VARIANCE OF A WEIGHTED MEAN*†

PAUL MEIER

School of Hygiene and Public Health
The Johns Hopkins University

1. *Introduction*

In many areas of statistical practice the problem arises of combining several estimates of an unknown quantity to obtain an estimate of improved precision.

For example, suppose two or more technicians have performed assays on several samples from a homogeneous material. It is desired to average their results in the best manner possible, allowing for the fact that technicians differ in precision. In general the relative precisions are not known exactly, but estimates are available from current or previous experimental data.

A similar problem arises in the analysis of incomplete block experiments. The "intra-block" and "inter-block" estimates of varietal means have different variances, and the recovery of "inter-block information" is an attempt to combine these estimates in the most efficient manner. Although similar methods are applicable, the experimental design problem differs from the simple case of weighted means in several important respects. An investigation of this problem in the case of simple lattice designs will be presented in a subsequent paper.

The model for the first problem may be described as follows. Let $u_1, \cdots, u_k$ be k estimates of a parameter μ, being independently and normally distributed with mean μ and variances $\sigma_1^2, \cdots, \sigma_k^2$. Also let $s_1^2, \cdots, s_k^2$ be independent unbiased estimates of the σ_i^2 having

*This paper is a revision of Part I of a thesis submitted to Princeton University in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

†Paper No. 287 from the Department of Biostatistics.

mean square distributions on $n_1, \cdots, n_k$ degrees of freedom respectively. We define notation as follows:

$$w_i = \frac{1}{\sigma_i^2}, \qquad w = \sum w_i, \qquad \theta_i = \frac{w_i}{w}.$$

Now if the σ_i^2 are given, the best estimate of μ is known to be $\sum \theta_i u_i$ and the variance of this estimate is $1/w$. For the case in which the variances are estimated we consider the analogous estimate, $\bar{\mu} = \sum \hat{\theta}_i u_i$, where a "$\hat{\ }$" denotes the replacement of σ_i^2 by s_i^2 in the formula for the indicated quantity. In particular we investigate $V(\bar{\mu})$, the variance of $\bar{\mu}$. It should be noted that the $\hat{\theta}_i$ are not the maximum likelihood weights. However, these weights provide an asymptotically efficient estimate of μ when the n_i are large, and have the advantage of simplicity of calculation, whereas the maximum likelihood weights require iterative procedures for their determination.

Our investigation gives first order asymptotic results for a fixed number, k, of estimates, with errors of order $\sum 1/n_i^2$. The basic results are

$$\text{(1)} \quad \text{Var}\,\{\bar{\mu}\} = \frac{1}{w}\left\{1 + 2\sum_{i=1}^{k} \frac{1}{n_i}\,\theta_i(1 - \theta_i) + 0\left(\sum_{i=1}^{k} \frac{1}{n_i^2}\right)\right\}$$

(2) An approximately unbiased estimate of $\text{Var}\{\bar{\mu}\}$ is given by

$$V^* = \frac{1}{\hat{w}}\left\{1 + 4\sum_{i=1}^{k} \frac{1}{n_i}\,\hat{\theta}_i(1 - \hat{\theta}_i)\right\}$$

(3) V^* is distributed approximately as a mean square with f degrees of freedom where

$$\frac{1}{f} = \sum_{i=1}^{k} \frac{\theta_i^2}{n_i}$$

The reader whose major interest is in applications should refer to the numerical example, Section 3, where the procedural details are described.

2. *Relation to Previous Investigations*

The problem of weighted means was investigated by Cochran in 1937 [5] for the case of a large number of estimates and constant $n_i = n$. It was shown that in this case $\text{Var}\{\bar{\mu}\} \approx 1/w\{1 + 2/(n - 4)\}$ and that an approximately unbiased estimate of this variance is given by $1/\hat{w}\{1 + 4/(n - 4)\}$. More recently numerical investigations of this case for different values of k have been carried out by Sarah Porter

Carroll [3] and by Carroll and Cochran [4]. They also give an empirical formula for estimating $\text{Var}\{\bar{\mu}\}$ when k is small.

