Won't you bar my neighbor[hood]?

brouwer's realizability and the bar principle

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Usually, we will work implicitly with the *universal spread*, which always says "yes".

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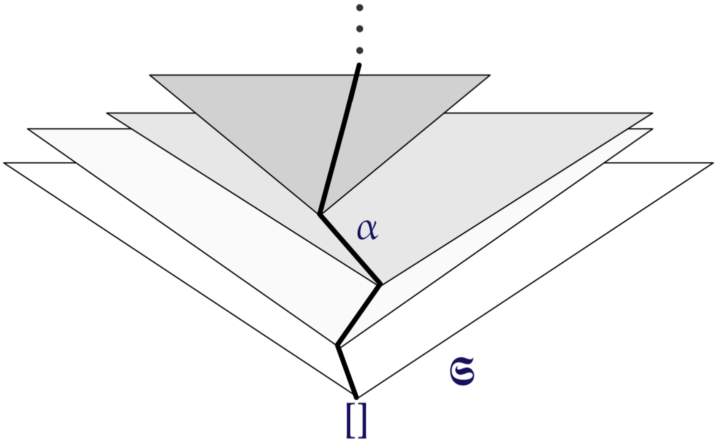
$$\vec{u} \prec \alpha \qquad (\vec{u} \text{ approximates } \alpha)$$

$$\alpha \in \vec{u} \qquad (\vec{u} \text{ is a neighborhood around } \alpha)$$

Spread Visualization



Bars and Securability

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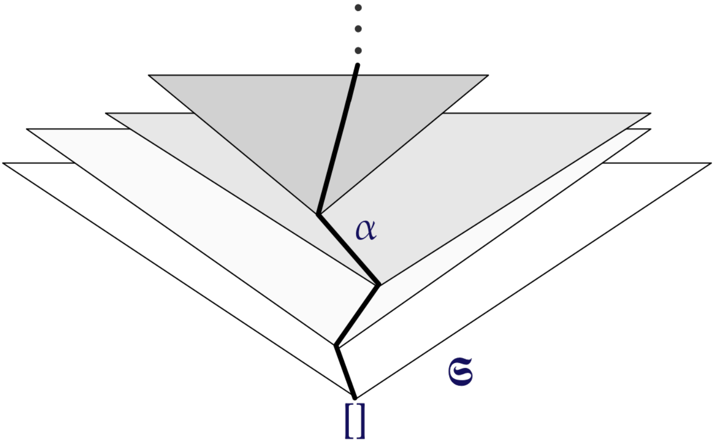
We say that a neighborhood is secured when it is in the bar, and that it is securable when every path out of it eventually hits the bar.

Visualizing Bars



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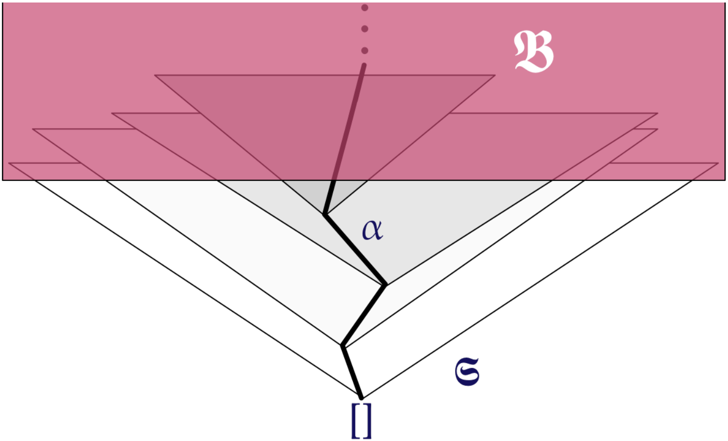

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(intuitionistic and classical, not constructive)

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2. If every immediate refinement of a neighborhood is in $\mathfrak{A}$, then that neighborhood is also in $\mathfrak{A}$.

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Then, every $\mathfrak{B}$-securable neighborhood is also in $\mathfrak{A}$. Or, equivalently:

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What is the constructive meaning of the bar principle?

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The proof of $\vec{u} \in \mathfrak{A}$ proceeds by considering the possible ways in which $\vec{u}$ could be $\mathfrak{B}$-securable:

- η ▶ $\vec{u}$ is $\mathfrak{B}$-secured.
- ζ ▶ $\vec{u} \equiv \vec{v} \frown x$ such that $\vec{v}$ is $\mathfrak{B}$-securable
- $\digamma$ ▶ For all immediate refinements x, $\vec{u} \frown x$ is $\mathfrak{B}$-securable

Normalizing securability

We have identified three different ways that $\vec{u}$ could be $\mathfrak{B}$-securable:

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Then, every η-inference is replaced with the base case, and every $\digamma$-inference is replaced with the inductive step, obtaining a proof that $\vec{u} \in \mathfrak{A}$.

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In fact, if we interpret this statement using propositions-as-types, then it is not the same! There is a procedure to convert a program realizing this statement into a well-founded $\eta, \zeta, \digamma$-tree, but to show that this procedure terminates, we need the bar induction principle already in the metatheory.

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So, what would Brouwer say to this?

Brouwerian Realizability: Neighborhood Functions

A point in the spread is an infinite stream of natural numbers. For Brouwer, a function that processes a stream is not a function from streams to results.

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Rather, it is a neighborhood function, a monotonic & continuous function from neighborhoods to partial results:

$$\left\{\phi \in \mathbb{N}^* \to (\mathbb{1} \oplus X) \mid P(\phi)\right\}$$

$$P(\phi) \equiv \forall \alpha \in \mathsf{stream}(\mathbb{N}).\ \exists k \in \mathbb{N}.\ \exists a \in X.\ \forall k' \geq k.\ \phi\,(\bar{\alpha}[k']) \equiv \mathsf{inr}(a)$$

Well-Ordering Neighborhood Functions

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The condition P on a neighborhood function ϕ induces a well-ordering on ϕ's graph. That is, each such function can be identified with some well-founded dialogue tree.

Neighborhood Functions for Securability

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As far as Brouwer is concerned, the evidence for this statement is a dependent neighborhood function:

$$\prod_{\vec{v} \succcurlyeq \vec{u}} \mathbb{1} \oplus \sum_{n \in \mathbb{N}} \vec{v}[|\vec{u}| + n] \in \mathfrak{B}$$

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Therefore, the Bar Principle is true under a Brouwerian explanation of the logical connectives!

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- There are models of type theory that refute the Bar Principle: recursive realizability (via Church's Thesis), etc.
- The Bar Principle relies on an open-ended notion of stream, i.e. one which does not require that all streams be computed by a recursive function.
- Adding the Bar Principle to Type Theory is harmless, but restricts the possible models.

Concluding Thoughts

To add the Bar Principle as an axiom to Type Theory is to formalize our intention that Type Theory shall be a semi-formal theory of constructions for Brouwer's mathematics.