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Proofs

Prove that: If x is an odd integer, then x + 1 is even. Formally, you must prove that $\forall x \in Z$ (x is odd $\Rightarrow$ x + 1 is even).

Two definitions:

- An odd integer is an integer that is not divisible by 2. Meaning, if y is an odd integer, y/2 will not be an integer, rather y/2 will have a decimal value of .5, or a half.
- An even integer is an integer that is divisible by 2. Meaning, if z is an even integer, z/2 will be an integer.

Proof:

- It is a given that x is an odd integer. I would like to take x+1 and divide it by two as follows: $\frac{x+1}{2}$.
- Based on the rules of fractions, I could rewrite $\frac{x+1}{2}$ as $\frac{x}{2} + \frac{1}{2}$. To write it as an equality, $\frac{x}{2} + \frac{1}{2} = \frac{x+1}{2}$.
- Given that x is an odd integer, $\frac{x}{2}$ must be a non-integer with a decimal value of .5 (based on the definition of an odd integer above). Additionally, $\frac{1}{2} = .5$.
- Therefore, $\frac{x}{2}$ $(which\ is\ a\ number\ which\ has\ a\ decimal\ value\ of\ .5) + \frac{1}{2}\ (which\ equals\ .5)$ must equal an integer, as the sum of the decimals of each of the operands are .5+.5=1, which is an integer. To elaborate on what was just said, the sum of the whole numbers of each operand (which is an integer) plus the sum of the decimals of each operand (which is 1, which is an integer), equals an integer. That means that $\frac{x+1}{2}$ is an integer.
- If $\frac{x+1}{2}$ is an integer, then x+1 must be an integer that is divisible by 2, as $\frac{x+1}{2}\ (which\ is\ an\ integer) * 2 = x + 1$. Because $\frac{x+1}{2}$ (which is an integer) was multiplied by 2, the result of that product (x+1) must be divisible by 2.
- Therefore, if x+1 is divisible by 2, x+1 is an even integer, given by the definition of an even integer above.

Theorem: $\forall n \in \mathbf{N}, 3 \mid (n^3 - n)$

Proving by induction:

- Before starting the proof, we can factor n^3-n into $n(n^2-1)$, and from there to $n(n+1)(n-1)$.

Base Case:

- Integer n=0

$$n^3-n=$$
$$(n)(n+1)(n-1)=$$
$$0(1)(-1)$$
$$=0$$

- Zero is divisible by 3, which supports that n^3-n is divisible by 3 for n=0.
- The 3 operands in the product are 3 consecutive integers (in this case, -1, 0, and 1).

Inductive Step:

- Integer y=x+1.
- Given we factored the formula to $n(n+1)(n-1)$. Plugging x into the formula would be

  $$x(x+1)(x-1),$$

  and y into the formula would be

  $$(x+1)((x+1)+1)((x+1)-1)=$$
  $$(x+1)(x+2)(x).$$

  - For example, when x=0, n^3-n would be

    $$0(1)(-1)=0,$$

    and y=1 would be

    $$0+1((0+1)+1)((0+1)-1)=$$
    $$(1)(0)(2)=0.$$

  - The three factors when we plug in x to the formula are x, x-1, and x+1, and when we plug in y they are x, x+1, and x+2 (which, in both cases, regardless of what integer is chosen for x, the three factors are three consecutive integers).
  - This is correct because when we factor n^3-n, the three factors are n-1, n, and n+1, which are each one integer apart from each other, making them consecutive integers.
- Given a set of 3 consecutive integers, one of those integers will be a multiple of 3.
  - This is correct because starting from 0, every three integers you move (whether forward or backward) will be a multiple of 3, so when you have a group of three consecutive integers, you can guarantee that one of them will be a multiple of three
- It is seen from here whether we plug in x or y (which is x+1), that the factors of the formula will be 3 consecutive integers, proving the inductive step.

Because the base case is correct, and the inductive step is correct (because n^3-n factors into 3 consecutive integers, and one of those integers is guaranteed to be a multiple of 3), then for all natural numbers, $3 \mid (n^3 - n)$.

Theorem: $\forall n \in \mathbb{N}$, for $n > 1$ we have $n! < n^n$

Proving by induction:

Base Case: Integer n=2

$$n! = n*(n-1) = 2*1 = 2,$$

and

$$n^n = n^2 = n*n = 2*2 = 4.$$

2 is less than 4, which holds true with the theorem.

Inductive Step: Integer y=x+1 (when x>1)

- n!
  - When n=x, the number of operands in the product to find the factorial is x. This is correct because assuming integer b is positive, the definition of b! is multiplying b by every integer between 1 and b, meaning there will be b operands in b!
    - For example, if x=3, 3!=3*2*1 (there are 3 operands). If x=4, 4!=4*3*2*1 (4 operands).
  - When n=y, the number of operands in the product to find the factorial is x+1=y, based on the same definition of factorial above.
    - For example, if x=3, y=4, so 4!=4*3*2*1 (4 operands, which is 3+1 operands, and x=3).
- n^n
  - When n=x, the number of operands in the product to find the exponent is x. This is correct because assuming integer b is positive, the definition of b^b is b multiplied by itself b number of times, meaning there will be b operands in b^b.
    - For example, if x=3, then $3^3 = 3*3*3$ (3 operands). If x=4, then $4^4=4*4*4*4$ (4 operands).
  - When n=y, the number of operands in the product to find the exponent is x+1=y, based on the same definition of powers above.
    - For example, if x=3, y=4, so $4^4=4*4*4*4$ (4 operands, which is 3+1 operands, and x=3).
- Assuming x>1, regardless of whether n=x or n=y, the number of operands in the product n! is equal to the number of operands in the product n^n (which is n operands each).
  - However, by n!, the highest valued operand is n, with every other operand being less than n. By n^n, the value of every operand in the product is equal to n.
  - All of the values of the operands in n^n are greater than all of the values of the operands in n!
    - The one exception is in n!, there is an operand that is equal to n, but that is only equal, not greater, than any operand in n^n.
- Because n! and n^n have the same number of operands in their products (n operands), and all of the operands in n^n are greater than all of the operands in n! (besides one which is equal to the operand in n^n, but not greater than, see bullet above), n! must be less than n^n when y=x+1.

Because the base case of n=2 is correct, and the inductive step of n=y=x+1 is correct for all integers x greater than 1, then for all n > 1, $n! < n^n$.