Mathematics. — *Demonstration that the concept of spreads of higher order does not come into consideration as a fundemental notion in intuitionistic mathematics.* By Prof. L. E. J. BROUWER[*]

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In my note, *"Towards the free development of spreads and functions"*,[1] the process M_σ was considered, through which the fundamental sequence F', which is enumerated in an arbitrary, predetermined way, is associated one-to-one with finite choice sequences of numbers and likewise an arbitrary element σ of the spread[2] M. We want to call this process M_σ a spread *of second order*, and the successive sequences of figures thus associated to the unrestricted choice sequences of numbers [we shall call] *the elements of the second-order spread M_σ.*

The assertion stated in my quoted note, that M_σ acts as a subspecies of a spread M_1 which is derivable from M, and that the union of all M_σ generated from M is identical with this M_1, shall be demonstrated as follows. First, we will deal with the construction of the spread M_1.

Let $\alpha_1 \, \alpha_2 \, \ldots \, \alpha_m$ be a finite choice sequences of numbers. We will notate the rank thereof in the fundamental sequence F' with $\varrho \, (\alpha_1 \, \alpha_2 \, \ldots \, \alpha_m)$ and the maximum of the numbers $\varrho \, (\alpha_1)$, $\varrho \, (\alpha_1 \, \alpha_2)$, $\ldots$ $\varrho \, (\alpha_1 \, \alpha_2 \, \ldots \, \alpha_m)$ with $\zeta \, (\alpha_1 \, \alpha_2 \, \ldots \, \alpha_m)$.

We will call the combination of an arbitrary number α_1 with $\varrho \, (\alpha_1)$ arbitrary numbers β_1, β_2, $\ldots$, $\beta_{\varrho(\alpha_1)}$ a K-combination. We enumerate the K-combinations through a fundamental sequence F. We notate any K-combination which receives the rank ν_1 in F with K_{ν_1}.

For a given ν_1, and thence also a given α_1, and arbitrary α_2, we call the number α_2 a K_{ν_1}-combination in case $\zeta \, (\alpha_1 \, \alpha_2) = \zeta \, (\alpha_1)$, and likewise, in case $\zeta \, (\alpha_1 \, \alpha_2) > \zeta \, (\alpha_1)$, we call the combination of α_2 with $\zeta \, (\alpha_1 \, \alpha_2) - \zeta \, (\alpha_1)$ arbitrary numbers $\beta_{\zeta(\alpha_1)+1}, \ldots \beta_{\zeta(\alpha_1 \, \alpha_2)}$ a K_{ν_1}-combination. For each ν_1 we enumerate the K_{ν_1}-combinations through a fundamental sequence F_{ν_1}. We notate each K_{ν_1}-combination, which receives rank ν_2 in F_{ν_1}, with $K_{\nu_1 \, \nu_2}$.

For arbitrary ν_1 and ν_2, and thence also given α_1 and α_2, and arbitrary α_3, we call the number α_3 a $K_{\nu_1 \, \nu_2}$-combination in case $\zeta \, (\alpha_1 \, \alpha_2 \, \alpha_3) = \zeta \, (\alpha_1 \, \alpha_2)$, and, in case $\zeta \, (\alpha_1 \, \alpha_2 \, \alpha_3) > \zeta \, (\alpha_1 \, \alpha_2)$, we likewise call the combination of α_3 with $\zeta \, (\alpha_1 \, \alpha_2 \, \alpha_3) - \zeta \, (\alpha_1 \, \alpha_2)$ arbitrary numbers $\beta_{\zeta(\alpha_1 \, \alpha_2)+1}, \ldots \beta_{\zeta(\alpha_1 \, \alpha_2 \, \alpha_3)}$ a $K_{\nu_1 \, \nu_2}$-combination. For

each pair of numbers ν_1, ν_2 we enumerate the $K_{\nu_1 \nu_2}$-combinations through a fundamental sequence $F_{\nu_1 \nu_2}$. We notate each $K_{\nu_1 \nu_2}$ combination which receives the rank ν_3 in $F_{\nu_1 \nu_2}$ with $K_{\nu_1 \nu_2 \nu_3}$.

Proceeding in this way, we define $K_{\nu_1 \nu_2 \ldots \nu_s}$ for each natural number s. In doing so, we take care from the outset to determine a law through which the fundamental sequences $F_{\nu_1 \nu_2 \ldots \nu_s}$, each enumerating $K_{\nu_1 \nu_2 \ldots \nu_s}$, shall be defined once and for all.

The construction of M_1 will now be now be carried out, in which we shall associate each sequence of figures with the finite sequence $\nu_1 \nu_2 \ldots \nu_s$, which is associated to the finite choice sequence $\beta_1 \beta_2 \ldots \beta_{\varrho(\alpha_1 \alpha_2 \ldots \alpha_s)}$ for the respective numbers $\alpha_1 \alpha_2 \ldots \alpha_s$, $\beta_1 \beta_2 \ldots \beta_{\varrho(\alpha_1 \alpha_2 \ldots \alpha_s)}$ in M.

Let σ be the element of M generated by the infinite choice sequence $\gamma_1 \gamma_2 \gamma_3 \ldots$. *Then, the second-order spread M_σ is identical with a subspecies $_\sigma M_1$ of M_1.* This $_\sigma M_1$ arises when for each s in M_1 only such ν_s may be chosen to which $K_{\nu_1 \nu_2 \ldots \nu_{s-1}}$-combinations correspond, in which each β_τ is equal to the γ_τ which carries the same index.

Conversely, let e be an arbitrary element of M_1. Then the unrestricted choice sequence $\nu_1 \nu_2 \nu_3 \ldots$ which generates e in M_1 is defined following the aforementioned definition of ν_s simultaneously with an unrestricted sequence of numbers $\beta_1 \beta_2 \beta_3 \ldots$, which in turn generates the element $\sigma(e)$ in M. The associated $M_{\sigma(e)}$ receives e as an element. *Consequently, M_1 is identical to the union of all M_σ generated from M.*

From the above, it follows that the concept of spreads of second order does not come into consideration as a fundamental notion in intuitionistic mathematics.

In order to define the concept of *spreads of higher order*, we accept that the concept of *spreads of nth order* may be defined already, and consider the process $\left(M^{(n)}\right)_\sigma$, through which the fundamental sequence F', which is enumerated in an arbitrary and predetermined way, will be associated one-to-one with finite choice sequences and likewise an element σ an element σ of the nth-order spread $M^{(n)}$. We will call this process $\left(M^{(n)}\right)_\sigma$ an *$(n+1)$th-order spread*, and we will call the successive sequences of figures which are thus associated to the unrestricted choice sequences of numbers *the elements of the $(n+1)$th-order spread $\left(M^{(n)}\right)_\sigma$*.

However, now a spread of second order M_σ is a subspecies of a spread M which takes M_σ as its basis, and is derivable from the spread M_1, i.e. an arbitrary element π of $M\sigma$ is simultaneously an element of M_1. Hence, it follows that the third-order spread $(M_\sigma)_\pi$ is identical with the second-order spread $(M_1)_\pi$, i.e. an arbitrary third-order spread is identical with a second-order spread. And hence it follows

moreover that also for arbitrary n an arbitrary nth-order spread is identical with a second-order spread.

Consequently, it emerges that also the concept of higher-order spreads, in contrast to the concept of higher-order species, does not come into consideration as a fundamental notion in intuitionistic mathematics.