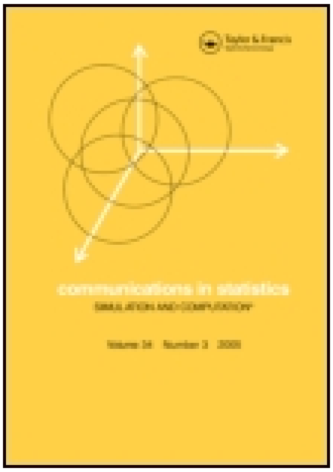

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Single-stage analysis of variance under heteroscedasticity

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SINGLE-STAGE ANALYSIS OF VARIANCE UNDER HETEROSCEDASTICITY

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ABSTRACT

The procedures of testing the equality of normal means in the conventional analysis of variance (ANOVA) are heavily based on the assumption of the equality of the error variances. Studies have shown that the distribution of the F-test depends heavily on the unknown variances and is not robust under the violation of equal error variances. When the variances are unknown and unequal, Bishop and Dudewicz (1978) developed a design-oriented two-stage procedure for ANOVA, which requires additional samples at the second stage. In this paper we use a single-stage sampling procedure to test the null hypotheses in ANOVA models under heteroscedasticity. The single-stage procedure for ANOVA has an exact distribution and it is a data-

analysis-oriented procedure. It does not require additional samples, and can reach a conclusion much earlier, save time and money. Simulation results indicate that the power of the single-stage procedure is better than the two-stage method when the initial sample size is smaller than 6, and performs well when n_0 is 6 or larger. Table of critical values and a numerical example are given.

1. Introduction

Suppose that I (≥ 2) independent populations $\pi_1, \ldots, \pi_I$ are available where observations taken from population π_i are normally distributed with mean μ_i and variance σ_i^2 $(1 \leq i \leq I)$. It is assumed that these variances are unknown and possibly unequal and that no prior knowledge about μ_i and σ_i^2 is available. The interest is a single-stage sampling procedure for the analysis of variance problem under heteroscedasticity.

The procedures of testing equal means in the conventional analysis of variance (ANOVA) are based on the assumptions of normality, independence, and equality of the error variances. Studies have shown that the distribution of the F-test depends heavily on the unknown variances and is not robust under the violation of equal error variances, especially if the sample sizes are not equal. (See Bishop (1976)). As pointed out by Bishop and Dudewicz (1981) that "in practice the assumption of equal error variances is often unjustified, and at the same time there is no exact theory to handle the case of heteroscedasticity. Historically transformations of the data, for example logarithm or arcsine transformations, have been used $\cdots$, they are only approximate in terms of equal variances, normality, and model specification. These approximations, inherent in the transformation technique, lead to test in the ANOVA setting which have only approximate level and power. Finally, if the errors associated with the original observations are normal, the errors associated with the transformed observations will not be normal." Furthermore, the transformed data may lose its practical meaning, and sometimes, the acceptance of the equality of means using transformed data does not automatically imply the acceptance of the equality of means in the original scale.

Several authors such as Brown and Forsythe (1974) and Welch (1947) have proposed procedures to deal with the inequality of variances in the one-way layout, but the exact distributions of the test statistics and their approximate distributions still depend on the unknown variances. Thus, the exact statistical table of critical values for testing an ANOVA problem is not available. When the variances are unknown and unequal, Bishop and Dudewicz (1978) developed an exact analysis of variance for the means of k independent normal populations by using a two-stage procedure as described by Dudewicz and Dalal (1975). The two-stage procedure requires additional samples, which can be large at the second stage may not be practicable in some real problems. Furthermore, the two-stage procedure is a design-oriented procedure in which a final sample size is to be determined with a preassigned power requirement. When working with statistical data analysis one often has only one single sample available. Chen and Lam (1989) developed a single-stage method for interval estimation of the largest normal mean. Lam (1992) adapted this idea to a subset selection procedure which was proven to be quite satisfactory under heteroscedasticity. Wen and Chen (1994) applied the single-stage procedure for multiple comparisons with the largest mean and with a control. The single-stage procedure has design and computational simplicity in practice. In this paper, we employ the exact single-stage sampling procedure for the analysis of variance problem. In Section 2, we developed an exact distribution of the test statistic for the one-way ANOVA problem, and under the null hypothesis the distribution of the test is completely free of the unknown variances. No additional samples are required in order to make an inference in the single-stage ANOVA test. Thus, the single-stage procedure is a data-analysis-oriented procedure, different from the design-oriented two-stage procedure, and it is easy to implement in practice. In Section 3 we give a supplementary table of critical points for small degrees of freedom which is not available elsewhere. A very short SAS computer program using a PC to find the critical values is also given in the Appendix. Section 4 presents a comparison of our new procedure with the two-stage procedure by simulation study. Simulation results indicate that the power of the single-stage procedure is better than the two-stage method when the two-stage procedure begins with a smaller initial sample size, and its performance is adequate for larger initial sample sizes. Section 5 illustrates the single-stage procedure with a numerical example. An extension to the two-way model and tables of its percentage points for small sample sizes are given in Section 6.

2. Single-Stage Sampling Procedure

The following is the single-stage sampling procedure in the context of the one-way layout. Let X_{ij} $(j = 1, \ldots, n_i)$ be an independent random sample of size n_i (≥ 3) from the normal population π_i $(i = 1, \ldots, I)$ with unknown mean μ_i and unknown and unequal variance σ_i^2. Employ the first (or randomly chosen) $n_i - 1$ observations to define the usual sample mean and sample variance, respectively, by

$$\bar{X}_i = \sum_{j=1}^{n_i-1} X_{ij}/(n_i - 1)$$

and

$$S_i^2 = \sum_{j=1}^{n_i-1} (X_{ij} - \bar{X}_i)^2/(n_i - 2).$$

Let the weights of the observations be

$$\begin{aligned} U_i &= \frac{1}{n_i} + \frac{1}{n_i}\sqrt{\frac{1}{n_i - 1}[S_{[k]}^2/S_i^2 - 1]} \\ V_i &= \frac{1}{n_i} - \frac{1}{n_i}\sqrt{(n_i - 1)[S_{[k]}^2/S_i^2 - 1]} \end{aligned} \tag{1}$$

where $S_{[k]}^2$ is the maximum of $S_1^2, \ldots, S_I^2$. Let the final weighted sample mean be defined by

$$\tilde{X}_{i.} = \sum_{j=1}^{n_i} W_{ij} X_{ij} \tag{2}$$

where

$$W_{ij} = \begin{cases} U_i & \text{for } 1 \leq j \leq n_i - 1 \\ V_i & \text{for } j = n_i \end{cases}$$

and U_i and V_i satisfy the following conditions

$$\begin{aligned} &(n_i - 1)U_i + V_i = 1, \\ &(n_i - 1)U_i^2 + V_i^2 = S_{[k]}^2/n_i S_i^2. \end{aligned} \tag{3}$$