Another investigation was made by Neyman and Scott [8]. They studied the maximum likelihood estimate of μ for the case of large k, but relaxed the requirement of equality of the n_i . It was shown that the maximum likelihood estimator is not efficient. A similar but more efficient estimator was exhibited. Estimates of the variance were not considered.

In addition it should be noted that our methods are similar to those used by Welch [11] to investigate the Behrens-Fisher problem.

3. *Numerical Example*

In an example given by Snedecor [10] the data from four experiments are used to estimate the percentage of albumin in the plasma protein of normal human subjects. The data are shown in table I.

TABLE I

Experimenter	Number of subjects (m_i)	Degrees of freedom (n_i)	Mean percentage (u_i)	Population variance estimate
A	12	11	62.3	12.986
B	15	14	60.3	7.840
C	7	6	59.5	33.433
D	16	15	61.5	18.513

If it be assumed that all four estimates have a common mean[1] and the same population variance (Bartlett's test for homogeniety of variances gives a value of $\chi^2 = 5.14$ on 3 degrees of freedom) the appropriate estimate of the mean obtained by pooling the data is 61.05%, with an estimated variance of 0.3178 on 46 degrees of freedom. However, it is evident that the results are consistent with a considerable divergence in the true population variances. If we assume only that the four populations have a common mean, but possibly different variances, we may proceed as follows.[2]

[1]It has been pointed out to me independently by Professors C. I. Bliss and Margaret Merrell that this assumption is likely to be invalid in experiments of this type. Experimenters with large variances may tend also to have large biases. The problem of weighted means when biases are present is not considered in this paper. This problem is discussed by Cochran [5], and by Yates and Cochran [12].

[2]The accuracy of the correction term has not been determined for the general case. Since one of the estimates is based on only 6 degrees of freedom the example should be viewed only as an illustration of the method.

(a) Find the estimated variances of the means by dividing the estimated population variances by the sample sizes,

$$s_i^2 = \frac{\text{pop'n. var. estimate}}{m_i}$$

(b) Find the weights by inverting these variance estimates,

$$w_i = \frac{1}{s_i^2}, \qquad w \text{ is the sum of the } w_i .$$

Steps (a) and (b) may be tabulated as shown in Table II.

TABLE II

Experimenter	Number of subjects (m_i)	Variance estimates for means (s_i^2)	Weights (w_i)
A	12	1.0822	0.9241
B	15	0.5227	1.9133
C	7	4.7761	0.2094
D	16	1.1571	0.8643
			w = 3.9110

(c) Calculate the weighted mean

$$\begin{aligned}\bar{\mu} &= \frac{\sum w_i u_i}{w} \\ &= \frac{1}{3.9110} [(0.9241)(62.3) + (1.9133)(60.3) \\ &\qquad + (0.2094)(59.5) + (0.8643)(61.5)] \\ &= 60.99\end{aligned}$$

(d) Calculate the estimated variance of $\bar{\mu}$ from (2), which may be written

$$\begin{aligned}V^* &= \frac{1}{w}\left[1 + \frac{4}{w^2} \sum \frac{1}{n_i} w_i(w - w_i)\right] \\ &= 0.2557\left\{1 + \frac{4}{(3.9110)^2}\left[\frac{1}{11}(0.9241)(2.9869)\right.\right. \\ &\quad \left.\left. + \frac{1}{14}(1.9133)(1.9977) + \frac{1}{6}(0.2094)(3.7016) + \frac{1}{15}(0.8643)(3.0467)\right]\right\} \\ &= 0.3111\end{aligned}$$

(e) Calculate the estimated equivalent degrees of freedom from (3), which may be written

$$\hat{f} = \frac{\hat{w}^2}{\sum \frac{\hat{w}_i^2}{n_i}} = \frac{(3.9110)^2}{\frac{1}{11}(0.9241)^2 + \frac{1}{14}(1.9133)^2 + \frac{1}{6}(0.2094)^2 + \frac{1}{15}(0.8643)^2}$$

$$= 38.6$$

(Note that in (d) and (e) the sample degrees of freedom rather than the sample sizes are used.)

