Comparison of Multiobjective Evolutionary Algorithms: Empirical Results

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Abstract

In this paper, we provide a systematic comparison of various evolutionary approaches to multiobjective optimization using six carefully chosen test functions. Each test function involves a particular feature that is known to cause difficulty in the evolutionary optimization process, mainly in converging to the Pareto-optimal front (e.g., multimodality and deception). By investigating these different problem features separately, it is possible to predict the kind of problems to which a certain technique is or is not well suited. However, in contrast to what was suspected beforehand, the experimental results indicate a hierarchy of the algorithms under consideration. Furthermore, the emerging effects are evidence that the suggested test functions provide sufficient complexity to compare multiobjective optimizers. Finally, elitism is shown to be an important factor for improving evolutionary multiobjective search.

1 Motivation

Evolutionary algorithms (EAs) have become established as *the* method at hand for exploring the Pareto-optimal front in multiobjective optimization problems that are too complex to be solved by exact methods, such as linear programming and gradient search. This is not only because there are few alternatives for searching intractably large spaces for multiple Pareto-optimal solutions. Due to their inherent parallelism and their capability to exploit similarities of solutions by recombination, they are able to approximate the Pareto-optimal front in a single optimization run. The numerous applications and the rapidly growing interest in the area of multiobjective EAs take this fact into account.

After the first pioneering studies on evolutionary multiobjective optimization appeared in the mid-eighties (Schaffer, 1984, 1985; Fourman,1985) several different EA implementations were proposed in the years 1991–1994 (Kursawe, 1991; Hajela and Lin, 1992; Fonseca

and Fleming, 1993; Horn et al., 1994; Srinivas and Deb, 1994). Later, these approaches (and variations of them) were successfully applied to various multiobjective optimization problems (Ishibuchi and Murata, 1996; Cunha et al., 1997; Valenzuela-Rendón and Uresti-Charre, 1997; Fonseca and Fleming, 1998; Parks and Miller, 1998). In recent years, some researchers have investigated particular topics of evolutionary multiobjective search, such as convergence to the Pareto-optimal front (Van Veldhuizen and Lamont, 1998a; Rudolph, 1998), niching (Obayashi et al., 1998), and elitism (Parks and Miller, 1998; Obayashi et al., 1998), while others have concentrated on developing new evolutionary techniques (Laumanns et al., 1998; Zitzler and Thiele, 1999). For a thorough discussion of evolutionary algorithms for multiobjective optimization, the interested reader is referred to Fonseca and Fleming (1995), Horn (1997), Van Veldhuizen and Lamont (1998b), and Coello (1999).

In spite of this variety, there is a lack of studies that compare the performance and different aspects of these approaches. Consequently, the question arises: which implementations are suited to which sort of problem, and what are the specific advantages and drawbacks of different techniques?

First steps in this direction have been made in both theory and practice. On the theoretical side, Fonseca and Fleming (1995) discussed the influence of different fitness assignment strategies on the selection process. On the practical side, Zitzler and Thiele (1998, 1999) used a NP-hard 0/1 knapsack problem to compare several multiobjective EAs. In this paper, we provide a systematic comparison of six multiobjective EAs, including a random search strategy as well as a single-objective EA using objective aggregation. The basis of this empirical study is formed by a set of well-defined, domain-independent test functions that allow the investigation of independent problem features. We thereby draw upon results presented in Deb (1999), where problem features that may make convergence of EAs to the Pareto-optimal front difficult are identified and, furthermore, methods of constructing appropriate test functions are suggested. The functions considered here cover the range of convexity, nonconvexity, discrete Pareto fronts, multimodality, deception, and biased search spaces. Hence, we are able to systematically compare the approaches based on different kinds of difficulty and to determine more exactly where certain techniques are advantageous or have trouble. In this context, we also examine further factors such as population size and elitism.

The paper is structured as follows: Section 2 introduces key concepts of multiobjective optimization and defines the terminology used in this paper mathematically. We then give a brief overview of the multiobjective EAs under consideration with special emphasis on the differences between them. The test functions, their construction, and their choice are the subject of Section 4, which is followed by a discussion about performance metrics to assess the quality of trade-off fronts. Afterwards, we present the experimental results in Section 6 and investigate further aspects like elitism (Section 7) and population size (Section 8) separately. A discussion of the results as well as future perspectives are given in Section 9.

2 Definitions

Optimization problems involving multiple, conflicting objectives are often approached by aggregating the objectives into a scalar function and solving the resulting single-objective optimization problem. In contrast, in this study, we are concerned with finding a set of optimal trade-offs, the so-called Pareto-optimal set. In the following, we formalize this

well-known concept and also define the difference between local and global Pareto-optimal sets.

A multiobjective search space is partially ordered in the sense that two arbitrary solutions are related to each other in two possible ways: either one dominates the other or neither dominates.

DEFINITION 1: *Let us consider, without loss of generality, a multiobjective minimization problem with m decision variables (parameters) and n objectives:*

$$
\begin{aligned}
\text{Minimize} \quad & \mathbf{y} = f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})) \\
\text{where} \quad & \mathbf{x} = (x_1, \ldots, x_m) \in X \\
& \mathbf{y} = (y_1, \ldots, y_n) \in Y
\end{aligned} \tag{1}
$$

and where $\mathbf{x}$ is called decision vector, X parameter space, $\mathbf{y}$ objective vector, *and Y* objective space. *A decision vector $\mathbf{a} \in X$ is said to* dominate *a decision vector $\mathbf{b} \in X$ (also written as $\mathbf{a} \prec \mathbf{b}$) if and only if*

$$
\begin{aligned}
\forall i \in \{1, \ldots, n\}: \quad & f_i(\mathbf{a}) \leq f_i(\mathbf{b}) \qquad \wedge \\
\exists j \in \{1, \ldots, n\}: \quad & f_j(\mathbf{a}) < f_j(\mathbf{b})
\end{aligned} \tag{2}
$$

Additionally, in this study, we say $\mathbf{a}$ covers $\mathbf{b}$ ($\mathbf{a} \preceq \mathbf{b}$) if and only if $\mathbf{a} \prec \mathbf{b}$ or $f(\mathbf{a}) = f(\mathbf{b})$.

Based on the above relation, we can define nondominated and Pareto-optimal solutions:

DEFINITION 2: *Let $\mathbf{a} \in X$ be an arbitrary decision vector.*

1. *The decision vector $\mathbf{a}$ is said to be* nondominated *regarding a set $X' \subseteq X$ if and only if there is no vector in X' which dominates $\mathbf{a}$; formally*

$$
\nexists \mathbf{a}' \in X' : \mathbf{a}' \prec \mathbf{a} \tag{3}
$$

   If it is clear within the context which set X' is meant, we simply leave it out.

2. *The decision vector $\mathbf{a}$ is* Pareto-optimal *if and only if $\mathbf{a}$ is nondominated regarding X.*

Pareto-optimal decision vectors cannot be improved in any objective without causing a degradation in at least one other objective; they represent, in our terminology, globally optimal solutions. However, analogous to single-objective optimization problems, there may also be local optima which constitute a nondominated set within a certain neighborhood. This corresponds to the concepts of global and local Pareto-optimal sets introduced by Deb (1999)[1]:

DEFINITION 3: *Consider a set of decision vectors $X' \subseteq X$.*

1. *The set X' is denoted as a* local Pareto-optimal set *if and only if*

$$
\forall \mathbf{a}' \in X' : \nexists \mathbf{a} \in X : \mathbf{a} \prec \mathbf{a}' \wedge \|\mathbf{a} - \mathbf{a}'\| < \epsilon \wedge \|f(\mathbf{a}) - f(\mathbf{a}')\| < \delta \tag{4}
$$

   where $\| \cdot \|$ is a corresponding distance metric and $\epsilon > 0$, $\delta > 0$.

---

[1]A slightly modified definition of local Pareto optimality is given here.

