

The CES Potential: Information Friction and Complementary Production

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Abstract

This paper derives the CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$ with $q = \rho$, connecting the CES aggregate—the unique constant-elasticity aggregator—to the Tsallis entropy—the unique entropy satisfying continuity, maximality, and pseudo-additivity. The companion paper ([Smirl, 2026a](#)) establishes that a single curvature parameter $K = (1 - \rho)(J - 1)/J$ simultaneously controls superadditivity, correlation robustness, strategic independence, and network scaling. The present paper shows how information friction degrades this curvature. The *effective curvature theorem* establishes that exploitable curvature is $K_{\text{eff}} = K \cdot (1 - T/T^*(\rho))^+$, where T^* is a critical information friction increasing in complementarity. Three of the four roles degrade non-uniformly—correlation robustness quadratically, superadditivity and strategic independence linearly—predicting a specific crisis sequence: diversification fails before production complementarities, which fail before strategic stability. The Tsallis generalization with $q = \rho$ replaces the Shannon chain rule with pseudo-additivity, producing q -exponential equilibrium distributions with compact support for complements and power-law tails for substitutes, a $1/(2 - q)$ correction to the variance-response identity, and a classification of all dynamical results into exact survivors, q -corrected, and structurally changed. Optimal firm scope and the integration boundary are derived in (ρ, T) space, nesting Williamson's governance structures as the qualitative limit. The CES potential serves as a welfare loss function with Lyapunov properties, and the architecture of the full (ρ, T) framework is established. A companion paper ([Smirl, 2026b](#)) demonstrates the framework by deriving key results from six areas of economic theory. Two empirical tests confirm the framework: a banking regulation test using 147 countries predicts 2007–2009 crisis severity via the $\sigma \times T$ interaction ($p = 0.016$), and a manufacturing tail distribution test confirms q -exponential superiority over exponential in 12 of 17 sectors.

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1 Introduction

Modern economics is divided into two broad traditions. The first—encompassing production theory, consumer choice, trade, and growth—concerns how inputs aggregate into outputs and how resources are allocated. Its canonical tool is the production or utility function, and the CES (Constant Elasticity of Substitution) aggregate is its workhorse.¹ The second tradition—encompassing information economics, mechanism design, contract theory, search theory, social choice, and behavioral economics—concerns who knows what, who can commit to what, and how agents behave under uncertainty. Its canonical tools are Bayesian updating, incentive compatibility constraints, and entropy measures of information.

These traditions developed largely independently. They use different mathematical machinery and publish in different journals. Yet a striking pattern emerges when one examines the mathematical structure: every major result in the information tradition can be expressed as a CES aggregate degraded by an entropy term, with severity controlled by the CES curvature parameter.

This paper proposes a unification. The argument is that economic theory is generated by two functions:

1. The **CES generating function** $\Phi(\rho)$, governing aggregation, allocation, and market structure;
2. The **Tsallis entropy** S_q with $q = \rho$, governing information, incentives, and bounded rationality;

connected through the **CES potential**:

$$\boxed{\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q, \quad q = \rho} \quad (1)$$

where $\rho \in (-\infty, 1]$ is the CES curvature parameter (equivalently, $\sigma = 1/(1 - \rho)$ is the elasticity of substitution), $T \geq 0$ is the information friction, and $S_q = (1 - \sum p_j^q)/(q - 1)$ is the Tsallis entropy with $q = \rho$ locked by the emergence theorem. Shannon entropy is the $q \rightarrow 1$ limit, applicable to perfect substitutes.

Standard economics is the $T = 0$ limit of this framework—perfect information, deterministic optimization, no search frictions, no behavioral anomalies. The rich structure of information economics, mechanism design, contract theory, social choice, and behavioral economics emerges at $T > 0$.

¹The CES function appears as: the Solow production function with capital-labor substitution ([Solow, 1956](#)); the Armington aggregate in trade ([Armington, 1969](#)); the Dixit-Stiglitz love-of-variety aggregate underlying new trade theory ([Dixit and Stiglitz, 1977](#)), new economic geography ([Krugman, 1991](#)), and endogenous growth ([Romer, 1990](#)); the price-setting aggregate in New Keynesian DSGE models; and the Atkinson inequality index in welfare economics ([Atkinson, 1970](#)).

The CES aggregate is not an arbitrary functional form. It is the *unique* constant-elasticity aggregator, and its curvature parameter ρ simultaneously controls at least four distinct economic properties (the quadruple role established in the companion paper (Smirl, 2026a)). The Tsallis entropy S_q with $q = \rho$ is not an arbitrary information measure—it is the *unique* entropy satisfying continuity, maximality, and pseudo-additivity (Tsallis, 1988; Santos, 1997), where pseudo-additivity replaces the Shannon chain rule to capture the non-additive information costs inherent in complementary production ($\rho < 1$). The identification $q = \rho$ is forced by the emergence theorem (Smirl, 2026a), not fitted as a free parameter. The framework is not a modeling choice—it is the only framework consistent with constant elasticity, axiomatic information measurement matched to production complementarity, and variational consistency simultaneously.

1.1 Summary of Results

The main results are:

1. **The CES potential** (Section 3): The unique framework satisfying constant elasticity, pseudo-additive information measurement ($q = \rho$), and variational consistency. The q -exponential equilibrium distribution exhibits compact support for complements ($\rho < 1$) and power-law tails for substitutes ($\rho > 1$).
2. **The effective curvature theorem** (Section 4): Information friction endogenously reduces exploitable complementarity to $K_{\text{eff}} = K(1 - T/T^*)^+$, with non-uniform degradation predicting a specific crisis sequence: diversification fails before production complementarities, which fail before strategic stability.
3. **The Tsallis generalization** (Section 5): All dynamical results are classified into exact survivors (Category A), q -corrected by $1/(2 - q)$ (Category B), and structurally changed with $\exp \rightarrow \exp_q$ (Category C). The correction amplifies fluctuations for complements and dampens them for substitutes.
4. **Optimal firm scope and integration boundaries** (Section 6–Section 6.2): Derived in (ρ, T) space, nesting Williamson's governance structures as the qualitative limit. Multi-tier supply chain architecture follows a monotone integration rule based on tier-specific K_ℓ/T_ℓ ratios.
5. **Welfare loss function** (Section 9): The CES potential has Lyapunov properties—bounded below, monotonically decreasing along adjustment paths—providing a welfare criterion for institutional design.
6. **The (ρ, T) architecture** (Section 10): The full framework is organized by two parameters, with ρ determining structure and T determining friction, and the interaction $\rho \times T$ determining institutional viability.

The companion paper (Smirl, 2026b) demonstrates the framework by deriving key results from six areas of economic theory: adverse selection (Akerlof), mechanism design (Myerson), social choice (Arrow), search and matching (DMP), contract theory (Hart-Moore/Williamson), and behavioral economics (Kahneman-Tversky). Each derivation yields new ρ -dependent predictions that the original theories cannot generate.

Two empirical tests confirm the framework’s predictions: a banking regulation test (Section 8.1) and a manufacturing tail distribution test (Section 8.2). Four additional predictions—management–technology complementarity, within-industry productivity dispersion, AI adoption priority, and crisis sequence timing—are derived and shown to be testable with existing data (Section 7).

1.2 Organization

The paper proceeds as follows. Section 2 restates the CES framework from the companion paper. Section 3 constructs the CES potential and derives the q -exponential equilibrium. Section 4 derives the effective curvature theorem and crisis sequence. Section 5 classifies all dynamical results under the Tsallis generalization. Section 6–Section 6.2 derive optimal firm scope and integration boundaries. Section 9 establishes the CES potential as a welfare loss function. Section 10 presents the unified architecture. Section 7–Section 8 develop predictions and empirical tests. Section 11 discusses methodological implications. Section 12 concludes.

1.3 Related Literature

The components of this framework—CES aggregation, Tsallis entropy, and the CES potential connection—exist independently across disparate literatures. The contribution is recognizing their unity: that CES and Tsallis entropy are the two generating functions of economics, connected through the CES potential, with the entire edifice parameterized by (ρ, T) . This subsection maps each body of existing work into the (ρ, T) parameter space.

CES as a generating function across fields. The CES aggregate has appeared independently in production theory (Arrow et al., 1961), trade (Armington, 1969; Ethier, 1982), industrial organization (Dixit and Stiglitz, 1977), and economic geography (Krugman, 1991). Sato (1967) showed that nested CES enables multi-level aggregation with different ρ at each tier. Anderson, de Palma, and Thisse (1992) established the formal duality between the CES demand system and the logit choice model, providing the micro-foundation connecting the two generating functions of this paper.

Entropy and information theory in economics. [Theil \(1967\)](#) first imported Shannon entropy into economics. The rational inattention program initiated by [Sims \(2003\)](#) derives logit choice from Shannon capacity constraints, interpreting T as the shadow price of attention. [Matějka and McKay \(2015\)](#) proved that the logit form is the unique solution to rational inattention problems with unrestricted prior, making $T > 0$ the canonical model of bounded rationality.

Statistical mechanics and economic equilibrium. [Jaynes \(1957\)](#) showed that the logit equilibrium distribution maximizes entropy subject to an energy constraint—the direct physical analogue of the economic CES potential minimization. [Foley \(1994\)](#) introduced statistical equilibrium into general equilibrium theory. [Smith and Foley \(2008\)](#) established the formal correspondence between the Helmholtz free energy in thermodynamics and economic general equilibrium. The econophysics program ([Yakovenko and Rosser, 2009](#); [Drăgulescu and Yakovenko, 2000](#); [Bouchaud and Mézard, 2000](#); [Gabaix, 2009](#)) demonstrates that statistical mechanics tools apply to economic systems; the present paper identifies the CES aggregation structure that determines *which* equilibrium the entropy dynamics select.

Active inference and the free energy principle. [Friston \(2010\)](#) introduced the free energy principle in neuroscience. The present paper shares the conviction that a single variational principle organizes complex adaptive behavior, but the two frameworks minimize their respective potentials over different state spaces. The key object that active inference lacks is the ρ parameter: two economies with identical information precision but different ρ will exhibit qualitatively different equilibria, regime shifts, and welfare losses.

Closest antecedents. [Caplin and Dean \(2015\)](#) provided the axiomatic foundations for the T axis through posterior-separable stochastic choice. On the ρ axis, [Costinot \(2009\)](#) showed that a single log-supermodularity condition—mathematically equivalent to the CES complementarity condition $\rho < 1$ —unifies Ricardian and Heckscher-Ohlin trade theory. What Caplin–Dean and Costinot leave separate, the CES potential framework joins: the log-supermodularity driving Costinot’s sorting *is* the CES curvature K , and the stochastic choice axiomatized by Caplin–Dean *is* the $T > 0$ departure from deterministic optimization.

The O-ring theory. [Kremer \(1993\)](#) established that complementary production (which he modeled as multiplicative, the limit $\rho \rightarrow -\infty$ of CES) creates skill sorting, wage inequality, and development traps. The CES potential framework generalizes Kremer’s insight from the multiplicative limit to the full CES family: the phenomena he identifies—

positive assortative matching, wage inequality, and development traps—all depend on K , and their intensity varies continuously with ρ . The O-ring model’s prediction that “a single low-skill worker destroys the entire output” is the extreme complementarity ($\rho \rightarrow -\infty$, Leontief) special case; for finite $\rho < 1$, the prediction is softer: a low-skill worker degrades output by $O(K \cdot \|\delta\|^2/c^2)$, which is continuous in K . This matters empirically because most production lies between Kremer’s extreme and perfect substitutability.

Endogenous growth and CES. The CES elasticity of substitution between capital and labor has been central to the growth theory debate since [Solow \(1956\)](#). [Jones \(1995\)](#) showed that balanced growth requires the capital-labor elasticity $\sigma = 1$ (Cobb-Douglas). When $\sigma > 1$: capital deepening alone drives growth (no need for technology). When $\sigma < 1$: diminishing returns to capital are too strong, and growth requires sustained technological progress. The CES potential framework adds the information dimension: even if $\sigma > 1$ at the production frontier, the effective elasticity $\sigma_{\text{eff}} = 1/(1 - \rho_{\text{eff}})$ depends on T . A high- T economy may have $\sigma_{\text{eff}} < 1$ even if the underlying technology has $\sigma > 1$, trapping the economy in a Solow-type convergence path despite frontier technology. This provides a CES-theoretic explanation for why technology transfer alone does not produce convergence—the information environment must support the exploitation of the transferred technology’s complementarity.

Each of these references uses one piece of the framework. The contribution of this paper is recognizing that these pieces are manifestations of a single object—the Tsallis CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$ with $q = \rho$. The CES quadruple role theorem is developed in the companion paper ([Smirl, 2026a](#)).

2 Preliminaries: The CES Framework

This section restates the key definitions and results from the companion paper ([Smirl, 2026a](#)) that are used throughout. Proofs are in that paper; here the results are stated without proof.

2.1 CES Production Technology

Consider $J \geq 2$ inputs $x_1, \dots, x_J > 0$. The CES aggregate is:

$$F = \left(\frac{1}{J} \sum_{j=1}^J x_j^\rho \right)^{1/\rho}, \quad \rho \in (-\infty, 1], \rho \neq 0 \tag{2}$$

with the convention $F = (\prod x_j)^{1/J}$ at $\rho = 0$ (Cobb-Douglas) and $F = \min_j x_j$ as $\rho \rightarrow -\infty$ (Leontief).

The elasticity of substitution is $\sigma = 1/(1 - \rho)$. The CES curvature at symmetric equilibrium is:

$$K = \frac{(1 - \rho)(J - 1)}{J} \quad (3)$$

The CES potential (the generating function of the associated conservative-dissipative system) is:

$$\Phi = - \sum_n \log F_n \quad (4)$$

The Hessian of $\log F$ at the symmetric point $\bar{\mathbf{x}} = c \cdot \mathbf{1}$ has eigenvalue $-1/(Jc^2)$ on $\text{span}\{\mathbf{1}\}$ and eigenvalue

$$\lambda_{\perp} = -\frac{(1 - \rho)}{Jc^2} = -\frac{K}{(J - 1)c^2} \quad (5)$$

with multiplicity $J - 1$ on the tangent space $\mathbf{1}^{\perp} = \{\mathbf{v} : \sum_j v_j = 0\}$.

2.2 The Quadruple Role of ρ

The companion paper ([Smirl, 2026a](#)) establishes that K (equivalently ρ) simultaneously controls four properties. They are stated here without proof.

Theorem 2.1 (CES Quadruple Role, [Smirl 2026a](#)). *At symmetric equilibrium with $\bar{x}_j = c$ for all j , the CES curvature K defined in (3) simultaneously determines:*

- (a) **Superadditivity.** For any perturbation δ with $\sum \delta_j = 0$:

$$F(\bar{x} + \delta) \leq F(\bar{x}) - \frac{K}{2(J - 1)} \cdot \frac{\|\delta\|^2}{c}$$

The gap between homogeneous and heterogeneous allocation is proportional to K .

- (b) **Correlation robustness.** If $x_j = \mu + \tau Z_j$ with $\text{Var}[Z_j] = 1$:

$$\frac{\text{Var}[Y]}{\tau^2} = 1 - \frac{K^2}{(J - 1)} + O(K^3)$$

CES nonlinearity extracts idiosyncratic variation with bonus proportional to K^2 .

- (c) **Strategic independence.** For any coalition $S \subset \{1, \dots, J\}$ deviating by δ :

$$\Delta v(S) \leq -\frac{K}{2(J - 1)} \cdot \frac{\|\delta\|^2}{c^2}$$

Balanced allocation is a Nash equilibrium; coalition deviations are penalized proportionally to K .

- (d) **Network scaling.** The unnormalized CES aggregate $G = (\sum x_j^\rho)^{1/\rho}$ with J symmetric inputs satisfies $G(J) = J^{1/\rho} \cdot c$. The network scaling exponent is $1/\rho$.

The key structural fact: superadditivity and strategic independence are first-order curvature effects (proportional to K), while correlation robustness is a second-order effect (proportional to K^2). This distinction drives the non-uniform degradation result in Section 4.4.

2.3 Micro-Foundations: Why CES Is Forced

CES aggregation is the unique functional form satisfying the following axiom sets (proved in Smirl, 2026a):

- Proposition 2.2** (CES from primitives). (a) *Consistent aggregation (Kolmogorov–Nagumo).* Consistent sub-aggregation + homogeneity forces the CES mean (Aczél, 1966; Hardy, Littlewood, and Pólya, 1952).
- (b) *Optimal selection (Fréchet).* Buyers with i.i.d. Fréchet taste shocks choosing quality-maximizing varieties produce CES market shares and welfare indices (Anderson, de Palma, and Thisse, 1992; Eaton and Kortum, 2002).
- (c) *Inequality-averse social evaluation (Atkinson).* Weak Pareto + anonymity + homotheticity + Pigou–Dalton uniquely selects the Atkinson–CES family (Atkinson, 1970).

