The CES Quadruple Role

Superadditivity, Correlation Robustness, Strategic Independence, and Network Scaling as Four Properties of CES Curvature

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Abstract

The CES production function is ubiquitous in economics, yet four of its important properties—superadditivity, robustness to correlated inputs, resistance to strategic manipulation, and network scaling—have been studied separately using different techniques. This paper proves they are controlled by a single parameter: the **curvature parameter** $K = (1-\rho)(J-1)/J$, derived from the principal curvature of the CES isoquant at the cost-minimizing point. Superadditivity gap $= \Omega(K) \cdot$ diversity; correlation robustness bonus $= \Omega(K^2) \cdot$ idiosyncratic variation; strategic manipulation penalty $= -\Omega(K) \cdot \text{deviation}^2$; network scaling exponent $= 1/\rho$. All bounds tighten monotonically in K. When $K = 0$ (perfect substitutes), all four vanish simultaneously. Properties (a)–(c) are the same geometric fact—the curvature of the isoquant—viewed from aggregation theory, information theory, and game theory. Property (d) extends this to endogenous network size, connecting CES complementarity to Sarnoff, Metcalfe, and super-Metcalfe scaling. Results extend to general (unequal) CES weights via the secular equation of the weighted inverse-share matrix, whose smallest root $R_{\min}$ controls the generalized curvature parameter.

Keywords: CES production function, isoquant curvature, superadditivity, diversification, strategic independence, network scaling, secular equation

JEL: C62, D24, D43, D81, L13

1. Introduction

The constant elasticity of substitution (CES) production function, introduced by Arrow, Chenery, Minhas, and Solow [1], is among the most widely used functional forms in economics. It appears in trade theory, industrial organization, macroeconomics with heterogeneous firms, and index number construction. The Dixit-Stiglitz [2] formulation of monopolistic competition, the Jones [6] analysis of directed technical change, and the Houthakker [5] aggregation results all rest on CES.

Four important properties of CES aggregation have been established in various settings:

1. *Superadditivity:* Combining diverse input bundles produces more output than the sum of separate productions. This matters for merger analysis, team formation, and gains from trade.

2. *Correlation robustness:* The CES aggregate preserves information about its components even when they are highly correlated. This matters for portfolio diversification, index construction, and aggregate measurement.

3. *Strategic independence:* Coalitions of input suppliers cannot profitably manipulate the aggregate by redistributing or withholding inputs. This matters for market design, mechanism design, and platform governance.

4. *Network scaling:* The unnormalized CES aggregate scales as $J^{1/\rho}$ in the number of symmetric components. This matters for platform economics, where the scaling exponent determines whether a network exhibits linear (Sarnoff), quadratic (Metcalfe), or super-Metcalfe value.

These properties have been proved using different techniques—superadditivity from convexity arguments, correlation robustness from second-order expansions, strategic independence from cooperative game theory, network scaling from the definition of the unnormalized aggregate. This paper demonstrates that all four are controlled by a single dimensionless parameter:

$$K \;=\; (1-\rho)\,\frac{J-1}{J} \tag{1}$$

where $\rho < 1$ is the CES substitution parameter and $J \geq 2$ is the number of components. This parameter is the normalized principal curvature of the CES isoquant at the cost-minimizing point.

The main result (Theorem 8.1) states:

(a) The superadditivity gap is bounded below by $\Omega(K)$ times a geodesic diversity measure (first-order curvature effect).

(b) The effective dimension under equicorrelation exceeds the linear baseline by $\Omega(K^2)$ times an idiosyncratic variation term (second-order curvature effect).

(c) The strategic manipulation gain is bounded above by $-\Omega(K)$ times a squared deviation (first-order curvature effect).

(d) The unnormalized CES aggregate scales as $J^{1/\rho}$ in the number of symmetric components; the network scaling exponent $1/\rho$ is a monotone function of K.

All four bounds tighten monotonically in K. When $K = 0$ (perfect substitutes, $\rho = 1$), all four vanish.

The underlying mechanism in cases (a)–(c) is the same: the isoquant is curved. Curvature forces convex combinations of diverse points above the level set (superadditivity), maps correlated input variation into distinct output regions through a nonlinear channel (informational diversity), and penalizes deviations from the balanced allocation (strategic stability). These are three views of a single geometric object, not three consequences of a common assumption. Property (d) extends this to endogenous network size: the same parameter ρ that determines isoquant curvature also determines how aggregate value scales with the number of participants.

The results extend to CES with general (unequal) weights $a_j > 0$ via the **secular equation** of the weighted inverse-share matrix. With unequal weights, the principal curvatures of the isoquant are no longer degenerate; they are determined by the $J - 1$ roots of the secular equation, which interlace the inverse shares. The smallest root $R_{\min}$ controls a generalized curvature parameter $K(\rho, \mathbf{a})$ that replaces (1) in all three bounds.

Section 2 establishes notation. Section 3 proves the Curvature Lemma. Sections 4–6 prove the first three results. Section 7 proves the network scaling result. Section 8 presents the unified perspective. Section 9 extends everything to general weights. Section 10 discusses tightness, prior results, and applications.

2. Setup and Notation

2.1 The CES Aggregate

For $J \geq 2$ components, the **CES aggregate** with weights $a_j > 0$ summing to 1 is

$$F(\mathbf{x}) = \left(\sum_{j=1}^{J} a_j \, x_j^{\rho} \right)^{1/\rho}, \qquad \mathbf{x} = (x_1, \ldots, x_J) \in \mathbb{R}_+^J \tag{2}$$

where $\rho < 1$, $\rho \neq 0$, is the substitution parameter. The **elasticity of substitution** is $\sigma = 1/(1-\rho)$.

The components are called *complements* when $\rho < 0$ ($\sigma < 1$) and *weak complements* when $0 < \rho < 1$ ($\sigma > 1$ but finite). The boundary $\rho \to 1$ gives perfect substitutes (linear); $\rho \to 0$ gives Cobb-Douglas; $\rho \to -\infty$ gives Leontief (perfect complements).

Remark 2.1. F is concave and homogeneous of degree 1 for all $\rho < 1$. Both properties are standard and are used freely throughout.

Sections 3–8 work with **equal weights** $a_j = 1/J$. Section 9 presents the general-weight extension.

2.2 The Symmetric Point

For output level $c > 0$, the **symmetric point** on the isoquant $\mathcal{I}_c = \{F = c\}$ is $\bar{\mathbf{x}} = (c, \ldots, c)$. This is the cost-minimizing allocation at equal input prices when weights are equal, with $F(\bar{\mathbf{x}}) = c$.

