COMPUTATIONAL FINANCE & RISK MANAGEMENT

UNIVERSITY *of* WASHINGTON

Department of Applied Mathematics

CRFM 541: Investment Science

2. Including a Risk-Free Asset

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Outline

Including a Risk-Free Asset

- Possible to borrow and lend at the risk-free rate r_f
- Returns on risk-free asset assumed to be certain $\implies \sigma^2_{r_f} = 0$
- Assume short-selling allowed

Intuition

- Invest a fraction α of the value of the portfolio in risky assets
  - Recall: the feasible set of portfolios of all risky assets is bounded by a hyperbola
  - Let r_A be the random rate of return on a portfolio A in the feasible set
- Invest the remaining fraction $(1-\alpha)$ in the risk-free asset
- Portfolio return

$$r_P = (1-\alpha) r_f + \alpha r_A$$

- Portfolio expected return and variance

$$\mu_P = (1-\alpha) r_f + \alpha \mu_A \qquad \sigma_P^2 = \alpha^2 \sigma_A^2$$

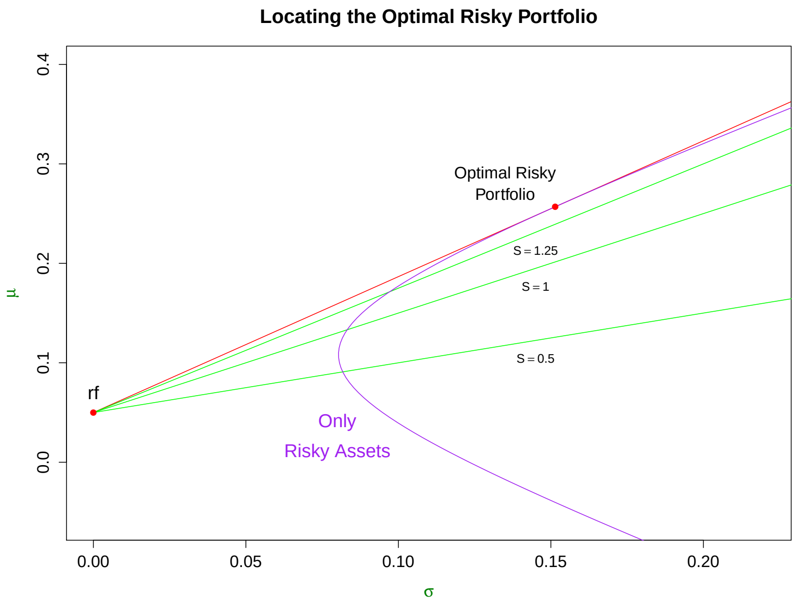
- Portfolio expected return in slope/y-intercept form

$$\mu_P = r_f + \left(\frac{\mu_A - r_f}{\sigma_A} \right) \sigma_P$$

Intuition

- Linear relationship between the expected portfolio return and risk
- Many risky portfolios A to choose from
  - Pick one that maximizes μ_P for a given σ_P
  - Equivalent to finding the risky portfolio A that maximizes

$$S = \frac{\mu_A - r_f}{\sigma_A}$$


Locating the Optimal Risky Portfolio
Optimal Risky
Portfolio
S = 1.25
S = 1
S = 0.5
rf
Only
Risky Assets
0.0
0.1
0.2
0.3
0.4
0.00
0.05
0.10
0.15
0.20
μ
σ

Try #1: Minimum Variance Portfolio

- Motivation: the risk-free asset is still an asset
- For an all risky asset portfolio, if

$$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix} \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \dots & \sigma_N^2 \end{bmatrix}$$

then $\mathbf{w}_{\text{MVP}} = \dfrac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^{\intercal}\boldsymbol{\Sigma}^{-1}\mathbf{1}}$

- What happens when the risk-free asset is included
  - $\mathbb{E}(r_f) = r_f$
  - $\text{Var}(r_f) = 0$
  - $\text{Cov}(r_f, r_i) = 0 \quad i = 1, \dots, N$

Try #1: Minimum Variance Portfolio

- Include the risk-free asset at index 0

$$\boldsymbol{\mu}^\star = \begin{bmatrix} r_f \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix} \qquad \boldsymbol{\Sigma}^\star = \begin{bmatrix} 0 & \mathbf{0}^\intercal \\ \mathbf{0} & \boldsymbol{\Sigma} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_1^2 & \sigma_{12} & \dots & \sigma_{1N} \\ 0 & \sigma_{21} & \sigma_2^2 & \dots & \sigma_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \sigma_{N1} & \sigma_{N2} & \dots & \sigma_N^2 \end{bmatrix}$$