Given the sample variances S_i^2, $i = 1, \ldots, I$, the weighted sample mean $\tilde{X}_{i.}$ has a conditional normal distribution with mean μ_i and variance $\sum_j W_{ij}^2 \sigma_i^2$. Furthermore, given S_i^2, the transformation

$$t_i = \frac{\tilde{X}_{i.} - \mu_i}{\sqrt{S_i^2 \sum_{j=1}^{n_i} W_{ij}^2}}$$

has a conditional normal distribution with mean zero and variance σ_i^2/S_i^2.

It is easy to see (but the following simple proof has not been seen elsewhere) that the conditional normal distributions of t_i given S_i^2, $i = 1, \ldots, I$, are unconditional and independent Student's t distributions with $n_i - 2$ ($i = 1, \ldots, I$) degrees of freedom as shown by the following expressions,

$$\begin{aligned} f(t_1, \ldots, t_I) &= \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{I} \left[\phi(0, \sigma_i^2/s_i^2) g_i(s_i^2)\right] \prod_{i=1}^{I} ds_i^2 \\ &= \prod_{i=1}^{I} \int_0^\infty \phi(0, \sigma_i^2/s_i^2) g_i(s_i^2) ds_i^2 \\ &= \prod_{i=1}^{I} f_{n_i-2}(t_i). \end{aligned}$$

where $f(t_1, \ldots, t_I)$ is the joint p.d.f. of $t_1, \ldots, t_I$; $\phi(0, \sigma_i^2/s_i^2)$ is the p.d.f. of the conditional normal distribution with mean 0 and variance σ_i^2/s_i^2, and $g_i(s_i^2)$ is the p.d.f. of $\sigma_i^2/(n_i - 2) \cdot \chi^2_{n_i-2}$, and $f_{n_i-2}(t_i)$ is the p.d.f. of Student's t with $n_i - 2$ degrees of freedom. Equivalently, using condition (3), t_i can be written as

$$t_i = \frac{\tilde{X}_{i.} - \mu_i}{S_{[k]}/\sqrt{n_i}}, \qquad i = 1, \ldots, I$$

which are distributed as independent Student's t with $n_i - 2$ degrees of freedom.

The model we are considering for the one-way layout is the following:

$$X_{ij} = \mu + \alpha_i + e_{ij}, \qquad i = 1, \ldots, I, \quad j = 1, \ldots, n_i$$

where the e_{ij}'s are independent random variables with $e_{ij} \sim N(0, \sigma_i^2)$, and assuming $\sum_{i=1}^{I} \alpha_i = 0$. We may denote mean μ_i by $\mu_i = \mu + \alpha_i$. The null hypothesis we want to test is that the population means μ_i are all equal, or equivalently that $H_0 : \alpha_1 = \alpha_2 = \cdots = \alpha_I = 0$. We assume that for $i = 1, \ldots, I$ the single-stage sampling procedure has been conducted and that the final weighted sample means $\tilde{X}_{i.}$ have been computed as in (2), and let $\tilde{X}_{..} = \sum_{i=1}^{I} \tilde{X}_{i.}/I$. We consider the following test statistic:

$$\tilde{F}^1 = \sum_{i=1}^{I} \left(\frac{\tilde{X}_{i.} - \tilde{X}_{..}}{S_{[k]}/\sqrt{n_i}} \right)^2. \tag{4}$$

Let $\bar{\mu} = \sum_{i=1}^{I} \mu_i / I$ and $\bar{t} = \sum_{i=1}^{I} t_i / I$, then we have

$$\tilde{F}^1 = \sum_{i=1}^{I} \left(\frac{\tilde{X}_{i.} - \mu_i}{S_{[k]}/\sqrt{n_i}} - \frac{\tilde{X}_{..} - \bar{\mu}}{S_{[k]}/\sqrt{n_i}} + \frac{\mu_i - \bar{\mu}}{S_{[k]}/\sqrt{n_i}} \right)^2, \tag{5}$$

where the second term of (5) in the parentheses can be written as

$$\frac{\tilde{X}_{..} - \bar{\mu}}{S_{[k]}/\sqrt{n_i}} = \frac{(1/I)\sum_{j=1}^{I} (\tilde{X}_{j.} - \mu_j)}{S_{[k]}/\sqrt{n_i}} = \frac{\sqrt{n_i}}{I} \sum_{j=1}^{I} \frac{t_j}{\sqrt{n_j}}.$$

Under the null hypothesis, it follows that $\tilde{F}^1$ is distributed as

$$\tilde{F}^1 = \sum_{i=1}^{I} \left(t_i - \frac{\sqrt{n_i}}{I} \sum_{j=1}^{I} \frac{t_j}{\sqrt{n_j}} \right)^2. \tag{6}$$

When, in a balanced design, the sample sizes from the populations are equal, i.e., $n_1 = \cdots = n_I = n$, $\tilde{F}^1$ under H_0 is then distributed as

$$Q = \sum_{i=1}^{I} (t_i - \bar{t})^2$$

which is a quadratic form in independent Student's t variates each with $n-2$ degrees of freedom. Thus, we reject the null hypothesis that the population means are all equal if

$$\tilde{F}^1 > \tilde{F}_{\alpha,I,n-2}$$

where $\tilde{F}_{\alpha,I,n-2}$ is the upper α percentage point of the null distribution of $\tilde{F}^1$ which is the same as that of the random variable Q. For $I = 2$, Q becomes $\frac{1}{2}(t_1 - t_2)^2$ and exact percentage points can be easily derived from Dudewicz, Ramberg and Chen (1975). For $I > 2$, tables of approximate upper α percentage points of Q are derived from several methods by Bishop, Dudewicz, Juritz, and Stephens (1978). Bishop and Dudewicz (1978) also derived the critical points of Q and the powers of $\tilde{F}^1$ under an alternative hypothesis by using Monte Carlo. Their tables were constructed mainly for degrees of freedom at least 10. The table for small degrees of freedom is limited. In

Section 3, we provide a simulation method to obtain the percentage points of $\tilde{F}^1$ in (6) for small degrees of freedom.

