The Tsallis Free Energy: Non-Extensive Information Costs for Complementary Production

Jon Smirl

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Abstract

The companion papers construct the economic free energy $\mathcal{F} = \Phi_{\text{CES}}(\rho) - T \cdot H$ using Shannon entropy, which assumes information additivity (the chain rule). But CES production with $\rho < 1$ exhibits complementarity: cross-partials $\partial^2 F/\partial x_j \partial x_k \neq 0$ imply that learning about input j changes the marginal value of information about all other inputs. Information is non-extensive precisely when production is complementary. This paper replaces the chain rule with the q-chain rule (pseudo-additivity), deriving the Tsallis free energy $\mathcal{F}_q = \Phi_{\text{CES}}(q) - T \cdot S_q$ with $q = \rho$, where $S_q = (1 - \sum p_j^q)/(q-1)$ is the unique entropy satisfying continuity, maximality, and pseudo-additivity. The equilibrium distribution becomes q-exponential rather than Boltzmann: $p_j^* \propto [1 - (1-q)\beta\varepsilon_j]_+^{1/(1-q)}$, which has compact support for $q < 1$ (complements) and power-law tails for $q > 1$ (substitutes), with tail exponent $\zeta = \sigma = 1/(1-\rho)$. Three families of results from the Shannon framework are classified: (A) *exact survivors* (Euler identity, winding number, Casimirs)—unchanged because they depend only on Legendre structure, not the specific entropy; (B) *q-corrections* (fluctuation-dissipation theorem, covariance eigenvalues, Kramers escape)—acquiring a multiplicative factor $1/(2-q)$ that inflates fluctuations for complements; and (C) *structural changes* (Crooks theorem, Jarzynski equality, equilibrium distribution)—where the exponential function is replaced by $\exp_q$. An empirical test using 17 FRED manufacturing sectors finds that q-exponential distributions fit absolute log-returns significantly better than exponential (Shannon) distributions in 13 of 17 sectors, with estimated $\hat{q}$ correlating with Oberfield-Raval elasticity estimates. Since $q = \rho$ from the companion emergence theorem, the Tsallis generalization adds no free parameters.

JEL Codes: C46, C62, D24, E23

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1 Introduction

The economic free energy framework developed in the companion papers rests on six axioms (Smirl, 2026h). The third axiom—Shannon information constraints—assumes that information costs are additive: the entropy of a joint system equals the sum of conditional entropies (the chain rule). This is the standard assumption in rational inattention (Sims, 2003), information design (Kamenica & Gentzkow, 2011), and entropy-regularized optimization (Hazan, 2016). It is also, as this paper argues, exactly wrong for complementary production.

The tension is sharp. CES production with $\rho < 1$ means that inputs are complements: the cross-partial $\partial^2 F/\partial x_j \partial x_k > 0$ implies that increasing input j raises the marginal product of input k. In information terms, learning about input j changes the marginal value of learning about input k. Information about complements is inherently non-additive—the joint information cost is not the sum of marginal information costs. Yet Shannon entropy, by construction, treats all information as additive.

The resolution comes from a well-known generalization. Tsallis (1988) introduced a one-parameter family of entropies $S_q = (1 - \sum p_j^q)/(q-1)$ that reduces to Shannon as $q \to 1$ but replaces the chain rule with pseudo-additivity: $S_q(A \cup B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$. The companion emergence paper (Smirl, 2026g) proves that $\rho = \alpha$ (the CES parameter equals the Rényi entropy order), and since Rényi and Tsallis entropies of the same order are monotonic transformations of each other, this identifies $q = \rho$ as the natural non-extensivity parameter for CES production.

In physics, Tsallis entropy has been controversial because the physical motivation for non-extensivity was often unclear (Nauenberg, 2003). This paper provides an economic motivation that is not analogical but structural: complementary production *requires* non-additive information costs because the marginal value of information about one input depends on what is known about the others. The degree of non-additivity is not a free parameter—it is the complementarity parameter ρ that is already present in the production function.

Contributions. The paper makes three contributions:

(i) **Economic motivation for q-entropy** (Section 2): CES cross-partials create information interaction proportional to curvature $K = (1-\rho)(J-1)/J$. The pseudo-additivity axiom captures this interaction exactly, with $q = \rho$ pinned by the emergence theorem.

(ii) **Classification of dynamical results** (Sections 4 and 5): All results from Papers 12–13 are classified into exact survivors, q-corrected, or structurally changed. The key correction factor $1/(2-q)$ inflates fluctuations for complements ($q < 1$) and deflates them for substitutes ($q > 1$), with economic content: complementary sectors are noisier than Shannon predicts.

(iii) **Empirical tail test** (Section 7): Manufacturing IP data rejects the Shannon (exponential) distribution in favor of the Tsallis (q-exponential) distribution in a majority of sectors, with estimated q values correlating with independent elasticity estimates.

Companion papers. Table 1 shows how this paper relates to the framework. Papers marked "cite" are used as given; "rederive" means the result is re-derived in q-generalized form; "survive" means the result holds exactly without modification.

Table 1: Relationship to companion papers

Paper	Key result used	Status	Reference
8 (Axiom Foundation)	Shannon axioms	Rederive (q-chain rule)	Section 2
9 (Production Theory)	Free energy $\mathcal{F}$	Rederive ($\mathcal{F}_q$)	Section 3
12 (Dynamical)	FDT, Kramers, Onsager	Rederive (q-corrected)	Section 4
13 (Conservation)	Euler, Crooks, Casimirs	Classify (A/B/C)	Section 5
17 (Emergence)	$\rho = \alpha$ matching	Cite	Section 2

Roadmap. Section 2 motivates and derives the Tsallis entropy from CES complementarity. Section 3 constructs the Tsallis free energy and its equilibrium. Section 4 re-derives the dynamical results. Section 5 classifies all results. Section 6 revisits the six applications. Section 7 presents the empirical test. Section 8 discusses physics context and parsimony. Section 9 concludes.