- Intuition: can compute $\mathbf{w}^\star_{\text{MVP}}$ without doing any math

$$\mathbf{w}^\star_{\text{MVP}} = \frac{\left(\boldsymbol{\Sigma}^\star\right)^{-1}\mathbf{1}}{\mathbf{1}^\intercal\left(\boldsymbol{\Sigma}^\star\right)^{-1}\mathbf{1}}$$

- Lets give it a try

Try #1: Minimum Variance Portfolio

```
> mu <- c(0.10, 0.08, 0.06, 0.2)
> N <- length(mu)
> h <- rep(1, N)
> Rho <- rbind(c(1.0, 0.70, 0.30, 0.5),
+              c(0.7, 1.00, 0.55, 0.1),
+              c(0.3, 0.55, 1.00, 0.4),
+              c(0.5, 0.10, 0.40, 1.0))
> vol <- c(0.1, 0.09, 0.12, 0.15)
> Sigma <- Rho * (vol %o% vol)

> Sigma.star <- rbind(0, cbind(0, Sigma))
> h1 <- rep(1, N + 1)

> Sigma.star.inv <- solve(Sigma.star)

Error in solve.default(Sigma.star) :
  Lapack routine dgesv:  system is exactly singular...

> w.star <- Sigma.star.inv %*% h1 / (sum(Sigma.star.inv))
```

Try #1: Minimum Variance Portfolio

```
> e <- eigen(Sigma.star, symmetric = TRUE, only.values = TRUE)
> print(e$values, digits = 4)

[1] 0.0318618 0.0135526 0.0086739 0.0009118 0.0000000

> if(any(e$values <= 0))
+   stop("covariance matrix is not positive definite")

Error:  covariance matrix is not positive definite
```

The Spectral Theorem

Try #2: Minimum Variance Portfolio

- Include the risk-free asset at index 0
- Let w_0 be the fraction of wealth invested in the risk-free asset
- Let $\sum_{i=1}^{N} = \mathbf{w}^\intercal \mathbf{1}$ be the fraction of wealth invested in risky assets
- The portfolio expected return becomes

$$\mu_P(\mathbf{w}) = w_0 r_f + \mathbf{w}^\intercal \mu$$

- The portfolio variance becomes

$$\sigma_P^2 = \mathrm{Var}(w_0 r_f + \mathbf{w}^\intercal \mathbf{r}) = \mathbf{w}^\intercal \mathbf{\Sigma} \mathbf{w}$$

Try #2: Minimum Variance Portfolio

- Let $\lambda > 0$ be a *risk aversion* parameter
- The optimization problem is to maximize the quadratic utility

$$\begin{aligned} \max_{\mathbf{w}} : \quad & w_0 r_f + \mathbf{w}^\intercal \boldsymbol{\mu} - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\ \text{subject to} : \quad & w_0 + \mathbf{w}^\intercal \mathbf{1} = 1 \end{aligned}$$

- Define the excess returns vector

$$\boldsymbol{\mu}_e = \boldsymbol{\mu} - r_f \mathbf{1}$$

- Rewrite the objective function in terms of excess returns

$$\begin{aligned} w_0 r_f + \mathbf{w}^\intercal \boldsymbol{\mu} - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} &= w_0 r_f + \mathbf{w}^\intercal (\boldsymbol{\mu}_e + r_f \mathbf{1}) - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\ &= (w_0 + \mathbf{w}^\intercal \mathbf{1}) r_f + \mathbf{w}^\intercal \boldsymbol{\mu}_e - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\ &= r_f + \mathbf{w}^\intercal \boldsymbol{\mu}_e - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \end{aligned}$$

Try #2: Minimum Variance Portfolio

- Since r_f is fixed it can be omitted from the objective function

$$\begin{aligned} \max_{\mathbf{w}} : & \quad \mathbf{w}^\intercal \boldsymbol{\mu}_e - \tfrac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\ \text{subject to} : & \quad w_0 + \mathbf{w}^\intercal \mathbf{1} = 1 \end{aligned}$$

- Solve using Lagrange's method

$$\mathcal{L}(w_0, \mathbf{w}, \gamma) = \mathbf{w}^\intercal \boldsymbol{\mu}_e - \frac{1}{2}\lambda \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} + \gamma(\mathbf{w}^\intercal \mathbf{1} + w_0 - 1)$$