2.4 Nested CES and Multi-Level Aggregation

Sato (1967) showed that CES aggregation can be nested: inputs are first grouped into clusters, each cluster is aggregated with its own CES parameter ρ_g , and the cluster-level outputs are then aggregated with a different parameter ρ_0 . The two-level nested CES takes the form:

$$F = \left(\frac{1}{G} \sum_{g=1}^G \left(\frac{1}{J_g} \sum_{j \in \mathcal{G}_g} x_j^{\rho_g} \right)^{\rho_0/\rho_g} \right)^{1/\rho_0} \quad (6)$$

where G is the number of groups, \mathcal{G}_g is the set of inputs in group g , $J_g = |\mathcal{G}_g|$, ρ_g governs within-group substitutability, and ρ_0 governs between-group substitutability. The consistency axiom (Proposition 2.2(a)) guarantees that this nesting is self-consistent: the aggregate does not depend on the order in which sub-aggregation is performed, a property that fails for non-CES aggregators.

Proposition 2.3 (Nested Curvature). *The effective curvature of a two-level nested CES at symmetric equilibrium is:*

$$K_{\text{nested}} = \underbrace{\frac{(1 - \rho_0)(G - 1)}{G}}_{K_{\text{between}}} + \underbrace{\sum_{g=1}^G \frac{(1 - \rho_g)(J_g - 1)}{J_g}}_{K_{\text{within}}} \quad (7)$$

The total curvature decomposes additively into between-group and within-group components.

Proof. At symmetric equilibrium ($x_j = c$ for all j), the nested CES reduces to $F = c$. The Hessian $\nabla^2 \log F$ has a block structure respecting the group partition. The eigenvalues decompose into: (i) $G - 1$ eigenvalues equal to $-(1 - \rho_0)/(Gc^2)$ for perturbations that shift resources between groups (holding within-group allocation fixed); (ii) for each group g , $J_g - 1$ eigenvalues equal to $-(1 - \rho_g)/(J_g c^2)$ for perturbations within group g (holding total group allocation fixed). The effective curvature $K_{\text{nested}} = -c^2 \sum_k \lambda_k$ gives (7) after summation: the $(G-1)$ between-group eigenvalues contribute $(G-1) \cdot (1 - \rho_0)/(Gc^2) \cdot c^2 = (1 - \rho_0)(G - 1)/G$, and each group g 's $(J_g - 1)$ within-group eigenvalues contribute $(1 - \rho_g)(J_g - 1)/J_g$. \square

Remark 2.4 (Hierarchical information friction). *The nested structure has a natural information-theoretic interpretation: the firm processes information at two levels. At the coarse level (between groups), decisions are made with friction T_0 ; at the fine level (within groups), with friction T_g . The CES potential decomposes correspondingly:*

$$\mathcal{F}_q = \underbrace{\mathcal{F}_0(K_{\text{between}}, T_0)}_{\text{inter-group}} + \underbrace{\sum_g \mathcal{F}_g(K_g, T_g)}_{\text{intra-group}}$$

This additive decomposition is a consequence of the CES nesting consistency and motivates the supply chain architecture results of Section 6.2: each tier of production has its own (ρ, T) pair, and the total CES potential is the sum.

Remark 2.5 (Connection to endogenous growth). *The nested CES structure is central to endogenous growth theory. In Romer (1990), variety expansion increases the number of inputs J while holding ρ fixed. The growth rate depends on the rate of variety creation, which in turn depends on the R&D sector's ability to reduce information friction (identify new varieties). The CES potential framework makes this precise: the growth rate is bounded by $\partial \mathcal{F}_q / \partial J = (1/\rho J) - T \cdot (\partial H^*/\partial J)$, the difference between the marginal diversity premium and the marginal coordination cost. Growth slows when coordination costs dominate, providing a CES-theoretic interpretation of semi-endogenous growth (Jones, 1995).*

3 The CES Potential and Tsallis Entropy

3.1 From Shannon to Tsallis: Why Complementarity Breaks Extensivity

For a discrete probability distribution $p = (p_1, \dots, p_N)$, Shannon entropy $H(p) = -\sum p_i \log p_i$ is uniquely characterized by three axioms (Khinchin, 1957): continuity, maximality at the uniform distribution, and the *chain rule* (additivity) for compound experiments. The chain rule assumes that information about different inputs is additive—learning about inputs A and B jointly costs the sum of learning about each separately.

This is exactly wrong for complementary production. When $\rho < 1$, the cross-partial $\partial^2 F / \partial x_j \partial x_k > 0$ means that learning about input j changes the marginal value of information about input k . At the symmetric point $x_j = \bar{x}$:

$$\frac{\partial^2 F}{\partial x_j \partial x_k} \Big|_{\text{sym}} = \frac{(1-\rho)}{J^2} \cdot \bar{x}^{\rho-2} \cdot F^{1-\rho} \quad (8)$$

which is positive (complements) when $\rho < 1$ and zero when $\rho = 1$ (perfect substitutes). Information about complements is inherently non-additive.

Replacing the chain rule with *pseudo-additivity*— $S(A \cup B) = S(A) + S(B|A) + (1-q)S(A)S(B|A)$, where the interaction term $(1-q)$ captures the non-additive component—while retaining continuity and maximality yields the Tsallis entropy (Tsallis, 1988; Santos, 1997; Suyari, 2004):

$$S_q(p) = \frac{1 - \sum_{i=1}^N p_i^q}{q-1} \quad (9)$$

with $S_q \rightarrow H$ (Shannon) as $q \rightarrow 1$. The interaction term has direct economic content:

- $q < 1$ (complements): joint information exceeds the sum—complementary inputs amplify each other's information value.
- $q = 1$ (neutral): Shannon chain rule recovered exactly.
- $q > 1$ (substitutes): joint information is less than the sum—redundant inputs diminish each other's information value.

Theorem 3.1 (Tsallis uniqueness). *Under continuity, maximality, and the q -chain rule replacing the Shannon chain rule, the unique entropy functional is the Tsallis entropy (9) for $q \neq 1$, with $S_q \rightarrow H$ as $q \rightarrow 1$.*

Proof. The result follows from the characterization theorems of Santos (1997), Abe (2000), and Suyari (2004). Define $g(S) = 1 + (1-q)S$. Pseudo-additivity with independent subsystems gives $g(S(AB)) = g(S(A)) \cdot g(S(B))$, so $g \circ S$ is multiplicative. The

unique continuous multiplicative functional on product distributions is $g(S) = \sum p_j^q$. Solving for S yields (9). \square

The identification $q = \rho$ is forced by the companion emergence theorem (Smirl, 2026a): CES with parameter ρ is the sufficient statistic for Rényi/Tsallis entropy of order ρ . Since $q = \rho$ from the production technology, the Tsallis generalization adds no free parameters.

3.2 The Information Friction

Following the rational inattention literature (Sims, 2003), the information friction T is the shadow price of information processing capacity. An agent with processing capacity κ has $T = 1/\kappa$. At $T = 0$: unlimited processing capacity, perfect rationality. As T increases: information is costlier, agents are more boundedly rational, decisions are noisier.

The agent's choice probability follows the q -exponential form:

$$P(\text{choose } a_i) = \frac{\exp_q(u(a_i)/T)}{\sum_k \exp_q(u(a_k)/T)} \quad (10)$$

where $\exp_q(x) = [1 + (1 - q)x]_+^{1/(1-q)}$ is the q -exponential and $q = \rho$. For $q \rightarrow 1$, this recovers the standard logit $P \propto \exp(u/T)$. For $q < 1$ (complements), the distribution has compact support—agents entirely ignore options beyond a cost threshold $T/(1 - q)$, producing the “all-or-nothing” patterns observed in complementary team production (Kremer, 1993). For $q > 1$ (substitutes), the distribution has power-law tails with exponent $\sigma = 1/(1 - \rho)$.

3.3 Construction of the CES Potential

Theorem 3.2 (Uniqueness of the CES Potential). *Let $\mathcal{F}[\Phi, \mathcal{H}, T]$ be a functional governing economic equilibrium, where Φ is an aggregator, \mathcal{H} is an information measure, and $T \geq 0$ is a scalar parameter. Suppose:*

- (A1) **Constant elasticity.** *The aggregator Φ has constant elasticity of substitution $\sigma = 1/(1 - \rho)$.*
- (A2) **Generalized Khinchin axioms.** *The information measure \mathcal{H} satisfies continuity, maximality, and pseudo-additivity with parameter q determined by the aggregator.*
- (A3) **Constrained optimality.** *Equilibrium maximizes Φ subject to an information capacity constraint $\mathcal{H} \leq \kappa$, where T is the shadow price of capacity.*
- (A4) **Self-consistency.** *The entropy order matches the aggregation parameter: $q = \rho$.*

Then, up to positive affine transformations:

- (i) Φ must be the CES aggregate (or its limits: Cobb-Douglas, Leontief);
- (ii) \mathcal{H} must be Tsallis entropy $S_q = (1 - \sum p_i^q)/(q - 1)$ with $q = \rho$;
- (iii) $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$.

Shannon entropy is recovered as the $q \rightarrow 1$ (perfect substitutes) special case.

Proof. Step 1: (A1) forces CES. By the functional equation theorem of Aczél (1966), the CES family is the unique class of linearly homogeneous aggregators with constant elasticity of substitution.

Step 2: (A2) forces Tsallis entropy. By the generalized Khinchin theorem (Santos, 1997; Abe, 2000; Suyari, 2004), S_q is the unique function satisfying the three axioms.

Step 3: (A4) forces $q = \rho$. The self-consistency axiom follows from the emergence theorem (Smirl, 2026a): CES with parameter ρ is the sufficient statistic for Rényi/Tsallis entropy of order ρ .

Step 4: Lagrangian duality forces $\mathcal{F}_q = \Phi - T \cdot S_q$. By (A3), equilibrium solves $\max_{\mathbf{x}, \mathbf{p}} \Phi(\mathbf{x})$ subject to $S_q(\mathbf{p}) \leq \kappa$. The Lagrangian is $\mathcal{L} = \Phi(\mathbf{x}) + T[\kappa - S_q(\mathbf{p})]$. Strong duality gives $\min_{T \geq 0} \max_{\mathbf{x}, \mathbf{p}} \Phi(\mathbf{x}) - T \cdot S_q(\mathbf{p})$. The additive structure is forced by Lagrangian duality; no alternative connector is consistent with constrained optimization. \square

3.4 The q -Exponential and q -Logarithm

The Tsallis CES potential generates a natural pair of deformed functions that replace the standard exponential and logarithm throughout the framework.

Definition 3.3 (q -exponential and q -logarithm). *The q -exponential and q -logarithm are:*

$$\exp_q(x) = [1 + (1 - q)x]_+^{1/(1-q)} \quad (11)$$

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \quad (12)$$

with $\exp_q \rightarrow \exp$ and $\ln_q \rightarrow \log$ as $q \rightarrow 1$. They satisfy $\ln_q(\exp_q(x)) = x$ and $\exp_q(\ln_q(x)) = x$ (inverse pair). The q -product rule is $\exp_q(a) \cdot \exp_q(b) = \exp_q(a + b + (1 - q)ab)$, which encodes pseudo-additivity: the product of two q -exponentials is not the q -exponential of the sum, but of the sum plus an interaction term proportional to $(1 - q)$.

Remark 3.4 (Economic meaning of the q -product rule). *The interaction term $(1 - q)ab$ in the q -product rule is the information-theoretic manifestation of complementarity. When two CES systems with energy costs a and b are combined, the joint cost is not $a + b$ (additive) but $a + b + (1 - q)ab$. For $q < 1$ (complements), $(1 - q)ab > 0$: combining complementary subsystems costs more than the sum, because coordination between*

subsystems requires additional information. For $q > 1$ (substitutes), $(1 - q)ab < 0$: combining substitutable subsystems costs less than the sum, because redundancy provides free information.

3.5 The q -Exponential Equilibrium

Proposition 3.5 (q -exponential equilibrium). *The equilibrium distribution that minimizes \mathcal{F}_q subject to normalization $\sum p_j = 1$ is:*

$$p_j^* = \frac{[1 - (1 - q)\beta\varepsilon_j]_+^{1/(1-q)}}{Z_q} \quad (13)$$

where $\beta = 1/T$, $[x]_+ = \max(x, 0)$, and $Z_q = \sum_j [1 - (1 - q)\beta\varepsilon_j]_+^{1/(1-q)}$ is the q -partition function.

Proof. The Lagrangian for minimizing $\mathcal{F}_q = \sum p_j\varepsilon_j - TS_q$ subject to $\sum p_j = 1$ is:

$$\mathcal{L} = \sum_j p_j\varepsilon_j - T \cdot \frac{1 - \sum_j p_j^q}{q - 1} - \lambda \left(\sum_j p_j - 1 \right)$$

The first-order condition gives:

$$\varepsilon_j + \frac{Tq}{q - 1} p_j^{q-1} = \lambda$$

Solving for p_j :

$$p_j = \left[\frac{(q - 1)(\lambda - \varepsilon_j)}{Tq} \right]^{1/(q-1)}$$

Defining the q -exponential $\exp_q(x) = [1 + (1 - q)x]_+^{1/(1-q)}$ and absorbing constants into the normalization yields $p_j^* \propto \exp_q(-\beta\varepsilon_j)$. \square

Proposition 3.6 (Properties of the q -equilibrium). (a) **Compact support for $q < 1$ (complements):** $p_j^* = 0$ for all j with $\varepsilon_j > T/(1 - q)$. High-cost inputs are excluded entirely, not just exponentially suppressed.

(b) **Power-law tails for $q > 1$ (substitutes):** $p_j^* \sim \varepsilon_j^{-\zeta}$ for large ε_j , with tail exponent $\zeta = 1/(q-1) = |\sigma|$, connecting CES theory to power-law economics (Gabaix, 2009).

(c) **Boltzmann recovery:** As $q \rightarrow 1$, $p_j^* \rightarrow e^{-\beta\varepsilon_j}/Z$ (the standard logit distribution).

Proof. (a) When $q < 1$, $1 - q > 0$, and the bracket $[1 - (1 - q)\beta\varepsilon_j]_+$ vanishes for $\varepsilon_j > 1/((1 - q)\beta) = T/(1 - q)$. The threshold $T/(1 - q)$ is finite, so the distribution has bounded support.

(b) When $q > 1$, $1 - q < 0$, so $\exp_q(-\beta\varepsilon_j) = (1 + (q-1)\beta\varepsilon_j)^{-1/(q-1)} \sim \varepsilon_j^{-1/(q-1)}$ for large ε_j . The tail exponent is $\zeta = 1/(q-1)$, and since $q = \rho$ and $\sigma = 1/(1-\rho)$, the result is $\zeta = |\sigma|$ for $\rho > 1$ (noting $\sigma < 0$ when $\rho > 1$).

(c) Standard: $\lim_{q \rightarrow 1} [1 + (1-q)x]^{1/(1-q)} = e^x$ by L'Hôpital's rule. \square

Remark 3.7 (Economic content of compact support). *Compact support for complements ($q < 1$) has direct economic meaning: when inputs are strong complements, the producer entirely ignores inputs whose cost exceeds the threshold $T/(1-q)$. This is sharper than exponential suppression—it is a hard cutoff. In a labor market with complementary skills, firms do not hire workers above a salary threshold; they simply exclude them from the candidate pool. This matches the “all-or-nothing” hiring patterns observed in complementary team production (Kremer, 1993): surgical teams do not hire cheaper but less qualified surgeons at a discount—they hire qualified surgeons or not at all.*

Remark 3.8 (Power-law tails and the size distribution of firms). *The power-law tail for substitutes ($q > 1$) with exponent $\sigma = 1/(1-\rho)$ connects to the empirical size distribution of firms. In industries with high ρ (substitutable inputs), the equilibrium firm size distribution should follow a power law with exponent related to σ . This provides a CES-theoretic foundation for the well-documented Zipf-like distributions of firm sizes (Gabaix, 2009), where the tail exponent is determined by the elasticity of substitution in the underlying production technology.*

3.6 The Two Limits and the Breakdown Threshold

The CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$ has two canonical limits that bracket the full framework:

$T = 0$ (Perfect information): $\mathcal{F}_q = \Phi_{\text{CES}}$. Pure CES optimization. This is standard economics: deterministic allocation, no behavioral anomalies, no information rents. The allocation is the first-best (15), the mechanism is fully efficient (zero PoIC), markets function at all quality levels (no adverse selection), and choice is deterministic ($P(j) \in \{0, 1\}$).

$T > 0$ (Information friction): The entropy term degrades the CES aggregate. Higher T means costlier information, coarser decisions, worse allocation. The severity depends on ρ : low ρ (high curvature K) is more robust because the value of precise allocation is higher.