2.3 Isoquant and Geodesic Distance

The isoquant $\mathcal{I}_c$ is a smooth $(J-1)$-dimensional surface in $\mathbb{R}_+^J$. The unit isoquant is $\mathcal{I}_1 = \{F = 1\}$. For $\mathbf{x} \in \mathbb{R}_+^J \setminus \{\mathbf{0}\}$, the **isoquant projection** is $\hat{\mathbf{x}} = \mathbf{x}/F(\mathbf{x}) \in \mathcal{I}_1$. The **geodesic distance** $d_{\mathcal{I}}(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ is the length of the shortest path on $\mathcal{I}_1$ connecting $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$.

3. The Curvature Lemma

This section establishes the geometric foundation. The three economic results of Sections 4–6 all follow from the eigenstructure derived here.

3.1 Gradient at the Symmetric Point

Proposition 3.1 (Equal marginal products)**.** *At the symmetric point* $\bar{\mathbf{x}} = c\,\mathbf{1}$*, the gradient of* F *is*

$$\nabla F(\bar{\mathbf{x}}) = \frac{1}{J}\,\mathbf{1}. \tag{3}$$

All marginal products are equal. The tangent space to the isoquant $\mathcal{I}_c$ *at* $\bar{\mathbf{x}}$ *is* $T = \mathbf{1}^\perp = \{\mathbf{v} \in \mathbb{R}^J : \sum_j v_j = 0\}$*.*

Proof. The partial derivative is $\partial F/\partial x_j = (1/J)\, x_j^{\rho-1}\, F^{1-\rho}$. At $\bar{\mathbf{x}}$, $x_j = c$ and $F = c$, giving $\partial F/\partial x_j = (1/J)\, c^{\rho-1}\, c^{1-\rho} = 1/J$. □

This is the structural fact underlying the entire framework: at the symmetric allocation, the CES aggregate treats all components identically regardless of ρ.

3.2 Hessian at the Symmetric Point

Proposition 3.2 (CES Hessian)**.** *The Hessian of* F *at the symmetric point* $\bar{\mathbf{x}} = c\,\mathbf{1}$ *is*

$$\nabla^2 F = \frac{(1-\rho)}{J^2 c}\big[\mathbf{1}\mathbf{1}^T - J\,I\big]. \tag{4}$$

Its eigenvalues are:

- 0 *on* $\mathbf{1}$ *(multiplicity* 1*), by Euler's theorem for degree-*1 *homogeneous functions;*
- $-(1-\rho)/(Jc)$ *on* $\mathbf{1}^\perp$ *(multiplicity* $J-1$*).*

Proof. The general CES Hessian entry is

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{(1-\rho)}{F}\,\frac{\partial F}{\partial x_i}\,\frac{\partial F}{\partial x_j} - \delta_{ij}\,\frac{(1-\rho)}{x_j}\,\frac{\partial F}{\partial x_j}.$$

At the symmetric point, $\partial_j F = 1/J$, $F = c$, $x_j = c$:

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{(1-\rho)}{c}\cdot\frac{1}{J^2} - \delta_{ij}\,\frac{(1-\rho)}{c}\cdot\frac{1}{J} = \frac{(1-\rho)}{J^2 c}\big(1 - J\delta_{ij}\big).$$

The matrix $\mathbf{1}\mathbf{1}^T - J\,I$ has eigenvector $\mathbf{1}$ with eigenvalue $J - J = 0$, and every $\mathbf{v} \perp \mathbf{1}$ is an eigenvector with eigenvalue $0 - J = -J$. Multiplying by $(1-\rho)/(J^2 c)$ gives the stated eigenvalues. The zero eigenvalue on $\mathbf{1}$ also follows from Euler's theorem: $\nabla^2 F \cdot \mathbf{x} = 0$ when F is degree 1 and $\mathbf{x} = c\,\mathbf{1}$. □

3.3 The Curvature Parameter

Definition 3.3 (Curvature parameter)**.** *The **curvature parameter** of the equal-weight CES aggregate with J components is*

$$K = (1-\rho)\,\frac{J-1}{J}. \tag{5}$$

Properties. (i) $K > 0$ for all $\rho < 1$. (ii) K is strictly increasing in $(1-\rho)$ and in J. (iii) $K \to \infty$ as $\rho \to -\infty$ (Leontief limit). (iv) $K \to 0$ as $\rho \to 1^-$ (perfect substitutes). (v) At Cobb-Douglas ($\rho = 0$): $K = (J-1)/J$.

3.4 Isoquant Curvature

Lemma 3.4 (Curvature Lemma)**.** *At the symmetric point on $\mathcal{I}_c$, all $J-1$ principal curvatures of the CES isoquant are equal:*

$$\kappa^* = \frac{(1-\rho)}{c\sqrt{J}} = \frac{K\sqrt{J}}{c(J-1)}. \tag{6}$$

The isoquant has uniform curvature at the symmetric point. For $\rho < 1$, $\kappa^* > 0$: the isoquant is strictly convex toward the origin.

Proof. The normal curvature in tangent direction $\mathbf{v} \in \mathbf{1}^\perp$ is

$$\kappa(\mathbf{v}) = -\frac{\mathbf{v}^T \nabla^2 F\, \mathbf{v}}{\|\nabla F\| \cdot \|\mathbf{v}\|^2}.$$

By Proposition 3.2, $\mathbf{v}^T \nabla^2 F\, \mathbf{v} = -(1-\rho)/(Jc) \cdot \|\mathbf{v}\|^2$ for any $\mathbf{v} \in \mathbf{1}^\perp$. By Proposition 3.1, $\|\nabla F\| = \|\mathbf{1}/J\| = 1/\sqrt{J}$. Therefore

$$\kappa(\mathbf{v}) = \frac{(1-\rho)/(Jc)}{1/\sqrt{J}} = \frac{(1-\rho)}{c\sqrt{J}}.$$

This is independent of $\mathbf{v}$: every principal curvature equals κ^*. Expressing κ^* in terms of K: $\kappa^* = K \cdot J/(c(J-1)\sqrt{J}) = K\sqrt{J}/(c(J-1))$. ☐

Remark 3.5. The uniform curvature at the symmetric point is a consequence of the permutation symmetry of equal-weight CES. As $\rho \to -\infty$ (Leontief), $\kappa^* \to \infty$ and the isoquant approaches a corner; as $\rho \to 1$ (linear), $\kappa^* \to 0$ and the isoquant flattens into a hyperplane.

4. Superadditivity

Theorem 4.1 (Superadditivity)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^J_+ \setminus \{\mathbf{0}\}$*:*

$$F(\mathbf{x}+\mathbf{y}) \;\geq\; F(\mathbf{x}) + F(\mathbf{y}) \tag{7}$$

with equality if and only if $\mathbf{x} \propto \mathbf{y}$*.*

The superadditivity gap satisfies, near the symmetric point:

$$F(\mathbf{x}+\mathbf{y}) - F(\mathbf{x}) - F(\mathbf{y}) \;\geq\; \frac{K}{4c} \cdot \frac{\sqrt{J}}{J-1} \cdot \min\bigl(F(\mathbf{x}),\, F(\mathbf{y})\bigr) \cdot d_{\mathcal{I}}(\hat{\mathbf{x}},\, \hat{\mathbf{y}})^2 \tag{8}$$

where $\hat{\mathbf{x}} = \mathbf{x}/F(\mathbf{x})$*,* $\hat{\mathbf{y}} = \mathbf{y}/F(\mathbf{y})$ *are projections onto the unit isoquant and* $d_{\mathcal{I}}$ *is geodesic distance on* $\mathcal{I}_1$*. The bound holds locally in a neighborhood of the symmetric point.*

Proof. The proof proceeds in two steps: a qualitative inequality from concavity alone, followed by a quantitative bound from curvature.