3. The Critical Values of The Null Distribution of $\tilde{F}^1$

The critical values of the null distribution of $\tilde{F}^1$ in (6) for a balanced or an unbalanced design can be obtained from a very short SAS computer simulation program given in the Appendix. The program can be run on a 486 or Pentium personal computer with a SAS PC software. Numerical results indicate that there is a difference in the critical values between the equal and unequal sample size cases even when their total sample sizes are the same. Therefore, it is not advisable to use the average sample size to replace the unequal sample sizes in computing the critical values. When the sample sizes are equal, the critical values of Q were obtained in Table 1 by Monte Carlo simulation for various combinations of $I = 3, 4, 5, 6, 8$, and small degrees of freedom (df $= 2, 3, 4, 5, 6, 8$). These critical values are not seen elsewhere. In each simulation run, a Student's $t = z/\sqrt{u/r}$ variate was calculated where z is the random variate of the standard normal distribution generated from the random number generator NORMAL and u is the chi-squared random variate with r degrees of freedom generated by the gamma random number generator, RANGAM respectively, in SAS 6.12 (SAS Institute Inc., 1990). Then the Q value was computed for each run. After 10,000 simulation runs, all of the Q values were ranked in ascending order. The $75th, 90th, 95th, 97.5th$, and $99th$ percentiles were used to estimate the upper α percentage points $25\%, 10\%, 5\%, 2.5\%$, and 1%, respectively. This process was replicated 16 times. The average values of the 16 critical points and their corresponding standard errors (in the parentheses) are listed in Table 1. The simulation

Table 1. The Average of 16 Upper α-Percentage Points of $Q = \sum(t_i - \bar{t})^2$ and their Standard Deviations in Parentheses.

I	df	25%	10%	5%	2.5%	1%
3	2	8.05 (.04)	21.76 (.23)	43.13 (.62)	85.62 (1.8)	206.3 (6.3)
	3	5.36 (.02)	11.61 (.07)	19.00 (.12)	30.15 (.25)	54.17 (.91)
	4	4.47 (.02)	8.82 (.05)	13.38 (.08)	19.47 (.13)	30.84 (.38)
	5	4.05 (.02)	7.67 (.04)	11.10 (.08)	15.24 (.10)	22.71 (.20)
	6	3.77 (.02)	6.99 (.04)	9.89 (.05)	13.28 (.09)	18.54 (.16)
	8	3.50 (.01)	6.26 (.02)	8.58 (.04)	11.26 (.07)	15.28 (.14)
4	2	13.16 (.08)	33.50 (.27)	65.08 (.84)	128.6 (2.4)	307.5 (8.7)
	3	8.40 (.03)	17.03 (.06)	26.86 (.14)	42.01 (.32)	73.99 (.93)
	4	6.85 (.02)	12.67 (.03)	18.52 (.08)	26.14 (.25)	40.67 (.49)
	5	6.15 (.02)	10.72 (.04)	14.94 (.05)	20.01 (.15)	29.03 (.32)
	6	5.74 (.02)	9.74 (.03)	13.27 (.06)	17.37 (.09)	24.23 (.22)
	8	5.25 (.02)	8.59 (.03)	11.42 (.04)	14.51 (.06)	19.34 (.14)
5	2	18.76 (.08)	46.81 (.34)	90.02 (.84)	173.4 (2.7)	427.8 (8.6)
	3	11.51 (.03)	22.19 (.09)	34.59 (.26)	52.76 (.38)	92.92 (.95)
	4	9.23 (.03)	16.16 (.05)	23.05 (.13)	32.04 (.23)	48.81 (.58)
	5	8.28 (.03)	13.69 (.04)	18.77 (.10)	25.03 (.15)	36.00 (.38)
	6	7.60 (.02)	12.18 (.04)	16.13 (.07)	20.56 (.12)	27.70 (.21)
	8	6.94 (.02)	10.86 (.04)	14.03 (.05)	17.57 (.06)	22.89 (.13)
6	2	24.74 (.16)	59.65 (.45)	112.6 (.89)	219.0 (3.5)	521.2 (12)
	3	14.66 (.04)	27.44 (.14)	42.29 (.21)	64.10 (.51)	113.2 (1.2)
	4	11.59 (.04)	19.61 (.08)	27.46 (.14)	37.67 (.25)	57.50 (.59)
	5	10.25 (.03)	16.43 (.05)	22.07 (.10)	28.60 (.20)	39.87 (.41)
	6	9.47 (.03)	14.69 (.04)	19.16 (.07)	24.28 (.13)	32.54 (.27)
	8	8.62 (.02)	12.95 (.03)	16.44 (.04)	20.27 (.05)	25.88 (.15)
8	2	36.93 (.16)	85.86 (.47)	161.0 (1.1)	301.1 (3.7)	730.6 (14)
	3	21.05 (.06)	37.39 (.13)	56.05 (.28)	82.71 (.62)	144.1 (2.0)
	4	16.32 (.05)	26.32 (.08)	35.93 (.17)	47.90 (.33)	72.47 (.53)
	5	14.32 (.03)	21.87 (.07)	28.56 (.13)	36.47 (.18)	50.42 (.45)
	6	13.18 (.03)	19.47 (.06)	24.72 (.08)	30.81 (.16)	40.16 (.29)
	8	11.84 (.02)	16.96 (.04)	20.87 (.08)	25.15 (.11)	31.21 (.15)

errors of the percentage points mostly occur in the second decimal when α is large and in the first decimal when α is small. This is due to the long tail of the t distribution when the df are small. When $I = 2$, the exact critical points can be derived from Table 4 of Dudewicz, Ramberg and Chen (1975).