Thus, finally, $\bar{\mu} = 60.99$ and $V^* = 0.3111$ on 38.6 degrees of freedom.

We may now compare three possible methods of treating this problem, assuming no biases are present.

(1) Assume all population variances are equal, i.e. the four samples come from a single population (equal variance method). This is the treatment used by Snedecor.

(2) Allow for the possibility of different population variances, and use the four variance estimates as if they were exactly equal to the true variances (unequal variance, uncorrected method).

(3) Allow for the possibility of different population variances, but use the above corrections to allow for the sampling variability of the weights (unequal variance, corrected method).

The results of the three methods may be summarized as follows.

TABLE III

Method	Estimate of μ	Variance of estimate	Degrees of freedom
(1) equal variance	61.05	0.3178	46
(2) uncorrected	60.99	0.2557	—
(3) corrected	60.99	0.3111	38.6

We see that the estimates of μ differ only trivially; the estimate for the equal variance method is 0.06 greater than the estimate for methods (2) and (3), a small fraction of the estimated standard deviation.

However, the variance estimates differ substantially. Methods (1) and (3) agree fairly closely, differing by approximately 2%, but the uncorrected estimate is 0.0554 less than the corrected version, a deficiency of 18%.

With respect to the stability of the variance estimate the uncorrected method yields no measure, although the 46 DF for method (1) might be taken for an upper bound. The drop from 46 DF to 38.6 DF indicated by method (3) has a negligible effect on the 5% and 1% levels of Student's t distribution.

The above example illustrates the fact that in the analysis for the case of unequal variances the required correction term may be substantial. The near equality of the results of methods (1) and (3) may not be surprising in view of the fact that Bartlett's test provided no evidence against the hypothesis of equal variances (see p. 61). Method (3), however, is the more conservative procedure and is applicable whenever the assumption of equal variances is doubtful, whether or not Bartlett's test gives a significant result.

The remainder of this paper is devoted to proving the basic relations stated in the introduction (section 4) and to an examination of the special case of two samples (section 5). The results are summarized in section 6.

4. *The General Case*

We are concerned with the variance of $\bar{\mu} = \sum_{i=1}^{k} \hat{\theta}_i u_i$, where the u_i are independently and normally distributed with mean μ and variances σ_i^2 ·

$$\hat{\theta}_i = \frac{(s_i^2)^{-1}}{\sum_{i=1}^{k} (s_i^2)^{-1}}$$

where $s_1^2, \cdots, s_k^2$ are unbiased estimates of $\sigma_1^2, \cdots, \sigma_k^2$ independently distributed as mean squares on $n_1, \cdots, n_k$ degrees of freedom. We shall be interested also in the distribution of estimates of Var $\{\bar{\mu}\}$.

The main tool of our investigation is the following theorem, an analogue of which is used by Welch [11].

Theorem If $x_1, \cdots, x_k$ are independently distributed with density functions

$$f_{n_i}(x_i) = \frac{\left(\frac{n_i}{2}\right)^{n_i/2}}{\Gamma\left(\frac{n_i}{2}\right)} x_i^{(n_i/2)-1} e^{-n_i x_i/2} \quad (0 \leq x_i < \infty)$$

and $R(x_1, \cdots, x_k)$ is a rational function with no singularities for $0 < x_1, \cdots, x_k < \infty$, then Ave $\{R(x_1, \cdots, x_k)\}$ can be expanded in an asymptotic series in the $1/n_i$. In particular

$$\text{Ave}\,\{R(x_1, \cdots, x_k)\} = R(1, \cdots, 1) + \sum_{i=1}^{k} \frac{1}{n_i} \frac{\partial^2 R}{\partial x_i^2}\Big]_{(1,\cdots,1)} + 0\left(\sum \frac{1}{n_i^2}\right).$$

The theorem is proved by means of the method of steepest descents. The function $R(x_1, \cdots, x_k)$ need not, in fact, be rational. If it has a Taylor series valid in the neighborhood of $(1, \cdots, 1)$ and does not go to infinity too rapidly the theorem remains valid. All the functions which we shall consider are rational.