- Solve the following system of equations to find the critical point

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial w_0} &= \gamma \overset{\text{set}}{=} 0 \\ \frac{\partial \mathcal{L}}{\partial \mathbf{w}} &= \boldsymbol{\mu}_e - \lambda \boldsymbol{\Sigma} \mathbf{w} + \gamma \mathbf{1} \overset{\text{set}}{=} \mathbf{0} \\ \frac{\partial \mathcal{L}}{\partial \gamma} &= \mathbf{w}^\intercal \mathbf{1} + w_0 - 1 \overset{\text{set}}{=} 0 \end{aligned}$$

Try #2: Minimum Variance Portfolio

- Since $\gamma = 0$

$$
\begin{aligned}
\lambda \boldsymbol{\Sigma} \mathbf{w} &= \boldsymbol{\mu}_e \\
\mathbf{w} &= \frac{1}{\lambda} \boldsymbol{\Sigma}^{-1} \mu_e
\end{aligned}
$$

- The final equation gives

$$
w_0 = 1 - \mathbf{w}^\intercal \mathbf{1}
$$

- The portfolio expected return is

$$
\begin{aligned}
\boldsymbol{\mu}_{P,e} &= \mu_P(\mathbf{w}) - r_f \\
&= \mathbf{w}^\intercal \boldsymbol{\mu}_e \\
&= \lambda^{-1} \boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e
\end{aligned}
$$

Try #2: Minimum Variance Portfolio

- $\lambda \sim$ risk aversion $\implies \dfrac{1}{\lambda} \sim$ risk *tolerance*
- Compute implied risk tolerance given target portfolio rate of return

$$\frac{1}{\lambda} = \frac{1}{\boldsymbol{\mu}_e^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}\, \mu_{P,e}$$

- Substitute the implied risk aversion into the optimal weights formula

$$\begin{aligned} \mathbf{w} &= \frac{1}{\lambda} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e \\ &= \frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^{\mathsf{T}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}\, \mu_{P,e} \end{aligned}$$

Try #2: Minimum Variance Portfolio

- Optimal portfolio variance given target rate of return $\mu_{P,e}$

$$
\begin{aligned}
\sigma_P^2 &= \mathbf{w}^\intercal \boldsymbol{\Sigma} \mathbf{w} \\
&= \left[\frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \, \mu_{P,e} \right]^\intercal \boldsymbol{\Sigma} \left[\frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \, \mu_{P,e} \right] \\
&= \frac{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \, \boldsymbol{\Sigma} \, \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\left(\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e \right)^2} \, \mu_{P,e}^2 \\
&= \frac{1}{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \, \mu_{P,e}^2
\end{aligned}
$$

Theorem Quadratic utility optimal portfolios that contain cash and risky assets have the following linear return versus volatility relationship

$$
\mu_P = r_f + \sqrt{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \, \sigma_P
$$

Sharpe Ratio

- The *Sharpe ratio* of a portfolio P is the ratio of the portfolio expected excess return to the portfolio standard deviation

$$SR_P = \frac{\mu_{P,e}}{\sigma_P} = \frac{\mu_P - r_f}{\sigma_P}$$

- The Sharpe ratio is constant along the efficient frontier

$$SR_{opt} = \frac{\mu_{opt,e}}{\sigma_{opt}} = \frac{\mu_{opt} - r_f}{\sigma_{opt}} = \sqrt{\boldsymbol{\mu}_e^{\intercal} \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}$$

Question

- Is there a no-cash position on the efficient frontier?
- Requires $\mathbf{1}^\top \mathbf{w} = 1$, it follows that

  a) $w_0 = 0$

  b) $\mathbf{1}^\top \mathbf{w} = \lambda^{-1} \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e = 1$

- Part b) implies that the risk aversion corresponding to an all risky asset portfolio is

$$\lambda = \mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e$$

- Reminder: λ is a risk aversion parameter $\implies \lambda > 0$
- There is an all risky asset portfolio on the efficient frontier when the condition

$$\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e > 0$$

  is satisfied

Interpretation

- In practice, the condition does not always hold
- Consider the expected excess return of the global minimum variance portfolio

$$
\begin{aligned}
\mu_{GMV,e} = \mu_{GMV} - r_f &= \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} - r_f \\
&= \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} - \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}\, r_f \\
&= \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}} (\boldsymbol{\mu} - r_f \mathbf{1}) \\
&= \frac{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}}
\end{aligned}
$$

- There is an all risky asset portfolio on the efficient frontier when the expected excess return of the global minimum variance portfolio is positive