2. *The set X' is called a* global Pareto-optimal set *if and only if*

$$\forall \mathbf{a}' \in X' : \nexists \mathbf{a} \in X : \mathbf{a} \prec \mathbf{a}' \tag{5}$$

Note that a global Pareto-optimal set does not necessarily contain all Pareto-optimal solutions. If we refer to the entirety of the Pareto-optimal solutions, we simply write "Pareto-optimal set"; the corresponding set of objective vectors is denoted as "Pareto-optimal front".

3   Evolutionary Multiobjective Optimization

Two major problems must be addressed when an evolutionary algorithm is applied to multiobjective optimization:

1. How to accomplish fitness assignment and selection, respectively, in order to guide the search towards the Pareto-optimal set.

2. How to maintain a diverse population in order to prevent premature convergence and achieve a well distributed trade-off front.

Often, different approaches are classified with regard to the first issue, where one can distinguish between criterion selection, aggregation selection, and Pareto selection (Horn, 1997). Methods performing criterion selection switch between the objectives during the selection phase. Each time an individual is chosen for reproduction, potentially a different objective will decide which member of the population will be copied into the mating pool. Aggregation selection is based on the traditional approaches to multiobjective optimization where the multiple objectives are combined into a parameterized single objective function. The parameters of the resulting function are systematically varied during the same run in order to find a set of Pareto-optimal solutions. Finally, Pareto selection makes direct use of the dominance relation from Definition 1; Goldberg (1989) was the first to suggest a Pareto-based fitness assignment strategy.

In this study, six of the most salient multiobjective EAs are considered, where for each of the above categories, at least one representative was chosen. Nevertheless, there are many other methods that may be considered for the comparison (cf. Van Veldhuizen and Lamont (1998b) and Coello (1999) for an overview of different evolutionary techniques):

- Among the class of criterion selection approaches, the Vector Evaluated Genetic Algorithm (VEGA) (Schaffer, 1984, 1985) has been chosen. Although some serious drawbacks are known (Schaffer, 1985; Fonseca and Fleming, 1995; Horn, 1997), this algorithm has been a strong point of reference up to now. Therefore, it has been included in this investigation.

- The EA proposed by Hajela and Lin (1992) is based on aggregation selection in combination with fitness sharing (Goldberg and Richardson, 1987), where an individual is assessed by summing up the weighted objective values. As weighted-sum aggregation appears still to be widespread due to its simplicity, Hajela and Lin's technique has been selected to represent this class of multiobjective EAs.

- Pareto-based techniques seem to be most popular in the field of evolutionary multiobjective optimization (Van Veldhuizen and Lamont, 1998b). In particular, the

algorithm presented by Fonseca and Fleming (1993), the Niched Pareto Genetic Algorithm (NPGA) (Horn and Nafpliotis, 1993; Horn et al., 1994), and the Nondominated Sorting Genetic Algorithm (NSGA) (Srinivas and Deb, 1994) appear to have achieved the most attention in the EA literature and have been used in various studies. Thus, they are also considered here. Furthermore, a recent elitist Pareto-based strategy, the Strength Pareto Evolutionary Algorithm (SPGA) (Zitzler and Thiele, 1999), which outperformed four other multiobjective EAs on an extended 0/1 knapsack problem, is included in the comparison.

4 Test Functions for Multiobjective Optimizers

Deb (1999) has identified several features that may cause difficulties for multiobjective EAs in 1) converging to the Pareto-optimal front and 2) maintaining diversity within the population. Concerning the first issue, multimodality, deception, and isolated optima are well-known problem areas in single-objective evolutionary optimization. The second issue is important in order to achieve a well distributed nondominated front. However, certain characteristics of the Pareto-optimal front may prevent an EA from finding diverse Pareto-optimal solutions: convexity or nonconvexity, discreteness, and nonuniformity. For each of the six problem features mentioned, a corresponding test function is constructed following the guidelines in Deb (1999). We thereby restrict ourselves to only two objectives in order to investigate the simplest case first. In our opinion, two objectives are sufficient to reflect essential aspects of multiobjective optimization. Moreover, we do not consider maximization or mixed minimization/maximization problems.

Each of the test functions defined below is structured in the same manner and consists itself of three functions f_1, g, h (Deb, 1999, 216):

$$
\begin{aligned}
\text{Minimize} \quad \mathcal{T}(\mathbf{x}) &= (f_1(x_1), f_2(\mathbf{x})) \\
\text{subject to} \quad f_2(\mathbf{x}) &= g(x_2, \ldots, x_m) h(f_1(x_1), g(x_2, \ldots, x_m)) \\
\text{where} \quad \mathbf{x} &= (x_1, \ldots, x_m)
\end{aligned}
\tag{6}
$$

The function f_1 is a function of the first decision variable only, g is a function of the remaining $m-1$ variables, and the parameters of h are the function values of f_1 and g. The test functions differ in these three functions as well as in the number of variables m and in the values the variables may take.

DEFINITION 4: *We introduce six test functions $\mathcal{T}_1, \ldots, \mathcal{T}_6$ that follow the scheme given in Equation 6:*

- *The test function $\mathcal{T}_1$ has a convex Pareto-optimal front:*

$$
\begin{aligned}
f_1(x_1) &= x_1 \\
g(x_2, \ldots, x_m) &= 1 + 9 \cdot \sum_{i=2}^{m} x_i/(m-1) \\
h(f_1, g) &= 1 - \sqrt{f_1/g}
\end{aligned}
\tag{7}
$$

  where $m = 30$, and $x_i \in [0, 1]$. The Pareto-optimal front is formed with $g(\mathbf{x}) = 1$.

- *The test function $\mathcal{T}_2$ is the nonconvex counterpart to $\mathcal{T}_1$:*

$$
\begin{aligned}
f_1(x_1) &= x_1 \\
g(x_2, \ldots, x_m) &= 1 + 9 \cdot \sum_{i=2}^{m} x_i/(m-1) \\
h(f_1, g) &= 1 - (f_1/g)^2
\end{aligned}
\tag{8}
$$

where $m = 30$, and $x_i \in [0, 1]$. The Pareto-optimal front is formed with $g(\mathbf{x}) = 1$.

- The test function $\mathcal{T}_3$ represents the discreteness feature; its Pareto-optimal front consists of several noncontiguous convex parts:

$$
\begin{array}{rcl}
f_1(x_1) & = & x_1 \\
g(x_2, \ldots, x_m) & = & 1 + 9 \cdot \sum_{i=2}^{m} x_i/(m-1) \\
h(f_1, g) & = & 1 - \sqrt{f_1/g} - (f_1/g) \sin(10\pi f_1)
\end{array}
\tag{9}
$$

where $m = 30$, and $x_i \in [0, 1]$. The Pareto-optimal front is formed with $g(\mathbf{x}) = 1$. The introduction of the sine function in h causes discontinuity in the Pareto-optimal front. However, there is no discontinuity in the parameter space.

- The test function $\mathcal{T}_4$ contains 21^9 local Pareto-optimal fronts and, therefore, tests for the EA's ability to deal with multimodality:

$$
\begin{array}{rcl}
f_1(x_1) & = & x_1 \\
g(x_2, \ldots, x_m) & = & 1 + 10(m-1) + \sum_{i=2}^{m}(x_i^2 - 10\cos(4\pi x_i)) \\
h(f_1, g) & = & 1 - \sqrt{f_1/g}
\end{array}
\tag{10}
$$

where $m = 10$, $x_1 \in [0, 1]$, and $x_2, \ldots, x_m \in [-5, 5]$. The global Pareto-optimal front is formed with $g(\mathbf{x}) = 1$, the best local Pareto-optimal front with $g(\mathbf{x}) = 1.25$. Note that not all local Pareto-optimal sets are distinguishable in the objective space.