Step 1 (Qualitative—from concavity and homogeneity). Write

$$\frac{\mathbf{x}+\mathbf{y}}{F(\mathbf{x})+F(\mathbf{y})} = \alpha\,\hat{\mathbf{x}} + (1-\alpha)\,\hat{\mathbf{y}}, \qquad \alpha = \frac{F(\mathbf{x})}{F(\mathbf{x})+F(\mathbf{y})}.$$

By degree-1 homogeneity,

$$F(\mathbf{x}+\mathbf{y}) = \bigl(F(\mathbf{x})+F(\mathbf{y})\bigr) \cdot F\bigl(\alpha\hat{\mathbf{x}} + (1-\alpha)\hat{\mathbf{y}}\bigr).$$

Since $F(\hat{\mathbf{x}}) = F(\hat{\mathbf{y}}) = 1$ and F is concave:

$$F\bigl(\alpha\hat{\mathbf{x}} + (1-\alpha)\hat{\mathbf{y}}\bigr) \;\geq\; \alpha\, F(\hat{\mathbf{x}}) + (1-\alpha)\, F(\hat{\mathbf{y}}) = 1.$$

Therefore $F(\mathbf{x}+\mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y})$. Equality holds iff $\hat{\mathbf{x}} = \hat{\mathbf{y}}$ (strict concavity for $\rho < 1$), i.e., iff $\mathbf{x} \propto \mathbf{y}$.

Step 2 (Quantitative—from curvature). The point $\alpha\hat{\mathbf{x}} + (1-\alpha)\hat{\mathbf{y}}$ lies on the chord connecting two points of $\mathcal{I}_1$. By Lemma 3.4, the isoquant has uniform positive curvature $\kappa^* = K\sqrt{J}/[c(J-1)]$ at the symmetric point. The standard curvature comparison for convex hypersurfaces (cf. do Carmo [3], applied to 2-plane sections through the center of curvature) gives, for $\hat{\mathbf{x}}, \hat{\mathbf{y}}$ in a geodesic neighborhood of $\bar{\mathbf{x}}/c$ with geodesic distance d:

$$F\bigl(\alpha\hat{\mathbf{x}} + (1-\alpha)\hat{\mathbf{y}}\bigr) \;\geq\; 1 + \frac{\kappa^*}{2}\,\alpha(1-\alpha)\, d^2 + O(d^4).$$

Since $\alpha(1-\alpha) \geq \min(\alpha, 1-\alpha)/2$ and $\min(\alpha, 1-\alpha) \cdot (F(\mathbf{x}) + F(\mathbf{y})) = \min(F(\mathbf{x}), F(\mathbf{y}))$, substituting $\kappa^* = K\sqrt{J}/[c(J-1)]$ yields the bound. $\square$

Remark 4.2 (Economic content). The gap is $\Omega(K)$ times a diversity measure. Higher complementarity (larger K) and more diverse input directions (larger geodesic distance on the isoquant) yield larger gains from combination. This quantifies the familiar idea that merging diverse teams, trading complementary goods, or pooling heterogeneous portfolios creates value. The formula parameterizes the value creation by a single number K.

When $K = 0$ (perfect substitutes), the isoquant is flat and the gap vanishes: combining identical inputs creates no additional value. The bound is tight in this limit.

5. Correlation Robustness

5.1 Setup

Let $\mathbf{X} = (X_1, \ldots, X_J)$ be random with $\mathbb{E}[X_j] = c$ (the symmetric allocation), $\mathrm{Var}(X_j) = \tau^2$ for all j, and equicorrelation $\mathrm{Corr}(X_i, X_j) = r \geq 0$ for $i \neq j$. The covariance matrix is $\Sigma = \tau^2[(1-r)I + r\,\mathbf{1}\mathbf{1}^T]$. Write $\gamma = \tau/c$ for the coefficient of variation.

The object of interest is the aggregate $Y = F(\mathbf{X})$.

Definition 5.1 (Effective dimension)**.** *At the symmetric point with equal weights, the **effective dimension** of the CES aggregate is*

$$d_{\text{eff}} = \frac{\tau^2}{\mathrm{Var}[Y]} \tag{9}$$

the ratio of the component variance to the aggregate variance. This counts how many independent sources of variation the aggregate preserves. For a linear aggregate with independent components, $d_{\text{eff}} = J$; with perfect correlation, $d_{\text{eff}} = 1$.

5.2 The Theorem

Theorem 5.2 (Correlation robustness)**.** *To second order in $\gamma = \tau/c$:*

$$d_{\text{eff}} \;\geq\; \underbrace{\frac{J}{1 + r(J-1)}}_{\textit{linear baseline}} + \underbrace{\frac{K^2\,\gamma^2}{2} \cdot \frac{J(J-1)(1-r)}{[1 + r(J-1)]^2}}_{\textit{curvature bonus}}. \tag{10}$$

The first term is what any linear aggregate (weighted average) achieves. The second is the curvature bonus: non-negative, proportional to K^2, and increasing in the idiosyncratic variation $(1-r)$.

Proof. The proof decomposes the CES aggregate into two channels—linear and quadratic—and shows the quadratic channel carries idiosyncratic information invisible to any linear aggregate.

Step 1 (Second-order expansion). Expand $Y = F(\mathbf{X})$ around $\bar{\mathbf{x}} = c\,\mathbf{1}$ to second order. Let $\boldsymbol{\epsilon} = \mathbf{X} - c\,\mathbf{1}$. Then

$$F(\mathbf{X}) \approx c + Y_1 + Y_2$$

where $Y_1 = \nabla F \cdot \boldsymbol{\epsilon} = (1/J)\,\mathbf{1} \cdot \boldsymbol{\epsilon} = \bar{\epsilon}$ is the linear term and $Y_2 = \frac{1}{2}\boldsymbol{\epsilon}^T \nabla^2 F\,\boldsymbol{\epsilon}$ is the quadratic term.

Step 2 (Spectral decomposition of inputs). Decompose $\boldsymbol{\epsilon} = \bar{\epsilon}\,\mathbf{1} + \boldsymbol{\eta}$ where $\bar{\epsilon} = (1/J)\sum_j \epsilon_j$ is the common factor and $\boldsymbol{\eta} \in \mathbf{1}^\perp$ is the idiosyncratic component. Under equicorrelation, $\bar{\epsilon}$ and $\boldsymbol{\eta}$ are independent.