Outline

Finding the Tangent Portfolio

- Possible to borrow and lend at the risk-free rate r_f
- Returns on risk-free asset assumed to be certain $\implies \sigma = 0$
- Assumption: short-selling allowed
- w_0 is the fraction of the endowment allocated to the risk-free asset
- Full-investment constraint

$$1 = w_0 + \mathbf{w}^\top \mathbf{1}$$

- Target rate of return

$$\mu_P = w_0 r_f + \mathbf{w}^\top \boldsymbol{\mu}$$

- Substitute to eliminate w_0

$$\mu_P - r_f = \mathbf{w}^\top (\boldsymbol{\mu} - r_f \mathbf{1})$$

- After solving for $\mathbf{w}$, can recover $w_0 = 1 - \mathbf{w}^\top \mathbf{1}$

Finding the Tangent Portfolio

- Lagrangian

$$\mathcal{L}(\mathbf{w}, \boldsymbol{\lambda}) = \frac{1}{2}\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} - \lambda[\mathbf{w}^\top(\boldsymbol{\mu} - r_f \mathbf{1}) - (\mu_P - r_f)]$$

- Necessary conditions ($N + 1$ equations)

$$\frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \boldsymbol{\Sigma}\mathbf{w} - \lambda(\boldsymbol{\mu} - r_f \mathbf{1}) = \mathbf{0}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{w}^\top(\boldsymbol{\mu} - r_f \mathbf{1}) - (\mu_P - r_f) = 0$$

- Follows that

$$\mathbf{w} = \lambda \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1})$$

Finding the Tangent Portfolio

- Define the *excess return vector*

$$\boldsymbol{\mu}_e = \boldsymbol{\mu} - r_f \mathbf{1}$$

- Solve for λ

$$\lambda = \frac{\mu_P - r_f}{\boldsymbol{\mu}_e^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}$$

- The portfolio weights for the N risky assets are

$$\mathbf{w} = (\mu_P - r_f) \frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}$$

- Portfolio variance

$$\sigma_P^2 = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} = \frac{(\mu_P - r_f)^2}{\boldsymbol{\mu}_e^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}$$

Finding the Tangent Portfolio

- Rewrite the weights vector

$$\mathbf{w} = \frac{(\mu_P - r_f)^2}{(\mu_P - r_f)} \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e} = \sigma_P^2 \frac{1}{(\mu_P - r_f)} \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e$$

Premultiply by $\boldsymbol{\mu}_e^\intercal$

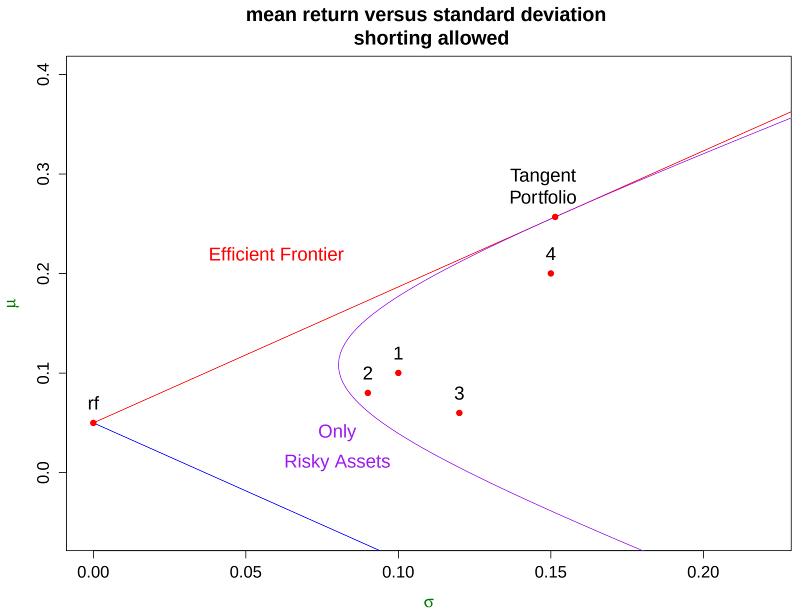
$$\boldsymbol{\mu}_e^\intercal \mathbf{w} = \mu_P - r_f = \frac{\sigma_P^2}{(\mu_P - r_f)} \boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e$$

Linear relationship between the expected portfolio return μ_P and portfolio standard deviation σ_P

$$\mu_P = r_f + \sigma_P \cdot \text{sign}(\mu_P - r_f)\ \sqrt{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}$$