4. Numerical Studies

We compare the single-stage test $\tilde{F}^1$ with the two-stage test $\tilde{F}^2$ proposed by Bishop and Dudewicz (1978,1981). For the one-way layout with four independent populations, we choose the asymptotic least favorable configuration of means, $\mu_1 = -\sqrt{\delta/2}, \mu_2 = \sqrt{\delta/2}, \mu_3 = \mu_4 = 0$, subject to $\sum_{i=1}^{4}(\mu_i - \bar{\mu})^2 = \delta$ as proposed by Bishop and Dudewicz (1978) and let $\delta = 1$ for simplicity. The power of the test statistics $\tilde{F}^2$ and $\tilde{F}^1$ is calculated using Monte Carlo simulation for various combinations of the initial sample size n_0 $(3 \leq n_0 \leq 10)$ in the two-stage, the level of significance α $(\alpha = .10, .05, .01)$, variances $((\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2) = (1,1,1,1), (1,4,4,9), (.1,.4,.4,.9))$ and the power-related z $(z = .04, .10)$. For each simulation run, under the various α-nominal levels, the configurations of variances, the initial sample sizes and the values of z, observations were drawn from the standard normal distribution (additional sample is required for the two-stage test $\tilde{F}^2$.) using the random number generator NORMAL in SAS 6.12. The values $\mu_1 = -\sqrt{1/2}, \mu_2 = \sqrt{1/2}, \mu_3 = 0$ and $\mu_4 = 0$ were added to the observations from the first, second, third and the fourth population, respectively. The values of the statistics $\tilde{F}^2$ and $\tilde{F}^1$ were calculated, and compared with the critical values using Table 1 and the table of Bishop and Dudewicz (1978). Then the power of each statistic was found after 10000 simulation runs and was reported in the main body of

Table 2. The first column of Table 2 is the initial sample size n_0 employed by the two-stage procedure. The second column is the total expected sample size of the two-stage and is also the total sample size of the single-stage procedure, which makes the comparison comparable. From the simulation results in Table 2, we obtained the following facts:

(a) For variances being equal, i.e. variances=(1,1,1,1), and for the power-related $z = .04$ which corresponds to a high power, the single-stage procedure produced a larger power (.977,.954,.850) than that (.975,.945,.824) of the two-stage procedure when the initial sample size n_0 is smaller than 9 for $\alpha = .10, .05$, and when n_0 is 10 or less for $\alpha = .01$. When $z = .10$, corresponding to a moderate or small power, the single-stage procedure gave a larger power (.608,.457,.231) than that (.525,.408,.210) of the two-stage procedure if n_0 is 5 or less for $\alpha = .10$, n_0 is 6 or less for $\alpha = .05$, and n_0 is 9 or less for $\alpha = .01$. Generally speaking, when variances are equal, the single-stage procedure performs almost as good as the two-stage procedure or loses little if n_0 is 6 or more and it performs better if n_0 is less than 6.

(b) For variances being unequal, i.e. variances=(1,4,4,9), and for $z = .04$ which corresponds a high power, the single-stage procedure produced a larger power (.908,.848,.665) than that (.832,.799,.622) of the two-stage procedure when n_0 is 4 or less for $\alpha = .10$, n_0 is 5 or less for $\alpha = .05$, and when n_0 is 7 or less for $\alpha = .01$. When $z = .10$, corresponding to a moderate or small power, the single-stage procedure gave a larger power (.569,.441,.208) than that (.528, .402,.176) of the two-stage procedure if n_0 is 5 or less for $\alpha = .10$, n_0 is 6 or less for $\alpha = .05$, and n_0 is 8 or less for $\alpha = .01$. The simulation results reveal that, if n_0 is larger than 6, the single-stage procedure loses little power relative to the two-stage procedure, however, it performs relatively good for small n_0. For variances being (.1,.4,.4,.9) a similar conclusion can be reached.

Table 2. The Power Comparison of $\tilde{F}^2$ and $\tilde{F}^1$

		$\tilde{F}^2$			$\tilde{F}^1$		
	total	nominal size			nominal size		
n_0	sample size	10%	5%	1%	10%	5%	1%
		variances: 1,1,1,1.			$z = 0.04$		
10	103	.985	.964	.824	.978	.955	.850
9	103	.981	.955	.792	.978	.954	.849
8	103	.975	.945	.729	.977	.954	.848
7	102	.972	.928	.628	.977	.951	.850
6	103	.957	.890	.461	.977	.951	.848
5	103	.927	.802	.186	.977	.952	.848
4	103	.838	.562	.036	.977	.954	.850
3	103	.422	.113	.011	.977	.952	.848
		variances: 1,1,1,1.			$z = 0.10$		
10	51	.741	.595	.249	.666	.518	.230
9	49	.730	.562	.210	.664	.518	.231
8	47	.704	.528	.178	.661	.520	.228
7	46	.674	.476	.126	.611	.460	.190
6	45	.620	.408	.077	.610	.457	.187
5	45	.525	.283	.034	.608	.451	.182
4	44	.382	.160	.016	.606	.450	.183
3	44	.181	.070	.010	.609	.450	.184
		variances: 1,4,4,9.			$z = 0.04$		
10	454	.985	.964	.832	.912	.852	.664
9	454	.983	.957	.795	.910	.846	.662
8	452	.978	.945	.731	.912	.852	.665
7	451	.971	.924	.622	.908	.845	.665
6	451	.956	.891	.456	.908	.846	.666
5	452	.927	.799	.193	.910	.848	.667
4	452	.832	.550	.037	.908	.845	.661
3	453	.424	.112	.012	.908	.846	.662

Table 2. (Continued)

