# <u>Gale-Shapely Drill #4</u>

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## Drill

| | Days | | | | |
|---|---|---|---|---|---|
| Women | Day 1 | Day 2 | Day 3 | Day 4 | Day 5 |
| A | 1, 3 | 1 | 1, 2 | 2 | 2 |
| B | 2 | 2, 3 | 3 | 1, 3 | 1 |
| C | | | | | 3 |

## Proofs

### 1. In <u>any</u> execution of the algorithm, if a woman receives a proposal on day i , then she receives some proposal on every subsequent day until the algorithm terminates.

To prove this statement, I will be using a <u>direct proof</u>. Let's assume Woman A gets a proposal on day i. The algorithm (from the requirements document) states, "Each afternoon, every woman collects the offers she received in the morning. Based on her preferences list, one of the men is the most highly ranked. She accepts the proposal for this man...the man will continue to propose until the algorithm terminates." Therefore, if A got at least one proposal, she will be put with her most preferable man who proposed (if there's only one proposal, she'll be put with that man), and that man that she accepted (let's call him Man 1) will "continue to propose until the algorithm terminates." The only possibility of her not being with Man 1 in the end is if another man (let's call him Man 2) proposes who is a more favorable candidate than Man 1. In that case, she will end up with Man 2. Regardless of her ending up with Men 1 or 2, on all days after "day i", she will be receiving at least one proposal: certainly from the man she accepted the day before, or possibly from another man (who may be more or less favorable).

### 2. In <u>any</u> execution of the algorithm, if a woman receives no proposal on day i, then she receives no proposal on any previous day j, $1 \leq j < i$.

To prove this statement, I will be using a <u>proof by contraposition</u>, proving that if a woman receives a proposal on day j ($1 \leq j < i$), she then receives a proposal on day i. From Proof #1, we learned that if "a woman receives a proposal on day j , then she receives some proposal on every subsequent day until the algorithm terminates." Therefore, if a woman received a proposal on day j, and day i comes after day j, then she will receive a proposal on day i. Because the contrapositive is proven, it is also proven that "if a woman receives no proposal on day i, then she receives no proposal on any previous day j, $1 \leq j < i$."

### 3. In <u>any</u> execution of the algorithm, there is at least one woman who only receives a single proposal.

To prove this statement, I will be using a <u>direct proof</u>. As the algorithm states (from the requirements document), "at that point (i.e. the last day), every man and woman are paired." Let's call the last day of the algorithm Day N. Because in this algorithm there are the same number of men and women, and every man and woman are paired on Day N, that means that every woman will have received exactly one proposal on Day N. This is because a man can only send one proposal on a given day, and every day, every woman must accept a proposal (if she receives at least one proposal on a given day). Therefore, the only way for all men and women to be paired is if every man sent a proposal to a different woman, so *every woman will have received exactly one proposal on Day N*.

Because "at that point (i.e. the last day), every man and woman are paired", on Day N-1, not every man and woman are paired. The only possible way for a given woman (let's call her Woman A) to not be paired on a given day (let's call this Day N-1) is for Woman A to not receive any proposals on all days up to and including Day N-1. This is because "Each afternoon, every woman collects the offers she received in the morning…She accepts the proposal for this man." Any day where a given woman has at least one proposal, she continues to "receive some proposal on every subsequent day until the algorithm terminates" (Proof #1). The contrapositive of this statement is Proof #2, which states that "if a woman receives no proposal on day i, then she receives no proposal on any previous day j, $1 \leq j < i$." Because Woman A was not paired on Day N-1, it must be because she received no proposals yet (from Day 1 until Day N-1). Because on Day N-1, not every man and woman were paired, *there must be at least one woman who has not received any proposals yet (from Day 1 until Day N-1)*. Because *on Day N all women will have received exactly one proposal*, and *all days before Day N there will be at least one woman who has not received any proposals*, "there is at least one woman who only receives a single proposal."