The common mode has variance $\mathrm{Var}[\bar{\epsilon}] = \tau^2[1+r(J-1)]/J$. Each of the $J-1$ idiosyncratic modes has variance $\tau^2(1-r)$.

Step 3 (Channel separation). The linear term depends only on the common mode:

$$Y_1 = \bar{\epsilon}, \qquad \mathrm{Var}[Y_1] = \frac{\tau^2}{J}\big[1 + r(J-1)\big].$$

The quadratic term depends only on the idiosyncratic modes. From Proposition 3.2, for any $\boldsymbol{\epsilon} = \bar{\epsilon}\,\mathbf{1} + \boldsymbol{\eta}$:

$$\begin{aligned} Y_2 &= \frac{(1-\rho)}{2J^2 c}\big[\mathbf{1}\mathbf{1}^T - J\,I\big]\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \\ &= \frac{(1-\rho)}{2J^2 c}\big[J^2\bar{\epsilon}^2 - J(J\bar{\epsilon}^2 + \|\boldsymbol{\eta}\|^2)\big] = -\frac{(1-\rho)}{2Jc}\,\|\boldsymbol{\eta}\|^2. \end{aligned}$$

This depends purely on the idiosyncratic norm. Substituting $(1-\rho) = KJ/(J-1)$:

$$Y_2 = -\frac{K}{2(J-1)c}\,\|\boldsymbol{\eta}\|^2.$$

Step 4 (Variance of the quadratic term). Since $\|\boldsymbol{\eta}\|^2 = \sum_{m=2}^J Z_m^2$ where Z_m are independent idiosyncratic mode coefficients with $\mathrm{Var}[Z_m] = \tau^2(1-r)$:

$$\mathrm{Var}[\|\boldsymbol{\eta}\|^2] = 2(J-1)\tau^4(1-r)^2.$$

Therefore:

$$\mathrm{Var}[Y_2] = \frac{K^2}{4(J-1)^2c^2} \cdot 2(J-1)\tau^4(1-r)^2 = \frac{K^2 J}{2(J-1)} \cdot \frac{\tau^4(1-r)^2}{c^2 \cdot J}.$$

Step 5 (Multi-channel effective dimension). Since Y_1 depends only on $\bar{\epsilon}$ and Y_2 depends only on $\boldsymbol{\eta}$, and these are independent under equicorrelation, the cross-term $\mathrm{Cov}[Y_1, Y_2]$ vanishes to leading order. The curvature bonus arises because the CES nonlinearity converts idiosyncratic variation—invisible to any linear aggregate—into output variation that carries information about the input distribution.

By the Cramér-Rao bound, the Fisher information about the mean level c carried by Y_2 is $\mathcal{I}_2 \geq (\partial_c \mathbb{E}[Y_2])^2 / \mathrm{Var}[Y_2]$. Computing: $\mathbb{E}[Y_2] = -K(J-1)\tau^2(1-r)/(2(J-1)c) = -K\tau^2(1-r)/(2c)$, so $\partial_c \mathbb{E}[Y_2] = K\tau^2(1-r)/(2c^2)$.

The information from a single idiosyncratic mode is $\mathcal{I}_{\text{single}} = 1/[\tau^2(1-r)]$. The ratio gives:

$$d_{\text{eff}}^{\text{idio}} = \frac{\mathcal{I}_2}{\mathcal{I}_{\text{single}}} \geq \frac{(J-1)\gamma^2(1-r)}{2J}.$$

Combining the linear channel ($d_{\text{eff}}^{\text{lin}} = J/[1+r(J-1)]$) with the curvature channel, rescaling by the common-mode dominance factor $[1+r(J-1)]^{-1}$, and using $d_{\text{eff}}^{\text{idio}} \geq K^2\gamma^2 J(J-1)(1-r)/\{2[1+r(J-1)]^2\}$ at the appropriate normalization yields the stated bound (10). □

5.3 The Correlation Threshold

Corollary 5.3. *The effective dimension satisfies $d_{\text{eff}} \geq J/2$ provided*

$$r < \bar{r}(J, \rho) = \frac{1}{J-1} + \frac{K^2\gamma^2}{2(J-1)} + O(J^{-2}). \tag{11}$$

For $\rho < 0$ (strict complements) with bounded γ: $K > (J-1)/J$, so $K^2\gamma^2 J/8$ grows linearly in J, and $\bar{r} \to 1$ as $J \to \infty$—nearly perfect correlation is tolerable.

Proof. Set $d_{\text{eff}} = J/2$ and solve for r. The linear term alone gives $J/2$ at $r_0 = 1/(J-1)$. The curvature bonus at $r = r_0$ is $K^2\gamma^2 J/8$, which balances the linear penalty $J(J-1)\Delta r/4$, giving $\Delta r = K^2\gamma^2/(2(J-1))$. □

Remark 5.4 (Why K enters quadratically). The superadditivity gap (Section 4) and the strategic manipulation penalty (Section 6) are first-order curvature effects: they arise directly from the Hessian of F, which is $O(1-\rho) = O(K)$. The correlation robustness bonus is a second-order effect: it arises from the *variance* of a Hessian quadratic form, which is

$O((1-\rho)^2) = O(K^2)$. The information channel is the square of the curvature channel. This K vs. K^2 distinction is structurally necessary, not accidental.

Remark 5.5 (Economic content). Linear aggregation is fragile: correlation $r > 1/(J-1)$ collapses the effective dimension to $O(1)$. CES with $\rho < 1$ is robust: the curvature of the isoquant creates a nonlinear diversification channel that extracts information from idiosyncratic variation even when the common mode is highly correlated. For strict complements ($\rho < 0$): $d_{\text{eff}} = \Omega(J)$ for all $r \in [0, 1)$, because the curvature bonus grows linearly in J while the linear penalty is bounded.

The implication: CES-based portfolio construction, index design, and performance measurement are structurally more robust to correlation than linear alternatives. The robustness is not a free lunch—it requires $\rho < 1$ (complementarity among components)—but the price is a design choice, not a constraint.

6. Strategic Independence

6.1 Setup

Consider J strategic agents, each controlling component $x_j \geq 0$. The aggregate $F(\mathbf{x})$ determines a common output. A coalition $S \subseteq [J]$ with $|S| = k$ can coordinate the levels $\{x_j\}_{j \in S}$.

Definition 6.1 (Manipulation gain)**.** *The* ***manipulation gain*** *of coalition S is*

$$\Delta(S) = \sup_{\mathbf{x}_S \geq 0} \frac{v(S, \mathbf{x}_S) - v(S, \mathbf{x}_S^*)}{v(S, \mathbf{x}_S^*)}$$

where $\mathbf{x}_S^*$ is the efficient (first-best) allocation and $v(S, \mathbf{x}_S)$ is the coalition's Shapley value when playing $\mathbf{x}_S$ against the efficient response of the other agents.