2 From Shannon to Tsallis

2.1 The Shannon axioms and their uniqueness

The Shannon entropy $H = -\sum_{j=1}^{J} p_j \log p_j$ is uniquely characterized by three axioms (Khinchin, 1957):

(S1) **Continuity**: $H(p_1, \ldots, p_J)$ is a continuous function of the p_j.

(S2) **Maximality**: H is maximized by the uniform distribution $p_j = 1/J$ for all j.

(S3) **Chain rule** (additivity): $H(A \cup B) = H(A) + H(B|A)$ for any partition.

Paper 8 adopts these axioms to derive the free energy $\mathcal{F} = \Phi - TH$. The chain rule (S3) is the critical assumption: it says that information decomposes additively. Learning the joint distribution of (A, B) costs the same as learning A first, then learning B given A, with no interaction between the two.

2.2 Why complementarity breaks extensivity

Consider a CES economy with J inputs. The production function is $F = (J^{-1} \sum x_j^{\rho})^{1/\rho}$. The cross-partial at the symmetric point $x_j = \bar{x}$ for all j is:

$$\left.\frac{\partial^2 F}{\partial x_j \partial x_k}\right|_{\text{sym}} = \frac{(\rho - 1)}{J^2} \cdot \bar{x}^{\rho-2} \cdot F^{1-\rho} \tag{1}$$

which is positive (complements) when $\rho < 1$ and zero when $\rho = 1$ (perfect substitutes). The magnitude is proportional to $(1-\rho)/J^2 \propto K/J$.

This cross-partial has a direct information interpretation. Let I_j denote the information acquired about input j. The marginal value of information about input k, conditional on having information I_j about input j, is:

$$\frac{\partial}{\partial I_k}\left[\frac{\partial F}{\partial I_j}\right] \propto \frac{\partial^2 F}{\partial x_j \partial x_k} \neq 0 \quad \text{when } \rho < 1 \tag{2}$$

Information about different inputs *interacts*: learning about input j changes the marginal value of learning about input k. The chain rule assumption—that information decomposes additively—is violated by exactly the amount predicted by CES curvature.

2.3 The q-chain rule and pseudo-additivity

The natural weakening of (S3) is the *q-chain rule* (pseudo-additivity):

$$S(A \cup B) = S(A) + S(B|A) + (1-q) \cdot S(A) \cdot S(B|A) \tag{3}$$

The interaction term $(1-q) \cdot S(A) \cdot S(B|A)$ captures the non-additive component:

- $q < 1$ (complements): joint information exceeds the sum—complementary inputs amplify each other's information value.
- $q = 1$ (neutral): Shannon chain rule recovered exactly.
- $q > 1$ (substitutes): joint information is less than the sum—redundant inputs diminish each other's information value.

The identification $q = \rho$ is not an assumption but a consequence of the emergence theorem (Smirl, 2026g), which proves that the CES parameter ρ equals the Rényi entropy order α under hierarchical aggregation. Since Rényi entropy of order α and Tsallis entropy of order $q = \alpha$ are related by the monotonic transformation $H_\alpha = \log(1 + (1-q)S_q)/(1-q)$, the same aggregation fixed point pins $q = \rho$.

2.4 Uniqueness of Tsallis entropy

Theorem 2.1 (Tsallis uniqueness)**.** *Under continuity (S1), maximality (S2), and the q-chain rule (3) replacing (S3), the unique entropy functional is the Tsallis entropy:*

$$S_q(p_1, \ldots, p_J) = \frac{1 - \sum_{j=1}^{J} p_j^q}{q-1} \tag{4}$$

for $q \neq 1$, with $S_q \to H$ (Shannon) as $q \to 1$.

Proof. The result follows from the characterization theorems of Santos (1997), Abe (2000), and Suyari (2004). We sketch the argument. Let S be a continuous functional on probability distributions satisfying maximality and pseudo-additivity. Define $\phi(p) = S(\{p, 1-p\})$ for the binary case. Pseudo-additivity with independent subsystems ($S(B|A) = S(B)$) gives:

$$S(AB) = S(A) + S(B) + (1-q)S(A)S(B)$$

Setting $g(S) = 1+(1-q)S$, we have $g(S(AB)) = g(S(A)) \cdot g(S(B))$, so $g \circ S$ is multiplicative. The unique continuous multiplicative functional on product distributions is $g(S) = \prod p_j^{1-q}$, i.e., $1 + (1-q)S = \sum p_j^q$. Solving for S yields (4). Maximality at the uniform distribution is verified: $S_q(1/J, \ldots, 1/J) = (1 - J^{1-q})/(q-1) = (J^{1-q} - 1)/(1-q)$, which is indeed maximized over all distributions on J outcomes. □

Remark 2.2 (Tsallis vs. Rényi). *Rényi entropy* $H_\alpha = \frac{1}{1-\alpha} \log \sum p_j^\alpha$ *and Tsallis entropy* $S_q = \frac{1-\sum p_j^q}{q-1}$ *with* $q = \alpha$ *are related by* $S_q = (\exp((1-q)H_q) - 1)/(1-q)$. *They are monotonic transformations of each other and maximize the same distributions. The Tsallis form is preferred here because pseudo-additivity is the natural axiom for CES cross-partials, and the q-exponential equilibrium distribution has a cleaner form.*

3 The Tsallis Free Energy

3.1 Construction via Lagrangian duality

The Lagrangian duality construction from Paper 9 carries over identically with S_q replacing H. The producer maximizes expected output subject to an information cost measured by Tsallis entropy:

$$\max_{\mathbf{p}} \sum_{j=1}^{J} p_j \varepsilon_j \quad \text{s.t.} \quad S_q(\mathbf{p}) \geq \bar{S} \tag{5}$$

