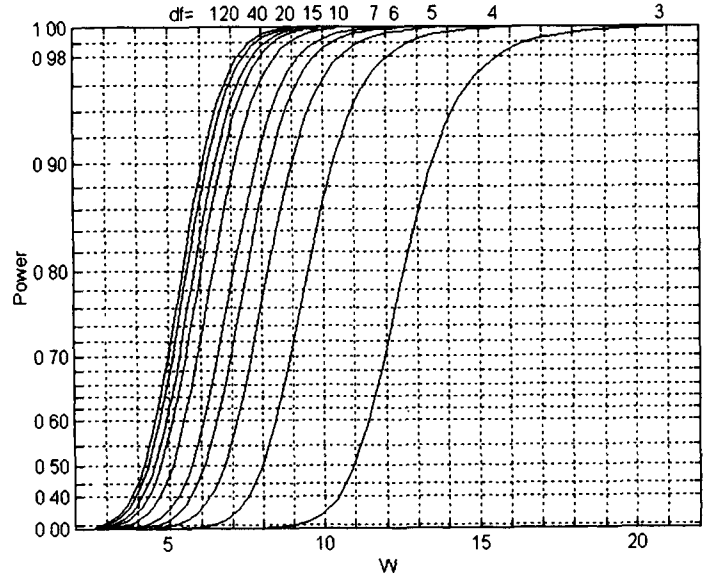

Figure A26 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 3$.

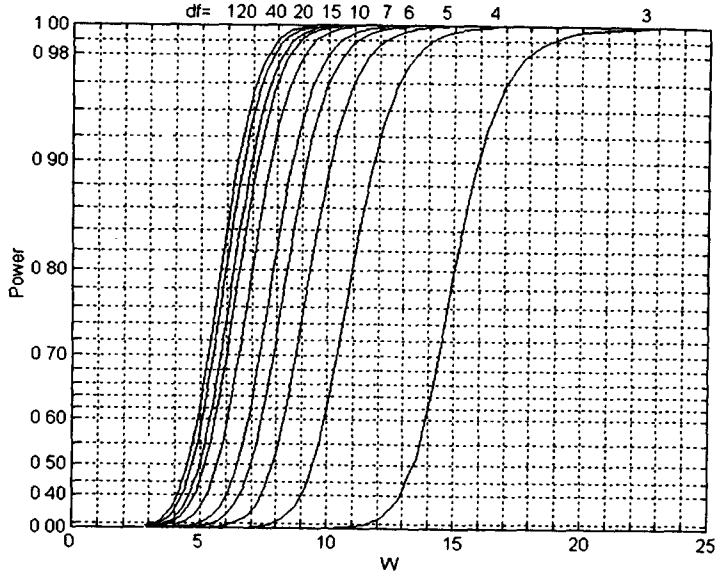

Figure A27 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 4$.

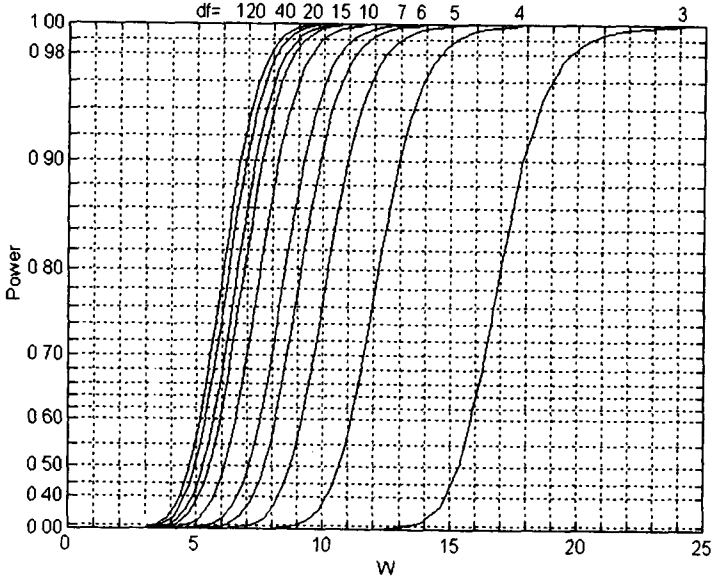

Figure A28 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 5$.