When $\mu_P - r_f > 0$, we obtain the Efficient Frontier

$$\mu_P = r_f + \sigma_P \cdot \sqrt{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e} \qquad \mu_P - r_f > 0$$


mean return versus standard deviation
shorting allowed
Tangent
Portfolio
Efficient Frontier
4
1
2
3
rf
Only
Risky Assets
0.4
0.3
0.2
0.1
0.0
μ
0.00
0.05
0.10
0.15
0.20
σ

Tangent Portfolio

- By construction, the portfolio weights satisfy the full investment and target return constraints (since $w_0 = 1 - \mathbf{w}^\top \mathbf{1}$)
- The portfolio corresponding to $w_0 = 0$ will also be a minimum variance portfolio.

$$w_0 = 0 \quad \Longrightarrow \quad \mathbf{w}_T^\top \mathbf{1} = 1$$

- Tangent Portfolio: the portfolio corresponding to the single point in risk-return space at which the linear Efficient Frontier intersects the universe of portfolios with full allocation in risky assets

Tangent Portfolio Weights

- $\mathbf{w} = (\mu_P - r_f)\dfrac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^{\mathsf{T}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}$
- $\mathbf{1}^{\mathsf{T}}\mathbf{w}_T = 1$

Tangent Portfolio

- The weights vector for the Tangent Portfolio is

$$\mathbf{w}_T = \frac{\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}$$

$$\mu_T \equiv \mu_P\Big|_{w_0=0} = \frac{\boldsymbol{\mu}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e} \qquad \sigma_T^2 = \frac{\boldsymbol{\mu}_e^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e}{\left(\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}_e\right)^2}$$

- The Tangent Portfolio (for $\mu_{GMV} > r_f$) also corresponds to the maximum Sharpe Ratio, subject to $\mathbf{w}^\intercal\mathbf{1} = 1$ and (σ_A, μ_A) in the feasible set of portfolios of all risky assets

$$\max_A \left\{\frac{\mu_A - r_f}{\sigma_A}\right\} = \frac{\mu_T - r_f}{\sigma_T}$$

Tangent Portfolio

- Rewrite the excess return of the tangent portfolio

$$
\begin{aligned}
\mu_T - r_f &= \frac{\boldsymbol{\mu}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} - r_f \\
&= \frac{\boldsymbol{\mu}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} - \frac{r_f \mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \\
&= (\boldsymbol{\mu} - r_f \mathbf{1})^\intercal \frac{\boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} \\
&= \frac{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}
\end{aligned}
$$

- Can the Tangent Portfolio return be less than the risk-free rate?

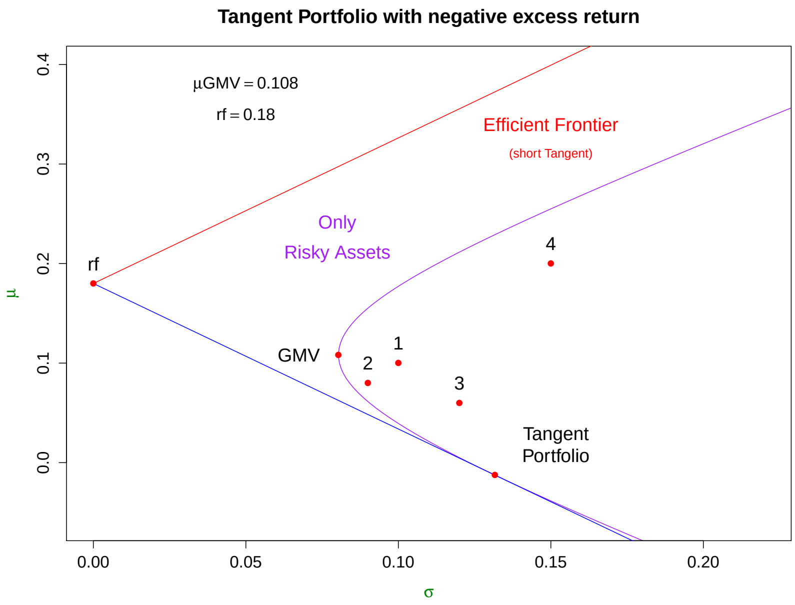
Tangent Portfolio ($r_f > \mu_{GMV}$)

In terms of the expected return of the global minimum variance (GMV) portfolio of risky assets

$$\mu_{GMV} - r_f = \frac{\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \boldsymbol{\mu}}{\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \mathbf{1}} - r_f = \frac{\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \boldsymbol{\mu}_e}{\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \mathbf{1}}$$

Proposition: if the GMV portfolio has a lower expected return than the risk-free asset, then the Tangent Portfolio expected return will also be less than r_f.