		$\tilde{F}^2$			$\tilde{F}^1$		
	total	nominal size			nominal size		
n_0	sample size	10%	5%	1%	10%	5%	1%
		variances: 1,4,4,9.			$z = 0.10$		
10	185	.739	.591	.246	.574	.444	.214
9	185	.722	.562	.219	.574	.443	.215
8	184	.705	.530	.176	.571	.443	.208
7	184	.671	.479	.118	.572	.442	.212
6	183	.618	.402	.076	.568	.441	.211
5	183	.528	.290	.036	.569	.438	.210
4	181	.371	.155	.017	.566	.440	.208
3	184	.172	.068	.010	.568	.441	.210
		variances: 0.1,0.4,0.4,0.9.			$z = 0.04$		
10	60	.986	.963	.827	.920	.860	.667
9	58	.984	.958	.791	.903	.838	.625
8	56	.979	.946	.738	.902	.835	.620
7	54	.969	.928	.623	.882	.802	.562
6	53	.959	.891	.463	.880	.800	.561
5	52	.928	.805	.195	.877	.798	.559
4	52	.836	.556	.037	.876	.793	.559
3	50	.425	.108	.012	.847	.756	.502
		variances: 0.1,0.4,0.4,0.9.			$z = 0.10$		
10	45	.743	.589	.251	.804	.693	.410
9	42	.726	.563	.210	.755	.633	.333
8	38	.698	.520	.174	.700	.558	.244
7	35	.668	.474	.118	.699	.557	.240
6	32	.617	.404	.078	.612	.444	.149
5	30	.523	.290	.036	.580	.409	.104
4	27	.382	.154	.014	.532	.365	.089
3	25	.180	.070	.011	.407	.235	.039

(c) For other combinations of k, z, variances, and configurations of means, the conclusions drawn in (a) and (b) generally hold. That is, when the initial sample size n_0 is 5 or less, the two-stage procedure is not as good as the single-stage procedure in terms of the power.

The conclusion is that when n_0 is smaller than 6, the single-stage procedure performs better, and it loses little in power when n_0 is 6 or larger. The advantage of the single-stage is that it requires no additional samples. Hence, it save money, time and energy because, in practice, additional samples are very costly.

5. A Numerical Example

The data in Table 3 is from an experiment reported in Bishop and Dudewicz (1978) for studying the bacterial killing ability of four solvents. The percentage of fungus destroyed was recorded. The aim of the experiment was to test the hypothesis that the mean percentages of fungus destroyed are all equal,

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

where μ_i denotes the mean percentage of fungus destroyed by solvent i. In the first stage of experiment $n_i = 15$ observations were run with each solvent. Wen and Chen (1994) have shown a significant difference among the variances by using Bartlett's χ^2 test for equality of variance. They also discovered that the data are not normally distributed. However, Dudewicz and van der Meulen (1983) have shown robustness results which applies to the single-stage procedure for general non-normal distributions. So we use the single-stage procedure with $n_i = 15$ observations from each population; the first 14 observations for initial estimation and the remaining one for use in the

Table 3. Bacterial killing ability data ($n = 15$)

	Solvent 1	Solvent 2	Solvent 3	Solvent 4
Data:	96.44	93.63	93.58	97.18
	96.87	93.99	93.02	97.42
	97.24	94.61	93.86	97.65
	95.41	91.69	92.90	95.90
	95.29	93.00	91.43	96.35
	95.61	94.17	92.68	97.13
	95.28	92.62	91.57	96.06
	94.63	93.41	92.87	96.33
	95.58	94.67	92.65	96.71
	98.20	95.28	95.31	98.11
	98.29	95.13	95.33	98.38
	98.30	95.68	95.17	98.35
	98.65	97.52	98.59	98.05
	98.43	97.52	98.00	98.25
	98.41	97.37	98.79	98.12
Intermediate statistics:				
$\bar{X}_i$	96.7300	94.4943	94.0686	97.2764
S_i^2	2.06962	2.82104	4.73647	0.78858
U_i	0.08689	0.08135	0.06667	0.10653
V_i	-0.21649	-0.13888	0.06667	-0.49146
$\tilde{X}_{i.}$	96.3663	94.0949	94.3833	96.8618

final computation. The intermediate statistics $\bar{X}_i, S_i^2, U_i, V_i$ and the final weighted sample means $\tilde{X}_{i.}$ are given at the bottom of Table 3. We found $\tilde{F}^1 = 18.38$, which exceeds the critical point of 9.69 at $df = 13$ and 5% level, so H_0 is rejected. The critical value was obtained by simulation in the Appendix or by linear interpolation method using the table by Bishop, Dudewicz, Juritz and Stephens (1978). Note that the two-stage procedure was applied to these data by Bishop and Dudewicz (1978). They rejected H_0

with additional observations required at the second stage. The final sample sizes they used for each solvent were 27, 40, 74 and 16, respectively. The advantage of the single-stage procedure is that it doesn't require additional observations (additional samples can be very costly in practice), and thus perhaps it can reach a conclusion much earlier, save time and money, and provide simplicity and ease. Follow-up multiple comparisons when a null hypothesis is rejected have been considered by Wen (1991) and Wen and Chen (1994).

6. The Two-Way Model

The two-way model of analysis of variance we consider is defined by:

$$X_{ijk} = \mu + \alpha_i + \beta_j + \alpha\beta_{ij} + e_{ijk} \tag{7}$$

where $i = 1, \ldots, I, j = 1, \ldots, J, k = 1, \ldots, n_{ij}$; the e_{ijk}'s are independent random variables with $e_{ijk} \sim N(0, \sigma_{ij}^2)$, where σ_{ij}^2 are unknown and possibly unequal, and we assume that

$$\sum_{i=1}^{I} \alpha_i = \sum_{j=1}^{J} \beta_j = \sum_{i=1}^{I} \alpha\beta_{ij} = \sum_{j=1}^{J} \alpha\beta_{ij} = 0.$$

The null hypotheses we want to test are

$$H^1 : \alpha_i = 0 \qquad \text{for all } i,$$
$$H^2 : \beta_j = 0 \qquad \text{for all } j, \tag{8}$$

and

$$H^3 : \alpha\beta_{ij} = 0 \qquad \text{for all } i \text{ and } j.$$

There are $I * J$ possible treatment combinations in the model (7). We denote cell (i, j) as the combination of level i of the first factor and level j of the second factor. The sample size of each cell is n_{ij} (≥ 3). The single-stage sampling procedure for testing the hypotheses of (8) proceeds as follows. Initially we employ the first (or randomly chosen) $n_{ij} - 1$ observations within each cell and compute the usual sample mean and sample variance, respectively,