$T \rightarrow \infty$ (No information): $\mathcal{F}_q = -T \cdot S_q \rightarrow 0$. The CES contribution vanishes. Allocation is uniform ($x_j = C/J$ for all j), choice is random ($P(j) = 1/J$), no market can sustain screening, and all behavior is noise. This is the “zero-intelligence” limit studied in agent-based computational economics.

The rich structure of economic theory—information rents, behavioral biases, institutional failure—resides entirely in the intermediate regime $0 < T < \infty$. The CES potential makes this precise by specifying exactly how each phenomenon depends on the position in (ρ, T) space.

The breakdown threshold. For each economic institution, there exists a breakdown threshold T^* above which the institution fails. The critical surface $T^* = T^*(v, \rho)$ satisfies:

$$\frac{\partial T^*}{\partial v} > 0, \quad \frac{\partial T^*}{\partial K} > 0 \quad (14)$$

Higher aggregate value v and higher curvature K both raise the failure threshold. Below T^* : the institution functions. Above T^* : it collapses via a first-order regime shift.

Domain-specific breakdown thresholds. Different economic institutions have different T^* values because they depend on different components of the CES potential:

Institution	T^* depends on	Failure mode
Markets	v, K, \bar{q}	Market unraveling
Mechanisms	K, σ_θ	Excessive information rents
Democracy	ρ, J	Ordering reversals
Search / matching	K, L	Matching breakdown
Contracts	ρ, τ	Hold-up paralysis
Choice	u_{\max}, J	Choice overload
Production	K, d^2, C	Effective substitutability

Domain-specific derivations showing how each institution maps to the CES potential are presented in the companion paper ([Smirl, 2026b](#)).

The ordering of T^* values across institutions determines which institutions fail first during an information crisis. This is the generalization of the non-uniform degradation result (Section 4.4) from three of the four curvature roles to the full institutional landscape.

4 Effective Curvature Under Information Friction

When a firm with CES technology (structural parameter ρ) operates under information friction (T), the complementarity it can *exploit* depends on both parameters jointly.

4.1 The Firm's Problem

A firm operates CES technology (2) with J input types and structural parameter ρ . Input j has productivity $\theta_j > 0$, drawn i.i.d. with mean μ and variance σ_θ^2 . The firm has total budget C and must allocate $x_j \geq 0$ with $\sum_j x_j = C$.

Under perfect information ($T = 0$), the optimal allocation is the CES demand system:

$$x_j^* = C \cdot \frac{\theta_j^{\sigma-1}}{\sum_{k=1}^J \theta_k^{\sigma-1}}, \quad \sigma = \frac{1}{1-\rho} \quad (15)$$

Under information frictions ($T > 0$), the firm does not observe $\boldsymbol{\theta}$ directly. It acquires information subject to a capacity constraint $I(\boldsymbol{\theta}; \mathbf{s}) \leq \kappa = 1/T$, where \mathbf{s} is the signal vector. The firm's problem is:

$$\max_{\mathbf{x}(\mathbf{s})} \mathbb{E}_{\boldsymbol{\theta}, \mathbf{s}}[\log F(\mathbf{x}(\mathbf{s}); \boldsymbol{\theta})] - T \cdot I(\boldsymbol{\theta}; \mathbf{s}) \quad (16)$$

subject to $\sum_j x_j(\mathbf{s}) = C$ for all \mathbf{s} . The objective is the production CES potential: expected log-output minus the entropy cost of information acquisition.

Remark 4.1. Under symmetric prior (all θ_j identically distributed) and the logit solution (10), the firm's allocation converges to the equal-share allocation $\hat{x}_j = C/J$ as $T \rightarrow \infty$, and to the first-best x_j^* as $T \rightarrow 0$. At intermediate T , the allocation is a “softened” version of the first-best: the firm leans toward the optimal allocation but with noise proportional to T .

The allocation error $\boldsymbol{\delta} = \hat{\mathbf{x}} - \mathbf{x}^*$ lies in the tangent space $\mathbf{1}^\perp$ (the budget constraint $\sum_j \delta_j = 0$ is always satisfied). Under the logit allocation, the per-component error variance is:

$$\mathbb{E}[\delta_j^2] = c^2 \cdot \frac{\sigma_\theta^2}{\mu^2} \cdot \frac{T}{T + |\lambda_\perp|^{-1} \sigma_\theta^2 / \mu^2} \quad (17)$$

where $|\lambda_\perp| = K/((J-1)c^2)$ is the curvature of $\log F$ on the tangent space. In the regime $T \ll c^2/K$, this simplifies to:

$$\mathbb{E}[\delta_j^2] \approx \frac{T \cdot K}{J-1} \quad (18)$$

The error variance is jointly proportional to T (noisier information, larger errors) and K (more curvature amplifies the cost of misallocation): the same complementarity that makes precise allocation valuable also amplifies the penalty from imprecise allocation.

4.2 Output Loss Under Friction

At intermediate T , the allocation is a “softened” version of the first-best. The allocation error $\boldsymbol{\delta} = \hat{\mathbf{x}} - \mathbf{x}^*$ lies in the tangent space $\mathbf{1}^\perp$ (the budget constraint is always satisfied).

Taylor-expanding $\log F$ around the first-best:

$$\log F(\hat{\mathbf{x}}) = \log F(\mathbf{x}^*) + \nabla(\log F)^\top \boldsymbol{\delta} + \frac{1}{2} \boldsymbol{\delta}^\top \nabla^2(\log F) \boldsymbol{\delta} + O(\|\boldsymbol{\delta}\|^3)$$

At the first-best, $\nabla(\log F)^\top \boldsymbol{\delta} = 0$ since $\boldsymbol{\delta} \in \mathbf{1}^\perp$. The expected output loss is dominated by the quadratic term with eigenvalue $|\lambda_\perp| = K/((J-1)c^2)$ from (5):

$$\mathbb{E}[\log F^* - \log F(\hat{\mathbf{x}})] \approx \frac{K^2 \cdot T}{2(J-1)c^2} \quad (19)$$

The loss is quadratic in K (more complementary technologies suffer more from misallocation) and linear in T (costlier information means larger errors). This is the key intermediate result connecting production complementarity to information friction.

4.3 The Effective Curvature Theorem

Theorem 4.2 (Effective Curvature). *Consider a CES production technology with J inputs, structural curvature K , operated by a firm at information friction T . The effective curvature is*

$$K_{\text{eff}}(\rho, T) = K \cdot \left(1 - \frac{T}{T^*(\rho)}\right)^+ \quad (20)$$

where

$$T^*(\rho) = \frac{2(J-1)c^2 d^2}{K} \quad (21)$$

is the critical production friction, d^2 is the squared geodesic diversity of the input mix, and $(z)^+ = \max(z, 0)$.

Proof. The superadditivity gap under perfect information is $\Delta_{\text{FB}} = \Omega(K) \cdot d^2$. Under information friction T , the expected gap is reduced by the output loss (19):

$$\Delta_{\text{eff}} = K \cdot d^2 \left(1 - \frac{KT}{2(J-1)c^2 d^2}\right).$$

Setting $\Delta_{\text{eff}} = K_{\text{eff}} \cdot d^2$ and solving gives (20). Above T^* , the output loss from misallocation exceeds the diversity premium, so no complementarity benefit remains. \square

Remark 4.3 (Interpretation). *Information friction acts as an endogenous source of apparent substitutability. A firm with low ρ but high T behaves as if its inputs were more substitutable. This resolves a puzzle in development economics: why firms in developing countries use simpler production processes even when the same technology is available (Bloom et al., 2013). The answer is not that the technology is inappropriate, but that the informational environment (T too high relative to $T^*(\rho)$) makes the complementarity unexploitable.*

Remark 4.4 (Properties of T^*). *The critical production friction has economically natural comparative statics:*

- (i) $\partial T^*/\partial(1 - \rho) > 0$ for fixed d^2 : more complementary technologies tolerate more information friction before the premium vanishes. This follows because $T^* \propto 2(J - 1)c^2d^2/K$ and, for realistic diversity levels, the diversity d^2 grows faster with complementarity than K does.
- (ii) $\partial T^*/\partial d^2 > 0$: more diverse input mixes provide a larger buffer against information degradation. A firm with homogeneous inputs ($d^2 = 0$) has $T^* = 0$ —there is no complementarity to degrade.
- (iii) $\partial T^*/\partial C > 0$: larger firms (higher budget C , hence higher $c = C/J$) are more robust to information frictions. Scale provides a buffer because the output loss is a fraction of a larger base. This provides a rationale for firm size beyond standard returns-to-scale arguments: large firms can exploit complementarity that small firms cannot.

4.4 Non-Uniform Degradation and the Crisis Sequence

Proposition 4.5 (Non-Uniform Degradation). *Under information friction $T < T^*(\rho)$:*

- (a) **Superadditivity:** Linear degradation in T/T^* . The effective gap is $\Omega(K(1 - T/T^*)) \cdot d^2$.
- (b) **Correlation robustness:** Quadratic degradation in T/T^* . The effective bonus scales as $K^2(1 - T/T^*)^2$.
- (c) **Strategic independence:** Linear degradation in T/T^* . The effective penalty is $\Omega(K(1 - T/T^*)) \cdot \|\delta_S\|^2/c^2$.

Proof. Parts (a) and (c) follow from replacing K with $K_{\text{eff}} = K(1 - T/T^*)$ in the linear bounds. Part (b): the K^2 bound receives two factors of $(1 - T/T^*)$ —one from the curvature channel operating on a noisier signal, one from the reduced correlation contrast. \square

Corollary 4.6 (Crisis Sequence). *As T rises toward T^* :*

1. **Correlation robustness fails first.** Since the bonus scales as $(1 - T/T^*)^2$, it falls below economic significance at $T \approx T^*/2$.
2. **Superadditivity degrades next.** The diversity premium erodes linearly, reaching zero at $T = T^*$.

3. *Strategic independence breaks last.* The manipulation penalty erodes linearly, but defection is a threshold event that occurs only after the diversity premium has substantially eroded.

Example 4.7 (The 2008 Global Financial Crisis). *The crisis sequence matches observed GFC dynamics:*

1. *Diversification failed first:* asset correlations spiked from $r \approx 0.3$ to $r > 0.8$ across supposedly diversified portfolios.
2. *Production complementarities degraded next:* supply chain disruptions spread as trade finance froze.
3. *Strategic stability broke last:* Lehman's counterparties defected, money market funds broke the buck, interbank lending ceased.

The sequence is forced by the mathematics of non-uniform degradation.

4.5 Pre-Crisis Deceleration

The effective curvature theorem has a dynamical consequence: as T approaches T^* from below, the system's adjustment speed decreases.

Proposition 4.8 (Pre-Crisis Deceleration). *Consider a CES economy adjusting toward equilibrium after a perturbation. The relaxation time $\tau(T)$ —the time for the perturbation to decay by a factor of $1/e$ —satisfies:*

$$\tau(T) = \frac{\tau_0}{1 - T/T^*(\rho)} \quad (22)$$

where τ_0 is the relaxation time at $T = 0$. As $T \rightarrow T^*$: $\tau \rightarrow \infty$ (the system ceases to adjust).

Proof. The linearized dynamics around the symmetric equilibrium are $\dot{\mathbf{x}} = -\mathbf{L}\nabla^2\mathcal{F}_q \cdot (\mathbf{x} - \mathbf{x}^*)$, where \mathbf{L} is the (symmetric, positive definite) friction matrix. The eigenvalues of $\mathbf{L}\nabla^2\mathcal{F}_q$ on the tangent space $\mathbf{1}^\perp$ are proportional to $K_{\text{eff}} = K(1 - T/T^*)$. The relaxation rate (inverse of relaxation time) is the smallest eigenvalue: $1/\tau = K_{\text{eff}} \cdot L_{\min}/c^2 = (K/c^2)(1 - T/T^*) \cdot L_{\min}$. Setting $\tau_0 = c^2/(K \cdot L_{\min})$ gives (22). \square

Remark 4.9 (Early warning signals). *The divergence of τ as $T \rightarrow T^*$ is a Category A result (Table 1): it depends only on the eigenstructure of the CES potential Hessian, not on the specific entropy. This means that pre-crisis deceleration provides a reliable early warning signal under both Shannon and Tsallis frameworks. Observable consequences:*

1. **Increased autocorrelation:** the autocorrelation coefficient $\rho(\Delta t) = e^{-\Delta t/\tau}$ approaches 1 as $\tau \rightarrow \infty$.

2. **Increased variance:** by the q -variance-response identity, $\text{Var} = T\tau/(2-q)$, which diverges as $\tau \rightarrow \infty$.
3. **Flickering:** near T^* , small perturbations cause temporary excursions to the alternative (collapsed) equilibrium before the system returns.

All three signals have been documented empirically in pre-crisis financial data, providing indirect support for the effective curvature mechanism.

5 q -Generalized Dynamical Results

This section derives the key dynamical results under Tsallis entropy, identifying what changes relative to the Shannon ($q = 1$) framework and what survives.

5.1 q -Variance-Response Identity

Theorem 5.1 (q -Variance-Response Identity (FDT)). *At equilibrium of the Tsallis CES potential \mathcal{F}_q , the covariance matrix of fluctuations Σ and the linear response matrix χ satisfy:*

$$\Sigma = \frac{T}{2-q} \cdot \chi \quad (23)$$

Proof. The Hessian of the Tsallis entropy at the uniform distribution is $\partial^2 S_q / \partial p_j \partial p_k|_{\text{unif}} = -qJ^{2-q}\delta_{jk}$ (Lemma A.3), differing from Shannon's $-J\delta_{jk}$ by a factor that evaluates to $1/(2-q)$ at symmetric equilibrium. At the symmetric equilibrium ($p_j = 1/J$), the linear response from the q -exponential gives $\chi_{jk} = \beta J^{2-q}(\delta_{jk} - 1/J)/q$, and the equilibrium covariance is $\Sigma_{jk} = J^{2-q}(\delta_{jk} - 1/J)/(q(2-q))$. The ratio $\Sigma_{jk}/\chi_{jk} = T/(2-q)$ establishes (23). \square

Corollary 5.2 (q -information friction measurement). *The information friction can be measured from observables as:*

$$T_n = (2-q) \cdot \frac{\sigma_n^2}{\chi_n} \quad (24)$$

For $q < 1$ (complements), $2 - q > 1$, so T is higher than the Shannon estimate—complementary sectors appear noisier than Shannon predicts.

5.2 q -Covariance Structure

Theorem 5.3 (q -covariance). *At q -exponential equilibrium of a CES(ρ) economy with J symmetric inputs, the covariance matrix has the permutation $(J-1, 1)$ structure:*

$$\Sigma = \sigma^2 \mathbf{I} + \gamma \mathbf{1} \mathbf{1}^\top \quad (25)$$

with eigenvalues:

$$\lambda_1 = \frac{T}{(2-q)(K_{\text{eff}} + J^{-1})} \quad (1\text{-dimensional, along } \mathbf{1}) \quad (26)$$

$$\lambda_{2,\dots,J} = \frac{T}{(2-q)K_{\text{eff}}} \quad ((J-1)\text{-dimensional}) \quad (27)$$

where $K_{\text{eff}} = K \cdot (1 - T/T^*)^+$ is the effective curvature.

Proof. The permutation symmetry of the CES aggregate at the symmetric equilibrium forces the $(J-1, 1)$ eigenstructure: any covariance matrix invariant under the symmetric group \mathfrak{S}_J must have the form $\Sigma = \sigma^2 \mathbf{I} + \gamma \mathbf{1}\mathbf{1}^\top$, with eigenvalue $\sigma^2 + J\gamma$ on $\text{span}\{\mathbf{1}\}$ (multiplicity 1) and eigenvalue σ^2 on $\mathbf{1}^\perp$ (multiplicity $J-1$). This depends only on the symmetry group, not on the specific entropy.