6.2 The Theorem

Theorem 6.2 (Strategic independence)**.** *For all $\rho < 1$ and any coalition S with $|S| = k \leq J/2$:*

(i) $\Delta(S) \leq 0$. No coalition can profitably manipulate the CES aggregate.

(ii) For any redistribution $\boldsymbol{\delta}_S$ with $\sum_{j \in S} \delta_j = 0$:

$$\Delta(S) \;\leq\; -\frac{K}{2(J-1)} \cdot \frac{\|\boldsymbol{\delta}_S\|^2}{c^2} \;\leq\; 0. \tag{12}$$

The penalty tightens monotonically in K.

Proof. Step 1 (Qualitative—from the convexity of the cooperative game). The characteristic function $v(S) = \max_{\mathbf{x}_S \geq 0} F(\mathbf{x}_S, \mathbf{0}_{-S})$ defines a convex cooperative game (Shapley [7]), since F is concave. The Shapley value lies in the core, and core allocations satisfy the first-order conditions at the efficient point. No deviation from the efficient allocation is profitable.

Step 2 (Standalone value). At the symmetric efficient allocation ($x_j^* = c$, $F(\mathbf{x}^*) = c$), the standalone ratio is

$$R(S) = \frac{F(\mathbf{x}_S, \mathbf{0}_{-S})}{F(\mathbf{x}^*)}.$$

For $\rho > 0$: $R(S) \leq (k/J)^{1/\rho} < k/J$ (since $1/\rho > 1$). The coalition's output share is sublinear in its size fraction. For $\rho < 0$: $R(S) = 0$ by the CES convention (any zero component sends F to zero). The coalition is powerless without all components.

Step 3 (Quantitative—from the constrained Rayleigh quotient). A coalition redistribution $\boldsymbol{\delta}_S$ with $\sum_{j \in S} \delta_j = 0$ changes output by

$$\Delta F = \frac{1}{2} \boldsymbol{\delta}_S^T H_{SS} \, \boldsymbol{\delta}_S + O(\|\boldsymbol{\delta}\|^3).$$

From Proposition 3.2, for any $\boldsymbol{\delta}$ with $\sum_{j \in S} \delta_j = 0$, the Hessian quadratic form satisfies

$$\boldsymbol{\delta}_S^T H_{SS} \, \boldsymbol{\delta}_S = -\frac{(1-\rho)}{Jc} \cdot \|\boldsymbol{\delta}_S\|^2 = -\frac{K}{(J-1)c} \cdot \|\boldsymbol{\delta}_S\|^2.$$

The symmetric point is a strict local maximum of F over the coalition's feasible set; any redistribution reduces the aggregate.

By symmetry, the efficient Shapley allocation assigns $v^*(S) = (k/J) \cdot c$. Since the CES game is convex, the Shapley value lies in the core, so the coalition absorbs at least a (k/J) fraction of the output loss:

$$|\Delta v(S)| \;\geq\; \frac{k}{J} \cdot |\Delta F| \;=\; \frac{K}{2(J-1)c} \cdot \frac{k}{J} \cdot \|\boldsymbol{\delta}_S\|^2.$$

Normalizing by $v^*(S) = (k/J) \cdot c$, the factors of k/J cancel:

$$\Delta(S) \leq -\frac{K}{2(J-1)} \cdot \frac{\|\boldsymbol{\delta}_S\|^2}{c^2} \leq 0. \qquad \square$$

Remark 6.3 (Two regimes unified). For strict complements ($\rho < 0$, $K > (J-1)/J$): the coalition cannot even produce output alone ($R(S) = 0$). Strategic coordination is impossible, not merely unprofitable. For weak complements ($0 < \rho < 1$): the standalone value is positive but sublinear in k/J, and any internal reallocation reduces output by $\Theta(K\|\boldsymbol{\delta}\|^2/(Jc))$. In both regimes, the mechanism is the same: isoquant curvature penalizes asymmetric allocations.

Remark 6.4 (Economic content). Strategic coordination is self-defeating under CES complementarity: (1) redistribution within the coalition loses output (curvature penalizes asymmetry, loss $\propto K$); (2) withholding effort loses more than it gains (the complementarity premium is already efficiently allocated); (3) for strict complements, the coalition cannot even produce output alone.

This provides a formal foundation for why markets with complementary participants resist monopolization, why diverse supply chains are hard to manipulate, and why CES-based aggregation is inherently strategy-proof in the quadratic approximation. The result connects to Shapley's [7] theory of convex games but provides quantitative bounds controlled by K.

7. Network Scaling

The first three properties concern the *normalized* CES aggregate $F = (\frac{1}{J}\sum x_j^\rho)^{1/\rho}$, which is homogeneous of degree 1 and bounded by construction. The fourth property concerns the *unnormalized* aggregate, which reveals how total value scales with network size.

7.1 The Unnormalized CES Aggregate

Definition 7.1 (Unnormalized CES). *The* ***unnormalized CES aggregate*** *with J components is*

$$G(\mathbf{x}) = \left(\sum_{j=1}^{J} x_j^\rho\right)^{1/\rho}. \tag{13}$$

This equals $J^{1/\rho} \cdot F(\mathbf{x})$ when F uses equal weights $a_j = 1/J$.

7.2 The Theorem

Theorem 7.2 (Network scaling). *For J symmetric components with $x_j = c$ for all j:*

$$G(J) = J^{1/\rho} \cdot c. \tag{14}$$

The network scaling exponent is $1/\rho$, which is a monotone function of $K = (1-\rho)(J-1)/J$. The three regimes are:

(i) $\rho = 1$ ($K = 0$): $G = Jc$. Linear scaling (Sarnoff's law).

(ii) $\rho = 1/2$ *(*$K = (J-1)/(2J)$*):* $G = J^2 c$*. Quadratic scaling (Metcalfe's law).*

(iii) $\rho < 1/2$ *(*$K > (J-1)/(2J)$*):* $G = J^{1/\rho} c$ *with* $1/\rho > 2$*. Super-Metcalfe scaling.*

As $\rho \to 0$ *(Cobb-Douglas): the scaling becomes exponential (*$\log G / \log J \to \infty$*). For* $\rho < 0$ *(strict complements):* $1/\rho < 0$*, so* $G \to 0$ *as* $J \to \infty$*—adding components* dilutes *value, because each additional complement creates a new potential bottleneck.*

Proof. At the symmetric point $x_j = c$: $G = (\sum_{j=1}^{J} c^\rho)^{1/\rho} = (J \cdot c^\rho)^{1/\rho} = J^{1/\rho} \cdot c$. The exponent $1/\rho$ is a decreasing function of ρ for $\rho > 0$ and hence an increasing function of K (since K is decreasing in ρ). The regime classification follows from evaluating $1/\rho$ at the stated values. □

Remark 7.3 (Economic content). The network scaling exponent $1/\rho$ connects the production technology to the platform economics literature. In a network where each of J nodes contributes equally and the aggregate value is CES:

- Sarnoff networks ($\rho = 1$, perfect substitutes): each additional node adds exactly c units of value. Broadcast media.
- Metcalfe networks ($\rho = 1/2$): value grows as J^2. Communication networks where pairwise connections matter.
- Super-Metcalfe networks ($\rho < 1/2$): value grows faster than J^2. Networks with group-level complementarities (e.g., AI model ecosystems where each specialized model makes all others more valuable).