- The test function $\mathcal{T}_5$ describes a deceptive problem and distinguishes itself from the other test functions in that x_i represents a binary string:

$$
\begin{array}{rcl}
f_1(x_1) & = & 1 + u(x_1) \\
g(x_2, \ldots, x_m) & = & \sum_{i=2}^{m} v(u(x_i)) \\
h(f_1, g) & = & 1/f_1
\end{array}
\tag{11}
$$

where $u(x_i)$ gives the number of ones in the bit vector x_i (unitation),

$$
v(u(x_i)) = \left\{ \begin{array}{ll} 2 + u(x_i) & \text{if } u(x_i) < 5 \\ 1 & \text{if } u(x_i) = 5 \end{array} \right\}
$$

and $m = 11$, $x_1 \in \{0, 1\}^{30}$, and $x_2, \ldots, x_m \in \{0, 1\}^5$. The true Pareto-optimal front is formed with $g(\mathbf{x}) = 10$, while the best deceptive Pareto-optimal front is represented by the solutions for which $g(\mathbf{x}) = 11$. The global Pareto-optimal front as well as the local ones are convex.

- The test function $\mathcal{T}_6$ includes two difficulties caused by the nonuniformity of the search space: first, the Pareto-optimal solutions are nonuniformly distributed along the global Pareto front (the front is biased for solutions for which $f_1(\mathbf{x})$ is near one); second, the density of the solutions is lowest near the Pareto-optimal front and highest away from the front:

$$
\begin{array}{rcl}
f_1(x_1) & = & 1 - \exp(-4x_1)\sin^6(6\pi x_1) \\
g(x_2, \ldots, x_m) & = & 1 + 9 \cdot ((\sum_{i=2}^{m} x_i)/(m-1))^{0.25} \\
h(f_1, g) & = & 1 - (f_1/g)^2
\end{array}
\tag{12}
$$

where $m = 10$, $x_i \in [0, 1]$. The Pareto-optimal front is formed with $g(\mathbf{x}) = 1$ and is nonconvex.

We will discuss each function in more detail in Section 6, where the corresponding Pareto-optimal fronts are visualized as well (Figures 1–6).

5   Metrics of Performance

Comparing different optimization techniques experimentally always involves the notion of performance. In the case of multiobjective optimization, the definition of quality is substantially more complex than for single-objective optimization problems, because the optimization goal itself consists of multiple objectives:

- The distance of the resulting nondominated set to the Pareto-optimal front should be minimized.

- A good (in most cases uniform) distribution of the solutions found is desirable. The assessment of this criterion might be based on a certain distance metric.

- The extent of the obtained nondominated front should be maximized, i.e., for each objective, a wide range of values should be covered by the nondominated solutions.

In the literature, some attempts can be found to formalize the above definition (or parts of it) by means of quantitative metrics. Performance assessment by means of weighted-sum aggregation was introduced by Esbensen and Kuh (1996). Thereby, a set X' of decision vectors is evaluated regarding a given linear combination by determining the minimum weighted-sum of all corresponding objective vectors of X'. Based on this concept, a sample of linear combinations is chosen at random (with respect to a certain probability distribution), and the minimum weighted-sums for all linear combinations are summed up and averaged. The resulting value is taken as a measure of quality. A drawback of this metric is that only the "worst" solution determines the quality value per linear combination. Although several weight combinations are used, nonconvex regions of the trade-off surface contribute to the quality more than convex parts and may, as a consequence, dominate the performance assessment. Finally, the distribution, as well as the extent of the nondominated front, is not considered.

Another interesting means of performance assessment was proposed by Fonseca and Fleming (1996). Given a set $X' \subseteq X$ of nondominated solutions, a boundary function divides the objective space into two regions: the objective vectors for which the corresponding solutions are not covered by X' and the objective vectors for which the associated solutions are covered by X'. They call this particular function, which can also be seen as the locus of the family of tightest goal vectors known to be attainable, the *attainment surface*. Taking multiple optimization runs into account, a method is described to compute a median attainment surface by using auxiliary straight lines and sampling their intersections with the attainment surfaces obtained. As a result, the samples represented by the median attainment surface can be relatively assessed by means of statistical tests and, therefore, allow comparison of the performance of two or more multiobjective optimizers. A drawback of this approach is that it remains unclear how the quality difference can be expressed, i.e., how much better one algorithm is than another. However, Fonseca and Fleming describe ways of meaningful statistical interpretation in contrast to the other studies considered here, and furthermore, their methodology seems to be well suited to visualization of the outcomes of several runs.

In the context of investigations on convergence to the Pareto-optimal front, some authors (Rudolph, 1998; Van Veldhuizen and Lamont, 1998a) have considered the distance of a given set to the Pareto-optimal set in the same way as the function $\mathcal{M}_1$ defined below. The distribution was not taken into account, because the focus was not on this

matter. However, in comparative studies, distance alone is not sufficient for performance evaluation, since extremely differently distributed fronts may have the same distance to the Pareto-optimal front.

Two complementary metrics of performance were presented in Zitzler and Thiele (1998, 1999). On one hand, the size of the dominated area in the objective space is taken under consideration; on the other hand, a pair of nondominated sets is compared by calculating the fraction of each set that is covered by the other set. The area combines all three criteria (distance, distribution, and extent) into one, and therefore, sets differing in more than one criterion may not be distinguished. The second metric is in some way similar to the comparison methodology proposed in Fonseca and Fleming (1996). It can be used to show that the outcomes of an algorithm dominate the outcomes of another algorithm, although, it does not tell how much better it is.[2] We give its definition here, because it is used in the remainder of this paper.

DEFINITION 5: *Let* $X', X'' \subseteq X$ *be two sets of decision vectors. The function* $\mathcal{C}$ *maps the ordered pair* (X', X'') *to the interval* $[0, 1]$:

$$\mathcal{C}(X', X'') := \frac{|\{\mathbf{a}'' \in X''; \exists \mathbf{a}' \in X' : \mathbf{a}' \preceq \mathbf{a}''\}|}{|X''|} \tag{13}$$

The value $\mathcal{C}(X', X'') = 1$ means that all solutions in X'' are dominated by or equal to solutions in X'. The opposite, $\mathcal{C}(X', X'') = 0$, represents the situation when none of the solutions in X'' are covered by the set X'. Note that both $\mathcal{C}(X', X'')$ and $\mathcal{C}(X'', X')$ have to be considered, since $\mathcal{C}(X', X'')$ is not necessarily equal to $1 - \mathcal{C}(X'', X')$.

In summary, it may be said that performance metrics are hard to define and it probably will not be possible to define a single metric that allows for all criteria in a meaningful way. Along with that problem, the statistical interpretation associated with a performance comparison is rather difficult and still needs to be answered, since multiple significance tests are involved, and thus, tools from analysis of variance may be required.

In this study, we have chosen a visual presentation of the results together with the application of the metric from Definition 5. The reason for this is that we would like to investigate 1) whether test functions can adequately test specific aspects of each multiobjective algorithm and 2) whether any visual hierarchy of the chosen algorithms exists. However, for a deeper investigation of some of the algorithms (which is the subject of future work), we suggest the following metrics that allow assessment of each of the criteria listed at the beginning of this section separately.

DEFINITION 6: *Given a set of pairwise nondominating decision vectors* $X' \subseteq X$, *a neighborhood parameter* $\sigma > 0$ *(to be chosen appropriately), and a distance metric* $\| \cdot \|$. *We introduce three functions to assess the quality of* X' *regarding the parameter space:*

1. The function $\mathcal{M}_1$ *gives the average distance to the Pareto-optimal set* $\overline{X} \subseteq X$:

$$\mathcal{M}_1(X') := \frac{1}{|X'|} \sum_{\mathbf{a}' \in X'} \min\{\|\mathbf{a}' - \overline{\mathbf{a}}\|; \overline{\mathbf{a}} \in \overline{X}\} \tag{14}$$

---

[2]Recently, an alternative metric has been proposed in Zitzler (1999) in order to overcome this problem.

