Exact Critical Values for Use with the Analysis of Means

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Exact critical values are given for the analysis of means of equal sample sizes for significance levels 0.10, 0.05, 0.01 and 0.001; number of means k equal to 3(1)20; and degrees of freedom equal to k(1)20, 24, 30, 40, 60, 120 and ∞. An approximate analysis for means of unequal sample sizes is discussed and exemplified. Approximations for the critical values are given and their derivation is discussed.

Introduction

THE analysis of means (ANOM) technique was introduced by Ott (1967) as a method of comparing k independent means based on the multiple significance test given by Halperin et al. (1955). Using Bonferroni inequalities they computed upper and lower bounds on the five percent and one percent critical values and conjectured that the true values were closer to the lower bound. In his first journal paper on this topic (reprinted in this issue) Ott (1967) used these upper bounds and denoted them by H_α. Later in his book, Ott (1975) used the average of the upper and lower bounds (for $\alpha = 0.05$ and 0.01) based on the Halperin et al. (1955) conjecture and again denoted the values by H_α. In order to avoid confusion, we will use H_α as in Ott (1967) to denote the upper bounds from Halperin et al. (1955).

Schilling (1973) extended the analysis of means technique to what he called analysis of means for treatment effects (ANOME) where the k means being compared were not necessarily independent. His critical values were again upper bounds which he denoted by h_α and are related by H_α by

$$h_\alpha = \sqrt{k/(k-1)}\, H_\alpha$$

Extensive tables of h_α for $\alpha = 0.10$, 0.05, 0.01, and 0.001 are given in L. S. Nelson (1974).

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When ANOM is applied to k means based on equal sample sizes, their deviations from the grand mean are all equicorrelated with correlation $-1/(k-1)$. This fact was used by P. R. Nelson (1982) to compute exact critical values for levels of significance $\alpha = 0.1$, 0.05, and 0.01. The present paper gives exact critical values for ANOM for significance levels of 0.10, 0.05, 0.01, and 0.001; numbers of means $k = 3(1)20$; and degrees of freedom equal to k(1)20, 24, 30, 60, 120 and ∞. Approximation functions are given that reproduce these critical values with an error of less than one in the third significant figure. ANOM for unequal sample sizes is also discussed.

The Tables

Values for significance levels of 0.10, 0.05 and 0.01 (Tables 1, 2 and 3) were taken from P. R. Nelson (1982). As a result of applying the approximations described later, a few table values were found that had been typed incorrectly. These were corrected and all values were rounded to three significant figures. It has been determined that only three significant figures can be guaranteed. Exact values for the 0.001 significance level (Table 4) were calculated using the algorithm described by P. R. Nelson (1982) and are given to three significant figures.

If we call the tabled values $h_{\alpha,k,\nu}$, then the ANOM is carried out by calculating decision lines and declaring significance whenever a mean lies outside these lines. The decision lines are given by