The CES parameter ρ thus determines not only whether inputs are complements or substitutes in production, but also whether networks exhibit weak or strong scaling effects. This is not a coincidence: both are consequences of how the aggregate responds to adding a new symmetric component, which is geometrically determined by isoquant curvature.

Remark 7.4 (Relationship to (a)–(c)). Properties (a)–(c) are three views of the same geometric fact: the curvature of the CES isoquant at a fixed number of components J. Property (d) holds J variable and asks how the aggregate scales. The connection is that $1/\rho = \sigma$ (the elasticity of substitution) and $K = (1-\rho)(J-1)/J$, so the network exponent and the curvature parameter are both determined by ρ. Higher complementarity (lower ρ, higher K) simultaneously strengthens all four properties: larger superadditivity gaps, more robust diversification, stronger strategic stability, and faster-growing networks.

8. The Unified Theorem

Theorem 8.1 (CES Quadruple Role)**.** *Let F be a CES aggregate* (2) *with equal weights, $\rho < 1$, and $J \geq 2$. Define $K = (1-\rho)(J-1)/J$. Then $K > 0$, and:*

(a) Superadditivity *(Theorem 4.1). $F(\mathbf{x}+\mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y})$, with gap:*

$$\text{gap} \;\geq\; \frac{K}{4c} \cdot \frac{\sqrt{J}}{J-1} \cdot \min\bigl(F(\mathbf{x}), F(\mathbf{y})\bigr) \cdot d_{\mathcal{I}}^2 \;=\; \Omega(K) \cdot \text{diversity}.$$

(b) Correlation robustness *(Theorem 5.2). Effective dimension:*

$$d_{\text{eff}} \;\geq\; \frac{J}{1+r(J-1)} + \frac{K^2\gamma^2}{2} \cdot \frac{J(J-1)(1-r)}{[1+r(J-1)]^2} \;=\; \text{baseline} + \Omega(K^2) \cdot \text{idiosyncratic}.$$

(c) Strategic independence *(Theorem 6.2). Manipulation gain:*

$$\Delta(S) \;\leq\; -\frac{K}{2(J-1)} \cdot \frac{\|\boldsymbol{\delta}\|^2}{c^2} \;=\; -\Omega(K) \cdot \text{deviation}^2.$$

(d) Network scaling *(Theorem 7.2). Unnormalized aggregate:*

$$G(J) = J^{1/\rho} \cdot c.$$

The exponent $1/\rho$ is monotone in K: linear at $K = 0$, Metcalfe at $K = (J-1)/(2J)$, super-Metcalfe for larger K.

All four properties tighten monotonically in K.

8.1 The Geometric Intuition

The four properties share a common origin: **the isoquant is not flat.**

$\rho < 1$ is precisely the condition for non-flatness. $K = (1-\rho)(J-1)/J$ is precisely the degree of non-flatness. Everything else is commentary.

Consider the unit isoquant $\mathcal{I}_1 = \{F = 1\}$ in $\mathbb{R}_+^J$.

For linear aggregation ($\rho = 1$, $K = 0$): $\mathcal{I}_1$ is a hyperplane. Convex combinations of points on $\mathcal{I}_1$ stay on $\mathcal{I}_1$. Correlated inputs project to the same output region. Coalitions can freely redistribute along the flat surface. All four properties vanish: gap $= 0$, curvature bonus $= 0$, manipulation penalty $= 0$, scaling exponent $= 1$ (linear).

For CES with $\rho < 1$ ($K > 0$): $\mathcal{I}_1$ curves toward the origin. The curvature has four simultaneous consequences:

1. **Superadditivity.** A chord between two points on $\mathcal{I}_1$ passes through the interior of $\{F > 1\}$. This is literally what $F(\alpha\hat{\mathbf{x}} + (1-\alpha)\hat{\mathbf{y}}) > 1$ means. The depth of penetration is $\Theta(K)$.

2. **Informational diversity.** Two inputs that are close in Euclidean distance (as when correlated) still lie on a curved surface. The curvature creates a gap between the correlated projection and the isoquant—a quadratic channel through which the aggregate extracts idiosyncratic information. The channel capacity is $\Theta(K^2)$.

3. **Strategic stability.** Moving along $\mathcal{I}_1$ away from the balanced point always moves toward the coordinate axes, where output is lower (for $\rho < 1$, the isoquant lies below the tangent hyperplane everywhere except at the tangent point). Any reallocation follows a curved path that loses altitude at rate $\Theta(K)$.

4. **Network scaling.** Adding a new symmetric component to the unnormalized aggregate increases total value by a factor $((J+1)/J)^{1/\rho}$, which exceeds $(J+1)/J$ (the linear baseline) whenever $\rho < 1$. The curvature that makes diverse inputs super-additive in production also makes each additional network participant super-linearly valuable.

8.2 Why K vs. K^2

K enters linearly in (a) and (c) because these are first-order consequences of curvature: they arise from the Hessian $\nabla^2 F$, which is $O(1-\rho) = O(K)$. K enters quadratically in (b) because the information channel is second-order: it arises from the *variance* of a Hessian quadratic form, which is $O((1-\rho)^2) = O(K^2)$. Property (d) is different: the scaling exponent $1/\rho$ depends on ρ directly (not through the Hessian), so it is a zeroth-order consequence of the functional form—but the same parameter ρ that determines isoquant curvature also determines network scaling.

This is consistent: (a) and (c) ask "how much does F change?" (first derivative of curvature). Part (b) asks "how much does F vary?" (second derivative of curvature). Part (d) asks "how does the aggregate grow with J?" (a global property of the functional form). The information channel is the square of the curvature channel; the scaling channel is the zeroth-order channel.

8.3 Relationship to Prior Results

Part (a) generalizes the folklore superadditivity result for CES (which states $F(\mathbf{x}+\mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y})$ without quantitative bounds) to a K-dependent lower bound on the gap.

Part (b) extends the variance-ratio diversification literature by providing an explicit curvature bonus formula for nonlinear aggregation, with a computable threshold $\bar{r}(J, \rho)$ beyond which CES outperforms any linear alternative.

Part (c) resolves a question in mechanism design: strategic independence under CES is not an additional assumption but a *theorem*, derivable from the same curvature parameter that controls superadditivity and correlation robustness. The connection to Shapley's [7] convex game theory provides the qualitative result; the curvature provides the quantitative bound.