2. *The function $\mathcal{M}_2$ takes the distribution in combination with the number of nondominated solutions found into account:*

$$\mathcal{M}_2(X') := \frac{1}{|X'-1|} \sum_{\mathbf{a'} \in X'} |\{\mathbf{b'} \in X'; \|\mathbf{a'} - \mathbf{b'}\| > \sigma\}| \qquad (15)$$

3. *The function $\mathcal{M}_3$ considers the extent of the front described by X':*

$$\mathcal{M}_3(X') := \sqrt{\sum_{i=1}^{m} \max\{\|a_i' - b_i'\|; \mathbf{a'}, \mathbf{b'} \in X'\}} \qquad (16)$$

Analogously, we define three metrics $\mathcal{M}_1^*$, $\mathcal{M}_2^*$, and $\mathcal{M}_3^*$ on the objective space. Let $Y', \overline{Y} \subseteq Y$ be the sets of objective vectors that correspond to X' and $\overline{X}$, respectively, and $\sigma^* > 0$ and $\|\cdot\|^*$ as before:

$$\mathcal{M}_1^*(Y') := \frac{1}{|Y'|} \sum_{\mathbf{p'} \in Y'} \min\{\|\mathbf{p'} - \overline{\mathbf{p}}\|^*; \overline{\mathbf{p}} \in \overline{Y}\} \qquad (17)$$

$$\mathcal{M}_2^*(Y') := \frac{1}{|Y'-1|} \sum_{\mathbf{p'} \in Y'} |\{\mathbf{q'} \in Y'; \|\mathbf{p'} - \mathbf{q'}\|^* > \sigma^*\}| \qquad (18)$$

$$\mathcal{M}_3^*(Y') := \sqrt{\sum_{i=1}^{n} \max\{\|p_i' - q_i'\|^*; \mathbf{p'}, \mathbf{q'} \in Y'\}} \qquad (19)$$

While $\mathcal{M}_1$ and $\mathcal{M}_1^*$ are intuitive, $\mathcal{M}_2$ and $\mathcal{M}_3$ (respectively $\mathcal{M}_2^*$ and $\mathcal{M}_3^*$) need further explanation. The distribution metrics give a value within the interval $[0, |X'|]$ $([0, |Y'|])$ that reflects the number of σ-niches (σ^*-niches) in X' (Y'). Obviously, the higher the value, the better the distribution for an appropriate neighborhood parameter (e.g., $\mathcal{M}_2^*(Y') = |Y'|$ means that for each objective vector there is no other objective vector within σ^*-distance to it). The functions $\mathcal{M}_3$ and $\mathcal{M}_3^*$ use the maximum extent in each dimension to estimate the range to which the front spreads out. In the case of two objectives, this equals the distance of the two outer solutions.

6 Comparison of Different Evolutionary Approaches

6.1 Methodology

We compare eight algorithms on the six proposed test functions:

1. RAND: A random search algorithm.

2. FFGA: Fonseca and Fleming's multiobjective EA.

3. NPGA: The Niched Pareto Genetic Algorithm.

4. HLGA: Hajela and Lin's weighted-sum based approach.

5. VEGA: The Vector Evaluated Genetic Algorithm.

6. NSGA: The Nondominated Sorting Genetic Algorithm.

7. **SOEA**: A single-objective evolutionary algorithm using weighted-sum aggregation.

8. **SPEA**: The Strength Pareto Evolutionary Algorithm.

The multiobjective EAs, as well as **RAND**, were executed 30 times on each test problem, where the population was monitored for nondominated solutions, and the resulting nondominated set was taken as the outcome of one optimization run. Here, **RAND** serves as an additional point of reference and randomly generates a certain number of individuals per generation according to the rate of crossover and mutation (but neither crossover and mutation nor selection are performed). Hence, the number of fitness evaluations was the same as for the EAs. In contrast, 100 simulation runs were considered in the case of **SOEA**, each run optimizing towards another randomly chosen linear combination of the objectives. The nondominated solutions among all solutions generated in the 100 runs form the trade-off front achieved by **SOEA** on a particular test function.

Independent of the algorithm and the test function, each simulation run was carried out using the following parameters:

Number of generations	:	250
Population size	:	100
Crossover rate	:	0.8
Mutation rate	:	0.01
Niching parameter σ_{share}	:	0.48862
Domination pressure t_{dom}	:	10

The niching parameter was calculated using the guidelines given in Deb and Goldberg (1989) assuming the formation of ten independent niches. Since **NSGA** uses genotypic fitness sharing on $\mathcal{T}_5$, a different value, $\sigma_{\text{share}} = 34$, was chosen for this particular case. Concerning **NPGA**, the recommended value for $t_{\text{dom}} = 10\%$ of the population size was taken (Horn and Nafpliotis, 1993). Furthermore, for reasons of fairness, **SPEA** ran with a population size of 80 where the external nondominated set was restricted to 20.

Regarding the implementations of the algorithms, one chromosome was used to encode the m parameters of the corresponding test problem. Each parameter is represented by 30 bits; the parameters $x_2, \ldots, x_m$ only comprise 5 bits for the deceptive function $\mathcal{T}_5$. Moreover, all approaches except **FFGA** were realized using binary tournament selection with replacement in order to avoid effects caused by different selection schemes. Furthermore, since fitness sharing may produce chaotic behavior in combination with tournament selection, a slightly modified method is incorporated here, named *continuously updated sharing* (Oei et al., 1991). As **FFGA** requires a generational selection mechanism, stochastic universal sampling was used in the **FFGA** implementation.