The eigenvalues are computed from $\Sigma = (T/(2-q))(\nabla^2 \mathcal{F}_q|_{\text{eq}})^{-1}$, where the q -variance-response identity (Theorem 5.1) provides the proportionality constant $T/(2-q)$. The Hessian of \mathcal{F}_q at the symmetric equilibrium inherits the CES Hessian structure: eigenvalue K_{eff} with multiplicity $J-1$ on $\mathbf{1}^\perp$ (from the production curvature) and eigenvalue $K_{\text{eff}} + J^{-1}$ with multiplicity 1 on $\text{span}\{\mathbf{1}\}$ (the additional J^{-1} arises from the entropy's contribution to the budget constraint direction). \square

Remark 5.4 (Observational consequence). *The ratio of eigenvalues $\lambda_{2,\dots,J}/\lambda_1 = (K_{\text{eff}} + J^{-1})/K_{\text{eff}} = 1 + 1/(JK_{\text{eff}})$ provides a direct estimate of effective curvature from observed cross-sectional covariances: $K_{\text{eff}} = 1/(J(\lambda_\perp/\lambda_1 - 1))$. The estimate is independent of T and q —the correction factor $1/(2-q)$ cancels in the ratio. This makes K_{eff} estimable from principal component analysis of sectoral fluctuations without knowledge of the information friction.*

5.3 q -Crooks Fluctuation Theorem and Jarzynski Equality

Theorem 5.5 (q -Crooks). *For a transition between states A and B of the Tsallis CES potential, the ratio of forward to reverse path probabilities satisfies:*

$$\frac{P_F(W)}{P_R(-W)} = \exp_q \left(\frac{W - \Delta \mathcal{F}_q}{T} \right) \quad (28)$$

where W is the work performed and $\Delta \mathcal{F}_q = \mathcal{F}_q(B) - \mathcal{F}_q(A)$.

Proof. The q -exponential equilibrium distribution $p^*(\varepsilon) \propto \exp_q(-\beta\varepsilon)$ satisfies q -detailed balance: the ratio of equilibrium occupancies at two energy levels is $p^*(\varepsilon_A)/p^*(\varepsilon_B) = \exp_q(-\beta\varepsilon_A)/\exp_q(-\beta\varepsilon_B)$. By the Crooks construction (Crooks, 1999), the ratio of path

probabilities under forward and time-reversed protocols is:

$$\frac{P_F[\gamma]}{P_R[\bar{\gamma}]} = \frac{p_{\text{eq},A}^*(\gamma_0)}{p_{\text{eq},B}^*(\gamma_\tau)} \cdot \exp_q(\beta W[\gamma])_{\text{path}}$$

where γ_0 and γ_τ are the initial and final microstates. Averaging over paths with fixed total work W and using the q -product rule ($\exp_q(a) \otimes_q \exp_q(b) = \exp_q(a + b + (1 - q)ab)$), the equilibrium ratio and path contribution combine to yield:

$$\frac{P_F(W)}{P_R(-W)} = \exp_q(\beta(W - \Delta\mathcal{F}_q))$$

The CES potential difference $\Delta\mathcal{F}_q$ absorbs the equilibrium free energy contribution, extending the standard Crooks identity (Crooks, 1999) to the Tsallis ($q \neq 1$) setting. The standard result is recovered as $q \rightarrow 1$ since $\exp_q \rightarrow \exp$. \square

Remark 5.6 (Economic interpretation). *The q -Crooks theorem governs the probability of “favorable” vs. “unfavorable” transitions in economic restructuring. For a firm transitioning from organizational mode A to mode B , the work W is the total resource cost of the transition, and $\Delta\mathcal{F}_q$ is the CES potential difference (the long-run efficiency gain). The theorem states that favorable transitions ($W < \Delta\mathcal{F}_q$) are exponentially more likely than unfavorable ones—but with q -exponential rather than exponential odds. For complements ($q < 1$), the q -exponential has compact support, meaning there is a maximum possible transition cost beyond which the transition simply does not occur. This matches the observation that firms with highly complementary production face hard ceilings on restructuring costs.*

Corollary 5.7 (q -Jarzynski equality). *Integrating the q -Crooks relation over work values:*

$$\langle \exp_q(-\beta W) \rangle = \exp_q(-\beta \Delta\mathcal{F}_q) \tag{29}$$

The minimum expected work for any transition is $\langle W \rangle \geq \Delta\mathcal{F}_q$, with equality for quasi-static processes.

5.4 The q -Log-Sum-Exp Dual

Proposition 5.8 (q -log-sum-exp). *The Legendre dual of the Tsallis CES potential is:*

$$\mathcal{F}_q^*(\boldsymbol{\varepsilon}) = -T \cdot \ln_q \left(\sum_{j=1}^J \exp_q(-\varepsilon_j/T) \right) \tag{30}$$

which reduces to the standard log-sum-exp as $q \rightarrow 1$.

Proof. The conjugate of $\mathcal{F}_q = \sum p_j \varepsilon_j - TS_q$ is computed by optimizing over \mathbf{p} :

$$\mathcal{F}_q^* = \min_{\mathbf{p} \in \Delta} \left[\sum p_j \varepsilon_j - T \cdot \frac{1 - \sum p_j^q}{q-1} \right]$$

Substituting the q -exponential optimum (13) and using the identity $\sum_j (\exp_q(-\beta \varepsilon_j))^q = Z_q^q \sum_j (p_j^*)^q$ yields (30) after algebraic simplification. \square

Remark 5.9 (Escort distributions). *The optimization naturally introduces the escort distribution $P_j = p_j^q / \sum_k p_k^q$, which reweights probabilities by their q -th power. For complements ($q < 1$), the escort distribution is more uniform than the original—emphasizing rare inputs. For substitutes ($q > 1$), it is more concentrated—emphasizing common inputs. The escort distribution appears in the q -expectation $\langle \varepsilon \rangle_q = \sum P_j \varepsilon_j$, which replaces the standard expectation in all q -thermodynamic identities. The q -internal energy is $U_q = \langle \varepsilon \rangle_q$, and the CES potential satisfies $\mathcal{F}_q = U_q - TS_q$ with the escort-averaged energy.*

5.5 q -Friction Bound

Corollary 5.10 (q -friction bound). *The expected friction loss $W_{\text{fric}} = \langle W \rangle - \Delta \mathcal{F}_q$ in any transition between CES potential equilibria satisfies:*

$$W_{\text{fric}} \geq \frac{q}{2T} \cdot \text{Var}(W) \tag{31}$$

For complements ($q < 1$), the coefficient $q/(2T)$ is smaller, but complementary systems exhibit higher outcome variance $\text{Var}(W)$, so the net friction cost is typically larger.

Proof. From the q -Jarzynski equality (Corollary 5.7), Jensen's inequality applied to the convex function $\exp_q(-\cdot)$ gives $\langle W \rangle \geq \Delta \mathcal{F}_q$. The tightened bound follows from a second-order expansion of \exp_q around the mean work: $\langle \exp_q(-\beta W) \rangle \approx \exp_q(-\beta \langle W \rangle) + \frac{1}{2} \exp_q''(-\beta \langle W \rangle) \cdot \beta^2 \text{Var}(W)$. Since $\exp_q''(x) = q \exp_q(x)^{2q-1} > 0$ (confirming convexity), evaluating at equilibrium where $\exp_q(-\beta \langle W \rangle)^{2q-1} \approx 1$ gives the coefficient $q\beta^2/2 = q/(2T^2)$, yielding $W_{\text{fric}} \geq q \text{Var}(W)/(2T)$. \square

Remark 5.11. *The q -friction bound has direct economic content. Consider a firm restructuring from governance mode A to mode B (e.g., vertical integration to outsourcing). The friction loss is the transition cost—resources consumed during restructuring that do not contribute to final output. The bound says that complementary production systems ($q < 1$) tolerate less restructuring waste for a given level of outcome uncertainty. This explains why restructuring complementary organizations (e.g., merging research labs) is more costly than restructuring substitutable operations (e.g., consolidating distribution centers).*

5.6 Euler Identity Survival

Proposition 5.12 (Euler identity survives). *At any equilibrium of any CES economy with any q -entropy, the CES Euler identity holds exactly:*

$$\mathbf{x}^* \cdot \nabla \Phi(\mathbf{x}^*) = -1 \quad (32)$$

where $\Phi = -\log F$. This follows from degree-1 homogeneity of F —it is a property of the CES aggregate alone, independent of the entropy functional or the information friction T .

Proof. Since F is homogeneous of degree 1 in \mathbf{x} , Euler's theorem gives $\mathbf{x} \cdot \nabla F = F$. Differentiating $\log F$: $\mathbf{x} \cdot \nabla \log F = \mathbf{x} \cdot (\nabla F/F) = 1$. Since $\Phi = -\log F$: $\mathbf{x} \cdot \nabla \Phi = -1$. This holds at every $\mathbf{x} > 0$, hence at \mathbf{x}^* in particular. The entropy S_q enters the first-order condition but not the identity: since $S_q(\mathbf{x}/\|\mathbf{x}\|_1)$ is homogeneous of degree 0, Euler's theorem gives $\mathbf{x} \cdot \nabla_{\mathbf{x}} S_q = 0$, so the entropy contribution drops out of any Euler-type identity. Consequently, the identity (32) holds for Shannon, Tsallis, or any other entropy consistent with the CES potential structure. \square

Remark 5.13 (Winding number and Casimir invariants). *By the same logic, the winding number of economic trajectories around the critical curve $T^*(\rho)$ is a topological invariant that depends on the homotopy class of the trajectory, not on the entropy functional. Similarly, the Casimir invariants of the conservative-dissipative formulation live in the kernel of the antisymmetric coupling matrix \mathbf{J} , which is determined by the trade network topology, not by the entropy. Both survive the Tsallis generalization exactly.*

5.7 q -Onsager Reciprocal Relations

The Onsager reciprocal relations state that the matrix of transport coefficients relating fluxes to forces is symmetric: $L_{ij} = L_{ji}$. In the Shannon framework, this follows from time-reversal symmetry at equilibrium. The Tsallis generalization preserves the symmetry but inflates the coefficients.

Proposition 5.14 (q -Onsager relations). *At Tsallis CES potential equilibrium, the transport matrix $\mathbf{L}^{(q)}$ relating economic fluxes $\mathbf{J} = (J_1, \dots, J_N)$ to forces $\mathbf{X} = (X_1, \dots, X_N)$ via $J_i = \sum_j L_{ij}^{(q)} X_j$ satisfies:*

- (i) **Symmetry:** $L_{ij}^{(q)} = L_{ji}^{(q)}$ (Onsager reciprocity survives exactly).
- (ii) **Inflation:** $L_{ij}^{(q)} = L_{ij}^{(1)} / (2 - q)$, where $L_{ij}^{(1)}$ is the Shannon transport coefficient.
- (iii) **Positivity:** $\mathbf{L}^{(q)}$ is positive semi-definite (the second law is preserved).

Proof. Symmetry follows from the time-reversal argument applied to the q -exponential equilibrium: the path probability ratio $P_F[\gamma]/P_R[\bar{\gamma}] = \exp_q(\beta(W_F - W_R))$ is symmetric

under exchange of i and j at equilibrium (where $W_F = W_R$). The inflation factor $1/(2-q)$ enters because the transport coefficients are proportional to the equilibrium covariance ($L_{ij} \propto \Sigma_{ij}$), which acquires the $1/(2-q)$ correction from the q -variance-response identity. Positivity follows from the positive semi-definiteness of Σ . \square

Remark 5.15 (Economic content). *In the economic context, the Onsager relations govern the cross-effects between sectors: how a price shock in sector i affects output in sector j . The symmetry condition $L_{ij}^{(q)} = L_{ji}^{(q)}$ means that the effect of sector i on sector j equals the reverse effect, at equilibrium. The inflation factor $1/(2-q)$ means that complementary economies ($q < 1$) have stronger cross-sector linkages than Shannon predicts: a shock in one sector propagates more strongly to others because of the non-additive information structure.*

5.8 q -Kramers Escape Rate

The Kramers escape rate governs the speed of first-order regime shifts—the rate at which an economy trapped in a suboptimal equilibrium can tunnel to the efficient one.

Proposition 5.16 (q -Kramers escape). *The escape rate from a metastable equilibrium A over a CES potential barrier $\Delta\mathcal{F}_q$ is:*

$$k_q = \nu \cdot \exp_q \left(-\frac{\Delta\mathcal{F}_q}{T} \right) \quad (33)$$

where ν is the attempt frequency (proportional to the curvature at the metastable minimum). This is the q -generalization of Kramers' classical escape-rate formula, obtained by replacing the Boltzmann weight $e^{-\Delta\mathcal{F}/T}$ with its Tsallis counterpart $\exp_q(-\Delta\mathcal{F}_q/T)$ in the standard saddle-point approximation (see Tsallis [Tsallis \(2009\)](#), Ch. 6; Abe and Okamoto [Abe and Okamoto \(2001\)](#)).

Remark 5.17. For complements ($q < 1$), the q -exponential has compact support: if $\Delta\mathcal{F}_q > T/(1-q)$, the escape rate is exactly zero. The economy is locked in the metastable state permanently—no amount of waiting produces a transition. This is sharper than the exponential case ($q = 1$), where escape is merely exponentially slow. Economically: institutions with strongly complementary components (e.g., a vertically integrated firm with highly specific assets) cannot restructure gradually; they require a discrete intervention (policy change, technology shock) that lowers the barrier below the critical threshold $T/(1-q)$.

5.9 Classification of Results

Every result from the Shannon framework falls into exactly one of three categories under the Tsallis generalization.

Table 1: Classification of Shannon-framework results under Tsallis generalization

Category	Result	Shannon form	Tsallis form
A: Exact	Euler identity	$\mathbf{x}^* \cdot \nabla H = -1/T$	$\mathbf{x}^* \cdot \nabla S_q = -1/T$
	Winding number	$w \in \mathbb{Z}$	$w \in \mathbb{Z}$ (unchanged)
	Casimir invariants	$\ker \mathbf{J}$	$\ker \mathbf{J}$ (unchanged)
	Pre-crisis deceleration	$\tau \sim T - T^* ^{-1}$	$\tau \sim T - T^* ^{-1}$ (unchanged)
B: q-corrected	Variance-response	$\Sigma = T\chi$	$\Sigma = (T/(2-q))\chi$
	Covariance eigenvalues	T/K_{eff}	$T/((2-q)K_{\text{eff}})$
	Kramers escape	$k \propto e^{-\Delta\mathcal{F}/T}$	$k \propto \exp_q(-\Delta\mathcal{F}_q/T)$
	Onsager coefficients	$L_{ij} = L_{ji}$	$L_{ij}^{(q)} = L_{ji}^{(q)}$ (inflated)
C: Structural	Crooks theorem	$P_F/P_R = e^{(W-\Delta\mathcal{F})/T}$	$P_F/P_R = \exp_q((W - \Delta\mathcal{F}_q)/T)$
	Jarzynski equality	$\langle e^{-\beta W} \rangle = e^{-\beta\Delta\mathcal{F}}$	$\langle \exp_q(-\beta W) \rangle = \exp_q(-\beta\Delta\mathcal{F}_q)$
	Equilibrium dist.	$p^* \propto e^{-\beta\varepsilon}$	$p^* \propto \exp_q(-\beta\varepsilon)$
	Log-sum-exp	$-T \log \sum e^{-\varepsilon/T}$	$-T \ln_q \sum \exp_q(-\varepsilon/T)$

Why the three categories exist. The classification reflects the mathematical origin of each result:

Category A results depend only on the Legendre structure of the CES potential—homogeneity, topology, and conservative-dissipative structure are properties of Φ and the network coupling, not the entropy. These survive any entropy generalization, not just Shannon→Tsallis.

Category B results depend on the *curvature* of the entropy at equilibrium. The Tsallis entropy has curvature qp^{q-2} versus Shannon’s $1/p$, differing by a factor that evaluates to $1/(2-q)$ at symmetric equilibrium. This multiplicative correction propagates to all second-order quantities: fluctuations, response, transport coefficients.

Category C results depend on the *functional form* of the equilibrium distribution. Replacing \exp with \exp_q changes the shape of the distribution—from exponential tails to power-law tails (or compact support)—affecting all results that reference the distribution directly.

Remark 5.18 (The $1/(2-q)$ correction factor). *The correction factor $1/(2-q)$ has a unified interpretation. For complements ($q < 1$), $1/(2-q) > 1$: fluctuations are amplified relative to Shannon. Complementary inputs interact, so a shock to one propagates to all others, amplifying the total variance. For substitutes ($q > 1$), $1/(2-q) < 1$: fluctuations are dampened. Substitutable inputs absorb shocks independently, reducing total variance. At $q = 1$ (Cobb-Douglas), $1/(2-q) = 1$ and Shannon is recovered exactly.*

The magnitude of the correction is:

$$\frac{1}{2-q} - 1 = \frac{q-1}{2-q} = \frac{\rho-1}{2-\rho} \quad (34)$$

For typical manufacturing values ($\sigma \approx 3-5$, i.e., $\rho \approx 0.67-0.80$), the correction is -25%

to -17% —a modest but systematic effect. For highly complementary production ($\rho \approx -2$, e.g., semiconductor fabrication), the correction is $+75\%$: Shannon-based models underestimate fluctuations by almost a factor of two in these sectors.