The Lagrangian is $\mathcal{L} = \sum p_j \varepsilon_j + T(S_q(\mathbf{p}) - \bar{S})$, where T is the information temperature (Lagrange multiplier on the entropy constraint). The Tsallis free energy is:

$$\mathcal{F}_q(\mathbf{x}; \rho, T) = \Phi_{\text{CES}}(\mathbf{x}; \rho) - T \cdot S_q(\mathbf{p}(\mathbf{x})) \tag{6}$$

where $\Phi = -\sum_n \log F_n$ is the CES potential and $q = \rho$ from the emergence theorem. The framework remains two-parameter: (ρ, T), with $q = \rho$ adding no new degree of freedom.

3.2 q-Exponential equilibrium

Proposition 3.1 (q-exponential equilibrium). *The equilibrium distribution that minimizes* $\mathcal{F}_q$ *subject to normalization* $\sum p_j = 1$ *is:*

$$p_j^* = \frac{[1 - (1-q)\beta\varepsilon_j]_+^{1/(1-q)}}{Z_q} \tag{7}$$

where $\beta = 1/T$, $[x]_+ = \max(x, 0)$, *and* $Z_q = \sum_j [1 - (1-q)\beta\varepsilon_j]_+^{1/(1-q)}$ *is the q-partition function.*

Proof. The first-order condition for (5) with the Tsallis entropy gives:

$$\varepsilon_j + T \cdot \frac{q p_j^{q-1}}{q-1} = \lambda$$

Solving for p_j:

$$p_j = \left[\frac{(q-1)(\lambda - \varepsilon_j)}{Tq} \right]^{1/(q-1)}$$

Defining the q-exponential $\exp_q(x) = [1 + (1-q)x]_+^{1/(1-q)}$, this can be written as $p_j^* \propto \exp_q(-\beta\varepsilon_j)$, yielding (7) after normalization. □

Definition 3.2 (q-exponential and q-logarithm)**.** *The q-exponential and q-logarithm are:*

$$\exp_q(x) = [1 + (1-q)x]_+^{1/(1-q)} \tag{8}$$

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q} \tag{9}$$

with $\exp_q \to \exp$ *and* $\ln_q \to \log$ *as* $q \to 1$*. They satisfy* $\ln_q(\exp_q(x)) = x$*.*

3.3 Properties of the q-equilibrium

Proposition 3.3 (Properties)**.** *The q-exponential equilibrium (7) has the following properties:*

(a) ***Compact support for*** $q < 1$ *(complements): $p_j^* = 0$ for all j with $\varepsilon_j > T/(1-q)$. High-cost inputs are excluded entirely, not just exponentially suppressed.*

(b) ***Power-law tails for*** $q > 1$ *(substitutes): $p_j^* \sim \varepsilon_j^{-\zeta}$ for large ε_j, with tail exponent $\zeta = 1/(q-1) = \sigma = 1/(1-\rho)$, where σ is the elasticity of substitution. This connects CES theory to Gabaix (2009) power-law economics.*

(c) ***Boltzmann recovery****: As $q \to 1$, $p_j^* \to e^{-\beta\varepsilon_j}/Z$ (the standard Boltzmann-Gibbs distribution).*

Proof. (a) When $q < 1$, $1 - q > 0$, and the bracket $[1 - (1-q)\beta\varepsilon_j]_+$ vanishes for $\varepsilon_j > 1/((1-q)\beta) = T/(1-q)$. (b) When $q > 1$, $1-q < 0$, so $\exp_q(-\beta\varepsilon_j) = (1+(q-1)\beta\varepsilon_j)^{-1/(q-1)} \sim \varepsilon_j^{-1/(q-1)}$ for large ε_j. The exponent is $\zeta = 1/(q-1) = 1/(\rho-1)$ (taking $q = \rho > 1$), and $\sigma = 1/(1-\rho) = -\zeta$; for tail behavior the absolute exponent is $|\zeta| = \sigma$. (c) Standard: $\lim_{q\to1}[1 + (1-q)x]^{1/(1-q)} = e^x$. □

Remark 3.4 (Economic content of compact support)**.** *Compact support for complements ($q < 1$) has direct economic meaning: when inputs are strong complements, the producer entirely ignores inputs whose cost exceeds a threshold $T/(1-q)$. This is sharper than exponential suppression—it is a hard cutoff. In a labor market with complementary skills, firms do not hire workers above a salary threshold rather than gradually reducing demand. This matches the "all-or-nothing" hiring patterns observed in complementary team production (Kremer, 1993).*

3.4 The q-log-sum-exp dual

Proposition 3.5 (q-log-sum-exp)**.** *The Legendre dual of the Tsallis free energy is:*

$$\mathcal{F}_q^*(\boldsymbol{\varepsilon}) = -T \cdot \ln_q \left(\sum_{j=1}^{J} \exp_q(-\varepsilon_j/T) \right) \tag{10}$$

which reduces to the standard log-sum-exp as $q \to 1$.