9. General Weights and the Secular Equation

With unequal weights $a_j > 0$ summing to 1, the symmetric point is replaced by the cost-minimizing point, the principal curvatures of the isoquant are no longer degenerate, and the curvature parameter K acquires a weight-dispersion factor. All three results generalize.

9.1 Effective Shares and the Cost-Minimizing Point

Define the **effective shares**

$$p_j = a_j^{\sigma} = a_j^{1/(1-\rho)}, \qquad \Phi = \sum_{j=1}^{J} p_j, \tag{15}$$

and the **inverse effective shares** $w_j = 1/p_j = a_j^{-\sigma}$. For output level $c > 0$, the **cost-minimizing point** on $\mathcal{I}_c$ at unit input prices is

$$x_j^* = \frac{c\, p_j}{\Phi^{1/\rho}}, \qquad j = 1, \ldots, J. \tag{16}$$

At this point, all marginal products are equal: $\partial F/\partial x_j\big|_{\mathbf{x}^*} = \Phi^{(1-\rho)/\rho} \equiv g$ for all j.

At equal weights: $p_j = J^{-\sigma}$, $\Phi = J^{1-\sigma}$, $x_j^* = c$, $g = 1/J$.

9.2 The Hessian at General Weights

Proposition 9.1. *At the cost-minimizing point* $\mathbf{x}^*$ *with general weights:*

$$(\nabla^2 F)_{ij}\big|_{\mathbf{x}^*} = \frac{(1-\rho)\, g\, \Phi^{1/\rho}}{c} \left[\frac{p_i p_j}{\Phi^2} - \frac{\delta_{ij}\, p_j}{\Phi} \right]. \tag{17}$$

In matrix form:

$$\nabla^2 F\big|_{\mathbf{x}^*} = \frac{(1-\rho)\, g\, \Phi^{1/\rho}}{c\, \Phi} \left[\frac{\mathbf{p}\mathbf{p}^T}{\Phi} - \mathrm{diag}(\mathbf{p}) \right] \tag{18}$$

where $\mathbf{p} = (p_1, \ldots, p_J)$.

Proof. The general CES Hessian is $H_{ij} = [(1-\rho)/F]\,(\partial_i F)(\partial_j F) - \delta_{ij}\,(1-\rho)/x_j \cdot \partial_j F$. At $\mathbf{x}^*$: $\partial_j F = g$, $F = c$, $x_j = c\,p_j/\Phi^{1/\rho}$. Substituting:

$$\begin{aligned} H_{ij} &= \frac{(1-\rho)}{c}\,g^2 - \delta_{ij}\,\frac{(1-\rho)}{c\,p_j/\Phi^{1/\rho}}\,g \\ &= \frac{(1-\rho)\,g}{c}\left[g - \delta_{ij}\,\frac{\Phi^{1/\rho}}{p_j}\right] \end{aligned}$$

where $g = \Phi^{(1-\rho)/\rho}$. Writing $g = \Phi^{1/\rho} \cdot \Phi^{-1}$ and factoring gives the result. □

9.3 The Secular Equation

The principal curvatures of $\mathcal{I}_c$ at $\mathbf{x}^*$ are determined by the constrained eigenvalues of the weighted inverse-share matrix.

Proposition 9.2 (Secular equation)**.** *Let $w_j = 1/p_j = a_j^{-\sigma}$ be the ordered inverse shares with $w_{(1)} \le w_{(2)} \le \cdots \le w_{(J)}$. The principal curvatures of the CES isoquant at $\mathbf{x}^*$ are determined by the constrained eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_{J-1}$ of $W = \mathrm{diag}(w_1, \ldots, w_J)$ restricted to $\mathbf{1}^\perp$, which satisfy the **secular equation***

$$\sum_{j=1}^{J} \frac{1}{w_j - \mu} = 0. \tag{19}$$

This equation has exactly $J-1$ roots, one in each interval $(w_{(k)}, w_{(k+1)})$ for $k = 1, \ldots, J-1$.

Proof. The Hessian (18) restricted to $\mathbf{1}^\perp$ (the tangent space to the isoquant) has the form $A - \lambda_0 B$ where A is a rank-1 perturbation of the diagonal matrix $\mathrm{diag}(p_1, \ldots, p_J)$. After a change of variables $y_j = \sqrt{p_j}\,v_j$, the constrained eigenvalue problem becomes finding the roots of $\det(W - \mu I) = 0$ subject to $\mathbf{1}^T\mathbf{v} = 0$, which reduces to the secular equation (19). This is a standard result from rank-1 perturbation theory: the function $f(\mu) = \sum_j 1/(w_j - \mu)$ is a sum of J hyperbolas with poles at w_j, and between consecutive poles it decreases from $+\infty$ to $-\infty$, so it has exactly one root in each interval. □

Remark 9.3 (Interlacing)*.* The roots μ_k strictly interlace the poles $w_{(k)}$:

$$w_{(1)} < \mu_1 < w_{(2)} < \mu_2 < \cdots < w_{(J-1)} < \mu_{J-1} < w_{(J)}.$$

This ensures all principal curvatures are positive (the isoquant is strictly convex toward the origin) for all $\rho < 1$ and all weight vectors.

9.4 The Generalized Curvature Parameter

Definition 9.4. *The* ***generalized curvature parameter*** *is*

$$K(\rho, \mathbf{a}) = (1-\rho)\,\frac{J-1}{J}\,\Phi^{1/\rho}\,R_{\min} \tag{20}$$

where $R_{\min} = \mu_1$ is the smallest root of the secular equation (19).

Proposition 9.5. *At equal weights ($a_j = 1/J$): $w_j = J^\sigma$ for all j, $R_{\min} = J^\sigma$, $\Phi^{1/\rho} = J^{-\sigma}$, and $K(\rho, \mathbf{a})$ reduces to $(1-\rho)(J-1)/J$.*

Proof. At equal weights, all w_j are equal, so the secular equation has $J-1$ roots all equal to $w = J^\sigma$. Then $K = (1-\rho)(J-1)/J \cdot J^{-\sigma} \cdot J^\sigma = (1-\rho)(J-1)/J$. □

9.5 General-Weight Versions of the Four Theorems

Theorem 9.6 (General-weight Quadruple Role)**.** *Let F be a CES aggregate with weights $\mathbf{a}$ and generalized curvature parameter $K(\rho, \mathbf{a})$ from (20). Then:*

(a) Superadditivity. *The qualitative result $F(\mathbf{x}+\mathbf{y}) \geq F(\mathbf{x}) + F(\mathbf{y})$ holds for all weight vectors (from concavity and homogeneity alone). The quantitative gap bound generalizes with $K(\rho, \mathbf{a})$ replacing K and the minimum curvature $\kappa_{\min}$ replacing κ^*.*

(b) Correlation robustness. *With heterogeneous variances $\mathrm{Var}[X_j] = \tau_j^2$ calibrated to the effective shares, the curvature bonus is bounded below by a term proportional to $K(\rho, \mathbf{a})^2$. The secular roots determine how the bonus distributes across the $J-1$ idiosyncratic modes: the mode corresponding to μ_k contributes proportionally to $(1-\rho)^2/\mu_k$.*

(c) Strategic independence. *For a coalition S with $|S| = k$, the manipulation penalty involves the* ***coalition curvature parameter***

$$K_S = (1-\rho)\,\frac{k-1}{k}\,\Phi_S^{1/\rho}\,R_{\min,S} \tag{21}$$

where $R_{\min,S}$ is the smallest root of the secular equation restricted to S. By the interlacing property, $K_S > 0$ for all coalitions of size $k \geq 2$, all $\rho < 1$, and all weight vectors.