The Efficient Frontier consists of shorting a fraction $|\alpha|$ of the Tangent Portfolio and lending $(1 - \alpha)$ at the risk-free rate



Tangent Portfolio with negative excess return
μGMV = 0.108
rf = 0.18
Efficient Frontier
(short Tangent)
Only
Risky Assets
rf
GMV
1
2
3
4
Tangent
Portfolio
0.0
0.1
0.2
0.3
0.4
μ
0.00
0.05
0.10
0.15
0.20
σ

Outline

One Fund Theorem

- Assuming that $\mu_{GMV} > r_f$, then the following is true

  A mean-variance investor need only allocate funds between the risk-free asset and the Tangent Portfolio. There is no need to consider any other portfolio of risky assets.

$$\mathbf{w} = \alpha \mathbf{w}_T \quad \text{where} \quad \alpha = (\mu_P - r_f)\frac{\mathbf{1}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e}{\boldsymbol{\mu}_e^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_e} = \frac{\mu_P - r_f}{\mu_T - r_f}$$

- $w_0 = 1 - \alpha$

$$\sigma_P = |\alpha| \sigma_T, \qquad \mu_P = r_f + \alpha(\mu_T - r_f)$$

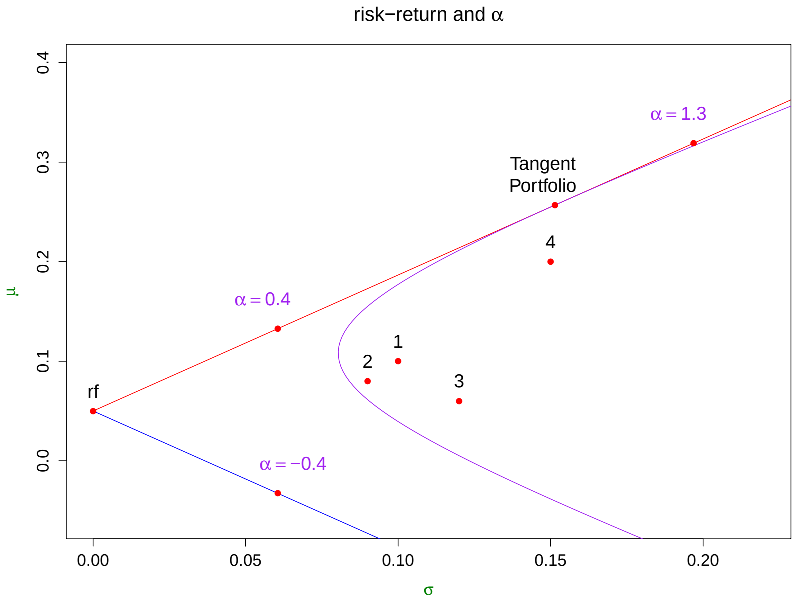
- Special cases

$$\alpha = \begin{cases} 0, & \mu_P = r_f \\ 1, & \mu_P = \mu_T \end{cases}$$

One Fund Theorem

- $0 \le \alpha \le 1$: lend $1 - \alpha$ at the risk-free rate, invest α in Tangent Portfolio
- $\alpha > 1$: borrow $|1 - \alpha|$ at the risk-free rate, invest α in Tangent Portfolio
- $\alpha < 0$: short the Tangent Portfolio and lend $1 - \alpha$ at the risk-free rate (no longer on Efficient Frontier)

One Fund Theorem



Constructing a Portfolio

1. Determine the weights of the Tangent Portfolio $\mathbf{w}_T$
   - $\mathbf{w}_T$ is independent of investor preferences, namely the investor-specified target rate of return μ_P
   - All mean-variance investors agree on the constitution of $\mathbf{w}_T$

2. Given the target rate of return μ_P, determine the fraction α allocated to the Tangent Portfolio

3. The investment allocation weights are

   $$\mathbf{w} = \alpha \mathbf{w}_T \qquad w_0 = 1 - \alpha$$

   and the portfolio standard deviation is:

   $$\sigma_P = |\alpha| \sigma_T$$

R Code

```
MVP_RFshort <- function(mu, Rho, vol, mu_R, rf)
{
  N <- length(mu)
  h <- rep(1, N)

  Sigma <- Rho * outer(vol, vol)

  eig <- eigen(Sigma)
  if(any(eig$values <= 0) > 0)
    stop("covariance matrix is not positive definite")

  Sinv <- solve(Sigma)
  a1 <- mu - rf

  a <- drop(t(h) %*% Sinv %*% a1)
```