$$\bar{X}_{ij} = \sum_{k=1}^{n_{ij}-1} X_{ijk}/(n_{ij} - 1)$$

and

$$S_{ij}^2 = \sum_{k=1}^{n_{ij}-1} (X_{ijk} - \bar{X}_{ij})^2/(n_{ij} - 2).$$

Then the weights of the observations in cell (i, j) are

$$\begin{aligned} U_{ij} &= \frac{1}{n_{ij}} + \frac{1}{n_{ij}}\sqrt{\frac{1}{n_{ij} - 1}[S_{[k]}^2/S_{ij}^2 - 1]} \\ V_{ij} &= \frac{1}{n_{ij}} - \frac{1}{n_{ij}}\sqrt{(n_{ij} - 1)[S_{[k]}^2/S_{ij}^2 - 1]} \end{aligned} \tag{9}$$

where $S_{[k]}^2$ is the maximum value of S_{ij}^2's, $i = 1, \ldots, I,\ j = 1, \ldots, J$. Let the final weighted sample mean for cell (i, j) be defined by

$$\tilde{X}_{ij.} = \sum_{k=1}^{n_{ij}} W_{ijk} X_{ijk} \tag{10}$$

where

$$W_{ijk} = \begin{cases} U_{ij} & \text{for } 1 \leq k \leq n_{ij} - 1 \\ V_{ij} & \text{for } k = n_{ij}. \end{cases}$$

Therefore, we compute

$$\tilde{X}_{i..} = \frac{1}{J}\sum_{j=1}^{J} \tilde{X}_{ij.}, \qquad \tilde{X}_{.j.} = \frac{1}{I}\sum_{i=1}^{I} \tilde{X}_{ij.}, \qquad \tilde{X}_{...} = \frac{1}{IJ}\sum_{i=1}^{I}\sum_{j=1}^{J} \tilde{X}_{ij.}$$

The test statistic we consider to use for H^1 is

$$\begin{aligned} \tilde{F}_1^1 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{i..} - \tilde{X}_{...}}{S_{[k]}/\sqrt{n_{ij}}}\right)^2 \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(t'_{i.} - t'_{..} + \frac{\alpha_i}{S_{[k]}/\sqrt{n_{ij}}}\right)^2; \end{aligned} \tag{11}$$

for H^2 :

$$\begin{aligned}\tilde{F}_2^1 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{.j.}-\tilde{X}_{...}}{S_{[k]}/\sqrt{n_{ij}}}\right)^2 \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(t'_{.j}-t'_{..}+\frac{\beta_j}{S_{[k]}/\sqrt{n_{ij}}}\right)^2; \end{aligned} \tag{12}$$

and for H^3 :

$$\begin{aligned}\tilde{F}_3^1 &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(\frac{\tilde{X}_{ij.}-\tilde{X}_{i..}-\tilde{X}_{.j.}+\tilde{X}_{...}}{S_{[k]}/\sqrt{n_{ij}}}\right)^2 \\ &= \sum_{i=1}^{I}\sum_{j=1}^{J}\left(t_{ij}-t'_{i.}-t'_{.j}+t'_{..}+\frac{\alpha\beta_{ij}}{S_{[k]}/\sqrt{n_{ij}}}\right)^2, \end{aligned} \tag{13}$$

where

$$t'_{i.}=\frac{\sqrt{n_{ij}}}{J}\sum_{l=1}^{J}\frac{t_{il}}{\sqrt{n_{il}}},\qquad t'_{.j}=\frac{\sqrt{n_{ij}}}{I}\sum_{m=1}^{I}\frac{t_{mj}}{\sqrt{n_{mj}}},\qquad t'_{..}=\frac{\sqrt{n_{ij}}}{IJ}\sum_{m=1}^{I}\sum_{l=1}^{J}\frac{t_{ml}}{\sqrt{n_{ml}}}.$$

It can be shown that

$$t_{ij}=\frac{\tilde{X}_{ij.}-(\mu+\alpha_i+\beta_j+\alpha\beta_{ij})}{S_{[k]}/\sqrt{n_{ij}}}, \tag{14}$$

for $i=1,\ldots,I, j=1,\ldots,J$, are distributed as independent Student's t with $n_{ij}-2$ degrees of freedom. This result is due to the fact that given the sample variances S_{ij}^2's, the weighted sample mean $\tilde{X}_{ij.}$ has a conditional normal distribution with mean $\mu+\alpha_i+\beta_j+\alpha\beta_{ij}$ and variance $\sum_k W_{ijk}\sigma_{ij}^2$, as described in Section 2.

Under the null hypotheses of (8) the test statistics in (11)-(13) reduce to

$$\tilde{F}_1^1 = J\sum_{i=1}^{I}\left(t'_{i.}-t'_{..}\right)^2$$

$$\tilde{F}_2^1 = I\sum_{j=1}^{J}\left(t'_{.j}-t'_{..}\right)^2 \tag{15}$$

and

$$\tilde{F}_3^1 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left(t_{ij}-t'_{i.}-t'_{.j}+t'_{..}\right)^2.$$

If cell sizes n_{ij} are all equal, i.e., $n_{ij} = n, i = 1, \ldots, I, j = 1, \ldots, J$, then the test statistics in (11)-(13) become

$$\tilde{F}_1^1 = J \sum_{i=1}^{I} \left(\bar{t}_{i.} - \bar{t}_{..} + \frac{\alpha_i}{S_{[k]}/\sqrt{n}} \right)^2,$$

$$\tilde{F}_2^1 = I \sum_{j=1}^{J} \left(\bar{t}_{.j} - \bar{t}_{..} + \frac{\beta_j}{S_{[k]}/\sqrt{n}} \right)^2, \tag{16}$$

and

$$\tilde{F}_3^1 = \sum_{i=1}^{I} \sum_{j=1}^{J} \left(t_{ij} - \bar{t}_{i.} - \bar{t}_{.j} + \bar{t}_{..} + \frac{\alpha\beta_{ij}}{S_{[k]}/\sqrt{n}} \right)^2,$$

where

$$\bar{t}_{i.} = \frac{1}{J} \sum_{j=1}^{J} t_{ij}, \qquad \bar{t}_{.j} = \frac{1}{I} \sum_{i=1}^{I} t_{ij}, \qquad \bar{t}_{..} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} t_{ij}.$$