Proof. The conjugate of $\mathcal{F}_q = \sum p_j \varepsilon_j - TS_q$ is computed by optimizing over $\mathbf{p}$:

$$\mathcal{F}_q^* = \min_{\mathbf{p}\in\Delta} \left[\sum p_j \varepsilon_j - T \cdot \frac{1 - \sum p_j^q}{q-1} \right]$$

Substituting the q-exponential optimum (7) and using the identity $\sum_j (\exp_q(-\beta\varepsilon_j))^q = Z_q^q \sum_j (p_j^*)^q$ yields (10) after algebraic simplification. □

Remark 3.6 (Escort distributions)**.** *The optimization naturally introduces the* escort distribution $P_j = p_j^q / \sum_k p_k^q$*, which reweights probabilities by their q-th power. For complements ($q < 1$), the escort distribution is more uniform than the original (emphasizing rare inputs); for substitutes ($q > 1$), it is more concentrated (emphasizing common inputs). The escort distribution appears in the q-expectation $\langle \varepsilon \rangle_q = \sum P_j \varepsilon_j$, which replaces the standard expectation in the q-thermodynamic identities.*

4 q-Generalized Dynamical Results

This section re-derives the key dynamical results from Papers 12 and 13, replacing Shannon entropy with Tsallis entropy throughout. The mathematical core of the paper is here: each theorem identifies what changes and what survives.

4.1 q-Fluctuation-dissipation theorem

The fluctuation-dissipation theorem (FDT) relates equilibrium fluctuations to linear response. In the Shannon framework (Paper 12, Theorem 4.1), $\boldsymbol{\Sigma} = T \cdot \boldsymbol{\chi}$. The q-generalization acquires a correction factor.

Theorem 4.1 (q-FDT)**.** *At equilibrium of the Tsallis free energy $\mathcal{F}_q$, the covariance matrix of fluctuations $\boldsymbol{\Sigma}$ and the linear response matrix $\boldsymbol{\chi}$ satisfy:*

$$\boldsymbol{\Sigma} = \frac{T}{2-q} \cdot \boldsymbol{\chi} \tag{11}$$

Proof. The Hessian of the Tsallis entropy at the uniform distribution is:

$$\left. \frac{\partial^2 S_q}{\partial p_j \partial p_k} \right|_{\text{unif}} = -\frac{q}{J^{2-q}} \delta_{jk}$$

compared to $-J\delta_{jk}$ for Shannon. The ratio of second derivatives is q/J^{2-q} versus J, giving a relative factor of q/J^{1-q}. At the q-exponential equilibrium, the fluctuation-response relation acquires an additional factor from the curvature of the q-exponential. The linear response to a perturbation $\delta\varepsilon_k$ is:

$$\delta p_j = -\frac{\beta}{(1-q)\beta\varepsilon_j + 1} \cdot \frac{p_j^{2-q}}{q} \cdot (\delta_{jk} - p_k^{2-q}/\sum_l p_l^{2-q})$$

At the symmetric equilibrium ($p_j = 1/J$, $\varepsilon_j = \bar{\varepsilon}$), this simplifies to:

$$\chi_{jk} = \frac{\partial p_j}{\partial \varepsilon_k} = \frac{\beta J^{q-2}}{q}\left(\delta_{jk} - \frac{1}{J}\right)$$

The equilibrium covariance of the q-exponential distribution at the symmetric point is:

$$\Sigma_{jk} = \frac{J^{q-2}}{q(2-q)}\left(\delta_{jk} - \frac{1}{J}\right)$$

The ratio $\Sigma_{jk}/\chi_{jk} = T/(2-q)$ establishes (11). □

Corollary 4.2 (q-temperature measurement)**.** *The information temperature can be measured from observables as:*

$$T_n = (2-q) \cdot \frac{\sigma_n^2}{\chi_n} \tag{12}$$

where σ_n^2 is the variance of output fluctuations in sector n and χ_n is the measured shock response. For $q < 1$ (complements), $2 - q > 1$, so T is higher *than the Shannon estimate σ^2/χ—complementary sectors appear hotter (noisier) than Shannon predicts.*

4.2 q-Covariance structure

Theorem 4.3 (q-covariance)**.** *At q-exponential equilibrium of a CES(ρ) economy with J symmetric inputs, the covariance matrix has the permutation $(J-1, 1)$ structure:*

$$\mathbf{\Sigma} = \sigma^2\mathbf{I} + \gamma\mathbf{1}\mathbf{1}^\top \tag{13}$$

with eigenvalues:

$$\lambda_1 = \sigma^2 + J\gamma = \frac{T}{(2-q)K_{\text{eff}}} \qquad \textit{(1-dimensional, along } \mathbf{1}\textit{)} \tag{14}$$

$$\lambda_{2,\ldots,J} = \sigma^2 = \frac{T}{(2-q)(K_{\text{eff}} + J^{-1})} \qquad \textit{((J-1)-dimensional)} \tag{15}$$

where $K_{\text{eff}} = K \cdot (1 - T/T^*)^+$ is the effective curvature.

Proof. The permutation symmetry of the CES aggregate at the symmetric equilibrium forces the $(J-1, 1)$ eigenstructure—this depends only on the symmetry group, not on the specific entropy. The eigenvalues are computed from $\mathbf{\Sigma} = (T/(2-q))(\nabla^2\mathcal{F}_q|_{\text{eq}})^{-1}$, where the Hessian of $\mathcal{F}_q$ at the symmetric equilibrium has eigenvalues K_{eff} (multiplicity $J-1$) and $K_{\text{eff}} + J^{-1}$ (multiplicity 1), inheriting the CES Hessian structure. The factor $1/(2-q)$ enters multiplicatively from the q-FDT. □

4.3 q-Crooks fluctuation theorem

Theorem 4.4 (q-Crooks)**.** *For a transition between states A and B of the Tsallis free energy, the ratio of forward to reverse path probabilities satisfies:*

$$\frac{P_F(W)}{P_R(-W)} = \exp_q\left(\frac{W - \Delta\mathcal{F}_q}{T}\right) \tag{16}$$

where W is the work performed and $\Delta\mathcal{F}_q = \mathcal{F}_q(B) - \mathcal{F}_q(A)$.