6.2 Simulation Results

In Figures 1–6, the nondominated fronts achieved by the different algorithms are visualized. Per algorithm and test function, the outcomes of the first five runs were unified, and then the dominated solutions were removed from the union set; the remaining points are plotted in the figures. Also shown are the Pareto-optimal fronts (lower curves), as well as additional reference curves (upper curves). The latter curves allow a more precise evaluation of the obtained trade-off fronts and were calculated by adding $0.1 \cdot |\max\{f_2(\mathbf{x})\} - \min\{f_2(\mathbf{x})\}|$ to the f_2 values of the Pareto-optimal points. The space between Pareto-optimal and
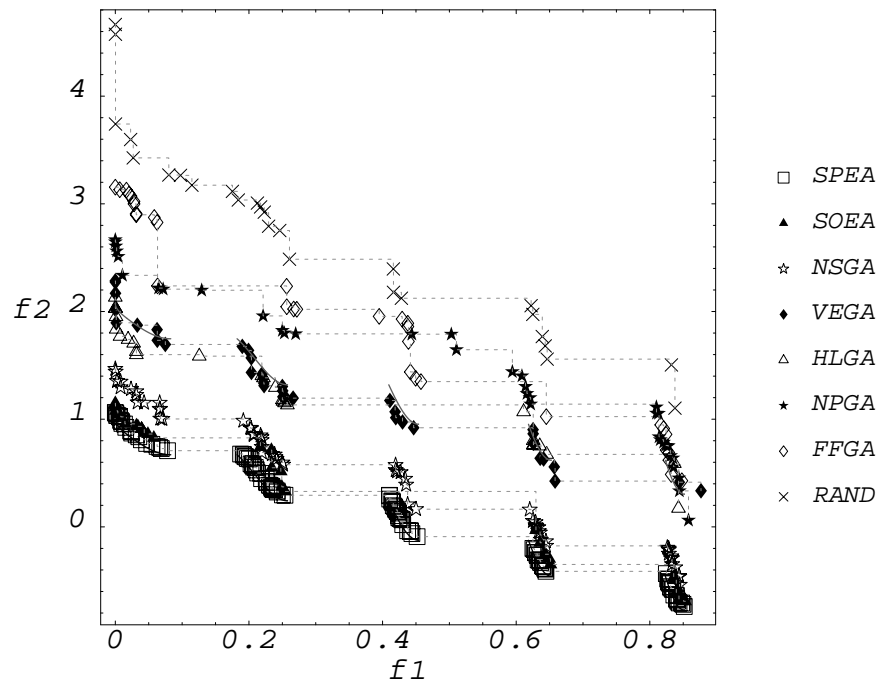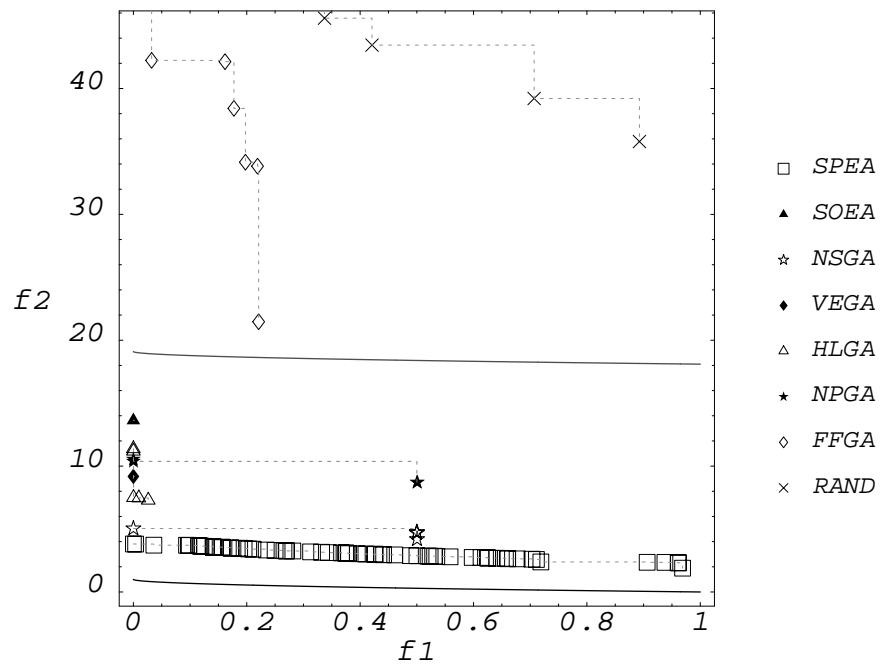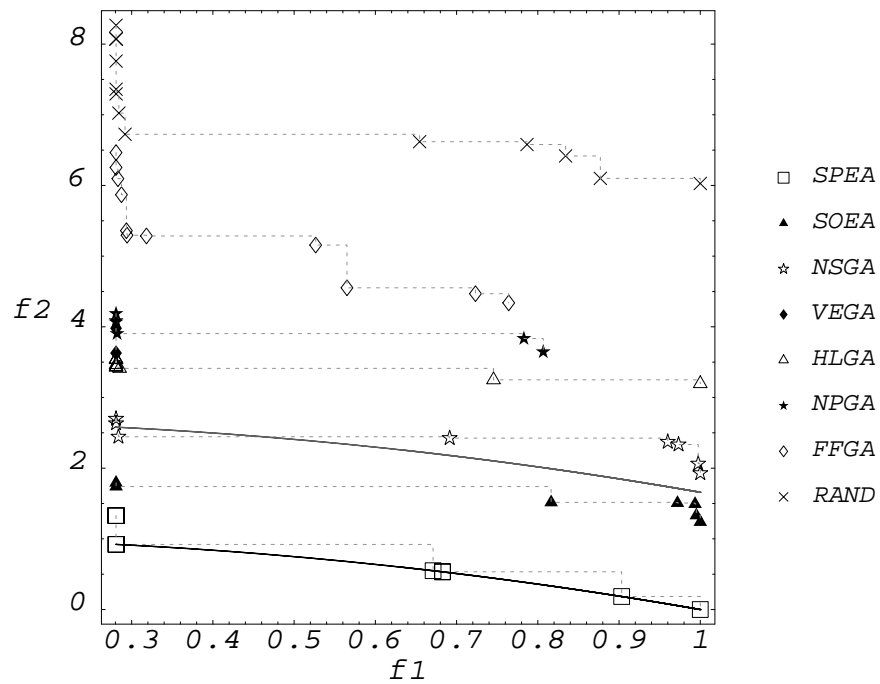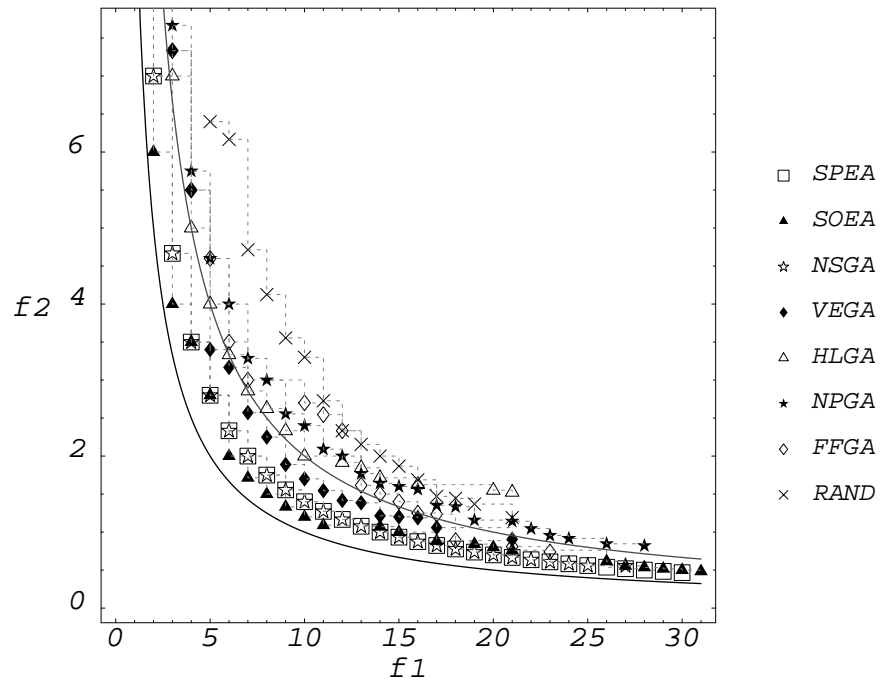
Figure 1: Test function $\mathcal{T}_1$ (convex).

