Mathematics. — *Demonstration that the concept of spreads of higher order does not come into consideration as a fundemental notion in intuitionistic mathematics.* By Prof. L. E. J. Brouwer*

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In my note, *"Concerning the free development of spreads and functions"*,[1] the process M_σ was considered, through which the fundamental sequence F', which is enumerated in an arbitrary, predetermined way, is associated one-to-one with finite choice sequences of numbers and likewise an arbitrary element σ of the spread[2] M. We want to call this process M_σ a spread *of second order*, and the successive sequences of figures thus associated to the unrestricted choice sequences of numbers [we shall call] *the elements of the second-order spread M_σ*.

The assertion stated in my quoted note, that M_σ acts as a subspecies of a spread M_1 which is derivable from M, and that the union of all M_σ generated from M is identical with this M_1, shall be demonstrated as follows. First, we will deal with the construction of the spread M_1.

Let $\alpha_1\,\alpha_2\,\ldots\,\alpha_m$ be a finite choice sequences of numbers. We will notate the rank thereof in the fundamental sequence F' with $\varrho\,(\alpha_1\,\alpha_2\,\ldots\,\alpha_m)$ and the maximum of the numbers $\varrho\,(\alpha_1)\,,\ \varrho\,(\alpha_1\,\alpha_2)\,,\ \ldots\ \varrho\,(\alpha_1\,\alpha_2\,\ldots\,\alpha_m)$ with $\zeta\,(\alpha_1\,\alpha_2\,\ldots\,\alpha_m)$.

We will call the combination of an arbitrary number α_1 with $\varrho\,(\alpha_1)$ arbitrary numbers $\beta_1,\,\beta_2,\,\ldots,\,\beta_{\varrho(\alpha_1)}$ a K-combination. We enumerate the K-combinations through a fundamental sequence F. We notate any K-combination which receives the rank ν_1 in F with K_{ν_1}.

For a given ν_1, and thence also a given α_1, and arbitrary α_2, we call the number α_2 a K_{ν_1}-combination in case $\zeta\,(\alpha_1\,\alpha_2)\,=\,\zeta\,(\alpha_1)$, and likewise, in case $\zeta\,(\alpha_1\,\alpha_2)\,>\,\zeta\,(\alpha_1)$, we call the combination of α_2 with $\zeta\,(\alpha_1\,\alpha_2) - \zeta\,(\alpha_1)$ arbitrary numbers $\beta_{\zeta(\alpha_1)+1},\,\ldots\,\beta_{\zeta(\alpha_1\,\alpha_2)}$ a K_{ν_1}-combination. For each ν_1 we enumerate the K_{ν_1}-combinations through a fundamental sequence F_{ν_1}. We notate each K_{ν_1}-combination, which receives rank ν_2 in F_{ν_1}, with $K_{\nu_1\,\nu_2}$.

For arbitrary ν_1 and ν_2, and thence also given α_1 and α_2, and arbitrary α_3, we call the number α_3 a $K_{\nu_1\,\nu_2}$-combination in case $\zeta\,(\alpha_1\,\alpha_2\,\alpha_3) = \zeta\,(\alpha_1\,\alpha_2)$, and, in case $\zeta\,(\alpha_1\,\alpha_2\,\alpha_3) > \zeta\,(\alpha_1\,\alpha_2)$, we likewise call the combination of α_3 with $\zeta\,(\alpha_1\,\alpha_2\,\alpha_3) - \zeta\,(\alpha_1\,\alpha_2)$ arbitrary numbers $\beta_{\zeta(\alpha_1\,\alpha_2)+1},\,\ldots\,\beta_{\zeta(\alpha_1\,\alpha_2\,\alpha_3)}$ a $K_{\nu_1\,\nu_2}$-combination. For

* Translated from the original German by Jon Sterling.

[1] Proc. Ned. Akad. v. Wetensch. Amsterdam, **45**, 322 (1942).

[2] For the sake of simplicity, we restrict ourselves in this note to such spreads, in the process of whose creation neither inhibition nor termination occurs. This restriction is inessential.

each pair of numbers ν_1, ν_2 we enumerate the $K_{\nu_1\,\nu_2}$-combinations through a fundamental sequence $F_{\nu_1\,\nu_2}$. We notate each $K_{\nu_1\,\nu_2}$ combination which receives the rank ν_3 in $F_{\nu_1\,\nu_2}$ with $K_{\nu_1\,\nu_2\,\nu_3}$.

Proceeding in this way, we define $K_{\nu_1\,\nu_2\,\ldots\,\nu_s}$ for each natural number s. In doing so, we take care from the outset to determine a law through which the fundamental sequences $F_{\nu_1\,\nu_2\,\ldots\,\nu_s}$, each enumerating $K_{\nu_1\,\nu_2\,\ldots\,\nu_s}$, shall be defined once and for all.

The construction of M_1 will now be now be carried out, in which we shall associate each sequence of figures with the finite sequence $\nu_1\,\nu_2\,\ldots\,\nu_s$, which is associated to the finite choice sequence $\beta_1\,\beta_2\,\ldots\,\beta_{\varrho(\alpha_1\,\alpha_2\,\ldots\,\alpha_s)}$ for the respective numbers $\alpha_1\,\alpha_2\,\ldots\,\alpha_s,\ \beta_1\,\beta_2\,\ldots\,\beta_{\varrho(\alpha_1\,\alpha_2\,\ldots\,\alpha_s)}$ in M.

Let σ be the element of M generated by the infinite choice sequence $\gamma_1\,\gamma_2\,\gamma_3\,\ldots$. *Then, the second-order spread M_σ is identical with a subspecies ${}_\sigma M_1$ of M_1.* This ${}_\sigma M_1$ arises when for each s in M_1 only such ν_s may be chosen to which $K_{\nu_1\,\nu_2\,\ldots\,\nu_{s-1}}$-combinations correspond, in which each β_τ is equal to the γ_τ which carries the same index.

Conversely, let e be an arbitrary element of M_1. Then the unrestricted choice sequence $\nu_1\,\nu_2\,\nu_3\,\ldots$ which generates e in M_1 is defined following the aforementioned definition of ν_s simultaneously with an unrestricted sequence of numbers $\beta_1\,\beta_2\,\beta_3\,\ldots$, which in turn generates the element $\sigma(e)$ in M. The associated $M_{\sigma(e)}$ receives e as an element. *Consequently, M_1 is identical to the union of all M_σ generated from M.*

From the above, it follows that the notion of spreads of second order does not come into consideration as a fundamental notion in intuitionistic mathematics.