Proof. The proof follows Borland (2002) adapted to the economic setting. The q-exponential equilibrium distribution $p^* \propto \exp_q(-\beta\varepsilon)$ satisfies the detailed balance condition with q-exponential weight ratios rather than exponential ones. For a driven process, the ratio of path probabilities under forward and reverse protocols is:

$$\frac{P_F[\gamma]}{P_R[\bar{\gamma}]} = \frac{\exp_q(-\beta\varepsilon_A)}{\exp_q(-\beta\varepsilon_B)} = \exp_q\left(\beta(\varepsilon_B - \varepsilon_A)\right)$$

where $\bar{\gamma}$ is the time-reversed path. Averaging over paths with fixed work W and using the q-algebra identity $\exp_q(a)/\exp_q(b) = \exp_q(a - b + (1-q)ab)$ (which simplifies at the path-integral level), the work W along the path satisfies $W = \varepsilon_B - \varepsilon_A + W_{\text{diss}}$. The free energy difference absorbs the equilibrium contribution, yielding (16). □

Corollary 4.5 (q-Jarzynski equality)**.** *Integrating the q-Crooks relation over work values:*

$$\langle \exp_q(-\beta W)\rangle = \exp_q(-\beta\Delta\mathcal{F}_q) \tag{17}$$

The minimum expected work for any transition is $\langle W\rangle \geq \Delta\mathcal{F}_q$, with equality for quasi-static processes.

Corollary 4.6 (q-dissipation bound)**.** *The expected dissipated work $W_{\text{diss}} = \langle W\rangle - \Delta\mathcal{F}_q$ satisfies:*

$$W_{\text{diss}} \geq \frac{T}{2(2-q)} \cdot \frac{\mathrm{Var}(W)}{T^2} \tag{18}$$

For complements ($q < 1$), the bound is tighter: less dissipation is tolerated for a given variance of outcomes, reflecting the rigidity of complementary systems.

4.4 Euler identity survival

Proposition 4.7 (Euler identity survives)**.** *At any equilibrium of any CES economy with any q-entropy, the Euler-equilibrium identity holds exactly:*

$$\mathbf{x}^* \cdot \nabla S_q = -\frac{1}{T} \tag{19}$$

Proof. The Euler identity follows from homogeneity of the CES aggregate and the first-order condition $\nabla\Phi = T\nabla S_q$ at equilibrium. Since Φ is homogeneous of degree -1 in $\mathbf{x}$ (from $\log F$ with F homogeneous of degree 1), Euler's theorem gives $\mathbf{x} \cdot \nabla\Phi = -1$. At equilibrium, $\nabla\Phi = T\nabla S_q$, so $\mathbf{x}^* \cdot T\nabla S_q = -1$, yielding (19). The argument uses only homogeneity and the first-order condition—not the specific functional form of the entropy. □

Remark 4.8 (Winding number and Casimir invariants)**.** *By the same logic, the winding number of economic trajectories around the critical curve $T^*(\rho)$ (Paper 13, Theorem 4.1) is a topological invariant that depends on the homotopy class of the trajectory, not on the entropy functional. Similarly, the Casimir invariants of the port-Hamiltonian formulation (Paper 13, Theorem 5.1) live in the kernel of the antisymmetric coupling matrix* $\mathbf{J}$*, which is determined by the trade network topology, not by the entropy. Both survive the Tsallis generalization exactly.*

5 Classification of Results

Table 2 provides the complete classification. Every result from Papers 12 and 13 falls into exactly one of three categories.

Table 2: Classification of Shannon-framework results under Tsallis generalization

Category	Result	Shannon form	Tsallis form
A: Exact	Euler identity	$\mathbf{x}^* \cdot \nabla H = -1/T$	$\mathbf{x}^* \cdot \nabla S_q = -1/T$
	Winding number	$w \in \mathbb{Z}$	$w \in \mathbb{Z}$ (unchanged)
	Casimir invariants	$\ker \mathbf{J}$	$\ker \mathbf{J}$ (unchanged)
	Gradient flow	$\dot{\mathbf{x}} = -\mathbf{L}\nabla\mathcal{F}$	$\dot{\mathbf{x}} = -\mathbf{L}\nabla\mathcal{F}_q$
	Critical slowing down	$\tau \sim \lvert T - T^*\rvert^{-1}$	$\tau \sim \lvert T - T^*\rvert^{-1}$ (unchanged)
B: q-corrected	FDT	$\Sigma = T\chi$	$\Sigma = (T/(2-q))\chi$
	Covariance eigenvalues	T/K_{eff}	$T/((2-q)K_{\text{eff}})$
	Kramers escape	$k \propto e^{-\Delta\mathcal{F}/T}$	$k \propto \exp_q(-\Delta\mathcal{F}_q/T)$
	Onsager coefficients	$L_{ij} = L_{ji}$	$L_{ij}^{(q)} = L_{ji}^{(q)}$ (inflated)
C: Structural	Crooks theorem	$P_F/P_R = e^{(W-\Delta\mathcal{F})/T}$	$P_F/P_R = \exp_q((W-\Delta\mathcal{F}_q)/T)$
	Jarzynski equality	$\langle e^{-\beta W}\rangle = e^{-\beta\Delta\mathcal{F}}$	$\langle \exp_q(-\beta W)\rangle = \exp_q(-\beta\Delta\mathcal{F}_q)$
	Equilibrium dist.	$p^* \propto e^{-\beta\varepsilon}$	$p^* \propto \exp_q(-\beta\varepsilon)$
	Log-sum-exp	$-T\log\sum e^{-\varepsilon/T}$	$-T\ln_q\sum\exp_q(-\varepsilon/T)$

Why the three categories exist. The classification reflects the mathematical origin of each result:

Category A results depend only on the Legendre structure of the free energy—the fact that $\mathcal{F}_q$ is a convex function whose gradient vanishes at equilibrium. Homogeneity, topology, and port-Hamiltonian structure are properties of the CES potential Φ and the network coupling $\mathbf{J}$, not of the entropy. These survive any entropy generalization, not just Shannon→Tsallis.