a. *Variance of* $\bar{\mu}$

It will now be convenient to define quantities x_i by $s_i^2 = \sigma_i^2 x_i$. The x_i are then distributed with the density functions $f_{n_i}(x_i)$. We then have

$$\bar{\mu} = \sum \hat{\theta}_i u_i = \frac{\sum w_i \dfrac{u_i}{x_i}}{\sum \dfrac{w_i}{x_i}}$$

Since the u_i and x_i are independent we may take average values successively—first with respect to the u_i holding the x_i fixed, and then with respect to the x_i. Therefore we may write the conditional variance of $\bar{\mu}$ given the x_i as

$$V(\bar{\mu} \mid x_i) = \frac{\sum \dfrac{w_i}{x_i^2}}{\left(\sum \dfrac{w_i}{x_i}\right)^2}$$

Now $V(\bar{\mu}|x_i)$ clearly satisfies the conditions of the theorem, so we can write the variance of $\bar{\mu}$ as

$$\begin{aligned} V(\bar{\mu}) &= V(\bar{\mu} \mid 1, \cdots, 1) + \sum \frac{1}{n_i} \frac{\partial^2 V(\bar{\mu} \mid x_i)}{\partial x_i^2}\Big]_{(1,\cdots,1)} + 0\left(\sum \frac{1}{n_i^2}\right) \\ &= \frac{1}{w}\left\{1 + 2 \sum \frac{1}{n_i}\, \theta_i(1 - \theta_i) + 0\left(\sum \frac{1}{n_i^2}\right)\right\} \end{aligned}$$

b. *Estimation of* $V(\bar{\mu})$

A natural estimate of $V(\bar{\mu})$ would be $1/\hat{w}\,\{1 + 2 \sum (1/n_i)\hat{\theta}_i(1 - \hat{\theta}_i)\}$. However, this estimate has, asymptotically, a negative bias which is approximately equal in magnitude to the correction term. This may be seen as follows. Application of the theorem to the first term of the above estimate, namely $1/\hat{w}$, yields.

$$\text{Ave}\left\{\frac{1}{\hat{w}}\right\} = \frac{1}{w}\left\{1 - 2\sum \frac{1}{n_i}\,\theta_i(1 - \theta_i) + 0\left(\sum \frac{1}{n_i^2}\right)\right\}$$

Obviously to obtain an estimate with bias of order $0(\sum 1/n_i^2)$ we must double the correction term. Hence

$$V^* = \frac{1}{\hat{w}}\left\{1 + 4\sum \frac{1}{n_i}\,\hat{\theta}_i(1 - \hat{\theta}_i)\right\}$$

is an estimate of $V(\bar{\mu})$ with bias of order $0(\sum 1/n_i^2)$.

c. *Stability of the estimate of* $V(\bar{\mu})$

One measure of the stability of V^*, our estimate of $V(\bar{\mu})$, is its variance. To first order terms in the $1/n_i$ it is clear that the variance of V^* is the same as that of $1/\hat{w}$. Since

$$\text{Ave}\left\{\left(\frac{1}{\hat{w}}\right)^2\right\} = \left(\frac{1}{w}\right)^2\left\{1 - 2\sum \frac{1}{n_i}\,\theta_i(2 - 3\theta_i) + 0\left(\sum \frac{1}{n_i^2}\right)\right\} \quad \text{and}$$

$$\text{Ave}^2\left(\frac{1}{\hat{w}}\right) = \left(\frac{1}{w}\right)^2\left\{1 - 2\sum \frac{2}{n_i}\,\theta_i(1 - \theta_i) + 0\left(\sum \frac{1}{n_i^2}\right)\right\}$$

we may write

$$\text{Var}\,\{V^*\} = 2\left(\frac{1}{w}\right)^2\left\{\sum \frac{\theta_i^2}{n_i} + 0\left(\sum \frac{1}{n_i^2}\right)\right\}$$

By analogy with a mean square distribution we are tempted to use the designation of *equivalent degrees of freedom* for the quantity

$$f = \frac{1}{\sum \frac{\theta_i^2}{n_i}}\,.$$

This quantity is the same as that which appears in the study of variance components [9]. To the order of accuracy considered here $\hat{f}$ is a suitable estimate of f.