R Code

```
    wT <- (Sinv %*% a1) / a
    mu_T <- drop(t(mu) %*% wT)
    alpha <- (mu_R - rf) / (mu_T - rf)
    w0 <- 1 - alpha
    vol_P <- alpha * sqrt(drop(t(wT) %*% Sigma %*% wT))
    mu_P <- alpha * drop(t(mu) %*% wT) + (1 - alpha)*rf

    list(mu = mu_P, vol = vol_P, w = drop(alpha * wT),
         w0 = 1 - alpha)
}
```

R Code

```
#specify target rate of return
mu_R <- 0.12

#risk-free return
rf <- 0.05

#risky assets
mu <- c(0.10, 0.08, 0.06, 0.2)
Rho <- rbind(c(1.0, 0.7,  0.3,  0.5),
             c(0.7, 1.0,  0.55, 0.1),
             c(0.3, 0.55, 1.0,  0.4),
             c(0.5, 0.1,  0.4,  1.0))
vol <- c(0.1, 0.09, 0.12, 0.15)
```

R Code

```
> MVP_RFshort(mu, Rho, vol, mu_R, rf)

$mu
[1] 0.12

$vol
[1] 0.05123593

$w
[1] -0.5450538  0.7852051 -0.4212889  0.5193962

$w0
[1] 0.6617414
```

Outline

Covariances as Risk Contributers

- The portfolio variance can be written

$$
\begin{aligned}
\sigma_P^2 = \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} &= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \mathrm{Cov}(r_i, r_j) \\
&= \sum_{i=1}^{N} w_i \,\mathrm{Cov}\left(r_i, \sum_{j=1}^{N} w_j r_j \right) \\
&= \sum_{i=1}^{N} w_i \,\mathrm{Cov}(r_i, r_P) \\
&= \mathbf{w}^\top \mathrm{Cov}(\mathbf{r}, r_P)
\end{aligned}
$$

- The portfolio variance is a linear combination (with weights **w**) of the covariances of the individual asset returns with the portfolio return

Outline

What is "Active Portfolio Management"?

Relative to a benchmark

- A portfolio manager selects a benchmark portfolio
- Goal: achieve high excess returns (relative to some benchmark), but with low variance

Terminology

- r_B: random rate of return on benchmark portfolio
- r_P: random rate of return on tracking portfolio
- $r_A = r_P - r_B$: "active" rate of return
- $\sigma_A^2 = \mathrm{Var}(r_A)$: tracking error variance (TEV)
- $\sigma_A = \sqrt{\mathrm{Var}(r_A)}$: tracking error

Terminology

Benchmark and tracking portfolio consider the same universe of risky assets, but with different allocations

- $\mathbf{r}$: random rate of return vector (risky assets)
- $\boldsymbol{\mu}$: expected rate of return vector (risky assets)
- $\mathbf{w}_B$: benchmark weights
- $\mathbf{w}_P$: tracking portfolio weights
- $\mathbf{w}_A = \mathbf{w}_P - \mathbf{w}_B$: active weights
- $\mathbf{w}_A^\top \mathbf{1} = 0$: (full allocation to benchmark weights)

The active random and expected rates of return:

$$r_A = \mathbf{w}_A^\top \mathbf{r} = \mathbf{w}_P^\top \mathbf{r} - \mathbf{w}_B^\top \mathbf{r}, \qquad \mu_A = \mathbf{w}_A^\top \boldsymbol{\mu} = \mu_P - \mu_B$$

Tracking error variance

$$\begin{aligned} \sigma_A^2 &= \mathrm{Var}(r_A) = \mathbb{E}[\mathbf{w}_A^\top (\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^\top \mathbf{w}_A] \\ &= \mathbf{w}_A^\top \mathbb{E}[(\mathbf{r} - \boldsymbol{\mu})(\mathbf{r} - \boldsymbol{\mu})^\top] \mathbf{w}_A = \mathbf{w}_A^\top \boldsymbol{\Sigma} \mathbf{w}_A \end{aligned}$$

TEV Optimization

Find optimal active weights through mean-variance optimization

$$\begin{aligned} \max_{\mathbf{w}_A} : \quad & \mathbf{w}_A^\intercal \boldsymbol{\mu} - \frac{1}{2}\lambda \mathbf{w}_A^\intercal \boldsymbol{\Sigma} \mathbf{w}_A \\ \text{subject to} : \quad & \mathbf{w}_A^\intercal \mathbf{1} = 0 \end{aligned}$$