$$\bar{\bar{X}} \pm s h_{\alpha,k,\nu} \sqrt{(k-1)/(kn)} \qquad (1)$$

TABLE 1. Exact Critical Values $h_{0.10}$ for the Analysis of Means

SIGNIFICANCE LEVEL = 0.10

NUMBER OF MEANS, K

DF	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	DF
3	3.16																		3
4	2.81	3.10																	4
5	2.63	2.88	3.05																5
6	2.52	2.74	2.91	3.03															6
7	2.44	2.65	2.81	2.92	3.02														7
8	2.39	2.59	2.73	2.85	2.94	3.02													8
9	2.34	2.54	2.68	2.79	2.88	2.95	3.01												9
10	2.31	2.50	2.64	2.74	2.83	2.90	2.96	3.02											10
11	2.29	2.47	2.60	2.70	2.79	2.86	2.92	2.97	3.02										11
12	2.27	2.45	2.57	2.67	2.75	2.82	2.88	2.93	2.98	3.02									12
13	2.25	2.43	2.55	2.65	2.73	2.79	2.85	2.90	2.95	2.99	3.03								13
14	2.23	2.41	2.53	2.63	2.70	2.77	2.83	2.88	2.92	2.96	3.00	3.03							14
15	2.22	2.39	2.51	2.61	2.68	2.75	2.80	2.85	2.90	2.94	2.97	3.01	3.04						15
16	2.21	2.38	2.50	2.59	2.67	2.73	2.79	2.83	2.88	2.92	2.95	2.99	3.02	3.05					16
17	2.20	2.37	2.49	2.58	2.65	2.72	2.77	2.82	2.86	2.90	2.93	2.97	3.00	3.03	3.05				17
18	2.19	2.36	2.47	2.56	2.64	2.70	2.75	2.80	2.84	2.88	2.92	2.95	2.98	3.01	3.03	3.06			18
19	2.18	2.35	2.46	2.55	2.63	2.69	2.74	2.79	2.83	2.87	2.90	2.94	2.96	2.99	3.02	3.04	3.06		19
20	2.18	2.34	2.45	2.54	2.62	2.68	2.73	2.78	2.82	2.86	2.89	2.92	2.95	2.98	3.00	3.03	3.05	3.07	20
24	2.15	2.32	2.43	2.51	2.58	2.64	2.69	2.74	2.78	2.82	2.85	2.88	2.91	2.93	2.96	2.98	3.00	3.02	24
30	2.13	2.29	2.40	2.48	2.55	2.61	2.66	2.70	2.74	2.77	2.81	2.84	2.86	2.89	2.91	2.93	2.96	2.98	30
40	2.11	2.27	2.37	2.45	2.52	2.57	2.62	2.66	2.70	2.73	2.77	2.79	2.82	2.85	2.87	2.89	2.91	2.93	40
60	2.09	2.24	2.34	2.42	2.49	2.54	2.59	2.63	2.66	2.70	2.73	2.75	2.78	2.80	2.82	2.84	2.86	2.88	60
120	2.07	2.22	2.32	2.39	2.45	2.51	2.55	2.59	2.62	2.66	2.69	2.71	2.74	2.76	2.78	2.80	2.82	2.84	120
INF	2.05	2.19	2.29	2.36	2.42	2.47	2.52	2.55	2.59	2.62	2.65	2.67	2.69	2.72	2.74	2.76	2.77	2.79	INF