Under the null hypotheses of (8), the tests in (16) become

$$\tilde{F}_1^1 = J \sum_{i=1}^{I} (\bar{t}_{i.} - \bar{t}_{..})^2,$$

$$\tilde{F}_2^1 = I \sum_{j=1}^{J} (\bar{t}_{.j} - \bar{t}_{..})^2, \tag{17}$$

and

$$\tilde{F}_3^1 = \sum_{i=1}^{I} \sum_{j=1}^{J} (t_{ij} - \bar{t}_{i.} - \bar{t}_{.j} + \bar{t}_{..})^2.$$

We reject H^1, H^2, or H^3 respectively if $\tilde{F}_1^1, \tilde{F}_2^1$, or $\tilde{F}_3^1$ exceed the upper α percentage point of the null distributions of (15) or (17). The statistics we used to test hypotheses H^1, H^2 or H^3 are distributed as quadratic functions of independent Student's t r.v.'s. Each of these statistics in (15) and (17) will have a level α test regardless of the values of the σ_{ij}^2. The percentage points of the null distributions of $\tilde{F}_1^1, \tilde{F}_2^1$, and $\tilde{F}_3^1$ in (15) require extensive simulation. However, when the sample sizes n_{ij} are large, following the argument of

Table 4. The Average of 16 Upper α-Percentage Points of $\tilde{F}_1^1$ and Their Standard Deviations in Parentheses

I	J	df	25%	10%	5%	2.5%	1%
2	2	2	4.35(.02)	12.44(.14)	24.70(.40)	46.31(.89)	109.9(2.9)
	3	2	4.94(.04)	13.67(.12)	26.15(.39)	48.90(.91)	112.7(3.3)
	4	2	5.31(.04)	14.24(.13)	26.81(.33)	49.10(.95)	113.4(2.6)
	6	2	5.88(.05)	15.46(.19)	28.36(.42)	50.95(1.1)	114.8(2.7)
	2	4	2.22(.02)	5.00(.03)	7.75(.05)	11.22(.09)	17.18(.24)
	3	4	2.32(.02)	5.10(.04)	7.73(.06)	10.85(.08)	16.00(.21)
	4	4	2.36(.02)	5.10(.04)	7.66(.06)	10.65(.14)	15.64(.27)
	6	4	2.44(.02)	5.19(.04)	7.64(.05)	10.55(.10)	15.12(.20)
	2	6	1.85(.01)	3.98(.03)	5.92(.05)	8.10(.08)	11.37(.14)
	3	6	1.90(.01)	4.01(.02)	5.89(.04)	7.97(.07)	11.14(.12)
	4	6	1.90(.01)	4.01(.02)	5.85(.04)	7.82(.07)	10.77(.13)
	6	6	1.93(.01)	4.02(.03)	5.84(.03)	7.76(.08)	10.49(.14)
3	2	2	10.08(.10)	25.49(.27)	48.76(.75)	93.36(1.6)	211.2(5.1)
	3	2	11.33(.08)	27.39(.22)	50.45(.79)	94.32(1.4)	211.7(5.9)
	4	2	12.06(.09)	28.55(.27)	51.96(.65)	96.11(1.7)	225.6(8.2)
	6	2	13.16(.09)	30.25(.29)	54.00(.69)	96.84(1.9)	226.5(8.0)
	2	4	4.81(.02)	8.99(.05)	12.84(.08)	17.69(.13)	26.29(.25)
	3	4	4.94(.02)	9.01(.04)	12.75(.07)	17.21(.14)	24.90(.36)
	4	4	5.04(.02)	9.12(.04)	12.69(.09)	16.84(.14)	23.56(.23)
	6	4	5.17(.03)	9.14(.05)	12.61(.09)	16.53(.16)	22.76(.18)
	2	6	3.92(.02)	6.97(.03)	9.55(.06)	12.44(.11)	16.93(.12)
	3	6	3.99(.02)	6.95(.04)	9.37(.06)	12.11(.06)	15.91(.16)
	4	6	4.03(.02)	6.93(.03)	9.31(.05)	11.81(.06)	15.46(.15)
	6	6	4.06(.02)	6.93(.03)	9.23(.05)	11.63(.08)	15.21(.14)

Exchange I and J to obtain the percentage points of $\tilde{F}_2^1$.

Table 4. (Continued)

I	J	df	25%	10%	5%	2.5%	1%
4	2	2	16.15(.07)	38.52(.40)	71.56(.95)	133.9(1.7)	324.0(7.2)
	3	2	17.93(.11)	41.12(.35)	75.20(.89)	139.9(2.4)	326.5(8.0)
	4	2	19.22(.11)	42.91(.46)	77.27(1.1)	142.4(2.0)	328.0(8.1)
	6	2	20.96(.14)	45.39(.41)	79.38(.97)	143.7(2.2)	332.5(10)
	2	4	7.30(.03)	12.59(.08)	17.57(.12)	23.73(.16)	34.86(.51)
	3	4	7.53(.03)	12.71(.05)	17.19(.12)	22.56(.20)	31.37(.33)
	4	4	7.65(.02)	12.63(.06)	16.93(.11)	22.06(.16)	30.55(.32)
	6	4	7.77(.04)	12.62(.07)	16.68(.08)	21.26(.11)	28.57(.22)
	2	6	5.89(.02)	9.54(.04)	12.64(.06)	15.99(.13)	20.98(.23)
	3	6	6.01(.03)	9.59(.04)	12.44(.05)	15.54(.11)	19.94(.16)
	4	6	6.02(.03)	9.51(.05)	12.22(.06)	15.15(.09)	19.31(.17)
	6	6	6.05(.02)	9.45(.03)	12.11(.06)	14.91(.09)	18.59(.15)
6	2	2	29.29(.16)	65.87(.80)	120.6(1.8)	223.4(5.0)	524.0(18)
	3	2	32.15(.15)	70.32(.54)	125.3(1.5)	232.4(5.4)	528.6(17)
	4	2	34.22(.19)	71.76(.60)	127.7(1.2)	233.1(3.1)	541.8(17)
	6	2	37.14(.23)	76.94(.60)	131.9(1.8)	234.2(5.8)	551.0(21)
	2	4	12.26(.04)	19.51(.09)	26.03(.11)	33.98(.31)	48.28(.79)
	3	4	12.46(.04)	19.32(.08)	25.35(.14)	32.51(.22)	44.86(.41)
	4	4	12.64(.05)	19.23(.07)	24.78(.14)	31.19(.23)	42.11(.46)
	6	4	12.74(.05)	18.97(.08)	24.14(.13)	29.73(.23)	38.74(.25)
	2	6	9.71(.03)	14.45(.04)	18.26(.09)	22.31(.14)	28.38(.26)
	3	6	9.75(.03)	14.27(.05)	17.80(.07)	21.55(.11)	27.02(.20)
	4	6	9.81(.03)	14.26(.05)	17.62(.08)	21.18(.14)	26.08(.18)
	6	6	9.85(.02)	14.18(.05)	17.33(.06)	20.45(.10)	24.90(.28)

Exchange I and J to obtain the percentage points of $\tilde{F}_2^1$.