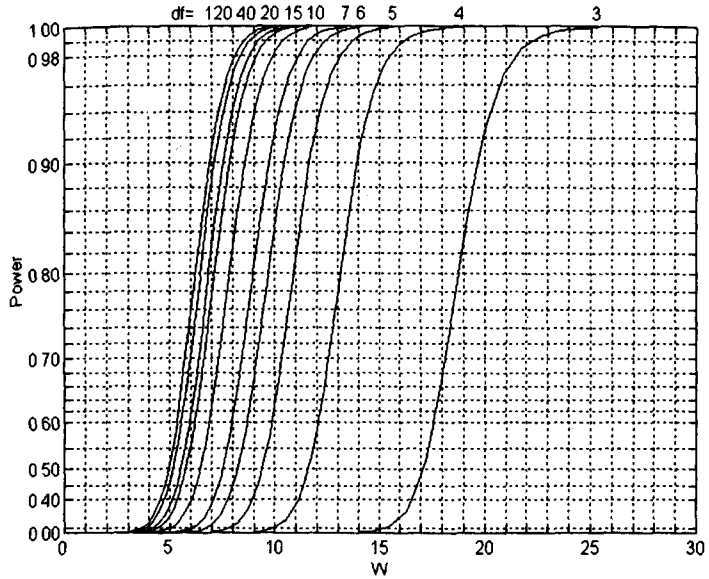

Figure A29 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 6$.

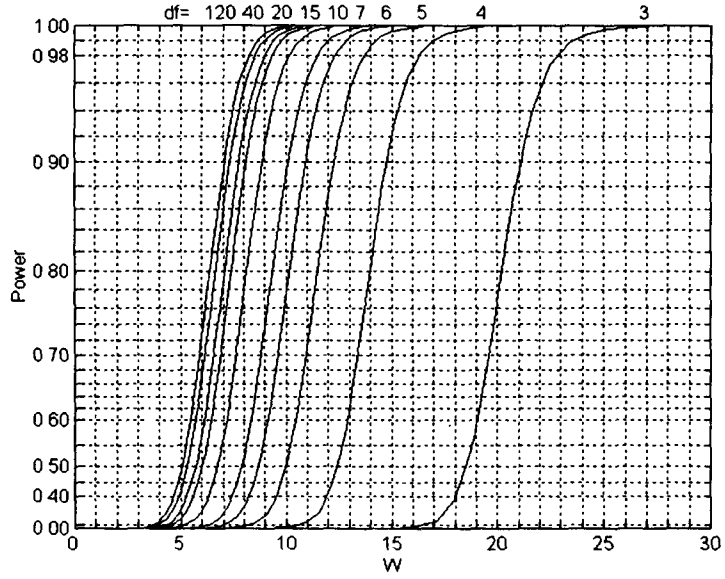

Figure A30 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 7$.

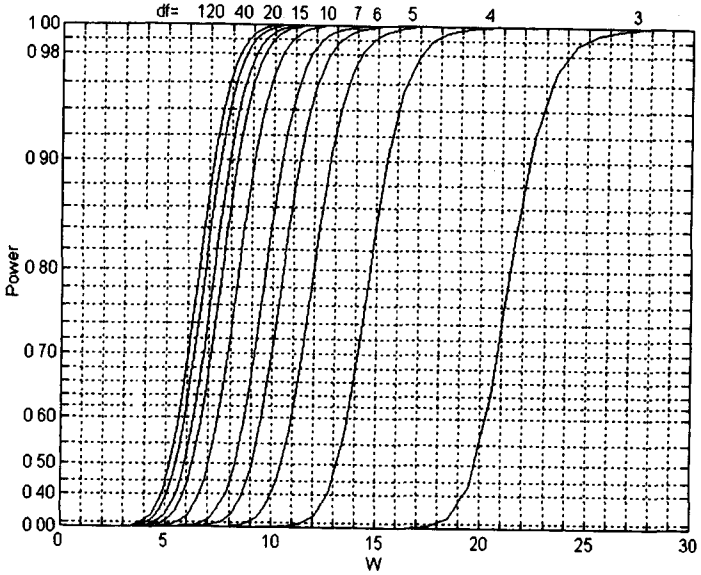

Figure A31 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 8$.