The obvious question which now arises is whether the distribution of V^* based on small n_i is satisfactorily approximated by a mean square distribution with f degrees of freedom. More particularly, will the tabulated percent points of t-tests based on V^* with $\hat{f}$ degrees of freedom be close to the true percent points? These questions cannot be answered satisfactorily in the absence of detailed investigation of the exact distributions. Pending such investigation we recommend treating

$\bar{\mu}/\sqrt{V^*}$ as a t variate with $\hat{f}$ degrees of freedom in preference to using the uncorrected quantity $\bar{\mu}/\sqrt{1/\hat{w}}$.

5. *Case of two means* ($k = 2$)

When only two estimates are to be combined the calculation of variances is simpler, permitting an exact evaluation of $V(\bar{\mu})$ and an investigation of the bias of V^*. The formula for $V(\bar{\mu} \mid x_i)$ reduces to

$$V(\bar{\mu} \mid x_i) = \frac{\dfrac{w_1}{x_1^2} + \dfrac{w_2}{x_2^2}}{\left(\dfrac{w_1}{x_1} + \dfrac{w_2}{x_2}\right)^2} = \frac{1}{w}\left\{1 + w_1 w_2 \frac{(x_1 - x_2)^2}{(w_1 x_2 + w_2 x_1)^2}\right\}$$

This is a function of the ratio x_1/x_2 which has the well known F distribution, enabling us to write

$$V(\bar{\mu}) = \text{Ave}\,\{V(\bar{\mu} \mid x_i)\}$$
$$= \frac{1}{w}\left\{1 + \frac{w_2/w_1}{\beta\left(\dfrac{n_1}{2}, \dfrac{n_2}{2}\right)} \int_0^\infty \frac{\left(1 - \dfrac{n_2}{n_1} t\right)^2}{(1 + \gamma t)^2} \frac{t^{(n_1/2)-1}}{(1 + t)^{(n_1+n_2)/2}}\, dt\right\}$$

where

$$\gamma = \frac{n_2 w_2}{n_1 w_1}$$

We refer to the second term within the braces as *the fractional correction to* $1/w$, i.e. the fraction by which $1/w$ must be increased to yield $V(\bar{\mu})$. This fractional correction to $1/w$ will be denoted by the letter C.

a. *Bounds on the increase in variance*

The variance of $\bar{\mu}$ is most conveniently described in terms of C, the fractional correction to $1/w$ which is defined above. We lose no generality if we assume that the notation has been assigned in such a manner that $\gamma \geq 1$. Using the relations

$$\frac{1}{\gamma^2(1 + t)^2} \leq \frac{1}{(1 + \gamma t)^2} \leq \frac{1}{(1 + t)^2}$$

we find that

$$\frac{2}{\gamma(n_1 + n_2 + 2)} \leq C \leq \frac{2\gamma}{n_1 + n_2 + 2}$$

with equality when $\gamma = 1$. These bounds are close only when γ is in the neighborhood of one. [Note: An upper bound to $V(\bar{\mu})$ when $k > 2$ is given by

$$V(\bar{\mu}) < \frac{1}{w}\left\{1 + \sum_{i<j} C_{ij}\right\} \qquad \text{where} \qquad C_{ij} = \frac{2\gamma_{ij}}{n_i + n_j + 2}$$

and

$$\gamma_{ij} = \max\left(\frac{n_i w_i}{n_j w_j}, \frac{n_j w_j}{n_i w_i}\right)\Bigg].$$

b. *Case with degrees of freedom proportional to variances*

A partial check on the accuracy of our approximation can be made by comparing the results with the exact values in the case $n_1/n_2 = \sigma_1^2/\sigma_2^2$, or equivalently, $n_1w_1 = n_2w_2$. In this case $\gamma = n_2w_2/n_1w_1 = 1$ and the above relation becomes an equality. Thus, for this case

$$V(\bar{\mu}) = \frac{1}{w}\left\{1 + \frac{2}{n_1 + n_2 + 2}\right\}$$

Our approximation is

$$V(\bar{\mu}) \approx \frac{1}{w}\left\{1 + 2\sum \frac{1}{n_i}\,\theta_i(1 - \theta_i)\right\} = \frac{1}{w}\left\{1 + \frac{2}{n_1 + n_2}\right\}$$