TABLE 2. Exact Critical Values $h_{0.05}$ for the Analysis of Means

SIGNIFICANCE LEVEL = 0.05

NUMBER OF MEANS, K

DF	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	DF
3	4.18																		3
4	3.56	3.89																	4
5	3.25	3.53	3.72																5
6	3.07	3.31	3.49	3.62															6
7	2.94	3.17	3.33	3.45	3.56														7
8	2.86	3.07	3.21	3.33	3.43	3.51													8
9	2.79	2.99	3.13	3.24	3.33	3.41	3.48												9
10	2.74	2.93	3.07	3.17	3.26	3.33	3.40	3.45											10
11	2.70	2.88	3.01	3.12	3.20	3.27	3.33	3.39	3.44										11
12	2.67	2.85	2.97	3.07	3.15	3.22	3.28	3.33	3.38	3.42									12
13	2.64	2.81	2.94	3.03	3.11	3.18	3.24	3.29	3.34	3.38	3.42								13
14	2.62	2.79	2.91	3.00	3.08	3.14	3.20	3.25	3.30	3.34	3.37	3.41							14
15	2.60	2.76	2.88	2.97	3.05	3.11	3.17	3.22	3.26	3.30	3.34	3.37	3.40						15
16	2.58	2.74	2.86	2.95	3.02	3.09	3.14	3.19	3.23	3.27	3.31	3.34	3.37	3.40					16
17	2.57	2.73	2.84	2.93	3.00	3.06	3.12	3.16	3.21	3.25	3.28	3.31	3.34	3.37	3.40				17
18	2.55	2.71	2.82	2.91	2.98	3.04	3.10	3.14	3.18	3.22	3.26	3.29	3.32	3.35	3.37	3.40			18
19	2.54	2.70	2.81	2.89	2.96	3.02	3.08	3.12	3.16	3.20	3.24	3.27	3.30	3.32	3.35	3.37	3.40		19
20	2.53	2.68	2.79	2.88	2.95	3.01	3.06	3.11	3.15	3.18	3.22	3.25	3.28	3.30	3.33	3.35	3.37	3.40	20
24	2.50	2.65	2.75	2.83	2.90	2.96	3.01	3.05	3.09	3.13	3.16	3.19	3.22	3.24	3.27	3.29	3.31	3.33	24
30	2.47	2.61	2.71	2.79	2.85	2.91	2.96	3.00	3.04	3.07	3.10	3.13	3.16	3.18	3.20	3.22	3.25	3.27	30
40	2.43	2.57	2.67	2.75	2.81	2.86	2.91	2.95	2.98	3.01	3.04	3.07	3.10	3.12	3.14	3.16	3.18	3.20	40
60	2.40	2.54	2.63	2.70	2.76	2.81	2.86	2.90	2.93	2.96	2.99	3.02	3.04	3.06	3.08	3.10	3.12	3.14	60
120	2.37	2.50	2.59	2.66	2.72	2.77	2.81	2.84	2.88	2.91	2.93	2.96	2.98	3.00	3.02	3.04	3.06	3.08	120
INF	2.34	2.47	2.56	2.62	2.68	2.72	2.76	2.80	2.83	2.86	2.88	2.90	2.93	2.95	2.97	2.98	3.00	3.02	INF

where

s = estimate of standard deviation of an individual observation with ν degrees of freedom (calculated using $n - 1$ in the denominator)

α = significance level

k = number of means each based on n measurements.

Equation (1) requires the use of s. Ott (1967) used an estimate of the population standard deviation based on the range. In particular, he used the d_2^*

TABLE 3. Exact Critical Values $h_{0.01}$ for the Analysis of Means

SIGNIFICANCE LEVEL = 0.01

NUMBER OF MEANS, K

DF	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	DF
3	7.51																		3
4	5.74	6.21																	4
5	4.93	5.29	5.55																5
6	4.48	4.77	4.98	5.16															6
7	4.18	4.44	4.63	4.78	4.90														7
8	3.98	4.21	4.38	4.52	4.63	4.72													8
9	3.84	4.05	4.20	4.33	4.43	4.51	4.59												9
10	3.73	3.92	4.07	4.18	4.28	4.36	4.43	4.49											10
11	3.64	3.82	3.96	4.07	4.16	4.23	4.30	4.36	4.41										11
12	3.57	3.74	3.87	3.98	4.06	4.13	4.20	4.25	4.31	4.35									12
13	3.51	3.68	3.80	3.90	3.98	4.05	4.11	4.17	4.22	4.26	4.30								13
14	3.46	3.63	3.74	3.84	3.92	3.98	4.04	4.09	4.14	4.18	4.22	4.26							14
15	3.42	3.58	3.69	3.79	3.86	3.92	3.98	4.03	4.08	4.12	4.16	4.19	4.22						15
16	3.38	3.54	3.65	3.74	3.81	3.87	3.93	3.98	4.02	4.06	4.10	4.14	4.17	4.20					16
17	3.35	3.50	3.61	3.70	3.77	3.83	3.89	3.93	3.98	4.02	4.05	4.09	4.12	4.14	4.17				17
18	3.33	3.47	3.58	3.66	3.73	3.79	3.85	3.89	3.94	3.97	4.01	4.04	4.07	4.10	4.12	4.15			18
19	3.30	3.45	3.55	3.63	3.70	3.76	3.81	3.86	3.90	3.94	3.97	4.00	4.03	4.06	4.08	4.11	4.13		19
20	3.28	3.42	3.53	3.61	3.67	3.73	3.78	3.83	3.87	3.90	3.94	3.97	4.00	4.02	4.05	4.07	4.09	4.12	20
24	3.21	3.35	3.45	3.52	3.58	3.64	3.69	3.73	3.77	3.80	3.83	3.86	3.89	3.91	3.94	3.96	3.98	4.00	24
30	3.15	3.28	3.37	3.44	3.50	3.55	3.59	3.63	3.67	3.70	3.73	3.76	3.78	3.81	3.83	3.85	3.87	3.89	30
40	3.09	3.21	3.29	3.36	3.42	3.46	3.50	3.54	3.58	3.60	3.63	3.66	3.68	3.70	3.72	3.74	3.76	3.78	40
60	3.03	3.14	3.22	3.29	3.34	3.38	3.42	3.46	3.49	3.51	3.54	3.56	3.59	3.61	3.63	3.64	3.66	3.68	60
120	2.97	3.07	3.15	3.21	3.26	3.30	3.34	3.37	3.40	3.42	3.45	3.47	3.49	3.51	3.53	3.55	3.56	3.58	120
INF	2.91	3.01	3.08	3.14	3.18	3.22	3.26	3.29	3.32	3.34	3.36	3.38	3.40	3.42	3.44	3.45	3.47	3.48	INF