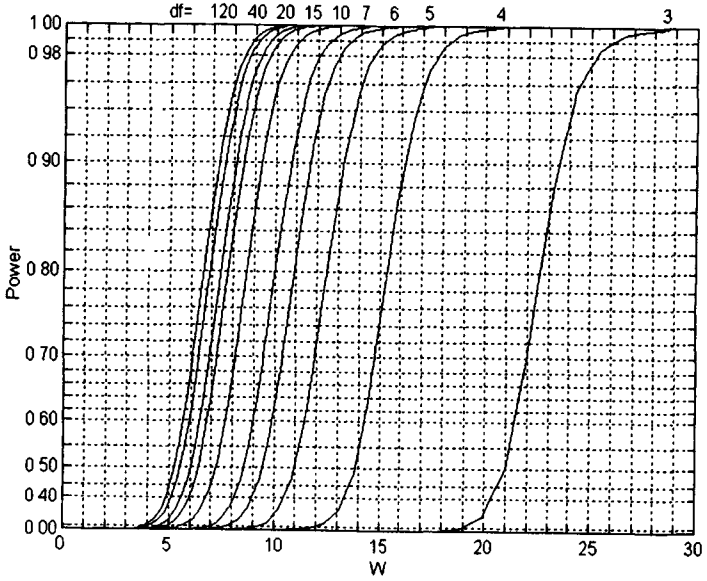

Figure A32 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 9$.

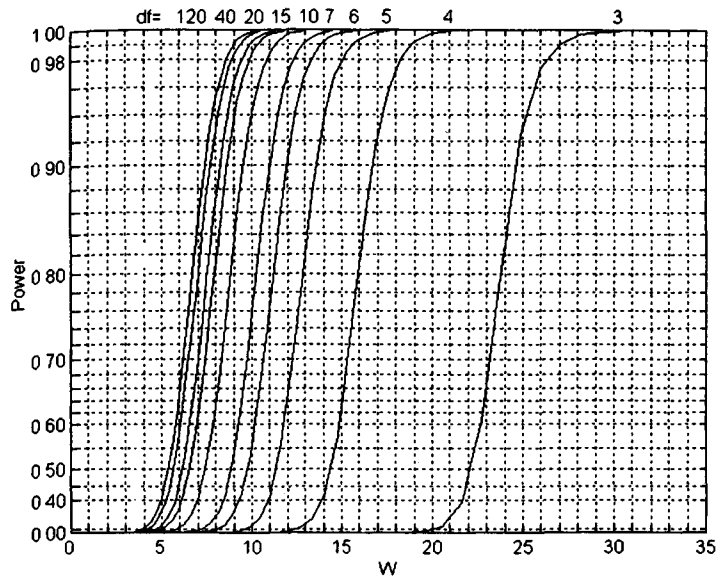

Figure A33 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 10$.

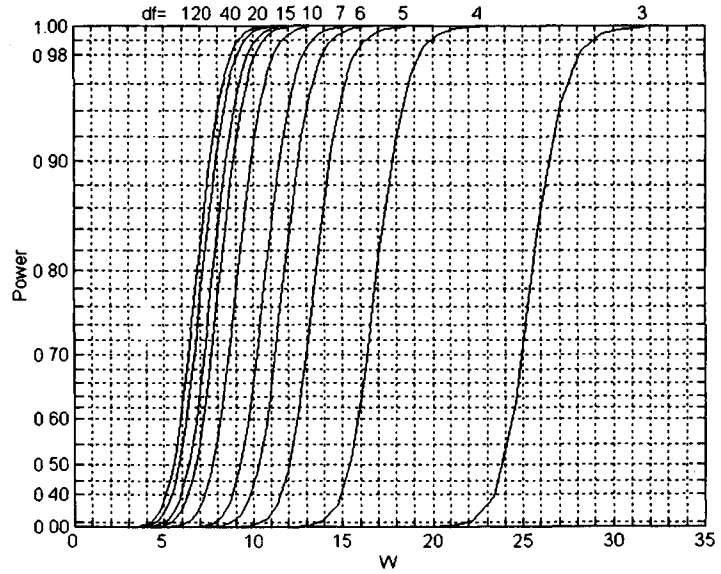

Figure A34 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 12$.

Figure A35 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 15$.

Figure A36 – Heteroscedastic ANOM Power Curves for $\alpha = 0.01$ and $I = 20$.

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