Figure 2: Test function $\mathcal{T}_2$ (nonconvex).
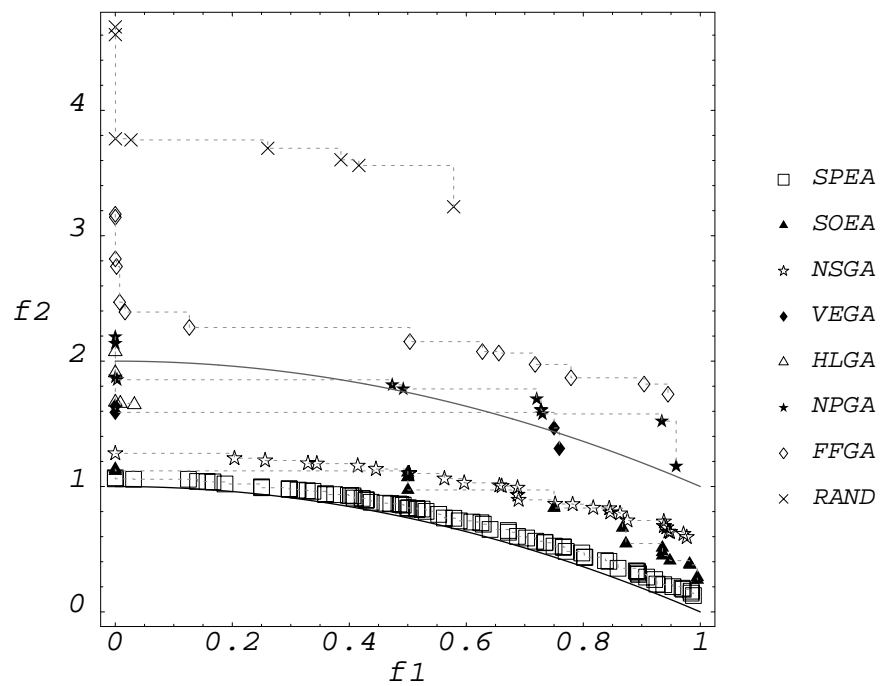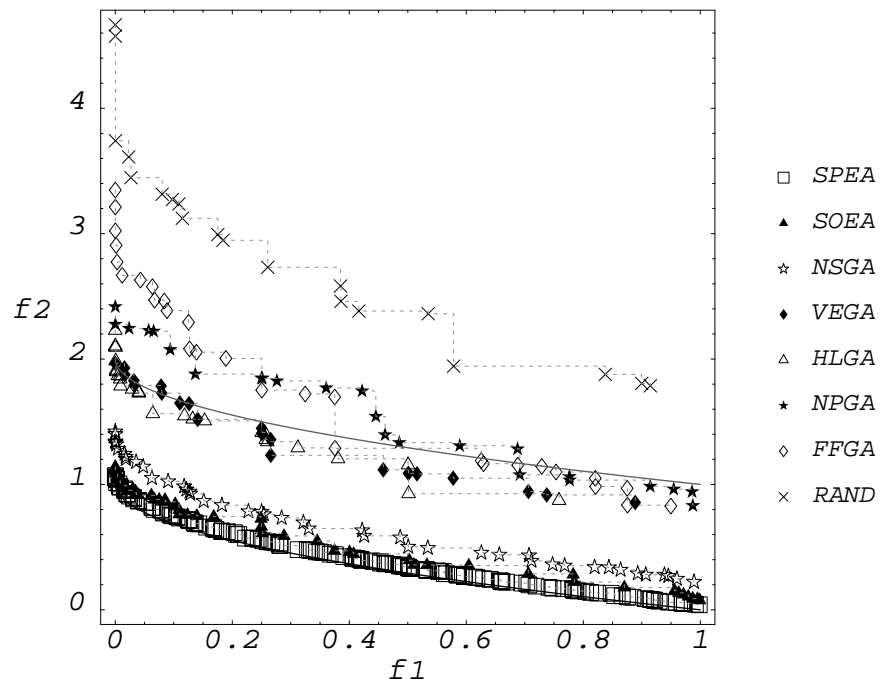
Figure 3: Test function $\mathcal{T}_3$ (discrete).

Figure 4: Test function $\mathcal{T}_4$ (multimodal).

Figure 5: Test function $\mathcal{T}_5$ (deceptive).

Figure 6: Test function $\mathcal{T}_6$ (nonuniform).

reference fronts represents about 10% of the corresponding objective space. However, the curve resulting from the deceptive function $\mathcal{T}_5$ is not appropriate for our purposes, since it lies above the fronts produced by the random search algorithm. Instead, we consider all solutions with $g(\mathbf{x}) = 10 \cdot 2$, i.e., for which the parameters are set to the deceptive attractors ($v(u(x_i)) = 2$ for $x_2, \ldots, x_{11}$).

In addition to the graphical presentation, the different algorithms were assessed in pairs using the $\mathcal{C}$ metric from Definition 5. For an ordered algorithm pair (A_1, A_2), there is a sample of 30 $\mathcal{C}$ values according to the 30 runs performed. Each value is computed on the basis of the nondominated sets achieved by A_1 and A_2 with the same initial population. Here, *box plots* are used to visualize the distribution of these samples (Figure 7). A box plot consists of a box summarizing 50% of the data. The upper and lower ends of the box are the upper and lower quartiles, while a thick line within the box encodes the median. Dashed appendages summarize the spread and shape of the distribution.[3] Furthermore, the shortcut REFS in Figure 7 stands for "reference set" and represents, for each test function, a set of 100 equidistant points that are uniformly distributed on the corresponding reference curve.

Generally, the simulation results prove that all multiobjective EAs do better than the random search algorithm. However, the box plots reveal that HLGA, NPGA, and FFGA do not always cover the randomly created trade-off front completely. Furthermore, it can be observed that NSGA clearly outperforms the other nonelitist multiobjective EAs regarding both distance to the Pareto-optimal front and distribution of the nondominated solutions. This confirms the results presented in Zitzler and Thiele (1998). Furthermore, it is remarkable that VEGA performs well compared to NPGA and FFGA, although some serious drawbacks of this approach are known (Fonseca and Fleming, 1995). The reason for this might be that we consider the off-line performance here in contrast to other studies that examine the on-line performance (Horn and Nafpliotis, 1993; Srinivas and Deb, 1994). On-line performance means that only the nondominated solutions in the final population are considered as the outcome, while off-line performance takes the solutions nondominated among all solutions generated during the entire optimization run into account. Finally, the best performance is provided by SPEA, which makes explicit use of the concept of elitism. Apart from $\mathcal{T}_5$, it even outperforms SOEA in spite of substantially lower computational effort and although SOEA uses an elitist strategy as well. This observation leads to the question of whether elitism would increase the performance of the other multiobjective EAs. We will investigate this matter in the next section.

Considering the different problem features separately, convexity seems to cause the least amount of difficulty for the multiobjective EAs. All algorithms evolved reasonably distributed fronts, although there was a difference in the distance to the Pareto-optimal set. On the nonconvex test function $\mathcal{T}_2$, however, HLGA, VEGA, and SOEA have difficulties finding intermediate solutions, as linear combinations of the objectives tend to prefer solutions strong in at least one objective (Fonseca and Fleming, 1995, 4). Pareto-based algorithms have advantages here, but only NSGA and SPEA evolved a sufficient number of nondominated solutions. In the case of $\mathcal{T}_3$ (discreteness), HLGA and VEGA are superior to both FFGA and NPGA. While the fronts achieved by the former cover about 25% of the reference set on average, the latter come up with 0% coverage. Among the considered test functions, $\mathcal{T}_4$ and $\mathcal{T}_5$ seem to be the hardest problems, since none of the algorithms was able to evolve a global Pareto-optimal set. The results on the multimodal problem indicate

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[3]Note that outside values are not plotted in Figure 7 in order to prevent overloading of the presentation.

Figure 7: Box plots based on the $\mathcal{C}$ metric. Each rectangle contains six box plots representing the distribution of the $\mathcal{C}$ values for a certain ordered pair of algorithms; the leftmost box plot relates to $\mathcal{T}_1$, the rightmost to $\mathcal{T}_6$. The scale is 0 at the bottom and 1 at the top per rectangle. Furthermore, each rectangle refers to algorithm A associated with the corresponding row and algorithm B associated with the corresponding column and gives the fraction of B covered by A ($\mathcal{C}(A, B)$). Consider, for instance, the top right box, which represents the fraction of solutions in the reference sets covered by the nondominated sets produced by the random search algorithm. For each test function and each optimization run, RAND covered 0% of the corresponding reference set.

that elitism is helpful here; SPEA is the only algorithm that found a widely distributed front. Remarkable is also that NSGA and VEGA outperform SOEA on $\mathcal{T}_4$. Again, the comparison with the reference set reveals that HLGA and VEGA (100% coverage) surpass NPGA (50% coverage) and FFGA (0% coverage). Concerning the deceptive function, SOEA is best, followed by SPEA and NSGA. Among the remaining EAs, VEGA appears to be preferable here, covering about 20% of the reference set, while the others cover 0% in all runs. Finally, it can be observed that the biased search space, together with the nonuniform represented Pareto-optimal front ($\mathcal{T}_6$), makes it difficult for the EAs to evolve a well-distributed nondominated set. This also affects the distance to the global optimum, as even the fronts produced by NSGA do not cover the points in the reference set.

Finally, it must be noted that the influence of the selection scheme in combination with the mutation rate has not been investigated here. This mainly concerns FFGA, which uses a different selection mechanism than the other EAs under consideration, and may provide better performance with lower mutation rates.