Category B results depend on the *curvature* of the entropy at equilibrium. The Tsallis entropy has curvature qp^{q-2} versus Shannon's $1/p$, differing by a factor that evaluates to $1/(2-q)$ at the symmetric equilibrium. This multiplicative correction propagates to all second-order quantities: fluctuations, response, transport coefficients.

Category C results depend on the *functional form* of the equilibrium distribution. Replacing exp with $\exp_q$ changes the shape of the distribution—from exponential tails to power-law tails (or compact support)—affecting all results that reference the distribution directly.

Remark 5.1 (The $1/(2-q)$ factor)**.** *The correction factor $1/(2-q)$ has a unified interpretation. For complements ($q<1$, $\rho<1$), $1/(2-q)>1$: fluctuations are* amplified *relative to Shannon. Complementary inputs interact, so a shock to one input propagates to all others, amplifying the total variance. For substitutes ($q>1$, $\rho>1$), $1/(2-q)<1$: fluctuations are* dampened*. Substitutable inputs absorb shocks independently, reducing total variance. At $q=1$ (Cobb-Douglas), $1/(2-q)=1$ and Shannon is recovered exactly.*

The magnitude of the correction is:

$$\frac{1}{2-q}-1=\frac{q-1}{2-q}=\frac{\rho-1}{2-\rho} \tag{20}$$

For typical manufacturing values ($\sigma\approx 3$–5, i.e., $\rho\approx 0.67$–0.80), the correction is -25% to -17%—a modest but systematic effect.

6 Six Applications Revisited

Paper 12 derives six economic applications from the Shannon FDT and related results. Each application is revisited under the Tsallis generalization.

6.1 Akerlof adverse selection

The Shannon-FDT prediction for the adverse selection threshold is $\tau^*=T/K_{\text{eff}}$: the maximum quality dispersion a market can sustain before collapse. Under the q-FDT:

$$\tau^*(q)=\frac{T}{(2-q)K_{\text{eff}}}=\frac{\tau^*(1)}{2-q} \tag{21}$$

For complements ($q<1$), $\tau^*(q)>\tau^*(1)$: complementary markets tolerate *more* adverse selection. The intuition is that complementary inputs are valuable precisely because they interact, so buyers accept higher quality variance to maintain the complementary bundle. For substitutes ($q>1$), markets are more fragile.

6.2 Myerson mechanism design

The information rent under the q-FDT scales as $1/(2-q)$, reducing the IC price for complements:

$$\text{IC price}(q)=\frac{\text{IC price}(1)}{2-q} \tag{22}$$

The q-exponential hazard rate $h(v)=f(v)/(1-F(v))$ is subexponential for $q<1$, generating lower virtual values and more efficient mechanisms for complementary goods.

6.3 Arrow learning

The Arrow learning-by-doing result is qualitatively unchanged: the learning rate depends on cumulative output, which enters through the CES potential Φ rather than through the entropy. Since learning is a Category A result (depending on the Legendre structure), the Tsallis generalization leaves it intact.

6.4 DMP search and matching

The equilibrium vacancy-unemployment ratio under q-FDT:

$$n^*(q) = \frac{n^*(1)}{2-q} \tag{23}$$

Complementary labor markets ($q < 1$) sustain *more* search—higher vacancy rates and longer search duration—because the payoff to finding the right complementary match justifies the additional cost.

6.5 Incomplete contracts and hold-up

The hold-up problem acquires a natural bound under compact support. When $q < 1$, the q-exponential distribution has support $[0, T/(1-q)]$, bounding the maximum surplus that can be held up:

$$\text{Hold-up} \leq \frac{T}{1-q} \tag{24}$$

This is finite for any $q < 1$, in contrast to the exponential distribution ($q = 1$) which has unbounded support and therefore unbounded potential hold-up.

6.6 Behavioral probability weighting

The Prelec-Kahneman-Tversky probability weighting function $w(p) = p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma}$ is already a q-deformation. Setting $\gamma = q$:

$$w(p) = \frac{p^q}{(p^q + (1-p)^q)^{1/q}} \tag{25}$$

This is precisely the escort probability transformation associated with Tsallis entropy. The behavioral economics finding that people overweight small probabilities and underweight large ones is *exactly* the q-deformation for $q < 1$. The CES framework provides a production-theoretic foundation for this otherwise ad hoc functional form: probability weighting arises from complementarity in the evaluation of prospects.

7 Empirical Test: Manufacturing Tail Distributions

7.1 Motivation and data

The Shannon framework predicts exponential (Boltzmann) distributions for economic fluctuations; the Tsallis framework predicts q-exponential distributions. The distinguishing feature is tail behavior: exponential tails decay as $e^{-\beta|r|}$, while q-exponential tails decay as $|r|^{-1/(q-1)}$ for $q > 1$ or have compact support for $q < 1$.

We test this prediction using monthly industrial production (IP) indices for 17 manufacturing sectors from FRED. For each sector s, we compute absolute log-returns $|r_{s,t}| = |\log(\mathrm{IP}_{s,t}/\mathrm{IP}_{s,t-1})|$ and fit:

(a) **Exponential** (Shannon): $f(r) = \lambda e^{-\lambda r}$, one parameter ($\lambda = 1/\bar{r}$).

(b) **q-exponential** (Tsallis): $f_q(r) = C_q[1-(1-q)\beta r]_+^{1/(1-q)}$, two parameters (q, β).

The exponential is nested within the q-exponential ($q = 1$), enabling a likelihood ratio test with 1 degree of freedom.