Remark 5.19 (Practical implications of the classification). *The classification has immediate practical consequences:*

1. **For estimation:** Category A results can be estimated using Shannon-based methods without correction. Category B results require the $1/(2-q)$ adjustment, which is computable from ρ alone. Category C results require fitting the full q -exponential distribution.
2. **For early warning:** Pre-crisis deceleration (Table 1, Category A) holds exactly under Tsallis: the relaxation time $\tau \sim |T - T^*|^{-1}$ diverges as $T \rightarrow T^*$, regardless of q . This means standard early-warning signals (increased autocorrelation, increased variance near the critical point) remain valid without q -correction.
3. **For policy:** Institutional design based on the CES potential can use Category A results (topology, conservation laws) confidently. Calibration exercises using variance-based estimates (Category B) must include the $1/(2-q)$ correction to avoid systematic bias.

6 Optimal Firm Scope and Integration Boundary

6.1 The Variety–Coordination Tradeoff

Adding input types creates value through the diversity premium but incurs coordination costs through the information entropy of managing more complex production. The firm's CES potential as a function of J is:

$$\mathcal{F}(J) = -\mathbb{E}[\log F(\hat{\mathbf{x}}(J); \boldsymbol{\theta})] + T \cdot H^*(J) \quad (35)$$

where $H^*(J)$ is the optimal entropy of the attention allocation across J input types.

The diversity premium (from superadditivity) scales as:

$$\Delta\Phi(J) \equiv \log F_J - \log F_1 \sim \frac{1}{\rho} \log J + O(1), \quad \rho < 0 \quad (36)$$

which grows logarithmically in J for complementary inputs. The coordination cost scales as:

$$T \cdot H^*(J) \sim T \cdot \log J \quad (37)$$

since the firm must process $O(\log J)$ bits to discriminate among J input types. The net benefit of adding the J -th input type is positive when $1/\rho > T$ (the diversity premium per additional variety exceeds the coordination cost per additional variety).

Proposition 6.1 (Optimal Firm Scope). *The optimal number of input types J^* satisfies the first-order condition:*

$$\frac{\partial \Delta\Phi}{\partial J} \Big|_{J^*} = T \cdot \frac{\partial H^*}{\partial J} \Big|_{J^*} \quad (38)$$

The marginal diversity premium equals the marginal coordination cost. The solution has the comparative statics:

- (i) $\partial J^*/\partial(1 - \rho) > 0$: more complementary technologies support more input types.
- (ii) $\partial J^*/\partial T < 0$: higher information costs reduce optimal variety.
- (iii) $\partial J^*/\partial C > 0$: larger firms support more input types.

Proof. At leading order, both the diversity premium ($\sim (1/\rho) \log J$) and coordination cost ($\sim T \log J$) scale as $\log J$, so the interior optimum J^* is determined by sub-leading corrections: the diversity premium has diminishing returns from input overlap (each additional variety contributes less when existing varieties partially substitute), while coordination costs grow slightly faster than $\log J$ due to pairwise interactions. The comparative statics follow from the leading-order structure.

Part (i): The diversity premium coefficient is $1/\rho$, which increases as ρ decreases. A higher coefficient means the diversity benefit exceeds the coordination cost for a wider range of J , shifting J^* rightward.

Part (ii): The coordination cost coefficient is T , which shifts the coordination cost curve upward proportionally, moving the intersection leftward.

Part (iii): Larger C implies larger $c = C/J$, which increases $T^*(\rho)$ (Remark 4.3), effectively lowering the ratio T/T^* and raising the net benefit of each additional input type. \square

Corollary 6.2 (Institutional Quality and Complexity). *Firms in high- T environments (weak institutions, poor monitoring, low management quality) optimally operate simpler production processes (lower J^*) even when the underlying technology has strong complementarity. This provides a micro-foundation for the observation that developing-country firms use less complex production methods than firms in advanced economies using identical machinery (Bloom et al., 2013).*

Example 6.3 (Concrete Comparative Statics). *Consider three production environments:*

<i>Environment</i>	ρ	T	J^*	<i>Organizational form</i>
<i>Pharmaceutical R&D</i>	-2	<i>Low</i>	~ 20	<i>Large diversified lab</i>
<i>Auto assembly</i>	-0.5	<i>Medium</i>	~ 8	<i>Tiered supply chain</i>
<i>Commodity agriculture</i>	0.5	<i>High</i>	~ 3	<i>Simple farm</i>

The ranking $J_{\text{pharma}}^* > J_{\text{auto}}^* > J_{\text{agri}}^*$ follows from both higher complementarity (lower ρ) and lower information friction in more complex industries. The pharmaceutical lab integrates chemistry, biology, clinical trial design, regulatory affairs, and multiple specialties—all highly complementary—because the institutional environment (trained scientists, peer review, FDA protocols) keeps T low enough to exploit the complementarity. A developing-country farm uses three inputs (land, labor, seed) not because the underlying agronomy lacks complementarity, but because T is too high to coordinate more complex production.

6.2 The Phase Diagram for Integration

A firm producing with CES technology faces two information environments: internal coordination at friction T_H (hierarchy) and market procurement at friction T_M (market). Hierarchy offers lower friction ($T_H < T_M$) because of direct monitoring, shared culture, and aligned incentives, but at a cost: bureaucratic overhead reduces output by a factor $(1-\beta)$ where $\beta \in (0, 1)$ captures the well-documented inefficiency of internal organization (Williamson, 1985).

The firm integrates input j when the hierarchical production CES potential is lower:

$$-\log F_j|_{T_H} + \beta \leq -\log F_j|_{T_M} \quad (39)$$

which, using the output loss formula (19), reduces to:

$$\frac{K^2(T_M - T_H)}{2(J-1)c^2} \geq \beta \quad (40)$$

Proposition 6.4 (Integration Boundary in (ρ, T) Space). *The firm integrates input j if and only if:*

$$K > K^*(\Delta T) \equiv \sqrt{\frac{2(J-1)c^2\beta}{\Delta T}} \quad (41)$$

where $\Delta T = T_M - T_H > 0$. Equivalently, in (ρ, T_M) space with T_H and β fixed:

$$\rho < \rho^*(T_M) = 1 - \frac{J}{J-1} \sqrt{\frac{2(J-1)c^2\beta}{T_M - T_H}} \quad (42)$$

This defines four regions in (ρ, T) space:

Proof. The integration condition (40) rearranges to $K^2 \geq 2(J-1)c^2\beta/\Delta T$. Taking square

roots (both sides positive) gives $K > K^*(\Delta T)$. Substituting $K = (1 - \rho)(J - 1)/J$ and solving for ρ gives (42). \square

Low ρ, Low T	High ρ, Low T
Large diversified firms	Liquid spot markets
Aerospace, pharma	Commodity exchanges
R&D, TSMC	
$K_{\text{eff}} \approx K$, full complementarity	$K \approx 0$, substitutability
Low ρ, High T	High ρ, High T
Vertical integration, simplified	Bilateral exchange, informal
Developing-country assembly	Bazaar economy, day labor
$K_{\text{eff}} < K$, partial complementarity	$K \approx 0$, no premium to exploit

Each quadrant corresponds to a distinct organizational form observed in practice:

- **Low ρ , Low T (upper-left):** The firm's technology has strong complementarity and the information environment is good. The full curvature K is exploitable. Optimal response: large, diversified firms managing many complementary inputs internally. Examples: TSMC (integrating chip design and fabrication), pharmaceutical R&D labs (integrating chemistry, biology, clinical trials).
- **High ρ , Low T (upper-right):** Inputs are near-substitutable and information is cheap. CES curvature provides no premium, so coordination costs dominate. Optimal response: liquid spot markets with minimal firm structure. Examples: commodity exchanges, wholesale markets, financial markets for standardized instruments.
- **Low ρ , High T (lower-left):** Strong technological complementarity but poor information. The full curvature cannot be exploited, but $K_{\text{eff}} > 0$ makes hierarchical coordination worthwhile. Optimal response: vertically integrated but simplified production. Examples: developing-country manufacturing using imported technology with simplified processes.
- **High ρ , High T (lower-right):** Neither complementarity premium nor information advantage. Optimal response: bilateral exchange with minimal institutional overhead. Examples: bazaar economies, informal day labor markets, subsistence agriculture.

Proposition 6.5 (Williamson as a Special Case). *The Williamson governance structures framework (Williamson, 1979) corresponds to the qualitative limit of the integration boundary:*

- (i) “Asset specificity” is low ρ : complementary inputs cannot be redeployed (the CES aggregate is sensitive to the specific input bundle).
- (ii) “Behavioral uncertainty” is high T : bounded rationality and opportunism create information friction.
- (iii) “Frequency” maps to effective diversity d^2 : repeated transactions build relationship-specific diversity, raising T^* .
- (iv) The “fundamental transformation”—in which a competitive procurement becomes a bilateral monopoly after investment—is the transition from high ρ (ex ante, many potential suppliers are substitutable) to low ρ (ex post, the chosen supplier’s investment is complementary to the firm’s assets).

The present framework adds quantitative content: the integration boundary (41) is a curve, not a qualitative typology.

6.3 Supply Chain Architecture

A production process comprises L tiers, each with its own CES technology (curvature K_ℓ) and information environment (friction T_ℓ). The total production CES potential is:

$$\mathcal{F}_{\text{total}} = \sum_{\ell=1}^L \mathcal{F}_\ell(K_\ell, T_\ell, G_\ell) \quad (43)$$

where $G_\ell \in \{M, H\}$ indicates market or hierarchical governance at tier ℓ . Market governance sets $T_\ell = T_M$ with no overhead; hierarchical governance sets $T_\ell = T_H$ with overhead β .

Proposition 6.6 (Optimal Supply Chain Architecture). *The optimal governance choice at each tier is:*

$$G_\ell^* = \begin{cases} H & (\text{integrate}) \quad \text{if } K_\ell > K^*(\Delta T) \\ M & (\text{outsource}) \quad \text{if } K_\ell \leq K^*(\Delta T) \end{cases} \quad (44)$$

where $K^*(\Delta T)$ is the threshold from the integration boundary. The architecture is monotone in K_ℓ , with highest- K tiers integrated and lowest- K tiers outsourced.

Proof. Each tier’s governance choice is independent given ΔT and β (the CES potentials are additively separable across tiers in (43)). Tier ℓ is integrated when the curvature

benefit of hierarchy exceeds the overhead cost, i.e., $K_\ell > K^*$. Since K^* is a single threshold independent of ℓ , the monotone structure follows. \square

Example 6.7 (Semiconductor Manufacturing).

Tier	ρ_ℓ	K_ℓ	Governance
Chip design	-3	High	Integrated (<i>TSMC in-house</i>)
Fabrication	-2	High	Integrated
Testing & packaging	0.2	Medium	Mixed (<i>OSAT partners</i>)
Commodity materials	0.8	Low	Market (<i>spot procurement</i>)

The monotone structure matches observed industry practice: design and fabrication are tightly integrated while commodity inputs are procured on spot markets.

6.4 Cross-Tier Information Spillovers

When tiers are vertically linked, integration at one tier can reduce T at adjacent tiers through shared information systems. Define the *information spillover* from tier ℓ to tier ℓ' :

$$\Delta T_{\ell'}^{(\ell)} = \alpha_{\ell\ell'} \cdot \mathbf{1}[G_\ell = H] \quad (45)$$

where $\alpha_{\ell\ell'} \geq 0$ measures the strength of informational complementarity between tiers. When spillovers are present, the optimization becomes a combinatorial problem with 2^L governance configurations. However, the monotone structure is preserved:

Corollary 6.8 (Spillover-Adjusted Architecture). *With non-negative spillovers ($\alpha_{\ell\ell'} \geq 0$), integrating a high- K tier weakly increases the incentive to integrate adjacent tiers. The optimal architecture remains monotone in K_ℓ , but the threshold K^* is lower for tiers adjacent to already-integrated tiers.*

This provides a formal basis for the empirical regularity that integration tends to cluster: firms that integrate one tier often integrate adjacent tiers as well (Grossman and Hart, 1986). Toyota's production system, for instance, integrates design and assembly (high K) while managing a tiered supplier network (lower K), with the most complementary suppliers brought closest to the integration boundary through long-term relationships that reduce their effective T .

6.5 Aggregate Supply Chain CES Potential

For the full supply chain, the aggregate effective curvature provides a summary statistic of supply chain fragility:

Proposition 6.9 (Supply Chain Fragility Index). *The aggregate CES potential of an L -tier supply chain is:*

$$\mathcal{F}_{\text{total}} = \sum_{\ell=1}^L w_\ell \cdot K_\ell \cdot \left(1 - \frac{T_\ell}{T_\ell^*(\rho_\ell)}\right)^+ \quad (46)$$

where $w_\ell = c_\ell^2 / \sum_{\ell'} c_{\ell'}^2$ is the cost weight of tier ℓ . The supply chain fragility index is:

$$\mathcal{S} = \max_{\ell} \frac{T_\ell}{T_\ell^*(\rho_\ell)} \quad (47)$$

The supply chain fails when $\mathcal{S} = 1$: the weakest tier determines the failure point.

The fragility index \mathcal{S} provides a natural metric for supply chain risk assessment. During the 2020–2021 semiconductor shortage, the highest T/T^* ratio occurred at the fabrication tier ($\ell = 2$, where $\rho \approx -2$ and information friction spiked due to demand uncertainty), consistent with the observed bottleneck location. The framework predicts that supply chain interventions should target the highest- T/T^* tier, not necessarily the highest-cost tier—a distinction relevant for policy.

7 Empirical Predictions

The framework generates several testable predictions that distinguish it from existing theories. These predictions arise from the interaction of ρ and T —a dimension absent from both standard production theory (which fixes $T = 0$) and standard information economics (which ignores ρ).

7.1 Management–Technology Complementarity

Proposition 7.1 (Returns to Management Quality). *The marginal return to reducing information friction (improving management quality) is:*

$$\frac{\partial \mathcal{F}}{\partial T} = H^*(J) + \frac{K^2}{2(J-1)c^2} \quad (48)$$

which is increasing in K (and hence in complementarity $1 - \rho$).

Proof. Differentiating the CES potential $\mathcal{F}(J) = -\mathbb{E}[\log F(\hat{\mathbf{x}}(J))] + T \cdot H^*(J)$ with respect to T : the direct effect is $H^*(J)$ (the entropy cost of coordination), and the indirect effect through the output loss (19) contributes $K^2/(2(J-1)c^2)$. Both terms are non-negative; the second is increasing in K . \square

Prediction 1. *The management quality premium is larger in industries with lower ρ .* Bloom, Sadun, and Van Reenen’s management practice scores (Bloom et al., 2013) predict

productivity differentials, but the present framework predicts that these differentials are largest in high-complementarity industries (e.g., pharmaceuticals, aerospace) and smallest in high-substitutability industries (e.g., commodity production, retail).

Test. Cross industry-level estimates of σ from [Oberfield and Raval \(2021\)](#) with management quality data. The interaction term management \times $(1/\sigma)$ should be positive and significant in explaining productivity.

7.2 Within-Industry Productivity Dispersion

Proposition 7.2 (Productivity Dispersion). *Within-industry log-productivity dispersion is:*

$$\text{Var}[\log F] \approx \frac{K^4}{4(J-1)^2 c^4} \cdot \text{Var}[T_i] \quad (49)$$

where $\text{Var}[T_i]$ is the cross-firm variance of information friction within the industry.

Prediction 2. *Within-industry productivity dispersion is increasing in $(1-\rho)$.* [Syverson \(2004\)](#) documents that the 90th-percentile plant produces nearly twice as much as the 10th-percentile plant with the same inputs. The prediction is that this dispersion is largest in high-complementarity industries, where differences in management quality (T) translate most strongly into output differences. Industries with ρ near 1 show compressed dispersion because $K \approx 0$ means T differences matter little for output.

7.3 AI Adoption Priority

AI-driven monitoring and management tools effectively reduce T by automating information processing. The marginal value of reducing T is highest where K is large and T is currently near $T^*(\rho)$.

Prediction 3. *AI adoption generates the largest productivity gains in industries with high technological complementarity and currently high information frictions.* Specifically:

- Healthcare (low ρ : complementary specialties; high T : fragmented information) — high AI returns.
- Legal services (low ρ : complementary expertise; high T : case-specific knowledge) — high AI returns.
- Complex manufacturing (low ρ : complementary inputs; medium T : partially automated) — moderate AI returns.
- Commodity production (high ρ : substitutable inputs) — low AI returns regardless of T .

The prediction distinguishes this framework from the common claim that AI benefits “knowledge-intensive” industries. The relevant distinction is not knowledge intensity but *complementarity under friction*: industries where diverse inputs create value but information frictions prevent full exploitation.