TABLE 4. Exact Critical Values $h_{0.001}$ for the Analysis of Means

SIGNIFICANCE LEVEL = 0.001

NUMBER OF MEANS, K

DF	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	DF
3	16.4																		3
4	10.6	11.4																	4
5	8.25	8.79	9.19																5
6	7.04	7.45	7.76	8.00															6
7	6.31	6.65	6.89	7.09	7.25														7
8	5.83	6.12	6.32	6.49	6.63	6.75													8
9	5.49	5.74	5.92	6.07	6.20	6.30	6.40												9
10	5.24	5.46	5.63	5.76	5.87	5.97	6.05	6.13											10
11	5.05	5.25	5.40	5.52	5.63	5.71	5.79	5.86	5.92										11
12	4.89	5.08	5.22	5.33	5.43	5.51	5.58	5.65	5.71	5.76									12
13	4.77	4.95	5.08	5.18	5.27	5.35	5.42	5.48	5.53	5.58	5.63								13
14	4.66	4.83	4.96	5.06	5.14	5.21	5.28	5.33	5.38	5.43	5.48	5.51							14
15	4.57	4.74	4.86	4.95	5.03	5.10	5.16	5.21	5.26	5.31	5.35	5.39	5.42						15
16	4.50	4.66	4.77	4.86	4.94	5.00	5.06	5.11	5.16	5.20	5.24	5.28	5.31	5.34					16
17	4.44	4.59	4.70	4.78	4.86	4.92	4.98	5.03	5.07	5.11	5.15	5.18	5.22	5.25	5.28				17
18	4.38	4.53	4.63	4.72	4.79	4.85	4.90	4.95	4.99	5.03	5.07	5.10	5.14	5.16	5.19	5.22			18
19	4.33	4.47	4.58	4.66	4.73	4.79	4.84	4.88	4.93	4.96	5.00	5.03	5.06	5.09	5.12	5.14	5.17		19
20	4.29	4.42	4.53	4.61	4.67	4.73	4.78	4.83	4.87	4.90	4.94	4.97	5.00	5.03	5.05	5.08	5.10	5.12	20
24	4.16	4.28	4.37	4.45	4.51	4.56	4.61	4.65	4.69	4.72	4.75	4.78	4.81	4.83	4.86	4.88	4.90	4.92	24
30	4.03	4.14	4.23	4.30	4.35	4.40	4.44	4.48	4.51	4.54	4.57	4.60	4.62	4.64	4.67	4.69	4.71	4.72	30
40	3.91	4.01	4.09	4.15	4.20	4.25	4.29	4.32	4.35	4.38	4.40	4.43	4.45	4.47	4.49	4.50	4.52	4.54	40
60	3.80	3.89	3.96	4.02	4.06	4.10	4.14	4.17	4.19	4.22	4.24	4.27	4.29	4.30	4.32	4.33	4.35	4.37	60
120	3.69	3.77	3.84	3.89	3.93	3.96	4.00	4.03	4.05	4.07	4.09	4.11	4.13	4.15	4.16	4.17	4.19	4.21	120
INF	3.58	3.66	3.72	3.76	3.80	3.84	3.87	3.89	3.91	3.93	3.95	3.97	3.99	4.00	4.02	4.03	4.04	4.06	INF