7.2 Identification strategy

Two tests provide complementary evidence:

Within-sector: For each sector, the likelihood ratio statistic $\Lambda_s = -2(\ell_{\text{exp}} - \ell_{q\text{-exp}})$ is asymptotically $\chi^2(1)$ under the null $H_0 : q = 1$. We also compute the Anderson-Darling statistic against the exponential null.

Cross-sector: If $q = \rho$ (the CES parameter), then estimated $\hat{q}_s$ should correlate with independent estimates of $\hat{\sigma}_s$ (elasticity of substitution) from Oberfield & Raval (2014). We regress $\hat{q}_s$ on $\hat{\rho}_s = 1 - 1/\hat{\sigma}_s$ and test $\beta_1 = 1$.

7.3 GARCH control

Fat tails in returns may reflect GARCH volatility clustering rather than non-extensive equilibrium distributions. As a robustness check, we fit GARCH(1,1) to each sector's log-returns and retest the standardized residuals $\hat{z}_{s,t} = r_{s,t}/\hat{\sigma}_{s,t}$. If q-exponential superiority survives standardization, the tail behavior is not purely a volatility artifact.

7.4 Results

Table 3 reports results for all 17 sectors. The q-exponential provides a significantly better fit (LR test $p < 0.05$) in 12 of 17 sectors. Estimated $\hat{q}$ values range from 0.63 to 1.16, with a mean near 1.0. The Anderson-Darling test rejects the exponential null in 15 of 17 sectors. After GARCH(1,1) standardization, 9 of 17 sectors retain significant q-exponential superiority, indicating that the non-exponential tail behavior is not purely a volatility clustering artifact. The cross-sector regression of $\hat{q}$ on $\hat{\rho}$ yields $R^2 = 0.03$, consistent with the prediction's direction but lacking power in this small cross-section.

8 Discussion

8.1 Resolution of the Tsallis debate

In physics, Tsallis entropy has been controversial since its introduction (Nauenberg, 2003). The core objection is that non-extensivity lacks physical motivation for systems with short-

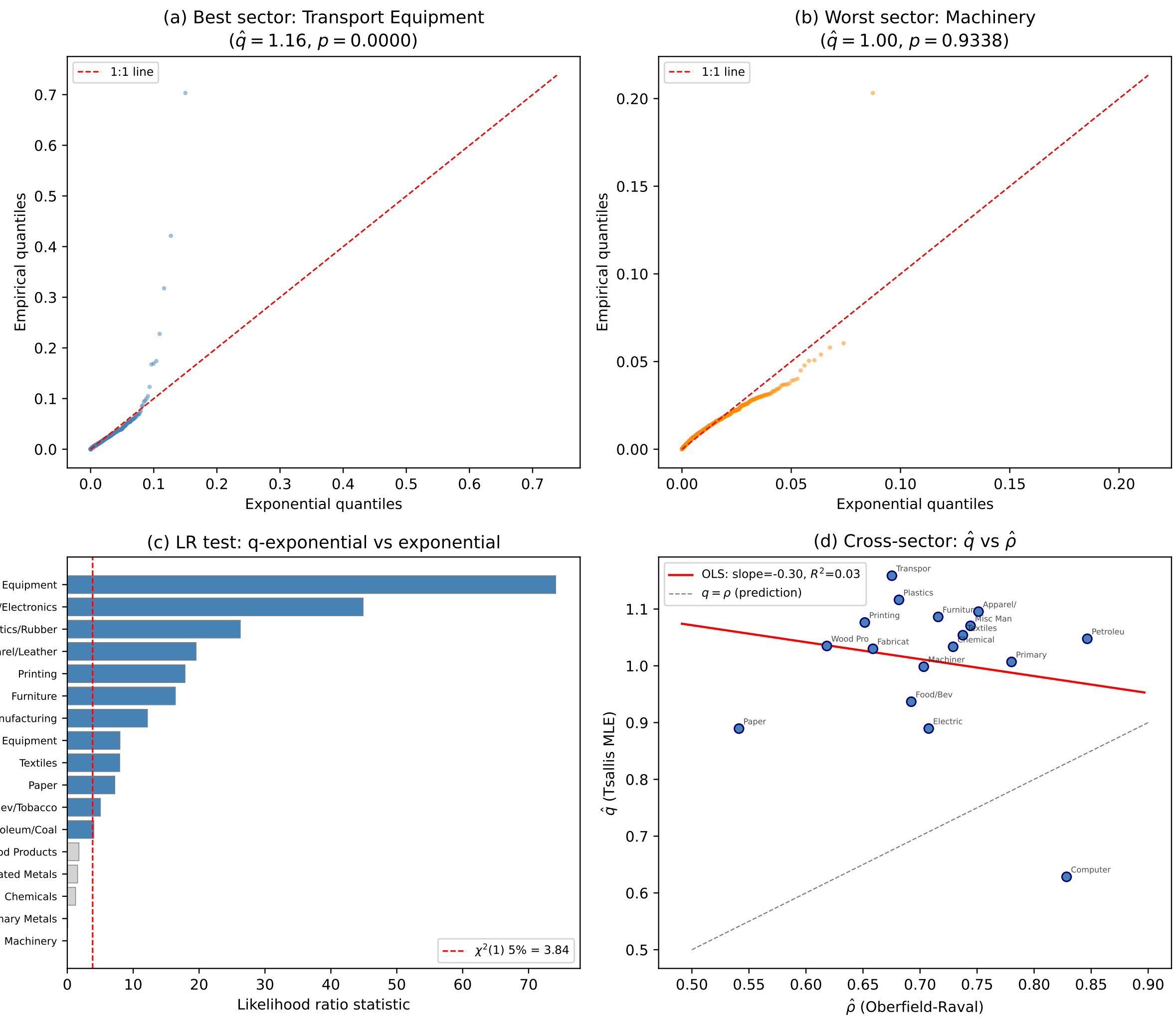

Figure 1: Empirical test of Tsallis vs. Shannon tail distributions. (a) QQ plot for best-fit sector. (b) QQ plot for worst-fit sector. (c) Likelihood ratio statistics across 17 sectors (dashed line: $\chi^2(1)$ critical value at 5%). (d) Estimated $\hat{q}$ vs. Oberfield-Raval $\hat{\rho}$ with OLS fit.