Calibration. The productivity gain from AI adoption in sector s is approximately:

$$\Delta \log F_s \approx \frac{K_s^2}{2(J_s - 1)c_s^2} \cdot (T_s^{\text{pre}} - T_s^{\text{post}}) \quad (50)$$

where T_s^{pre} is the pre-AI information friction and $T_s^{\text{post}} = T_s^{\text{pre}} \cdot (1 - \alpha_{\text{AI}})$ is the post-adoption friction, with $\alpha_{\text{AI}} \in (0, 1)$ measuring the fraction of information processing automated by AI. For healthcare ($K \approx 2$, $J = 8$, $\alpha_{\text{AI}} = 0.3$): the predicted gain is approximately $0.3 \times 4/(14c^2) \times T$, which for realistic T values implies 5–15% productivity improvement. For commodity agriculture ($K \approx 0.2$, $J = 3$, $\alpha_{\text{AI}} = 0.3$): the predicted gain is $0.3 \times 0.04/(4c^2) \times T$, an order of magnitude smaller. The K^2 amplification is the key: it makes the AI productivity gain quadratic in complementarity, creating a stark differential between complex and simple industries.

Test. Cross-industry regressions of AI adoption rates (from firm surveys) on estimated $K_s \times T_s$ should show positive coefficients. The prediction is sharper than generic “technology adoption” models because the relevant variable is the interaction $K \times T$, not either component alone.

7.4 Crisis Dynamics Prediction

Prediction 4. *In economic crises, diversification benefits fail before production complementarities, which fail before strategic stability.* This follows from Corollary 4.6 and is testable by examining the timing of:

- (i) Correlation spikes across asset classes (correlation robustness failure),
- (ii) Supply chain disruptions and trade volume declines (superadditivity failure),
- (iii) Counterparty defections and institutional breakdowns (strategic independence failure).

The prediction is that (i) precedes (ii) precedes (iii) in every major crisis, not just the 2008 GFC.

8 Empirical Tests

8.1 Banking Regulation and the Global Financial Crisis

The framework predicts that markets with higher substitutability (ρ close to 1, low K) are more vulnerable to information degradation, and that this vulnerability is *mediated* by information quality. The 2007–2009 GFC provides a natural experiment.

The BCL Puzzle. [Barth, Caprio, and Levine \(2006\)](#) documented that countries with stricter activity restrictions experienced worse financial outcomes. The CES potential framework resolves this: activity restrictions force banks into commodity lending (high ρ , low K), lowering the breakdown threshold T^* .

Mechanism. The causal chain has three links. First, activity restrictions (measured by the Barth-Caprio-Levine Activity Restrictions Index, ARI) constrain the range of activities banks can undertake—securities dealing, insurance, real estate, ownership of non-financial firms. This forces bank portfolios toward standardized, commodity-like assets (high ρ) rather than diversified, complementary asset bundles (low ρ). Second, the effective curvature theorem (Theorem 4.2) predicts that high- ρ portfolios have lower K_{eff} and hence lower breakdown thresholds T^* . Third, when the crisis hits (information friction T rises sharply as counterparty risk becomes opaque), high-ARI countries cross their lower T^* threshold sooner, triggering the crisis sequence of Corollary 4.6. The prediction is an interaction: ARI alone should not predict crisis severity (column 1), but ARI interacted with pre-crisis information quality (PMI, proxying for T^{-1}) should be significant, because the combination determines the initial distance T^*/T from the breakdown boundary.

Data. Pre-crisis BCL indices from BRSS Wave 2 (2003): Activity Restrictions Index (ARI, proxy for σ), Private Monitoring Index (PMI, proxy for T^{-1}). Outcome: cumulative GDP per capita change 2007–2009. Sample: 147 countries. The ARI ranges from 4 (least restrictive: Germany, UK) to 16 (most restrictive: China, many developing countries), constructed from survey responses on restrictions in four activity domains. The PMI ranges from 3 to 12, constructed from survey questions on disclosure requirements, external audit quality, credit ratings agencies, and subordinated debt requirements—each of which directly reduces information friction by making bank risk observable.

The interaction $\text{ARI} \times \text{PMI}$ enters with $\beta = +0.307$ ($p = 0.016$), with both constituent terms significant and correctly signed. At the median PMI of 7, the marginal effect of ARI is near zero; at low PMI (~ 4), it is -0.69 pp per ARI unit. This is the framework’s prediction: $T^*(\sigma)$ is low when σ is high (high ARI), so the system collapses unless T

Table 2: Cross-Country GFC Severity: Activity Restrictions \times Private Monitoring

	(1) Baseline	(2) +PMI	(3) Interaction
ARI (Activity Restrictions)	0.324 (0.212)	0.317 (0.210)	-1.928* (0.950)
PMI (Private Monitoring)		-0.391 (0.304)	-3.310** (1.246)
ARI \times PMI			0.307* (0.127)
log GDP per capita	-1.828*** (0.496)	-1.718*** (0.483)	-1.797*** (0.487)
<i>N</i>	147	147	147
<i>R</i> ²	0.110	0.117	0.138

Dependent variable: cumulative GDP per capita change 2007–2009 (%). Robust (HC1) standard errors in parentheses. Placebo interactions (all insignificant): CSI \times PMI $\beta = -0.09$, $p = 0.51$; SPI \times PMI $\beta = -0.26$, $p = 0.07$; EBI \times PMI $\beta = -0.24$, $p = 0.31$. *** $p < 0.001$; ** $p < 0.01$; * $p < 0.05$.

is also low (high PMI). Placebo tests using CSI, SPI, and EBI instead of ARI show no significant interactions, confirming specificity to the product substitutability dimension.

Interpretation. The regression results admit a precise interpretation through the CES potential framework. Column (1) shows that ARI alone has no significant effect—activity restrictions do not directly predict crisis severity. Column (2) adds PMI, which is also insignificant on its own. Column (3) adds the interaction, which enters significantly ($p = 0.016$) with the predicted sign: countries with both high ARI (commodity banking, high ρ) and low PMI (poor information quality, high T) experienced worse crises. The economic magnitude is substantial: moving from the 25th to 75th percentile of the ARI \times PMI interaction corresponds to a 2.4 percentage-point difference in cumulative GDP per capita change—roughly the difference between a mild recession and a severe contraction.

The placebo tests provide additional support. If the ARI \times PMI interaction reflected a generic “regulation intensity” channel, then other BCL indices interacted with PMI should also be significant. They are not: the Capital Stringency Index (CSI, measuring capital adequacy requirements), Supervisory Power Index (SPI, measuring supervisory authority), and Entry Barriers Index (EBI, measuring barriers to new bank entry) all produce insignificant interactions with PMI. The specificity to ARI—the index that directly maps to product substitutability—supports the CES potential interpretation over alternative channels.

Companion Test: Financial Development Panel. The GFC test uses a single crisis. A companion asks whether the same $\sigma \times T$ mechanism operates in normal times.

Using the full five-wave BCL panel (2001–2019, 154 countries, $N = 657$), with domestic credit to the private sector (% of GDP) as the dependent variable, wave fixed effects, and standard errors clustered by country: the ARI \times PMI interaction is positive in all specifications ($\hat{\beta} = +0.28$ to $+0.36$), consistent with the framework. The coefficient is not statistically significant ($p = 0.18$ – 0.29), reflecting the noisier panel setting. The placebo pattern replicates: CSI \times PMI, SPI \times PMI, and EBI \times PMI are all insignificant and smaller in magnitude. Only ARI—the index mapping to product substitutability—shows the predicted interaction.

8.2 Manufacturing Tail Distributions (Tsallis Test)

The Shannon framework predicts exponential (Boltzmann) distributions for economic fluctuations; the Tsallis framework predicts q -exponential distributions. The distinguishing feature is tail behavior: exponential tails decay as $e^{-\beta|r|}$, while q -exponential tails decay as $|r|^{-1/(q-1)}$ for $q > 1$ or have compact support for $q < 1$.

Data. Monthly industrial production (IP) indices for 17 manufacturing sectors from FRED. For each sector s , absolute log-returns $|r_{s,t}| = |\log(\text{IP}_{s,t}/\text{IP}_{s,t-1})|$ are computed and fit with:

- (a) **Exponential** (Shannon): $f(r) = \lambda e^{-\lambda r}$, one parameter ($\lambda = 1/\bar{r}$).
- (b) **q -exponential** (Tsallis): $f_q(r) = C_q [1 - (1 - q)\beta r]_+^{1/(1-q)}$, two parameters (q, β).

The exponential is nested within the q -exponential ($q = 1$), enabling a likelihood ratio test with 1 degree of freedom.

Identification strategy. Two tests provide complementary evidence:

Within-sector: For each sector, the likelihood ratio statistic $\Lambda_s = -2(\ell_{\text{exp}} - \ell_{q-\text{exp}})$ is asymptotically $\chi^2(1)$ under the null $H_0 : q = 1$. The Anderson-Darling statistic against the exponential null provides a complementary test.

Cross-sector: If $q = \rho$ (the CES parameter), then estimated \hat{q}_s should correlate with independent estimates of $\hat{\sigma}_s$ (elasticity of substitution) from [Oberfield and Raval \(2021\)](#). The test regresses \hat{q}_s on $\hat{\rho}_s = 1 - 1/\hat{\sigma}_s$ and tests $\beta_1 = 1$.

GARCH control. Fat tails in returns may reflect GARCH volatility clustering rather than non-extensive equilibrium distributions. As a robustness check, GARCH(1,1) models are fit to each sector's log-returns and the standardized residuals $\hat{z}_{s,t} = r_{s,t}/\hat{\sigma}_{s,t}$ are retested. If q -exponential superiority survives standardization, the tail behavior is not purely a volatility artifact.

Results. The q -exponential provides a significantly better fit (likelihood ratio test $p < 0.05$) in 12 of 17 sectors. The Anderson-Darling test rejects the exponential null in 15 of 17 sectors. Estimated \hat{q} values range from 0.63 to 1.16, with a mean near 1.0. After GARCH(1,1) standardization, 9 of 17 sectors retain significant q -exponential superiority, indicating that the non-exponential tail behavior is not purely a volatility clustering artifact. The cross-sector regression of \hat{q} on $\hat{\rho}$ yields $R^2 = 0.03$, consistent with the prediction's direction but lacking power in this small cross-section.

9 The CES Potential as a Welfare Loss Function

The CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$ has the mathematical structure of a Lyapunov function for economic dynamics: it is bounded below, continuous, and strictly decreasing along the adjustment path until equilibrium is reached.

Theorem 9.1 (Lyapunov Property). *For any CES economy with J inputs, structural curvature $K > 0$, and information friction $T < T^*(\rho)$, the CES potential \mathcal{F}_q satisfies:*

- (i) **Boundedness:** $\mathcal{F}_q \geq -\log F^* - T \cdot S_q^{\max}$, where $S_q^{\max} = (J^{1-q} - 1)/(1 - q)$ is the Tsallis entropy at the uniform distribution (bounded below by the first-best output less the maximum entropy contribution).
- (ii) **Monotone decay:** Along any gradient-descent adjustment path $\dot{\mathbf{x}} = -\mathbf{L}\nabla\mathcal{F}_q$, the CES potential satisfies $d\mathcal{F}_q/dt \leq 0$, with equality only at the equilibrium allocation.
- (iii) **Strict convexity on the tangent space:** The Hessian of \mathcal{F}_q restricted to $\mathbf{1}^\perp$ is positive definite with minimum eigenvalue $K_{\text{eff}}/(J - 1)c^2 > 0$.

Proof. (i) The CES aggregate satisfies $F \leq F^*$ (first-best is the maximum), so $-\log F \geq -\log F^*$. The Tsallis entropy is bounded above by $S_q^{\max} = (1 - J^{1-q})/(q - 1)$ (at the uniform distribution). Since the CES potential is $\Phi - T \cdot S_q$ with $\Phi = -\log F$ and $S_q \leq S_q^{\max}$:

$$\mathcal{F}_q = -\log F - TS_q \geq -\log F^* - T \cdot S_q^{\max}$$

For $q \rightarrow 1$: $S_q^{\max} \rightarrow \log J$, recovering the Shannon bound.

- (ii) Along $\dot{\mathbf{x}} = -\mathbf{L}\nabla\mathcal{F}_q$:

$$\frac{d\mathcal{F}_q}{dt} = (\nabla\mathcal{F}_q)^\top \dot{\mathbf{x}} = -(\nabla\mathcal{F}_q)^\top \mathbf{L}(\nabla\mathcal{F}_q) \leq 0$$

since \mathbf{L} is positive semi-definite. Equality holds only when $\nabla\mathcal{F}_q = 0$ (equilibrium) or $\nabla\mathcal{F}_q \in \ker \mathbf{L}$.

(iii) The Hessian on $\mathbf{1}^\perp$ is $\nabla^2\mathcal{F}_q|_{\mathbf{1}^\perp} = |\lambda_\perp| \cdot \mathbf{I}_{J-1} - T \cdot \nabla^2 S_q|_{\mathbf{1}^\perp}$. At the symmetric equilibrium, $|\lambda_\perp| = K/((J - 1)c^2)$ and $-T\nabla^2 S_q$ contributes a positive-definite correction

(since S_q is concave, $\nabla^2 S_q \prec 0$). The minimum eigenvalue is at least $K_{\text{eff}}/((J-1)c^2)$, which is positive for $T < T^*$. \square

Remark 9.2 (Economic interpretation). *The Lyapunov property means that the CES potential provides a welfare-loss accounting for all adjustments. Every reallocation that improves production efficiency (reducing $\Phi = -\log F$) or improves information utilization (increasing S_q toward the optimal entropy level) reduces \mathcal{F}_q . At equilibrium, these two forces are balanced: the marginal production gain from further information acquisition equals its marginal cost. The CES potential thus serves the same role in the economic framework that the Helmholtz free energy serves in thermodynamics: it is the quantity minimized at equilibrium, and its distance from the minimum measures the system's distance from efficiency.*

Corollary 9.3 (Convergence Rate). *The adjustment dynamics converge exponentially to equilibrium at rate:*

$$\mathcal{F}_q(t) - \mathcal{F}_q^* \leq (\mathcal{F}_q(0) - \mathcal{F}_q^*) \cdot \exp\left(-\frac{2L_{\min}K_{\text{eff}}}{(J-1)c^2} \cdot t\right) \quad (51)$$

where L_{\min} is the smallest eigenvalue of the friction matrix \mathbf{L} . The convergence rate is proportional to $K_{\text{eff}} \cdot L_{\min}$: faster convergence requires both high effective curvature (strong complementarity, low friction) and fast adjustment mechanisms.

Remark 9.4 (Connection to the eigenstructure bridge). *The eigenstructure bridge established in the hierarchical architecture paper shows that $\nabla^2 \Phi|_{\text{slow}} = \mathbf{W}^{-1} \cdot \nabla^2 V$, where V is the welfare loss function and \mathbf{W} is the institutional supply-rate matrix. Combined with the Lyapunov property, this yields: the rate of welfare improvement is $dV/dt = -(\nabla V)^\top \mathbf{W} \mathbf{L} \mathbf{W} (\nabla V)$, which factors through both the institutional adjustment speed (\mathbf{W}) and the informational adjustment speed (\mathbf{L}). This double dependence explains why welfare gains from institutional reform are non-additive: improving \mathbf{W} without improving \mathbf{L} (or vice versa) yields less than proportional welfare gains.*

10 The Architecture

The two parameters have clean domains: ρ determines **structure** (market form, network effects, asset specificity, mechanism format, political robustness); T determines **friction** (optimization precision, market clearing speed, information rents, behavioral deviations). The interaction $\rho \times T$ determines **institutional viability**: the breakdown threshold $T^* \propto K(\rho)$ is the institutional failure boundary.

Area	Key result derived	New ρ -dependent prediction
Industrial production	Effective curvature degrades under friction	Crisis sequence: robustness \rightarrow superadditivity \rightarrow strategic
Firm scope / integration	Variety–coordination tradeoff in (ρ, T) space	Monotone integration rule from tier-specific K_ℓ/T_ℓ
Welfare loss	CES potential as Lyapunov function	Convergence rate $\propto K_{\text{eff}} \cdot L_{\min}$
Tsallis extension	$q = \rho$ classifies all dynamical results	$1/(2 - q)$ correction; compact support vs. power-law tails
Six derivations	Akerlof, Myerson, Arrow, DMP, GHM, KT	See companion paper (Smirl, 2026b)

Table 3: Unification through the CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$ with $q = \rho$.

10.1 Welfare Implications of the Architecture

The CES potential framework yields a unified welfare criterion. Define the *welfare distance* from the first-best as:

$$W_{\text{loss}}(\rho, T) = \mathcal{F}_q(\rho, T) - \mathcal{F}_q(\rho, 0) = T \cdot S_q^* + \frac{K^2 T}{2(J-1)c^2} \quad (52)$$

where the first term is the direct information cost and the second is the production loss from misallocation. This decomposes welfare loss into two channels:

1. **Information channel** ($T \cdot S_q^*$): the resources spent on information acquisition and processing. This is a transfer from productive activity to information infrastructure (management, monitoring, auditing).
2. **Allocation channel** ($K^2 T / 2(J-1)c^2$): the output forgone because information friction prevents the first-best allocation. This loss is quadratic in K —complementary production systems lose disproportionately more from the same level of friction.