factor. Schilling (1973) likewise used the range and d_2^* to estimate the standard deviation in his explication of the method. (A table of d_2^* values can be found in L. S. Nelson (1975a)). It is important to realize that this procedure gives an approximate (although usually quite satisfactory) result. The exact procedure, for which $h_{\alpha,k,\nu}$ values are now available, calls for the use of s.

Approximations to the Critical Values

The values in each of the four tables can be approximated by

$$h_{\alpha,k,\nu} = B_1 + B_2K_1^{B_3} + (B_4 + B_5K_1)V_1 + (B_6 + B_7K_2 + B_8K_2^2)V_1^2 + \epsilon \quad (2)$$

where

k = number of means

$K_1 = \ln(k)$

$K_2 = \ln(k-2)$

ν = degrees of freedom for error

$V_1 = 1/(\nu - 1)$

and the appropriate B_i values are given for each significance level in Table 5. The maximum of the absolute differences between the approximate and the true values for all tables is less than one in the third significant digit. A discussion of how the approximations were obtained is given in the Appendix.

TABLE 5. Constants for Use in Approximating h_α

	α			
	0.1	0.05	0.01	0.001
B_1	1.2092	1.7011	2.3539	3.1981
B_2	0.7992	0.6047	0.5176	0.3619
B_3	0.6238	0.7102	0.7107	0.7886
B_4	0.4797	1.4605	4.3161	8.3489
B_5	1.6819	1.9102	2.3629	3.1003
B_6	-0.2155	0.2250	4.6400	27.7005
B_7	0.4529	0.6300	1.8640	5.1277
B_8	-0.6095	-0.2202	0.3204	0.7271

ANOM with Unequal Sample Sizes

For sets of means each of which is based on the same sample size, equation (1) together with the tabled values provided here will give exact results. When the means are based on unequal sample sizes their deviations from the grand mean are no longer equicorrelated and decision limits, equation (3) should be computed using critical values that are upper bounds on the true but not available values of h. The decision limits in this case are

$$\bar{\bar{X}} \pm sh^*_{\alpha,k,\nu}\sqrt{(N-n_i)/(Nn_i)} \quad (3)$$

where

N = total number of observations

n_i = number of observations in the ith mean.

Following Sidak (1967), the necessary values of h^* can be calculated as the upper $\alpha^*/2$ percentage points of a t distribution, where

$$\alpha^* = 1 - (1-\alpha)^{1/k} \quad (4)$$

α = desired significance level

k = number of means.

The upper bounds obtained by the use of equation (4) are slightly less than the factors given by L. S. Nelson (1974). Remember that in the use of equation (3) it is necessary to calculate as many pairs of decision lines as there are different sample sizes.

To illustrate the use of equation (4), suppose that four means, not all based on the same sample size, are to be judged using the 0.05 significance level with 10 degrees of freedom available for error. Since the sample size is not constant, the value of $h^*_{0.05,4,10}$ is required. Applying equation (4) gives $\alpha^* = 1 - (1-0.05)^{1/4} = 0.0127$. The required value of $h^*_{0.05,4,10}$ is the value of Student's t for 10 degrees of freedom and a significance level of 0.0127 (two-sided). This corresponds to a one-sided level of 0.00635 or a percentile of 0.99365. Interpolation in the usual t table is necessary and can be done linearly with satisfactory results using t values and the logarithms of the required significance level and the bracketing significance levels. Thus