(d) Network scaling. *The unnormalized aggregate $G(\mathbf{x}) = (\sum a_j x_j^\rho)^{1/\rho}$ at the cost-minimizing point satisfies $G = \Phi^{1/\rho} \cdot c$, where $\Phi = \sum p_j$ depends on the weight vector. For J equal-weight components, this reduces to $G = J^{1/\rho} \cdot c$ as in Theorem 7.2. For general weights, the effective network size is $\Phi^{1/\rho}$, which accounts for weight heterogeneity.*

Proof. (a) follows from the same concavity + homogeneity argument as in equal-weight case; the quantitative bound uses $\kappa_{\min}$ from Proposition 9.2.

(b) The expansion of $F(\mathbf{X})$ around $\mathbf{x}^*$ uses the general Hessian (18). The spectral decomposition of the idiosyncratic modes now uses the eigenvectors of the secular equation, which are no longer degenerate. Each mode k contributes $\mathrm{Var}[Y_{2,k}] \propto (1-\rho)^2/\mu_k^2$. Summing over modes and taking the minimum gives the $K(\rho, \mathbf{a})^2$ bound.

(c) The constrained Rayleigh quotient restricted to S yields

$$\boldsymbol{\delta}_S^T H_{SS}\, \boldsymbol{\delta}_S \leq -\frac{(1-\rho)\, g\, \Phi^{1/\rho}}{c}\, R_{\min,S}\, \|\boldsymbol{\delta}_S\|^2$$

from the spectral bound of $W_S = \mathrm{diag}(w_j)_{j \in S}$ restricted to $\mathbf{1}_S^\perp$, where $R_{\min,S}$ is the smallest root of the secular equation restricted to S.

(d) At the cost-minimizing point, $G = (\sum a_j (x_j^*)^\rho)^{1/\rho} = (\sum a_j (c\, p_j/\Phi^{1/\rho})^\rho)^{1/\rho} = \Phi^{1/\rho} \cdot c$ by direct substitution of (16). □

Remark 9.7 (The secular equation in applied work). For applied economists using CES with calibrated weights (trade models, IO, macro with heterogeneous firms), the secular equation provides a direct route to the curvature parameter. Given weight vector $\mathbf{a}$: compute the inverse shares $w_j = a_j^{-\sigma}$, find the smallest root μ_1 of $\sum 1/(w_j - \mu) = 0$ numerically, and evaluate $K(\rho, \mathbf{a})$. This K then enters all four bounds. The computation is $O(J)$ per root-finding iteration and is numerically stable because the secular function is a sum of hyperbolas with explicit poles.

10. Discussion

10.1 Tightness

All three bounds become equalities in limit cases:

- (a): Equality when $\hat{\mathbf{x}} = \hat{\mathbf{y}}$ (proportional inputs); the gap vanishes for zero diversity.
- (b): Curvature bonus $\to 0$ as $\rho \to 1$ ($K \to 0$) or $r \to 1$ (perfect correlation); the CES aggregate degenerates to a linear aggregate and loses its informational advantage.
- (c): Manipulation penalty $\to 0$ as $K \to 0$ (perfect substitutes allow free redistribution) or $k/J \to 0$ (small coalitions have negligible impact).

The bounds are local (valid near the symmetric/cost-minimizing point). Global bounds require additional assumptions on the curvature behavior away from the symmetric point; for $\rho < 0$, the curvature increases away from the symmetric point, so the local bounds are conservative.

10.2 Sufficiency of J

The qualitative results (a) and (c) hold for all $J \geq 2$. The quantitative result (b) requires J large enough that the curvature bonus exceeds the correlation penalty; specifically, $J \geq 2/(K^2\gamma^2)$ suffices for the threshold $\bar{r}$ to meaningfully exceed $1/(J-1)$. For applications with many components (diversified portfolios, large supply chains, broad indices), the condition is easily satisfied.

10.3 Connection to Other CES Results

The CES aggregate appears in several literatures where the quadruple role is economically relevant:

International trade. The Dixit-Stiglitz [2] formulation uses CES to aggregate differentiated varieties. Superadditivity (a) implies that gains from trade are largest when trading partners have the most diverse production profiles. The curvature parameter K quantifies these gains.

Directed technical change. Jones [6] studies how the elasticity of substitution determines the direction of technical change. Part (b) implies that CES economies with lower σ (higher K) are more robust to correlated technology shocks—the aggregate is less sensitive to common-factor variation.

Market power. Part (c) provides a formal foundation for why markets with complementary products resist monopolization more effectively than markets with substitute products. The penalty for manipulation grows monotonically with K.

Index construction. The Fisher, Törnqvist, and CES price indices all embed CES-type aggregation. Part (b) implies that CES indices with lower σ are more informationally efficient—they better represent the underlying distribution even when prices are correlated.

Platform economics. Part (d) provides a production-theoretic foundation for network effects. The empirical scaling literature distinguishes Sarnoff (linear), Metcalfe (quadratic), and Reed (exponential) networks but lacks a unified parameter. The CES exponent $1/\rho$ fills this gap: a platform's network scaling regime is determined by the complementarity among its participants, which is the same parameter that controls all three static properties (a)–(c).

10.4 What This Paper Does Not Cover

This paper proves static properties of a single CES aggregate.[1] It says nothing about:

[1] The quadruple role is not four properties of an assumed functional form: CES is the unique aggregation function compatible with constant returns to scale and scale consistency (aggregation invariant to partitioning). See the companion paper [8] for the proof, which shows that CES is the fixed point of the aggregation renormalization group—the production-theory analogue of the Central Limit Theorem.

- *Dynamics across multiple levels.* How the curvature parameter governs activation thresholds and transition dynamics in a hierarchical economy is a separate question requiring dynamical systems methods.
- *Endogenous ρ.* The substitution parameter is taken as exogenous. Whether and how ρ evolves endogenously is an open question.
- *Stochastic dynamics.* The correlation robustness result (b) uses a second-order expansion around the symmetric point. The behavior under large shocks or non-Gaussian inputs is not covered.

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