The welfare implication is sharp: reducing T yields larger welfare gains in high- K economies. This provides a welfare-theoretic justification for institutional investment (reducing T) that is larger for economies with more complementary production structures. The prediction aligns with the development economics literature: institutional quality matters more for complex economies ([Bloom et al., 2013](#)).

10.2 The (ρ, T) Classification of Economic Theories

The architecture table (Table 3) can be read as a classification of the economics literature by its position in (ρ, T) space:

- **Neoclassical economics** ($T = 0$, arbitrary ρ): perfect information, deterministic optimization. This is the first line of every textbook—production theory, consumer theory, general equilibrium. All behavioral phenomena vanish.
- **Information economics** ($T > 0$, $\rho = 1$): positive friction but linear aggregation. This is the domain of Shannon entropy, standard logit, exponential distributions, and the classical Akerlof-Myerson-Arrow-DMP results. The CES curvature plays no role because $K = 0$.
- **Industrial organization** ($T > 0$, $\rho < 1$): positive friction with complementary production. This is the domain of the present paper: effective curvature, firm scope, integration boundaries, and the crisis sequence all require both $T > 0$ and $\rho < 1$.
- **Behavioral economics** ($T \gg 0$, arbitrary ρ): high friction dominates the CES structure. The behavioral catalog emerges as the universal characterization of high- T behavior, independent of ρ (see [Smirl 2026b](#)).

Each “school” of economics thus corresponds to a region of (ρ, T) space, and the apparent disagreements between schools reflect the different assumptions about which region is empirically relevant. The CES potential framework does not adjudicate this empirical question—it provides the mathematical structure that connects the regions.

11 Discussion

11.1 Why Wasn’t This Seen Before?

The components are not new. CES aggregates have been used since [Arrow et al. \(1961\)](#). Tsallis entropy has been applied to complex systems since [Tsallis \(1988\)](#). The unity was obscured by the structure of the economics profession: trade economists use CES for Armington elasticities, macro economists for production functions, information economists use entropy for adverse selection, mechanism designers for incentive constraints, behavioral economists use noise models for bounded rationality. Each group uses its piece without recognizing it as a piece.

Three specific barriers prevented the recognition:

1. **The ordinal-cardinal divide.** Arrow’s impossibility theorem ([1951](#)) led welfare economics to insist on ordinal methods, while CES aggregation is inherently cardinal. The CES potential framework shows that the ordinal-cardinal divide is not an absolute boundary but a T -dependent transition: at high T , cardinal information is washed out and ordinal methods are appropriate; at low T , cardinal methods are superior. The divide is a consequence of the information friction, not a structural feature of welfare economics.

2. **The additivity assumption.** Shannon entropy's chain rule (additivity) is deeply embedded in information theory and its economic applications. Rational inattention models ([Sims, 2003](#)) assume Shannon entropy without questioning whether additivity is appropriate for complementary production. The Tsallis generalization relaxes this assumption, but the Tsallis literature in physics did not provide the economic motivation for non-additivity. The CES framework supplies this motivation: complementary production creates non-additive information costs, with the degree of non-additivity determined by ρ .
3. **Missing the identification $q = \rho$.** The Tsallis literature parameterizes non-extensivity by q and estimates it from data. The CES literature parameterizes complementarity by ρ and estimates it from production functions. Without the emergence theorem forcing $q = \rho$, these appear to be independent parameters requiring separate estimation. The identification collapses the two-parameter problem to a single parameter already estimated in the production literature, making the framework testable without new data.

11.2 Resolution of the Tsallis Debate

In physics, Tsallis entropy has been controversial because non-extensivity lacked physical motivation for systems with short-range interactions ([Nauenberg, 2003](#)). This objection is inapplicable to economics, where complementarity is the norm. The CES framework provides the missing motivation: non-extensivity is the information-theoretic consequence of complementary production, with $q = \rho$ determined by the production function rather than fitted as a free parameter. The classification in Table 1 resolves the secondary debate about which results generalize.

The resolution has four components:

1. **Motivation for $q \neq 1$.** In physics, the Hamiltonian is typically additive (particles interact locally), making Shannon entropy ($q = 1$) appropriate. In economics, the CES aggregate with $\rho < 1$ is inherently non-additive: the cross-partial $\partial^2 F / \partial x_j \partial x_k > 0$ means that inputs interact globally. This non-additivity maps directly to the Tsallis pseudo-additivity axiom with $q = \rho$.
2. **Determination of q .** The physics criticism that q is fitted rather than predicted is valid for many applications of Tsallis entropy. In the CES framework, $q = \rho$ is determined by the production technology, which is independently measurable. The identification is not a fit—it is forced by the emergence theorem.
3. **Classification of generalizations.** The physics literature contains hundreds of q -generalized formulas, many without clear economic content. The three-category

classification (Table 1) provides a principled criterion: a generalization is meaningful when it corresponds to an economic phenomenon that changes qualitatively under complementarity. Category A results (topological) are meaningful in all entropies; Category B results (second-order) are meaningful because the correction $1/(2 - q)$ has economic content (amplified fluctuations under complementarity); Category C results (distributional) are meaningful because compact support and power-law tails have distinct economic implications.

4. **Empirical discriminability.** The physics debate was partly unresolvable because exponential and q -exponential distributions are hard to distinguish in small samples. The economic framework provides additional structure: q is not a free parameter but equals ρ , which is estimated from production data. This additional constraint dramatically increases discriminating power: the manufacturing tail test (Section 8.2) tests not just whether tails are non-exponential, but whether the degree of non-exponentiality correlates with independently estimated complementarity.

11.3 The (ρ, T) Phase Diagram as a Unifying Framework

The CES potential parameterizes the entire space of economic theory by two continuous parameters. This is a strong claim—every economic theory is either a point or a region in (ρ, T) space—and it deserves scrutiny.

The claim rests on the uniqueness results of Theorem 3.2: given constant elasticity (forcing CES), pseudo-additive information (forcing Tsallis), and self-consistency ($q = \rho$), the CES potential is the unique framework. Any economic theory that satisfies these three axioms must be expressible as a special case. The axioms are restrictive: constant elasticity excludes translog and flexible functional forms; pseudo-additivity excludes Shannon and Rényi entropies (except in the $q \rightarrow 1$ limit); self-consistency excludes models where the information structure is independent of the production structure.

Are these restrictions empirically defensible? The constant elasticity assumption is standard in international trade (Armington 1969), industrial organization (Dixit and Stiglitz 1977), and growth theory (Romer 1990). The pseudo-additivity assumption is the new element, motivated by the cross-partial argument of Section 3.1: complementary inputs create non-additive information costs. The self-consistency assumption follows from the emergence theorem and is not separately testable—it is a mathematical consequence of the first two assumptions.

The framework is falsifiable through its *interaction predictions*: any finding where ρ and T affect outcomes independently (additively rather than multiplicatively) would violate the CES potential structure. The GFC banking regulation test (Section 8.1) is a direct test of this interaction, and its significance supports the framework.

11.4 Superstatistics and the Hierarchical Interpretation

Beck and Cohen (2003) show that a system with fluctuating information friction— T drawn from a χ^2 distribution with ν degrees of freedom—generates q -exponential marginals with $q = 1 + 2/(\nu + 1)$. In the economic framework, hierarchical production creates natural information friction fluctuations: the information friction T_n at each level depends on slow variables at lower levels, which drift on longer timescales. A sector experiencing technology transitions has a slowly varying T , and the time-averaged distribution of fluctuations is q -exponential even if the instantaneous distribution is Boltzmann.

This provides a complementary interpretation: the Tsallis distribution arises either from non-extensive entropy (the axiomatic route, Section 3.1) or from extensive entropy with fluctuating information friction (the superstatistic route). Both routes give $q = \rho$, since the information friction fluctuations are driven by the CES curvature that determines the hierarchical structure. The two interpretations are not competing but *dual*: non-extensivity at the macro level is equivalent to fluctuating extensivity at the micro level.

11.5 Two-Parameter Parsimony

The Tsallis generalization might appear to add a parameter (q) to the framework. But $q = \rho$ from the emergence theorem, so no new parameter is introduced. The framework remains two-parameter: (ρ, T) . The Tsallis generalization is not an extension but a *correction*—replacing an approximate axiom (Shannon additivity) with the exact axiom (q -additivity with $q = \rho$) appropriate for CES production.

The practical consequence is that all q -corrected results (Category B in Table 1) can be computed from the same two parameters already estimated. The correction factor $1/(2 - q) = 1/(2 - \rho)$ is known once ρ is known. No additional estimation is required.

To quantify the practical significance: for manufacturing industries with typical elasticities $\sigma \in [2, 5]$ (i.e., $\rho \in [0.5, 0.8]$), the correction factor $1/(2 - \rho)$ ranges from 0.83 to 0.67. This means that Shannon-based estimates of equilibrium variances systematically *overstate* fluctuations by 17–33% in these industries. For services with lower elasticities ($\sigma \in [0.5, 1.5]$, $\rho \in [-1, 0.33]$), the correction ranges from 1.0 to 0.60—the misestimation can be in either direction depending on whether ρ is above or below zero. For highly complementary sectors like semiconductor fabrication ($\sigma \approx 0.3$, $\rho \approx -2.3$), the correction is $1/(2 - (-2.3)) = 0.23$ —Shannon underestimates fluctuations by a factor of more than 4. These are not negligible corrections.

The parsimony also has implications for model selection. Competing frameworks that explain the same phenomena (e.g., GARCH for fat tails, prospect theory for behavioral biases, transaction cost economics for firm boundaries) each require their own parameter sets. The CES potential framework explains all of these with (ρ, T) alone, imposing

cross-phenomenon restrictions: the same ρ that determines production complementarity also determines tail behavior, behavioral anomaly severity, and integration boundaries. This parsimony is the framework’s strongest empirical lever: violations of the cross-phenomenon restrictions would falsify it.

11.6 Testability

The framework produces testable predictions at multiple levels: effective curvature and crisis sequence, Tsallis tail distributions (12/17 sectors favor q -exponential), and the $\rho \times T$ interaction tested in the GFC banking regulation analysis ($p = 0.016$). The companion paper (Smirl, 2026b) generates additional domain-specific predictions across six areas of economic theory. The management–technology complementarity prediction (Proposition 7.1) and the within-industry dispersion prediction (Proposition 7.2) are both testable with existing data.

The framework’s falsifiability is concentrated in the $\rho \times T$ interaction: any empirical finding where ρ and T operate independently rather than multiplicatively would reject the CES potential structure. The GFC banking test (Section 8.1) confirms the interaction.

Hierarchy of tests. The predictions admit a natural ordering by stringency:

1. **Qualitative predictions** (weakest): the sign of $\partial T^*/\partial\rho$ in each domain. These are confirmed by the banking regulation test.
2. **Cross-domain consistency**: the same ρ parameter should predict outcomes in adverse selection, mechanism design, and contract theory simultaneously. This requires estimating ρ from production data and testing it out-of-sample in information-economics settings.
3. **Quantitative predictions** (strongest): the $1/(2 - q)$ correction factor and the exact crisis sequence timing. These require high-frequency financial data during crisis episodes and precise estimates of both ρ and T .

The framework passes the first level. The second and third levels define a research agenda.

11.7 Endogenous Technology Choice

The framework treats ρ as exogenous. Endogenizing technology choice—allowing firms to choose their ρ given the information environment T —yields additional predictions.

Proposition 11.1 (Optimal Complementarity). *A firm choosing ρ to maximize output net of information costs selects:*

$$\rho^*(T) = 1 - \frac{Jd^2C_Kc^2}{T} \quad (53)$$

where d^2 is the input diversity, C_K is the production benefit per unit of curvature, and c is the symmetric input level. The optimal complementarity is increasing in T : firms in high-friction environments optimally choose more substitutable technologies (ρ^* closer to 1).

Proof. The firm chooses ρ (equivalently $K = (1 - \rho)(J - 1)/J$) to maximize the net production surplus. The diversity premium from curvature is $K \cdot d^2 \cdot C_K$ (Theorem 2.1(a)), while the output loss from information friction is $K^2 T / (2(J - 1)c^2)$ (Equation (19)). The net surplus is:

$$\Pi(K) = K \cdot d^2 C_K - \frac{K^2 T}{2(J - 1)c^2}$$

The first-order condition $\Pi'(K^*) = 0$ gives $d^2 C_K = K^* T / ((J - 1)c^2)$, so:

$$K^* = \frac{d^2 C_K (J - 1)c^2}{T}$$

The second-order condition $\Pi''(K) = -T / ((J - 1)c^2) < 0$ confirms a maximum. Substituting $K = (1 - \rho)(J - 1)/J$ and solving: $1 - \rho^* = JK^*/(J - 1) = Jd^2 C_K c^2 / T$, yielding (53). The optimal K^* is decreasing in T : higher friction reduces the net return to complementarity. The formula is valid for $T \geq Jd^2 C_K c^2$ (ensuring $\rho^* \leq 1$); below this threshold the firm is at the corner $\rho^* = 0$ (unit elasticity). \square

This provides a micro-foundation for the empirical regularity that firms in developing countries (high T) use simpler, more modular production processes (high ρ) even when the same complex technologies are available. It also predicts that technology adoption follows a specific path: as institutional quality improves (reducing T), firms adopt progressively more complementary technologies. The prediction is dynamic: an exogenous reduction in T (e.g., through management training interventions) should lead firms to adopt more complex production methods, not just to use existing methods more efficiently.

The endogenous ρ result also creates a feedback loop: high T induces high ρ^* (substitutable technology), which reduces the returns to institutional improvement ($\partial K_{\text{eff}} / \partial T$ is small when K is small), which reduces the incentive to invest in lowering T . This is a technology-information poverty trap: countries with poor information environments choose simple technologies that do not reward institutional improvement, perpetuating poor institutions. Breaking the trap requires a discrete intervention—reducing T below the threshold at which complementary technology becomes optimal—rather than marginal institutional reform.

11.8 Implications for Economic Methodology

1. The division of economics into “rational” and “behavioral” reflects the two terms of the CES potential, not a fundamental boundary. Rational economics is $\Phi_{\text{CES}}(\rho)$;

behavioral economics is $T \cdot S_q$.

2. “Perfect rationality” is the $T = 0$ limit, valid when information costs are negligible relative to stakes.
3. Behavioral “biases” are the variationally inevitable consequences of finite information processing capacity. They are not errors but optimal responses to information constraints.
4. Arrow’s impossibility demonstrates that non-dictatorial welfare judgments require cardinal information, which noisy optimization at $T > 0$ provides.
5. The Tsallis generalization shows that Shannon information theory is the special case of perfect substitutability ($\rho = 1$). For complementary production ($\rho < 1$), information costs are inherently non-additive, and the standard Shannon-based rational inattention framework systematically underestimates fluctuations by the factor $1/(2 - \rho)$.

11.9 Related Literature on Industrial Organization

Transaction cost economics. Williamson (1979, 1985) answered Coase’s question of why firms exist with a framework based on asset specificity, uncertainty, and frequency. The present paper provides micro-foundations: asset specificity *is* low ρ , behavioral uncertainty *is* high T , and the governance boundary is the quantitative curve $K^*(\Delta T)$ rather than a qualitative typology.

Organizational economics. The hierarchy model of Garicano (2000) treats communication costs as determining span of control. In the CES potential framework, communication cost is a component of T , and the hierarchy’s value depends on K : hierarchies are more valuable when inputs are complementary. This predicts that flat organizational structures are optimal when ρ is high (substitutable tasks) and steep hierarchies when ρ is low (complementary tasks).

Rational inattention in production. Sims (2003) introduced rational inattention; applications to production include price-setting under information constraints. The present paper extends rational inattention from individual choice to the aggregation problem: the CES structure determines how individual attention constraints compound into firm-level production losses.

Growth theory. The CES aggregate is central to growth theory, from Solow (1956) through Romer (1990) and the endogenous growth literature. The elasticity of substitution between capital and labor has macroeconomic implications: Jones (1995) showed that

balanced growth requires $\sigma = 1$ (Cobb-Douglas), but sectoral CES with $\sigma \neq 1$ generates structural transformation and Baumol’s cost disease. The nested CES (Section 2.4) provides a micro-foundation for multi-sector growth models: the between-sector parameter ρ_0 governs structural transformation, while within-sector parameters ρ_g govern technology adoption. The CES potential framework adds the information dimension: growth depends not only on the production structure (ρ) but on the information environment (T) that determines how much of the structural complementarity can be exploited.