The approximation is too large by the amount

$$\frac{4}{(n_1 + n_2)(n_1 + n_2 + 2)}$$

Of more direct interest is the bias in the observed quantity

$$V^* = \frac{1}{\hat{w}}\left\{1 + 4\sum \frac{1}{n_i}\,\hat{\theta}_i(1 - \hat{\theta}_i)\right\} = \frac{1}{\hat{w}}\left\{1 + 4\hat{\theta}_1\hat{\theta}_2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right\}$$

Ave $\{V^*\}$ can be evaluated straightforwardly, giving the result

$$\text{Ave}\{V^*\} = \frac{1}{w}\left\{1 + \frac{2}{n_1 + n_2 + 2} + 8\,\frac{(n_1 + n_2)^3 - 5n_1n_2(n_1 + n_2) + 2(n_1 + n_2)^2 - 12n_1n_2}{n_1n_2(n_1 + n_2 + 2)(n_1 + n_2 + 4)(n_1 + n_2 + 6)}\right\}$$

For $n_1 = n_2 = n$ this reduces to

$$\text{Ave}\{V^*\} = \frac{1}{w}\left\{\left(1 + \frac{1}{n + 1}\right) - \frac{2}{(n + 1)(n + 3)}\right\}$$

Thus the bias is seen to be rather small, being already less than 5% when $n_1 = n_2 = 4$.

c. *Case with large variance ratio*

For $\gamma > 1$ the quantity $V(\bar{\mu} \mid x_i)$ can be expanded in a power series in $1/\gamma$. Term by term integration to evaluate $V(\bar{\mu})$ gives an asymptotic series valid for large γ. Thus

$$C \sim \frac{2}{\gamma} \frac{n_1^2 + n_1 n_2 + 4n_2}{n_1(n_1 - 2)(n_1 - 4)} - \frac{4}{\gamma^2} \frac{n_1^2 n_2 + n_1 n_2^2 + 4n_1^2 + 12n_2^2 + 12n_1 n_2}{n_1(n_1 - 2)(n_1 - 4)(n_1 - 6)} + \cdots ,$$

the error being less than the first term ignored. Unfortunately for moderate size n_1, n_2 the series is useful only for rather large γ (e.g. for $n_1 = n_2 = 10$ we require $\gamma \approx 15$ to make the error less than 0.05).

The average value of V^* can be obtained similarly. The expressions are rather complex and uninteresting so we omit them. However, it appears that for large variance ratio V^* slightly underestimates $V(\bar{\mu})$.

d. *Exact calculation of* $V(\bar{\mu})$ *for* $n_1 = n_2 = 1, 2, 3, 4, 5, 6$.

For the sake of simplicity the detailed investigation of $V(\bar{\mu})$ will be restricted to the case $n_1 = n_2 = n$. (This restriction is not essential. A similar analysis can be made when $n_1 \neq n_2$). In this case the fractional correction to $1/w$ is

$$C_n(\alpha) = \frac{\alpha}{\beta\left(\frac{n}{2}, \frac{n}{2}\right)} \int_0^\infty \frac{(1 - t)^2}{(1 + \alpha t)^2} \frac{t^{(n/2)-1}}{(1 + t)^n} \, dt, \quad \text{where } \alpha = \frac{w_2}{w_1} = \frac{\sigma_1^2}{\sigma_2^2}$$

Now it is evident that if σ_1^2 and σ_2^2 are interchanged the fractional correction to $1/w$ will remain the same. Thus $C_n(1/\alpha) = C_n(\alpha)$, a fact which can be verified by direct substitution. It is sufficient, therefore, to calculate $C_n(\alpha)$ for $\alpha \geq 1$.