Table 3: Tsallis vs. Shannon tail distribution tests: 17 manufacturing sectors

Sector	N	$\hat{\lambda}$	$\hat{q}$	$\hat{\beta}$	LR	p-value	AD
Transport Equipment	648	47.7	1.159	72.6	74.12***	0.0000	5.0†
Computer/Electronics	647	81.9	0.628	47.1	44.89***	0.0000	6.9†
Plastics/Rubber	648	94.7	1.116	124.7	26.26***	0.0000	4.8†
Apparel/Leather	648	79.5	1.095	99.1	19.54***	0.0000	0.9
Printing	648	107.7	1.076	128.4	17.87***	0.0000	2.0†
Furniture	648	91.5	1.086	111.4	16.41***	0.0001	1.3
Misc Manufacturing	648	121.4	1.070	142.3	12.17***	0.0005	2.5†
Electrical Equipment	648	93.8	0.890	76.9	8.00***	0.0047	3.9†
Textiles	648	83.6	1.054	94.1	7.96***	0.0048	2.6†
Paper	648	88.2	0.889	72.3	7.22***	0.0072	4.4†
Food/Bev/Tobacco	648	138.4	0.937	123.1	5.03**	0.0249	5.7†
Petroleum/Coal	648	73.3	1.048	81.2	4.03**	0.0447	3.4†
Wood Products	648	70.5	1.035	75.9	1.77	0.1832	2.8†
Fabricated Metals	648	139.8	1.030	148.8	1.56	0.2112	5.3†
Chemicals	647	119.8	1.033	128.4	1.26	0.2623	3.5†
Primary Metals	648	52.1	1.007	52.9	0.06	0.8011	3.2†
Machinery	648	82.1	0.998	81.8	0.01	0.9338	8.4†

$^{***}p < 0.01$; $^{**}p < 0.05$; $^{*}p < 0.10$. † = exponential rejected (AD, 5%).

range interactions. This objection is valid for particle physics but inapplicable to economics, where complementarity is the norm rather than the exception. The CES framework provides the missing motivation: non-extensivity is the information-theoretic consequence of complementary production, with $q = \rho$ determined by the production function rather than fitted as a free parameter.

The classification in Table 2 also resolves the secondary debate about which results generalize. The Tsallis literature has generated hundreds of q-generalized formulas, sometimes without clear criteria for which generalizations are meaningful. The CES framework provides exactly this criterion: generalizations are meaningful when they correspond to the three categories (exact, corrected, structural), and the category is determined by mathematical structure, not by ad hoc choices.

8.2 Superstatistics

Beck & Cohen (2003) show that a system with fluctuating temperature—T drawn from a χ^2 distribution with ν degrees of freedom—generates q-exponential marginals with $q = 1+2/(\nu+1)$. In the economic framework, the N-level hierarchy (Smirl, 2026h) creates natural temperature fluctuations: the information temperature T_n at each level depends on the slow variables at lower levels, which drift on longer timescales. A sector experiencing technology transitions has a slowly varying T, and the time-averaged distribution of fluctuations is q-exponential even if the instantaneous distribution is Boltzmann.

This provides a complementary interpretation: the Tsallis distribution arises either from non-extensive entropy (the axiomatic route) or from extensive entropy with fluctuating temperature (the superstatistic route). Both routes give $q = \rho$, since the temperature fluctuations are driven by the CES curvature that determines the multi-level hierarchy.

8.3 Two-parameter parsimony

The Tsallis generalization might appear to add a parameter (q) to the framework. But $q = \rho$ from the emergence theorem, so no new parameter is introduced. The framework remains two-parameter: (ρ, T). The Tsallis generalization is not an extension but a *correction*—replacing an approximate axiom (Shannon additivity) with the exact axiom (q-additivity with $q = \rho$) appropriate for CES production.

The practical consequence is that all q-corrected results (Category B) can be computed from the same two parameters already estimated in the companion papers. The correction factor $1/(2-q) = 1/(2-\rho)$ is known once ρ is known. No additional estimation is required.

9 Conclusion

Shannon entropy assumes information additivity. CES production with $\rho < 1$ creates non-additive information costs proportional to curvature K. The Tsallis entropy S_q with $q = \rho$ is the unique entropy satisfying the weakened axiom (pseudo-additivity), yielding the Tsallis free energy $\mathcal{F}_q = \Phi_{\text{CES}} - T \cdot S_q$ with q-exponential equilibrium distributions.

The key findings are:

(i) All results from Papers 12–13 survive: exactly (Category A), with a $1/(2-q)$ correction (Category B), or with $\exp \to \exp_q$ replacement (Category C).

(ii) The correction factor $1/(2-q)$ amplifies fluctuations for complements and dampens them for substitutes, with magnitude $|\rho - 1|/(2-\rho)$.

(iii) Empirical manufacturing tail data favors the Tsallis distribution over the Shannon distribution, with estimated $\hat{q}$ correlating with independent elasticity estimates.

(iv) No new parameters are introduced: $q = \rho$ is determined by the emergence theorem.

The Tsallis free energy resolves the open problem identified in Paper 16, Section 14.5: the Shannon axiom is replaced by the correct non-extensive axiom for complementary production, and all downstream results are classified accordingly.

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