$\alpha^*/2$	$-\log(\alpha^*/2)$	t
0.005	2.30	3.17
0.00635	2.20	x
0.01	2.00	2.76

$$x = 2.76 + ((2.20-2.00)/(2.30-2.00))(3.17-2.76) = 3.03.$$

This is the required value of $h_{0.05,4,10}$. The corresponding value from the table of slightly more conservative upper bounds given in L. S. Nelson (1974) is 3.04. The upper bounds tabled by Nelson (1974) were based on $\alpha^* = \alpha/k$ instead of equation (4). The differences are smaller the smaller α^* is and the larger k is. For $h^*_{0.001}$ there is no difference out to the second decimal place. A nomograph of Student's t published by L. S. Nelson (1975b) is particularly well suited to finding values of t corresponding to $\alpha^*/2$ from equation (4). It covers single tail areas from 0.10 to 0.0001.

Equation (4) would also be used along with a table or nomograph of Student's t to obtain values of h^* for combinations of k and degrees of freedom outside the range of Tables 1-4, in particular for degrees of freedom less than the number of means.

Appendix

The development of the form of equation (2) and the assignment of values to the B's (given in Table 5) is outlined here.

1. Since Student's t values are roughly linear when plotted against the reciprocals of the degrees of freedom, it was thought that the same might be true for ANOM values. Plots of h vs. $1/\nu$ for various numbers of means at the two extremes of significance levels (0.10 and 0.001) showed slight curvature. It seemed reasonable to add a quadratic term to improve the fit. Thus the model went from

$$h = b_1 + b_2(1/\nu) + \epsilon \qquad (1A)$$

$$\text{to } h = b_1 + b_2(1/\nu) + b_3(1/\nu)^2 + \epsilon \qquad (2A)$$

where ϵ represents the error in the approximation.

2. It was hoped that further improvement could be made by adjusting the degrees of freedom and trying exponents other than two in the third term. Some experimentation showed that reducing the degrees of freedom by one improved the fit, but two turned out to be the best exponent for the third term. Then we had

$$h = b_1 + b_2\left(\frac{1}{\nu - 1}\right) + b_3\left(\frac{1}{\nu - 1}\right)^2 + \epsilon. \qquad (3A)$$

3. A best fit at $\nu = \infty$ requires that b_1 be equal to $h_{\alpha,k,\infty}$ because the second and third terms both become zero at that point.

4. Using model (3A) we could approximate the table values for each significance level with 18 different equations, one for each value of k. We then tried to condense these by expressing the coefficients b_1, b_2 and b_3 as functions of k. The first step in doing this was to plot the values of each coefficient as a function of k. A study of each plot followed by trials of various functions (with some luck) led to a satisfactory fit. The following sub-models were used.

$$b_1 = h_{\alpha,k,\infty} = c_1 + c_2(\ln(k))^{c_3} \qquad (4A)$$

$$b_2 = c_4 + c_5\ln(k) \qquad (5A)$$

$$b_3 = c_6 + c_7\ln(k - 2) + c_8(\ln(k - 2))^2 \qquad (6A)$$

5. Equations (4A)-(6A) were substituted into equation (3A) and the fitted values c_1 through c_8 were used as starting values in a search program designed to locate new values of c_1 through c_8 that minimized the sum of squares of the differences between the true critical values and those given by the approximation. There were four separate problems, one for each significance level.

6. The values of the eight coefficients found in Step 5 were then used as starting values with the objective function to be minimized changed from the sum of squares to the maximum absolute deviation. (When approximating table values, the most reasonable criterion of fit measures how far off the approximation is in the worst case. This is the maximum deviation taken without regard to sign.) The reason for this sequence of fitting is that experience has shown that the response surface associated with the maximum absolute deviation tends to be filled with local minima, and it is necessary to start the search as close as possible to the global minimum to avoid suboptimum solutions. Values associated with the minimum residual sum of squares have always provided satisfactory starting values.

The search routine used throughout this work was the Nelder-Mead simplex procedure as described in Olsson (1974). It is highly recommended as a general purpose minimization procedure, especially useful in curve fitting.

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