An indication of the behavior of $C_n(\alpha)$ in the neighborhood of $\alpha = 1$ is given by its derivatives. We have

$$C_n \Big]_{\alpha=1} = \frac{1}{n + 1}$$

$$\frac{dC_n}{d\alpha} \Big]_{\alpha=1} = 0$$

$$\frac{d^2C_n}{d\alpha^2} \Big]_{\alpha=1} = - \frac{(n - 6)}{2(n + 1)(n + 3)}$$

It appears that at $\alpha = 1$, $C_n(\alpha)$ has a local minimum when $n < 6$ and a maximum when $n > 6$. For $n = 6$

$$\frac{d^2C_6}{d\alpha^2} \Big]_{\alpha=1} = \frac{d^3C_6}{d\alpha^3} \Big]_{\alpha=1} = 0, \qquad \text{but} \qquad \frac{d^4C_6}{d\alpha^4} \Big]_{\alpha=1} = - \frac{2}{1155}$$

so $C_6(\alpha)$ also has a maximum at $\alpha = 1$.

Now to evaluate $C_n(\alpha)$ for small n we note that the integrand is a rational function of $\sqrt{t}$ and can be integrated by the methods of elementary calculus. This procedure becomes rather tedius for even moderate n and the form of the result, being rather complex, does not seem to warrant a considerable listing. The formulas and numerical results for $n = 1, 2, 3, 4, 5, 6$ are shown in Tables IV and V

TABLE IV
FRACTIONAL CORRECTION TO $1/w$

n	$C_n(\alpha)$	$\lim_{\alpha \to 1}$	Asymptotic value for large α
1	$\dfrac{\sqrt{\alpha}\,(\alpha+1)(\alpha^2-6\alpha+1)+8\alpha^2}{2\alpha(\alpha-1)^2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{\alpha}$
2	$\dfrac{(\alpha^2+6\alpha+1)}{(\alpha-1)^2}-\dfrac{4\alpha(\alpha+1)}{(\alpha-1)^3}\ln\alpha$	$\frac{1}{3}$	1
3	$\dfrac{4\sqrt{\alpha}\,(\alpha+1)(\alpha^2+10\alpha+1)}{(\alpha-1)^4}-\dfrac{4\alpha(5\alpha^2+14\alpha+5)}{(\alpha-1)^4}$	$\frac{1}{4}$	$\dfrac{4}{\sqrt{\alpha}}$
4	$\dfrac{6\alpha(\alpha+1)(\alpha^2+6\alpha+1)}{(\alpha-1)^5}\ln\alpha-\dfrac{4\alpha(5\alpha^2+14\alpha+5)}{(\alpha-1)^4}$	$\frac{1}{5}$	$6\,\dfrac{\ln\alpha}{\alpha}$
5	$\dfrac{4}{3}\dfrac{\alpha}{(\alpha-1)^6}(7\alpha^4+148\alpha^3+330\alpha^2+148\alpha+7)$ $-\dfrac{64}{3}\dfrac{\sqrt{\alpha}}{(\alpha-1)^6}\alpha(\alpha+1)(3\alpha^2+14\alpha+3)$	$\frac{1}{6}$	$\dfrac{28}{3\alpha}$
6	$4\dfrac{\alpha}{(\alpha-1)^6}(\alpha^4+41\alpha^3+96\alpha^2+41\alpha+1)$ $-60\dfrac{\alpha^2}{(\alpha-1)^7}(\alpha+1)(\alpha^2+4\alpha+1)\ln\alpha$	$\frac{1}{7}$	$\dfrac{4}{\alpha}$

It is interesting to note that for $n > 2, C_n(\alpha)$ is not very far from its value at $\alpha = 1$, even when α is as large as 20. Thus for $3 \leq n \leq 6$ or even for larger n the approximation $C_n(\alpha) \sim 1/(n+1)$ may be satisfactory for most purposes.