7 Elitism in Multiobjective Search

SPEA showed the best performance among the algorithms under consideration for the given parameter settings. As it is the only method which explicitly makes use of the concept of elitism, the question arises whether elitism is the reason for this gap in performance and whether the other EAs can be improved by the incorporation of elitism. We will briefly discuss this issue in the following.

As opposed to single-objective optimization, where the best solution is always copied into the next population, the incorporation of elitism in multiobjective EAs is substantially more complex. Instead of one best solution, we have an elite set whose size can be considerable compared to the population. This fact involves two questions which must be answered in this context:

- **Population $\Longrightarrow$ Elite Set:**
  Which solutions are kept for how long in the elite set?

- **Elite Set $\Longrightarrow$ Population:**
  When and how are which members of the elite set reinserted into the population?

Often used is the concept of maintaining an external set of solutions that are nondominated among all individuals generated so far. In each generation, a certain percentage of the population is filled up or replaced by members of the external set—these members are either selected at random (Ishibuchi and Murata, 1996) or according to other criteria, such as the period that an individual has stayed in the set (Parks and Miller, 1998). Another promising elitism approach provides the so-called ($\lambda + \mu$) selection, mainly used in the area of evolutionary strategies (Bäck, 1996), where parents and offspring compete against each other. Rudolph (1998) examines a simplified version of a multiobjective EA originally presented in Kursawe (1991) that is based on (1+1) selection.

In this study, the elitism mechanism proposed in Zitzler and Thiele (1999) was generalized and implemented in FFGA, NPGA, HLGA, VEGA, and NSGA as follows: Let P denote the current population of size N and $\overline{P}$ denote a second, external population that keeps the nondominated solutions found so far; the size of $\overline{P}$ is restricted to $\overline{N}$.
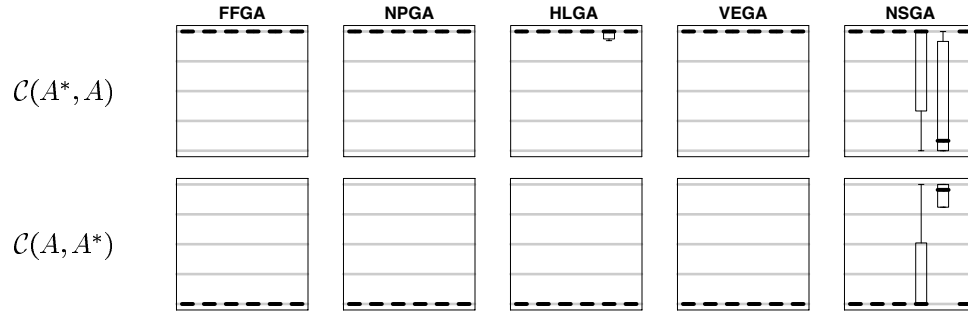
Figure 8: Box plots comparing each nonelitism algorithm A with its elitism-variant A^*.

Step 1: Generate the initial population P and set $\overline{P} = \emptyset$.

Step 2: Set $P' = P + \overline{P}$ (multiset union) and perform fitness assignment on the extended population P' of size $N' = N + \overline{N}$.

Step 3: Update external population by copying all nondominated members of P to $\overline{P}$ and afterwards removing double or dominated individuals from $\overline{P}$.

Step 4: If $|\overline{P}| > \overline{N}$, then calculate reduced nondominated set $\overline{P_r}$ of size $\overline{N}$ by clustering and set $\overline{P} = \overline{P_r}$.

Step 5: Select N individuals out of the N' individuals in P' and perform crossover and mutation to create the next population P''.

Step 6: Substitute P by P'' and go to Step 2 if the maximum number of generations is not reached.

The elitism variants of the algorithms are marked by an asterisk in order to distinguish them from the techniques originally proposed by the corresponding authors. Note that the clustering procedure in Step 4 requires a distance metric. In case of NSGA*, the phenotypic distance on the parameter space was considered, while the other algorithms used the phenotypic distance on the objective space.

The results for $\mathcal{T}_1$ and $\mathcal{T}_2$ are shown in Figures 9 and 10.[4] Obviously, elitism is helpful on these two functions, although the visual presentation has to be interpreted with care as only five runs are considered. For instance, NSGA* and SPEA seem to perform equally well here using those particular parameter settings. Moreover, the figures indicate that elitism can even help multiobjective EAs to surpass the performance of a weighted-sum single-objective EA in spite of significantly lower computational effort. However, both test functions and the metric used are not sufficient here to also compare the elitist variants with each other. Testing different elitist strategies and different elitist multiobjective EAs on more difficult test functions will be the subject of future work.

Nevertheless, we have compared each algorithm with its elitist variant based on the $\mathcal{C}$ metric. As can be seen in Figure 8, elitism appears to be an important factor to improve evolutionary multiobjective optimization. Only in one case (NSGA on the deceptive problem)

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[4]The experiments were performed as described in Section 6; however, N was set to 80 and $\overline{N}$ to 20, similar to SPEA.
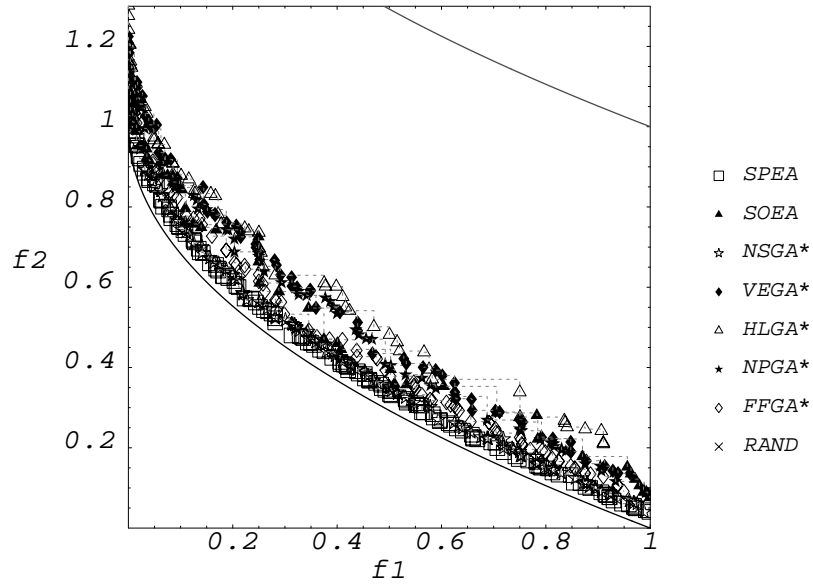
Figure 9: Results on the test function $\mathcal{T}_1$ using elitism.
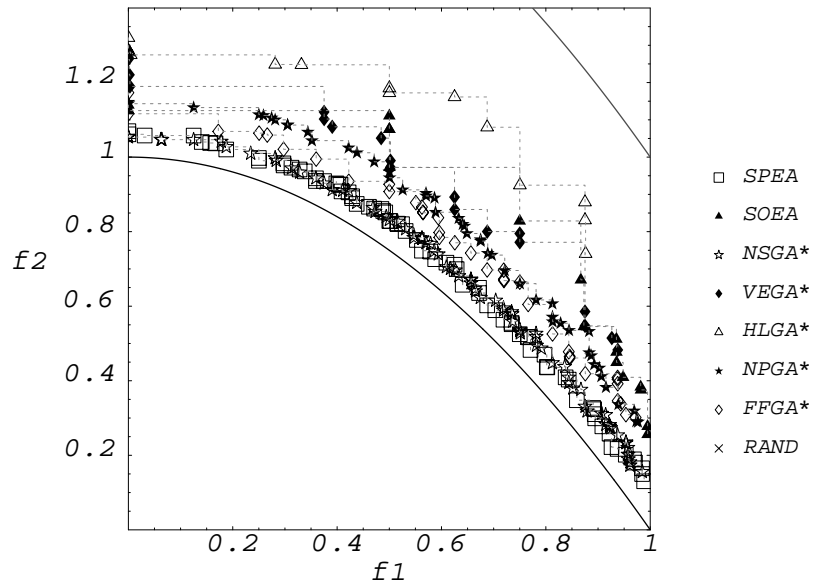

Figure 10: Results on the test function $\mathcal{T}_2$ using elitism.

was the performance of the elitist variant worse than the nonelitist version. Investigation of this matter will also be an important part of an elitism study.