Bishop and Dudewicz (1978), one can easily show that the limiting null distributions of $\tilde{F}_1^1, \tilde{F}_2^1$ and $\tilde{F}_3^1$ are chi-square with degrees of freedom $I-1, J-1$ and $(I-1)*(J-1)$, respectively. Furthermore, when the sample size are all equal to n, Bishop and Dudewicz also suggested that the distribution of $\tilde{F}_1^1$ can be better approximated by $(n-2)/(n-4)\cdot\chi^2_{I-1}$ if $n \geq 5$, and similar adjustments can be made for $\tilde{F}_2^1$ and $\tilde{F}_3^1$. The approximation is fairly good when n is larger than 8 and α is 5% or higher according to our simulation results. Therefore, for large n, the chi-square approximation can be used to find the critical values. In this article we only provide critical values for small sample size. The critical points of $\tilde{F}_1^1$ and $\tilde{F}_2^1$ in (17) for sample size $n = 4, 6, 8$ and $\alpha = .25, .10, .05, .025$ and .01 are given in Table 4 by Monte Carlo simulation for the combinations of $I, J = 2, 3, 4, 6$. The simulation process is the same as that described in Section 3. Similarly, the critical points of $\tilde{F}_3^1$ in (17) are given in Table 5 for $I, J = 2, 3, 4, 6$, $n = 4, 6, 8$ and $\alpha = .25, .10, .05, .025$ and .01. An extended table of these critical values is available in Chen and Chen (1996).

7. Discussion and Conclusion

The assumption of equal variances in ANOVA is often in doubt and the single-stage procedure eliminates the necessity of such an assumption. The single-stage procedure in one-way layout provides an exact distribution which is free of the unknown variances, and no additional samples are required. Extension to two-way layout is outlined and the single-stage procedure is applied to each cell and the test statistics are generated. Tables of percentage points for small sample sizes are given. The limiting distribution of the test statistic provides a very good approximation to critical point, and it does

Table 5. The Average of 16 Upper α-Percentage Points of $\tilde{F}_3^1$ and Their Standard Deviations in Parentheses

I	J	df	25%	10%	5%	2.5%	1%
2	2	2	4.35(.05)	12.45(.15)	24.50(.47)	46.14(.86)	108.4(2.1)
2	3	2	10.08(.05)	25.45(.18)	48.28(.49)	91.59(1.6)	213.8(4.5)
2	4	2	16.16(.06)	38.80(.22)	71.82(.69)	135.8(1.6)	314.8(8.3)
2	6	2	29.32(.15)	66.15(.56)	120.6(1.1)	221.6(5.1)	530.0(13)
3	3	2	22.37(.14)	51.44(.61)	94.41(1.2)	180.3(2.0)	407.5(4.2)
3	4	2	35.15(.24)	78.42(.72)	143.8(1.3)	272.4(5.0)	658.8(18)
3	6	2	63.02(.35)	135.4(1.2)	244.7(3.1)	452.8(10)	1080(42)
4	4	2	55.58(.21)	121.2(1.1)	218.1(3.3)	413.1(5.5)	937.1(15)
4	6	2	99.12(.49)	205.2(1.4)	367.5(3.9)	674.9(11)	1636(45)
6	6	2	176.4(.58)	357.4(4.0)	620.6(8.6)	1139(22)	2702(74)
2	2	4	2.25(.02)	5.03(.04)	7.73(.06)	11.15(.09)	17.18(.23)
2	3	4	4.83(.02)	9.00(.06)	12.91(.10)	17.81(.14)	26.48(.30)
2	4	4	7.30(.03)	12.60(.06)	17.47(.10)	23.50(.15)	34.46(.36)
2	6	4	12.24(.03)	19.49(.07)	26.06(.12)	34.30(.27)	48.44(.48)
3	3	4	9.73(.02)	16.01(.07)	21.74(.16)	28.96(.26)	41.98(.64)
3	4	4	14.50(.04)	22.76(.10)	30.15(.15)	39.30(.25)	55.66(.37)
3	6	4	23.94(.10)	35.22(.20)	45.09(.26)	57.79(.46)	80.19(.86)
4	4	4	21.54(.08)	32.23(.14)	41.80(.15)	53.58(.47)	76.79(1.2)
4	6	4	35.47(.07)	49.68(.13)	62.39(.24)	78.25(.42)	107.8(1.0)
6	6	4	58.05(.05)	77.54(.22)	95.73(.43)	116.0(.55)	152.4(.74)
2	2	6	1.85(.01)	3.99(.02)	5.88(.04)	8.01(.09)	11.35(.08)
2	3	6	3.93(.01)	6.96(.03)	9.51(.04)	12.40(.09)	16.69(.14)
2	4	6	5.91(.02)	9.63(.03)	12.69(.05)	16.19(.09)	21.24(.16)
2	6	6	9.72(.03)	14.46(.03)	18.29(.05)	22.47(.11)	28.52(.18)
3	3	6	7.79(.02)	12.04(.06)	15.65(.08)	19.55(.18)	25.75(.16)
3	4	6	11.52(.03)	16.84(.04)	21.08(.07)	25.81(.16)	32.76(.25)
3	6	6	18.70(.04)	25.60(.08)	31.16(.12)	36.87(.20)	45.62(.32)
4	4	6	16.87(.04)	23.34(.06)	28.44(.08)	33.91(.12)	42.28(.18)
4	6	6	27.41(.05)	35.87(.06)	42.29(.13)	49.04(.14)	59.29(.28)
6	6	6	44.42(.09)	55.44(.09)	63.74(.22)	72.73(.35)	85.30(.53)

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