Interpretation: maximize active returns with a penalty for large volatility of active returns

> *Performance fees induce an option-like pattern in the compensation of the manager, who may have an incentive to take on more risk to increase the value of the option. To control this behavior, institutional investors commonly impose a limit on the volatility of the deviation of the active portfolio from the benchmark, which is also known as tracking-error volatility (TEV). [Jorian 2003]*

TEV Optimization

- Find optimal active weights through mean-variance optimization

$$\begin{aligned} \max_{\mathbf{w}_A} : \quad & \mathbf{w}_A^\intercal \boldsymbol{\mu} - \frac{1}{2}\lambda \mathbf{w}_A^\intercal \boldsymbol{\Sigma} \mathbf{w}_A \\ \text{subject to} : \quad & \mathbf{w}_A^\intercal \mathbf{1} = 0 \end{aligned}$$

- Lagrangian and first order optimality conditions

$$\mathcal{L}(\mathbf{w}_A, \gamma) = \mathbf{w}_A^\intercal \boldsymbol{\mu} - \frac{1}{2}\lambda \mathbf{w}_A^\intercal \boldsymbol{\Sigma} \mathbf{w}_A + \gamma \mathbf{w}_A^\intercal \mathbf{1}$$

$$\mathbf{w}_A = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \gamma \mathbf{1}) \qquad \mathbf{w}_A^\intercal \mathbf{1} = 0$$

TEV Optimization

- Optimal active weights

$$\mathbf{w}_A = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - \frac{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\mathbf{1}}\mathbf{1}\right)$$

- The optimal active weights are independent of the benchmark - will be the same regardless of which index is being tracked
- Expected active return and variance

$$\mu_A = \frac{1}{\lambda}\frac{(\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\mathbf{1})(\boldsymbol{\mu}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - (\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

$$\sigma_A^2 = \frac{1}{\lambda^2}\frac{(\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\mathbf{1})(\boldsymbol{\mu}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}) - (\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})^2}{\mathbf{1}^\intercal\boldsymbol{\Sigma}^{-1}\mathbf{1}}$$

TEV Optimization

The Efficient Frontier is a straight line in risk-return space

$$\mu_A = \sigma_A \sqrt{\frac{(\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \mathbf{1})(\boldsymbol{\mu}^\intercal \mathbf{\Sigma}^{-1} \boldsymbol{\mu}) - (\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \boldsymbol{\mu})^2}{\mathbf{1}^\intercal \mathbf{\Sigma}^{-1} \mathbf{1}}}$$

The point $(\sigma_A, \mu_A) = (0, 0)$ represents no tracking error - the benchmark is perfectly replicated

- The slope of the Efficient Frontier is often referred to as the Information Ratio (IR), it represents the expected excess return per unit of risk

TEV Optimization

- High Information Ratios are desirable, portfolios on the efficient frontier offer the maximum IR among all TEV portfolios with the same level of risk
- Level of risk-aversion, i.e., the parameter λ, determines where the portfolio manager is on the Efficient Frontier
- Jorion (2003)

  > *The problem with this setup is that it induces the manager to optimize in only excess-return space while totally ignoring the investor's overall portfolio risk.*

Alternative Setup: TEV Optimization

Jorion (2003) proposed the following optimization problem

$$\begin{aligned}
\max_{\mathbf{w}_A} : \quad & \mathbf{w}_A^\intercal \boldsymbol{\mu} \\
\text{subject to} : \quad & \mathbf{w}_A^\intercal \mathbf{1} = 0 \\
& \mathbf{w}_A^\intercal \boldsymbol{\Sigma} \mathbf{w}_A = T \\
& \mathbf{w}_P^\intercal \boldsymbol{\Sigma} \mathbf{w}_P = \sigma_P^2
\end{aligned}$$

Interpretation: maximize active returns subject to a prescribed tracking error variance and overall portfolio variance

Alternative Setup: TEV Optimization

- Jorion (2003) derives an analytic expression for the active weights, expected return and TEV

Theorem

The constant-TEV frontier is an ellipse in the (σ_P, μ_P) space centered at μ_B and $\sigma_B^2 + T$. With the deviations from the center defined as $y = \sigma_P^2 - T$ and $z = \mu_P - \mu_B$.

- The portfolios constructed from this optimization problem exhibit higher total expected returns for a given level of risk compared to those constructed in the Quadratic Utility maximization

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