Trade theory. The CES framework has deep roots in trade: the Armington aggregate (Armington, 1969), the Eaton-Kortum model (Eaton and Kortum, 2002), and the Melitz model of heterogeneous firms all use CES to aggregate across varieties or source countries. Costinot (2009) showed that a single log-supermodularity condition—equivalent to $\rho < 1$ in CES—unifies Ricardian and Heckscher-Ohlin trade. The present paper extends this connection: sorting and assignment results (Smirl 2026b, Search and Matching derivation) apply directly to trade models where countries sort into production patterns based on comparative advantage. The information friction T corresponds to trade barriers—not just tariffs, but the information costs of cross-border contracting, regulatory compliance, and quality verification. The CES potential framework predicts that trade liberalization (reducing T) yields larger welfare gains for complementary trade relationships (low ρ) than for substitutable ones (high ρ), a prediction testable with bilateral trade data.

Sparsity-based bounded rationality. Gabaix (2014) proposes a model where agents optimally simplify their decision problems by ignoring dimensions of variation that contribute little to their payoff. In the CES potential framework, this is a special case of the q -exponential compact support property (Proposition 3.6(a)): for $q < 1$, the equilibrium distribution assigns exactly zero weight to options beyond the threshold $T/(1-q)$. Gabaix’s “default model” corresponds to the equal-share allocation $\hat{x}_j = C/J$ (the $T \rightarrow \infty$ limit), and the “attention” allocated to each dimension is the deviation from this default, governed by the CES potential gradient. The CES framework adds the prediction that sparsity (the number of ignored dimensions) depends on ρ : highly complementary decisions ($\rho \ll 1$) are less sparse because every dimension matters, while substitutable decisions ($\rho \approx 1$) are more sparse because dimensions are redundant.

12 Conclusion

This paper has proposed that economic theory is generated by two canonical functions—the CES aggregate and Tsallis entropy with $q = \rho$ —connected through the CES potential $\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q$. The framework is parameterized by ρ (controlling aggregation

structure and information non-additivity) and T (controlling information friction), with the interaction determining institutional viability.

The main results are:

1. **The CES potential** (Section 3) is the unique framework satisfying constant elasticity, pseudo-additive information measurement ($q = \rho$), and variational consistency. The q -exponential equilibrium distribution exhibits compact support for complements and power-law tails for substitutes.
2. **The effective curvature theorem** (Section 4) shows that information friction endogenously reduces exploitable complementarity to $K_{\text{eff}} = K(1 - T/T^*)^+$, with non-uniform degradation predicting a specific crisis sequence: diversification fails before production complementarities, which fail before strategic stability.
3. **The Tsallis generalization** (Section 5) with $q = \rho$ classifies all dynamical results into exact survivors (Category A), q -corrected by $1/(2 - q)$ (Category B), and structurally changed with $\exp \rightarrow \exp_q$ (Category C). The correction amplifies fluctuations for complements and dampens them for substitutes.
4. **Optimal firm scope and integration boundaries** (Section 6–Section 6.2) are derived in (ρ, T) space, nesting Williamson’s governance structures as the qualitative limit of quantitative predictions. Multi-tier supply chain architecture follows a monotone integration rule based on tier-specific K_ℓ/T_ℓ ratios.
5. **The welfare loss function** (Section 9) establishes that the CES potential has Lyapunov properties, providing a welfare criterion for institutional design with convergence rate proportional to $K_{\text{eff}} \cdot L_{\min}$.
6. **Two empirical tests** support the framework: the GFC banking regulation test ($p = 0.016$) and the manufacturing tail distribution test (12/17 sectors favor q -exponential over exponential). Four additional predictions—management–technology complementarity, within-industry dispersion, AI adoption priority, and crisis sequence timing—are testable with existing data (Section 7–Section 8). The companion paper (Smirl, 2026b) provides six additional derivations from information economics, mechanism design, social choice, search and matching, contract theory, and behavioral economics.

Three limitations deserve emphasis.

First, ρ is treated as exogenous throughout the main analysis. Proposition 11.1 in the discussion section sketches the endogenous technology choice extension, showing that firms in high- T environments optimally choose high- ρ (substitutable) technologies. A

full treatment would embed this choice in a general equilibrium model where aggregate T is determined by institutional investment and aggregate ρ is determined by the technology frontier. The resulting fixed-point problem— $\rho^*(T)$ and $T^*(\rho)$ determined simultaneously—may have multiple equilibria, corresponding to development traps (high T , high ρ , low institutional investment) and development takeoffs (low T , low ρ , high institutional returns). Formalizing this is left for future work.

Second, the supply chain results assume additively separable CES potentials across tiers; in practice, tiers interact through quality propagation and information spillovers (Section 6.4 sketches this extension). A full treatment would model the joint optimization over tier-specific governance choices $\{G_\ell\}_{\ell=1}^L$ with inter-tier spillovers, yielding a combinatorial optimization problem. The monotone structure (Proposition 6.6) survives under non-negative spillovers (Corollary 6.8), but the optimal thresholds shift. For negative spillovers (where integration at one tier increases friction at adjacent tiers due to bureaucratic complexity), the optimal architecture may be non-monotone—a prediction testable in supply chain data.

Third, the empirical predictions require industry-level estimates of both ρ and T , which are not yet available in combination. Oberfield and Raval (2021) provide $\hat{\sigma}$ (hence $\hat{\rho}$) for manufacturing industries from production data. Estimates of T are implicit in management quality data (Bloom et al., 2013) and in the rational inattention literature’s calibrations. Combining these into a joint $(\hat{\rho}, \hat{T})$ panel would enable direct tests of the effective curvature theorem and the crisis sequence prediction. The main obstacle is measurement: T is a latent variable inferred from behavioral patterns, not directly observable, and different proxies (management scores, forecast errors, price dispersion) may capture different components of information friction.

The deeper implication is methodological. The division of economics into “rational” (the Φ_{CES} term) and “behavioral” (the $T \cdot S_q$ term) reflects the two components of a single object, not a fundamental boundary. Production theory and information economics have developed as separate fields because they formalize different aspects of economic activity. The CES potential reveals that they are aspects of a single variational principle.

Several extensions and applications are immediate:

1. **Dynamic CES potential.** The present paper treats ρ and T as static. In reality, both evolve: ρ changes as technology shifts (modularization increases ρ ; integration decreases it) and T changes as information infrastructure develops. A fully dynamic model would track the joint trajectory $(\rho(t), T(t))$ and predict the timing of institutional regime shifts as the economy crosses the $T^*(\rho)$ boundary. The companion papers develop this extension for specific domains: endogenous decentralization, mesh equilibrium, autocatalytic training, and settlement feedback.
2. **Heterogeneous agents.** The framework assumes a representative firm with CES

technology. Extending to heterogeneous agents—firms with different ρ_i and T_i —creates a distribution over (ρ, T) space. The aggregate economy’s behavior depends on the joint distribution, not just the mean. A bimodal distribution (some firms at high K , some at low K) could produce aggregate behavior that no single CES economy can generate. The cross-sectional implications include: within-industry productivity dispersion (Proposition 7.2) reflects the dispersion of T_i for given ρ , while cross-industry productivity differences reflect the dispersion of ρ_i for given T .

3. **International trade with heterogeneous T .** The trade theory connection (Section 11) suggests that bilateral trade flows depend not only on the complementarity of traded goods (ρ) but on the information quality of the trading relationship (T). Gravity models of trade could be enriched with a CES potential term: the predicted bilateral trade flow from country i to country j in sector s is proportional to $K_{\text{eff}}(s) \cdot d_{ij}^2(s)/T_{ij}(s)$, where T_{ij} captures the bilateral information friction (language barriers, regulatory differences, trust). This would provide a micro-foundation for the well-documented persistence of the border effect in trade.
4. **Macroeconomic stabilization.** The crisis sequence (Corollary 4.6) has implications for macroeconomic policy. Stabilization policy that targets T (e.g., lender-of-last-resort facilities that maintain information flow during crises) is more effective than policy that targets ρ (e.g., structural reform during a crisis). The reason is that T can be changed quickly (information injection is fast) while ρ changes slowly (technology is sticky). This provides a CES-theoretic justification for the central bank’s role as information intermediary during crises, complementing its standard role as liquidity provider.

Several directions for future work emerge naturally:

1. **Estimation strategy.** A unified estimation approach would jointly estimate (ρ, T) from production data (output, input quantities, prices) and information data (forecast errors, decision latencies, bid-ask spreads). The CES potential imposes cross-equation restrictions that increase estimation efficiency: the same ρ that determines the production function curvature also determines the equilibrium distribution shape and the $1/(2-q)$ correction factor. Structural estimation methods from the IO literature could be adapted for this purpose.
2. **Computational implementation.** The q -exponential equilibrium distribution has a well-defined sampling algorithm (generalized inverse method with compact support or power-law tail), enabling Monte Carlo simulation of CES potential economies. Agent-based models with CES production and rational inattention agents would provide a computational laboratory for testing the crisis sequence and pre-crisis deceleration predictions.

3. Normative applications. The CES potential framework has normative content: the Lyapunov property (Theorem 9.1) provides a welfare criterion for institutional design. An institution that reduces \mathcal{F}_q toward \mathcal{F}_q^* is welfare-improving, and the magnitude of improvement is measurable from observable quantities (K_{eff} , T , output). This could inform cost-benefit analysis of institutional reforms by providing a theory-based welfare function that accounts for both production complementarity and information friction.

The equation is:

$$\mathcal{F}_q = \Phi_{\text{CES}}(\rho) - T \cdot S_q, \quad q = \rho$$

Two parameters. One equation. The rest is application.

A Technical Lemmas

This appendix collects technical results used in the main text.

Lemma A.1 (CES Hessian Eigenstructure). *Let $F(\mathbf{x}) = (J^{-1} \sum x_j^\rho)^{1/\rho}$ with $\rho \in (-\infty, 1] \setminus \{0\}$. At the symmetric point $\bar{\mathbf{x}} = c \cdot \mathbf{1}$, the Hessian $\mathbf{H} = \nabla^2 \log F$ has:*

- (i) *Eigenvalue $\lambda_1 = -1/(Jc^2)$ on $\text{span}\{\mathbf{1}\}$ (multiplicity 1).*
- (ii) *Eigenvalue $\lambda_\perp = -(1-\rho)/(Jc^2)$ on $\mathbf{1}^\perp$ (multiplicity $J-1$).*
- (iii) *The spectral decomposition: $\mathbf{H} = \lambda_\perp \mathbf{I} + (\lambda_1 - \lambda_\perp) J^{-1} \mathbf{1} \mathbf{1}^\top = -(1-\rho)/(Jc^2) \cdot \mathbf{I} - \rho/(J^2 c^2) \cdot \mathbf{1} \mathbf{1}^\top$.*

Under a budget constraint $\sum x_j = C$, the direction $\mathbf{1}$ is constrained out, and only the $(J-1)$ -dimensional eigenvalue λ_\perp governs the curvature of production on the feasible set.

Proof. At $\bar{\mathbf{x}} = c\mathbf{1}$: $F = c$, $\partial F / \partial x_j = c^{1-\rho} x_j^{\rho-1} / J|_{\text{sym}} = 1/J$. Then $\partial \log F / \partial x_j = (1/F)(\partial F / \partial x_j) = 1/(Jc)$. By Euler's theorem for degree-1 homogeneous functions: $\sum x_j \partial \log F / \partial x_j = 1$, confirming $\mathbf{1} \cdot \nabla \log F = 1/c$.

For the second derivatives:

$$\frac{\partial^2 \log F}{\partial x_j \partial x_k} = \frac{1}{F} \frac{\partial^2 F}{\partial x_j \partial x_k} - \frac{1}{F^2} \frac{\partial F}{\partial x_j} \frac{\partial F}{\partial x_k}$$

At symmetry: $\partial^2 F / \partial x_j^2|_{\text{sym}} = -(1-\rho)(J-1)/(J^2 c)$ and $\partial^2 F / \partial x_j \partial x_k|_{\text{sym}} = (1-\rho)/(J^2 c)$ for $j \neq k$ (both computed from the chain rule applied to $F = S^{1/\rho}$, $S = J^{-1} \sum x_i^\rho$). The Hessian of $\log F$ is:

$$\begin{aligned} H_{jj} &= \frac{1}{c} \cdot \frac{-(1-\rho)(J-1)}{J^2 c} - \frac{1}{J^2 c^2} = -\frac{J(1-\rho) + \rho}{J^2 c^2} \\ H_{jk} &= \frac{1}{c} \cdot \frac{(1-\rho)}{J^2 c} - \frac{1}{J^2 c^2} = -\frac{\rho}{J^2 c^2} \quad (j \neq k) \end{aligned}$$

Writing $\mathbf{H} = (H_{jj} - H_{jk})\mathbf{I} + H_{jk}\mathbf{1}\mathbf{1}^\top$: the coefficient of \mathbf{I} is $-(1-\rho)/(Jc^2)$ and of $\mathbf{1}\mathbf{1}^\top$ is $-\rho/(J^2 c^2)$. On $\mathbf{1}^\perp$: eigenvalue $-(1-\rho)/(Jc^2)$. On $\mathbf{1}$: eigenvalue $-(1-\rho)/(Jc^2) + J \cdot (-\rho/(J^2 c^2)) = -(1-\rho)/(Jc^2) - \rho/(Jc^2) = -1/(Jc^2)$. \square

Lemma A.2 (q -Exponential Normalization). *For $q \in (0, 2)$, $q \neq 1$, and energy levels $\varepsilon_1 \leq \dots \leq \varepsilon_J$ with $\varepsilon_1 = 0$, the q -partition function $Z_q = \sum_{j=1}^{J_q} [1 - (1-q)\beta\varepsilon_j]_+^{1/(1-q)}$ satisfies:*

- (i) *For $q < 1$: $J_q = |\{j : \varepsilon_j < T/(1-q)\}| \leq J$ (only inputs below the cost threshold contribute). Z_q is finite for finite J .*

(ii) For $q > 1$: $J_q = J$ (all inputs contribute). For large ε_J : $Z_q \sim \varepsilon_J^{1/(q-1)} \cdot C(q)$ (power-law growth).

(iii) For $q \rightarrow 1$: $Z_q \rightarrow Z = \sum_j e^{-\beta\varepsilon_j}$ (standard partition function recovery).

Proof. (i) When $q < 1$: $1 - q > 0$, so the bracket $[1 - (1 - q)\beta\varepsilon_j]_+$ vanishes for $\varepsilon_j > T/(1 - q)$. The number of contributing terms is $J_q = |\{j : (1 - q)\beta\varepsilon_j < 1\}|$. Each surviving term is bounded above by 1, so $Z_q \leq J_q \leq J$.

(ii) When $q > 1$: $1 - q < 0$, so $[1 - (1 - q)\beta\varepsilon_j]_+ = 1 + (q - 1)\beta\varepsilon_j > 0$ for all $\varepsilon_j \geq 0$. For large ε_J : $[1 + (q - 1)\beta\varepsilon_j]^{1/(1-q)} = [(q - 1)\beta\varepsilon_j]^{-1/(q-1)} \cdot (1 + o(1))$, which decays as a power law. The sum converges if and only if $1/(q - 1) > 1$, i.e., $q < 2$.

(iii) The limit $\lim_{q \rightarrow 1} [1 + (1 - q)x]^{1/(1-q)} = e^x$ is standard (defining the exponential as a limit of powers). \square

Lemma A.3 (Tsallis Entropy Hessian). *The Hessian of Tsallis entropy $S_q(p) = (1 - \sum p_j^q)/(q - 1)$ at the uniform distribution $p_j = 1/J$ is:*

$$\frac{\partial^2 S_q}{\partial p_j \partial p_k} \Big|_{\text{unif}} = -qJ^{2-q}\delta_{jk} \quad (54)$$

Compared to Shannon: $\partial^2 H / \partial p_j \partial p_k|_{\text{unif}} = -J\delta_{jk}$. The ratio is $qJ^{2-q}/J = qJ^{1-q}$. At $q \rightarrow 1$: $qJ^{1-q} \rightarrow J^0 = 1$, recovering the Shannon case. The effective curvature at symmetric equilibrium, entering the variance-response identity through the inverse Hessian, gives the $1/(2 - q)$ factor as stated in Theorem 5.1.

Proof. Direct computation: $\partial S_q / \partial p_j = -qp_j^{q-1}/(q-1)$ and $\partial^2 S_q / \partial p_j^2 = -q(q-1)p_j^{q-2}/(q-1) = -qp_j^{q-2}$. At $p_j = 1/J$: $\partial^2 S_q / \partial p_j^2 = -q(1/J)^{q-2} = -qJ^{2-q}$. At $q = 1$: $-1 \cdot J^1 = -J$, matching Shannon. The off-diagonal terms vanish since Tsallis entropy decomposes as a sum over individual p_j^q terms. The diagonal Hessian is $-qJ^{2-q}\mathbf{I}$, restricted to the simplex tangent space. \square

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