8 Influence of the Population Size

On two test functions ($\mathcal{T}_4$ and $\mathcal{T}_5$), none of the algorithms under consideration was able to find a global Pareto-optimal set regarding the chosen parameters. Therefore, several runs were performed in order to investigate the influence of the population size as well as the maximum number of generations converging towards the Pareto-optimal front.

In Figures 11 and 12, the outcomes of multiple NSGA runs are visualized. On the deceptive test function $\mathcal{T}_5$, NSGA found a subset of the globally optimal solutions using a population size of 1000. In contrast, $\mathcal{T}_4$ seems to be a difficult test problem, since even a population size of $10,000$ was not sufficient to converge to the optimal trade-off front after 250 generations. This also did not change when the maximum number of generations was increased substantially ($10,000$). In the later case, the resulting front was (using a population size of 500) almost identical to the one achieved by NSGA* running 1000 generations. However, the incorporation of elitism finally enabled NSGA to find a global Pareto-optimal set after $10,000$ generations.

To sum up, one may say that the choice of the population size strongly influences the EA's capability to converge towards the Pareto-optimal front. Obviously, small populations do not provide enough diversity among the individuals. Increasing the population size, however, does not automatically yield an increase in performance, as can be observed with the multimodal function. The same holds for the number of generations to be simulated. Elitism, on the other hand, seems to be an appropriate technique to prevent premature convergence. Even after 1000 generations, better solutions, and finally Pareto-optimal solutions, evolved with $\mathcal{T}_4$.

9 Conclusions

We have carried out a systematic comparison of several multiobjective EAs on six different test functions. Major results are:

- The suggested test functions provide sufficient complexity to compare different multiobjective optimizers. Multimodality and deception seem to cause the most difficulty for evolutionary approaches. However, nonconvexity is also a problem feature that causes difficulty primarily for weighted-sum based algorithms.

- For the chosen test problems and parameter settings, a clear hierarchy of algorithms emerges regarding the distance to the Pareto-optimal front in descending order of merit:

  1. SPEA
  2. NSGA
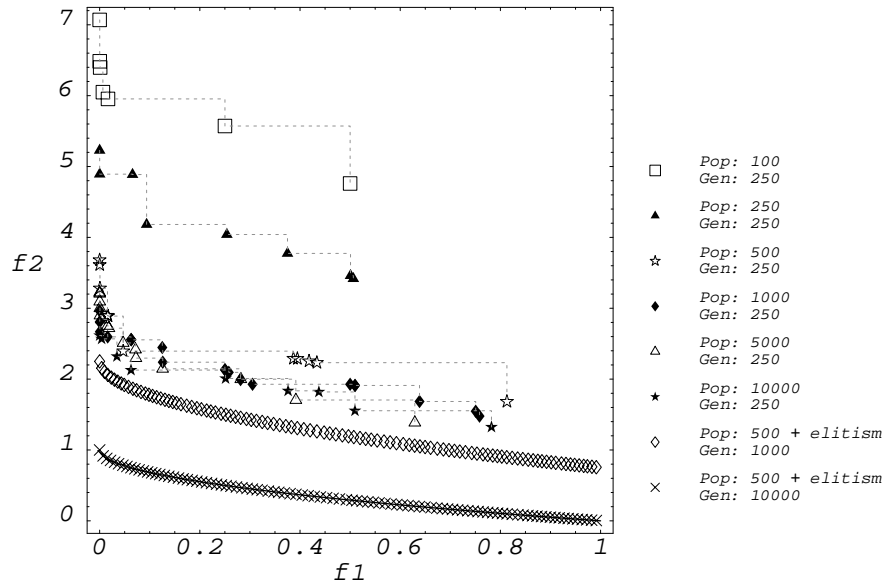 3. VEGA
  4. HLGA
  5. NPGA
  6. FFGA

Figure 11: Comparison of different population sizes on the test function $\mathcal{T}_4$ using NSGA. Two runs with elitism were performed for 1000 and 10,000 generations.
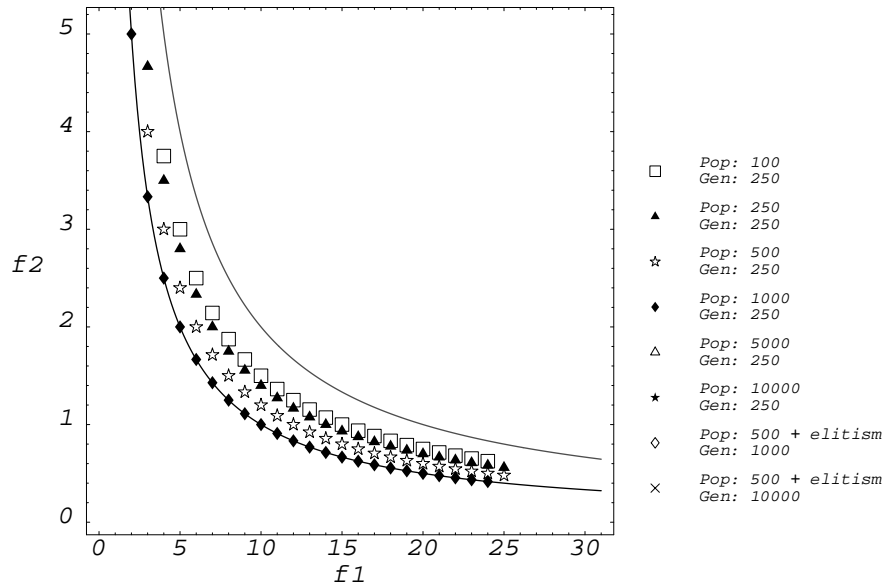

Figure 12: Comparison of different population sizes on the test function $\mathcal{T}_5$ using NSGA.

While there is a clear performance gap between SPEA and NSGA, as well as between NSGA and the remaining algorithms, the fronts achieved by VEGA, HLGA, NPGA, and FFGA are rather close together. However, the results indicate that VEGA might be slightly superior to the other three EAs, while NPGA achieves fronts closer to the global optimum than FFGA. Moreover, it seems that VEGA and HLGA have difficulties evolving well-distributed trade-off fronts on the nonconvex function. Nevertheless, the situation may be different for other parameter settings and other test problems.

- Elitism is an important factor in evolutionary multiobjective optimization. On one hand, this statement is supported by the fact that SPEA 1) clearly outperforms all algorithms on five of the six test functions and 2) is the only method among the ones under consideration that incorporates elitism as a central part of the algorithm. On the other hand, the performance of the other algorithms improved significantly when SPEA's elitist strategy was included (cf. Figure 8). Preliminary results indicate that NSGA with elitism equals the performance of SPEA.

  However, it also has to be mentioned that in certain situations, e.g., when preference information is included in the fitness assignment process and the preferences change over time, elitism may have its drawbacks. This issue has not been considered here.

This study forms a good basis for combining promising aspects of different algorithms into a new approach that shows good performance on all test problems. The experimental results suggest that such an algorithm may be constructed, where probably the nondominated sorting classification as well as elitism play a major role. Several issues must be addressed ranging from the question of how elitism is implemented most effectively, to the problem of whether distance metrics should operate on the parameter space or the objective space. In this context, the suggested performance metrics could be useful to compare techniques quantitatively, allowing a more accurate assessment than the $\mathcal{C}$ metric used here.

Finally, authors who are interested in comparing the performance of their own algorithms with those considered here can download the simulation results from `http://www.tik.ee.ethz.ch/~zitzler/testdata.html`.

Acknowledgments

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