e. *Bias of V^* for the case $n_1 = n_2 = 10$*

The calculation of Ave $\{V^*\}$ by the above method is considerably

TABLE V
FRACTIONAL CORRECTION TO $1/w$

α	$C_1(\alpha)$	$C_2(\alpha)$	$C_3(\alpha)$	$C_4(\alpha)$	$C_5(\alpha)$	$C_6(\alpha)$
1	.5000	.3333	.2500	.2000	.1667	.1429
2	.5754	.3645	.2641	.2061	.1686	.1424
3	.6906	.4083	.2820	.2126	.1694	.1402
4	.8056	.4480	.2963	.2163	.1683	.1367
5	.9146	.4823	.3070	.2179	.1659	.1326
6	1.0172	.5119	.3149	.2180	.1629	.1283
7	1.1137	.5376	.3207	.2171	.1595	.1240
8	1.2050	.5601	.3250	.2155	.1559	.1199
9	1.2917	.5801	.3281	.2135	.1523	.1159
10	1.3742	.5979	.3303	.2112	.1488	.1121
20	2.0492	.7095	.3297	.1858	.1193	.0841
50	3.3891	.8274	.2928	.1354	.0759	.0487
100	4.8847	.8899	.2486	.0970	.0487	.0292

more difficult than the calculation of $V(\bar{\mu})$. An alternative procedure which applies for any n is to expand $1/(1 + \alpha t)^2$ in a power series in $1/(1 + t)$. We thus obtain convergent series expansions for C and for Ave $\{V^*\}$. These series have the disadvantage of converging rather slowly when α is large. Calculations were made for the case $n_1 = n_2 = 10$ and are shown in Table VI. The quantity Ave $\{1/\hat{w}\}$ is also shown for comparison as $1/\hat{w}$ might be taken a priori as a reasonable estimate of $V(\bar{\mu})$.

It appears that for $n_1 = n_2 = 10$, the bias of the uncorrected variance estimate, $1/\hat{w}$, is in the neighborhood of 15% for small α and

TABLE VI
TRUE VARIANCE AND AVERAGE ESTIMATED VARIANCES OF $\bar{\mu}$; $1/w = 1$
($n_1 = n_2 = 10$)

α	True Variance $V(\bar{\mu})$	Average estimated variances		Biases percent error in	
		Ave $\{V^*\}$	Ave $\{1/\hat{w}\}$	Ave $\{V^*\}$	Ave $\{1/\hat{w}\}$
1	1.091	1.077	0.909	−1.3%	−16.7%
2	1.088	1.073	0.917	−1.3%	−15.7%
3	1.082	1.068	0.928	−1.3%	−14.3%
4	1.077	1.062	0.936	−1.4%	−13.1%
α large	$1 + 1/\alpha$	$1 + 7/10\alpha$	$1 - 1/2\alpha$	$-30/(1 + \alpha)\%$	$-150/(1 + \alpha)\%$

decreases rather slowly. The bias of V^* is under 2%. The amount by which the correction reduces the bias varies from about 90% for small α to 80% for large α.

6. *Summary*

In combining estimates of an unknown parameter it may be reasonable to assume that the individual estimates are unbiased, but are derived from populations with possibly different variances. This paper provides first order corrections to the estimated variance of a weighted mean when the weights are the reciprocals of the estimated variances of the individual estimates.

In addition to the general results, described in the introduction, a special investigation is made for the case in which only two estimates are to be combined. Upper and lower bounds for the variance of the weighted mean are determined. Exact formulas for the variance of the weighted mean are given for the case with degrees of freedom proportional to the weights and for the case of arbitrary weights having the same number of degrees of freedom when this is less than or equal to 6.

The biases of the corrected and uncorrected methods are compared for the case of two estimates with both weights based on 10 degrees of freedom. The correction reduces the maximum bias from 16% to under 2%.

ACKNOWLEDGEMENT

The writer would like to express his thanks to Professor John W. Tukey for suggesting this problem and for his advice and encouragement throughout the investigation.

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